Böhm, J. and Papadakis, S.A.
Osaka J. Math.
49 (2012), 81–100

ON THE STRUCTURE OF STANLEY–REISNER RINGS ASSOCIATED TO CYCLIC POLYTOPES

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(Received December 18, 2009, revised August 9, 2010)

Abstract

We study the structure of Stanley–Reisner rings associated to cyclic polytopes, using ideas from unprojection theory. Consider the boundary simplicial complex $\Delta(d, m)$ of the d-dimensional cyclic polytope with m vertices. We show how to express the Stanley–Reisner ring of $\Delta(d, m+1)$ in terms of the Stanley–Reisner rings of $\Delta(d,m)$ and $\Delta(d-2,m-1)$. As an application, we use the Kustin–Miller complex construction to identify the minimal graded free resolutions of these rings. In particular, we recover results of Schenzel, Terai and Hibi about their graded Betti numbers.

1. Introduction

Gorenstein commutative rings form an important class of commutative rings. For example, they appear in algebraic geometry as canonical rings of regular surfaces and anticanonical rings of Fano n-folds and in algebraic combinatorics as Stanley–Reisner rings of sphere triangulations. In codimensions 1 and 2 they are complete intersections and in codimension 3 they are Pfaffians [2], but, to our knowledge, no structure theorems are known for higher codimensions.

Unprojection theory [11], which analyzes and constructs complicated commutative rings in terms of simpler ones, began with the aim of partly filling this gap. The first kind of unprojection which appeared in the literature is that of type Kustin–Miller, studied originally by Kustin and Miller [8] and later by Reid and the second author [9, 10]. Starting from a codimension 1 ideal J of a Gorenstein ring R such that the quotient R/J is Gorenstein, Kustin–Miller unprojection uses the information contained in $\mathrm{Hom}_R(J, R)$ to construct a new Gorenstein ring S which is birational to R and corresponds to the contraction of $V(J) \subset \mathrm{Spec}\, R$. See Subsection 2.2 for a precise definition of Kustin–Miller unprojection and the introduction of [3] for references to applications.

In the paper [3], the authors proved that on the algebraic level of Stanley–Reisner rings, stellar subdivisions of Gorenstein* simplicial complexes correspond to Kustin–Miller unprojections and gave applications to Stanley–Reisner rings associated to stacked polytopes. In the present paper, we use unprojection theory to study the structure of

2010 Mathematics Subject Classification. Primary 13F55; Secondary 13H10, 13D25, 05E99.

J.B. supported by DFG (German Research Foundation) through Grant BO3330/1-1. S.P. was supported by the Portuguese Fundação para a Ciência e a Tecnologia through Grant SFRH/BPD/22846/2005 of POCI2010/FEDER and through Project PTDC/MAT/099275/2008.

Stanley–Reisner rings associated to cyclic polytopes. This setting is different from the one studied in [3] since here, except for some easy subcases, stellar subdivisions do not appear and the unprojection ideals are more complicated.

Our main result, which is stated precisely in Theorems 3.3 and 4.4, can be described as follows. Assume $d \geq 4$ and $d+1 < m$. Consider the cyclic polytope which has m vertices and dimension d, and denote by $\Delta(d, m)$ its boundary simplicial complex. We show how to express the Stanley–Reisner ring of $\Delta(d, m+1)$ in terms of the Stanley–Reisner rings of $\Delta(d, m)$ and $\Delta(d-2, m-1)$ via Kustin–Miller unprojection. Moreover, a similar result is also true for the remaining cases $d = 2, 3$ and $m = d+1$, see Subsections 3.1, 3.2, 4.1 and 4.2. In Section 5 we give a combinatorial interpretation of our construction.

As an application, in Section 6 we inductively identify the minimal graded free resolutions of the Stanley–Reisner rings $k[\Delta(d, m)]$. We use this identification in Proposition 6.6 to calculate the graded Betti numbers of these rings, recovering results originally due to Schenzel [12] for d even and Terai and Hibi [13] for d odd. Our derivation is more algebraic than the one in [13], and does not use Hochster's formula or Alexander duality. Finally, Subsection 6.2 contains examples and a link to related computer algebra code.

An interesting open question is whether there are other families of Gorenstein Stanley–Reisner rings related by unprojections in a similar way as cyclic polytopes, compare also the discussion in [3, Section 6].

2. Preliminaries

Assume k is a field, and m a positive integer. An (abstract) simplicial complex on the vertex set $\{1, \ldots, m\}$ is a collection Δ of subsets of $\{1, \ldots, m\}$ such that (i) all singletons $\{i\}$ with $i \in \{1, \ldots, m\}$ belong to Δ and (ii) $\sigma \subset \tau \in \Delta$ implies $\sigma \in \Delta$. The elements of Δ are called *faces* and those maximal with respect to inclusion are called *facets*. The dimension of a face σ is defined as one less than the cardinality of σ. The dimension of Δ is the maximum dimension of a face. Any abstract simplicial complex Δ has a geometric realization, which is unique up to linear homeomorphism.

For any subset W of $\{1, \ldots, m\}$, we denote by x_W the square-free monomial in the polynomial ring $k[x_1, \ldots, x_m]$ with support W, in other words x_W is the product of x_t for $t \in W$. The ideal I_Δ of $k[x_1, \ldots, x_m]$ which is generated by the square-free monomials x_W with $W \notin \Delta$ is called the *Stanley–Reisner ideal* of Δ. The *face ring*, or *Stanley–Reisner ring*, of Δ over k, denoted $k[\Delta]$, is defined as the quotient ring of $k[x_1, \ldots, x_m]$ by the ideal I_Δ.

Assume $R = k[x_1, \ldots, x_m]$ is a polynomial ring over a field k with the degrees of all variables x_i positive, and denote by $m = (x_1, \ldots, x_m)$ the maximal homogeneous ideal of R. Assume M is a finitely generated graded R-module. Denote by

$$0 \to F_g \to F_{g-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

the minimal graded free resolution of M as R-module, and write

$$F_i = \bigoplus_j R(-j)^{b_{ij}}.$$

The integer b_{ij} is called the *ij-th graded Betti number* of M and is also denoted by $b_{ij}(M)$. For fixed i we set $b_i(M) = \sum_j b_{ij}(M)$. The integer $b_i(M)$ is the rank of the free R-module F_i in the category of (ungraded) R-modules, and

$$b_i(M) = \dim_{R/m} \operatorname{Tor}_i^R(R/m, M), \tag{2.1}$$

cf. [7, Proposition 1.7]. For more details about free resolutions and Betti numbers see, for example, [6, Sections 19, 20].

Assume R is a ring. An element $r \in R$ will be called *R-regular* if the multiplication by r map $R \to R$, $u \mapsto ru$ is injective. A sequence $r_1, \ldots, r_n$ of elements of R will be called a *regular R-sequence* if r_1 is R-regular, and, for $2 \le i \le n$, we have that r_i is $R/(r_1, \ldots, r_{i-1})$-regular.

Assume k is a field, and a, m, n three positive integers with $m < n$ and $2a \le n - m + 2$. We define the ideal $I_{a,m,n} \subset k[x_m, x_{m+1}, \ldots, x_n]$ by

$$I_{a,m,n} = (x_{t_1} x_{t_2} \cdots x_{t_a} \mid m \le t_1,\ t_a \le n,\ t_j + 2 \le t_{j+1} \text{ for } 1 \le j \le a-1).$$

The assumption $2a \le n-m+2$ implies that there exists at least one monomial generator of $I_{a,m,n}$, namely $x_m x_{m+2} \cdots x_{m+2(a-1)}$. For example, we have $I_{2,3,6} = (x_3x_5, x_3x_6, x_4x_6)$.

2.1. Cyclic polytopes. Recall from [1, Section 5.2] the definition of cyclic polytopes. We fix two integers m, d, with $2 \le d < m$, and define the cyclic polytope $C_d(m) \subset \mathbb{R}^d$ as follows: Fix, for $1 \le i \le m$, $t_i \in \mathbb{R}$ with $t_1 < t_2 < \cdots < t_m$. By definition, the cyclic polytope $C_d(m) = C_d(t_1, \ldots, t_m)$ is the convex hull in $\mathbb{R}^d$ of the subset $\{f(t_1), f(t_2), \ldots, f(t_m)\} \subset \mathbb{R}^d$, where $f\colon \mathbb{R} \to \mathbb{R}^d$ with $f(t) = (t, t^2, \ldots, t^d)$ for $t \in \mathbb{R}$. We have that $C_d(m)$ is a simplicial d-polytope, which up to combinatorial equivalence does not depend on the choice of the points t_i. We denote by $\Delta(d, m)$ the boundary simplicial complex of $C_d(m)$, by definition $\Delta(d, m)$ has as elements the empty set and the sets of vertices of the proper faces of $C_d(m)$, cf. [1, Corollary 5.2.7].

Assume $W \subset \{1, \ldots, m\}$ is a proper nonempty subset. A nonempty subset $X \subset W$ is called contiguous if there exist i, j with $2 \le i \le j \le m-1$ such that $i - 1 \notin W$, $j + 1 \notin W$, $X = \{i, i+1, \ldots, j\}$. A contiguous $X \subset W$ is called odd contiguous if $\#X$ is odd. Assume W contains a contiguous subset, this is equivalent to the existence of $a \in W$ and $b_1, b_2 \in \{1, \ldots, m\} \setminus W$ with $b_1 < a < b_2$. Then, there exist a unique integer $t \ge 1$ and a unique decomposition

$$W = Y_1 \cup X_1 \cup X_2 \cup \cdots \cup X_t \cup Y_2,$$

such that Y_1 is either empty or of the form $\{1, 2, \ldots, i\}$ for some $i \geq 1$ with $i+1 \notin W$, Y_2 is either empty or of the form $\{j, j+1, \ldots, m\}$ for some $j \leq m$ with $j-1 \notin W$, each X_p, for $1 \leq p \leq t$, is a contiguous subset of W, and for $p_1 < p_2$ each element of X_{p_1} is strictly smaller than any element of X_{p_2}.

For a real number r we denote by $[r]$ the integral value of r, i.e., the largest integer which is smaller or equal than r. The following theorem characterizing the faces of $\Delta(d, m)$ is proven in [1, Theorem 5.2.13], compare also [13, Lemma 2.2].

Theorem 2.1. *Assume $W \subset \{1, \ldots, m\}$ is a nonempty subset with $\#W \leq d$. W is a face of $\Delta(d, m)$ if and only if the number of odd contiguous subsets of W is at most $d - \#W$. In particular, if $\#W \leq [d/2]$ then W is a face of $\Delta(d, m)$.*

2.2. Kustin–Miller unprojection. We recall the definition of Kustin–Miller unprojection from [10]. Assume R is a local (or graded) Gorenstein ring, and $J \subset R$ a codimension 1 ideal with R/J Gorenstein. Fix $\phi \in \mathrm{Hom}_R(J, R)$ such that $\mathrm{Hom}_R(J, R)$ is generated as an R-module by the subset $\{i, \phi\}$, where i denotes the inclusion morphism. The *Kustin–Miller unprojection ring* S of the pair $J \subset R$ is the quotient ring

$$S = \frac{R[T]}{(Tu - \phi(u) \mid u \in J)},$$

where T is a new variable. The ring S is, up to isomorphism, independent of the choice of ϕ. The original definition of Kustin and Miller [8] was using projective resolutions, compare Subsection 2.3 below.

2.3. The Kustin–Miller complex construction. The following construction, which is due to Kustin and Miller [8], will be important in Section 6, where we identify the minimal graded free resolution of $k[\Delta(d, m)]$.

Assume R is a polynomial ring over a field with the degrees of all variables positive, and $I \subset J \subset R$ are two homogeneous ideals of R such that both quotient rings R/I and R/J are Gorenstein and $\dim R/J = \dim R/I - 1$. We define $k_1, k_2 \in \mathbb{Z}$ such that $\omega_{R/I} = R/I(k_1)$ and $\omega_{R/J} = R/J(k_2)$, compare [1, Proposition 3.6.11], and assume that $k_1 > k_2$. We fix a graded homomorphism $\phi \in \mathrm{Hom}_{R/I}(J, R/I)$ of degree $k_1 - k_2$ such that $\mathrm{Hom}_{R/I}(J, R/I)$ is generated as an R/I-module by the subset $\{i, \phi\}$, where i denotes the inclusion morphism, compare Subsection 2.2. We denote by $S = R[T]/Q$ the Kustin–Miller unprojection ring of the pair $J \subset R/I$ defined by ϕ, where T is a new variable of degree $k_1 - k_2$. We have that $Q = (I, Tu - \phi(u) \mid u \in J)$ and that S is a graded algebra.

We denote by $g = \dim R - \dim R/J$ the codimension of the ideal J of R. Let

$$C_J : 0 \to R = A_g \to A_{g-1} \to \cdots \to A_1 \to R = A_0$$

and

$$C_I : 0 \to R = B_{g-1} \to \cdots \to B_1 \to R = B_0$$

be the minimal graded free resolutions of R/J and R/I respectively as R-modules. Due to the Gorensteiness of R/J and R/I they are both self-dual. We denote by $a_i\colon A_i \to A_{i-1}$ and $b_j\colon B_j \to B_{j-1}$ the differential maps. In the following, for an R-module M we denoted by M' the $R[T]$-module $M \otimes_R R[T]$.

Kustin and Miller constructed in [8] a graded free resolution C_S of S as $R[T]$-module of the form

$$C_S\colon 0 \to F_g \to F_{g-1} \to \cdots \to F_1 \to F_0 \to S \to 0,$$

where, when $g \geq 3$,

$$\begin{aligned} &F_0 = B_0', \quad F_1 = B_1' \oplus A_1'(k_2 - k_1), \\ &F_i = B_i' \oplus A_i'(k_2 - k_1) \oplus B_{i-1}'(k_2 - k_1), \quad \text{for} \quad 2 \leq i \leq g-2, \\ &F_{g-1} = A_{g-1}'(k_2 - k_1) \oplus B_{g-2}'(k_2 - k_1), \quad F_g = B_{g-1}'(k_2 - k_1), \end{aligned}$$

cf. [8, p. 307, Equation (3)]. When $g = 2$ we have

$$F_0 = B_0', \quad F_1 = A_1'(k_2 - k_1), \quad F_2 = B_1'(k_2 - k_1).$$

We will now describe the differentials of the complex C_S. We denote the rank of the free R-module A_1 by t_1, since C_J is self-dual t_1 is also the rank of the free R-module A_{g-1}. We fix R-module bases $e_1, \ldots, e_{t_1}$ of A_1 and $\hat{e}_1, \ldots, \hat{e}_{t_1}$ of A_{g-1}. We define, for $1 \leq i \leq t_1$, $c_i, \hat{c}_i \in R$ by $a_1(e_i) = c_i 1_R$ and $a_g(1_R) = \sum_{i=1}^{t_1} \hat{c}_i \hat{e}_i$. By Gorensteiness we have that $c_i, \hat{c}_i \in J$ for all $1 \leq i \leq t_1$. For $1 \leq i \leq t_1$, let $l_i \in R$ be a lift in R of $\phi(c_i)$ and let $\hat{l}_i \in R$ be a lift in R of $\phi(\hat{c}_i)$. For an R-module A we set $A^* = \mathrm{Hom}_R(A, R)$. For an R-basis $f_1, \ldots f_t$ of A we denote by $f_1^*, \ldots, f_t^*$ the basis of A^* dual to it.

Denote by $\tilde{\alpha}_{g-1}^d\colon A_{g-1}^* \to R = B_{g-1}^*$ the R-homomorphism with $\tilde{\alpha}_{g-1}^d(\hat{e}_i^*) = \hat{l}_i 1_R$ for $1 \leq i \leq t_1$. Taking into account the self-duality of C_I, C_J, we have that $\tilde{\alpha}_{g-1}^d$ extends to a chain map $\tilde{\alpha}^d\colon C_J^* \to C_I^*$. We denote by $\tilde{\alpha}\colon C_I \to C_J$ the chain map dual to $\tilde{\alpha}^d$. The map $\tilde{\alpha}_0\colon B_0 = R \to R = A_0$ is multiplication by an invertible element, say w, of R, cf. [9], and we set $\alpha = \tilde{\alpha}/w$.

We will now define a chain map $\beta\colon C_J \to C_I[-1]$. We first define $\beta_1\colon A_1 \to R = B_0$ by $\beta_1(e_i) = -l_i 1_R$. We obtain a chain map $\beta\colon C_J \to C_I[-1]$ by extending β_1. Moreover, $\beta_g\colon A_g = R \to R = B_{g-1}$ is multiplication by a nonzero constant $u \in R$. By [8, p. 308] there exists a homotopy map $h\colon C_I \to C_I$ with $h_0\colon B_0 \to B_0$ and $h_{g-1}\colon B_{g-1} \to B_{g-1}$ being the zero maps and

$$\beta_i \alpha_i = h_{i-1} b_i + b_i h_i,$$

for $1 \leq i \leq g$.

Finally, following [8, p. 307], we have that the differential maps $f_i\colon F_i \to F_{i-1}$ of the complex C_S are given in block format by the following formulas

$$f_1 = \begin{bmatrix} b_1 & \beta_1 + Ta_1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} b_2 & \beta_2 & h_1 + TI_1 \\ 0 & -a_2 & -\alpha_1 \end{bmatrix},$$

$$f_i = \begin{bmatrix} b_i & \beta_i & h_{i-1} + (-1)^i T I_{i-1} \\ 0 & -a_i & -\alpha_{i-1} \\ 0 & 0 & b_{i-1} \end{bmatrix} \quad \text{for} \quad 3 \le i \le g-2,$$

$$f_{g-1} = \begin{bmatrix} \beta_{g-1} & h_{g-2} + (-1)^{g-1} T I_{g-2} \\ -a_{g-1} & -\alpha_{g-2} \\ 0 & b_{g-2} \end{bmatrix},$$

$$f_g = \begin{bmatrix} -\alpha_{g-1} + (-1)^g u^{-1} T a_g \\ b_{g-1} \end{bmatrix},$$

where I_t denotes the identity rank $B_t \times$ rank B_t matrix.

The resolution C_S is, in general, not minimal [3, Example 5.2]. However, in the cases of stacked and cyclic polytopes it is minimal, see [3] and Theorem 6.1. In the following we will call C_S the *Kustin–Miller complex construction*. We refer the reader to Subsection 6.2 for explicit examples of this construction.

3. The main theorem for d even

We fix a field k, and assume that d, m are integers with d even and $2 \le d < m-1$. (The case $m = d+1$ is discussed in Subsection 3.2.) We set $a = (d+2)/2$, and denote by $k[\Delta(d, m)]$ the Stanley–Reisner ring of the simplicial complex $\Delta(d, m)$.

The following lemma is an almost immediate consequence of Theorem 2.1.

Lemma 3.1. *We have*

$$k[\Delta(d, m)] \cong k[x_1, \ldots, x_m]/(I_{a,1,m-1}, I_{a,2,m}).$$

Proof. Denote by $\mathcal{A}$ the set of minimal monomial generators of the ideal $(I_{a,1,m-1}, I_{a,2,m})$. We first show that if $x_V \in \mathcal{A}$, then V is not a face of $\Delta(d, m)$. Assume x_V is a monomial generator of $I_{a,1,m-1}$, the case x_V is a monomial generator of $I_{a,2,m}$ follows by the same arguments. Since $\#V = a$, we have that the number of odd contiguous subsets of V is at least $a-1$. Since $a - 1 = d/2 > d/2 - 1 = d - a$, by Theorem 2.1 V is not a face of $\Delta(d, m)$.

Assume now $W \subset \{1, \ldots, m\}$ is a subset with $\#W \le d$. We will show that if W is not a face of $\Delta(d, m)$ then there exists a monomial generator $x_V \in \mathcal{A}$ with $V \subset W$. By Theorem 2.1 $\#W \ge a$. We will argue by induction on the cardinality of W.

Denote by p the number of the odd contiguous subsets of W considered as a subset of $\{1, \ldots, m\}$, and, for $w \in W$, by p_w the number of the odd contiguous subsets

of $W \setminus \{w\}$ also considered as a subset of $\{1, \ldots, m\}$. By Theorem 2.1 $p > d - \#W$. If $\#W = a$, then $p > d - \#W$ implies that W has at least $d - a + 1 = a - 1 = \#W - 1$ odd contiguous subsets, and we set $V = W$.

Assume for the rest of the proof that $\#W > a$. By the inductive hypothesis it is enough to show that there exists $w \in W$ such that $W \setminus \{w\}$ is not a face of $\Delta(d, m)$. Hence, by Theorem 2.1 it is enough to show that there exists $w \in W$ with $p_w > d - \#W + 1$.

We call a nonempty $X \subset W$ a gc-subset if there exist $i \leq j$ with $i - 1 \notin W$, $j + 1 \notin W$ such that $X = \{i, i + 1, \ldots, j\}$. It is obvious that a contiguous subset of W is a gc-subset, and that a gc-subset of W is contiguous if and only if contains neither 1 nor m.

If W contains a gc-subset of even cardinality, say $\{i, i + 1, \ldots, j\}$ we set $w = m$ if $j = m$, while if $j \neq m$ we set $w = i$. In the first case, since $i = 1$ contradicts $\#W \leq d$, we have that $p_w = p + 1$, so $p_w > d - \#W + 1$ follows. Similarly, for the second case again $p_w = p + 1$ and $p_w > d - \#W + 1$ follows.

Assume for the rest of proof that all gc-subsets of W are of odd cardinality. First assume that W contains a gc-subset $\{i, i + 1, \ldots, j\}$ of odd cardinality at least 3, and set $w = i + 1$. Since $(i, j) = (1, m)$ is impossible by $\#W \leq d$, it is clear that $p_w = p + 1$, so again $p_w > d - \#W + 1$.

So we can assume for the rest of the proof that all gc-subsets of W are of cardinality 1. We either set $w = m$ if $m \in W$, or if $m \notin W$ we set w to be the smallest element of W. If $m \in W$ and $1 \in W$ we have $p_w = p = \#W - 2$, and $p > d - \#W$ implies $2\,\#W - 2 > d$, so since d is even $2\,\#W > d + 3$, hence $p_w > d - \#W + 1$. If $m \in W$ and $1 \notin W$, we have $p_w = p = \#W - 1$, and $p_w > d - \#W + 1$ is equivalent to $2\,\#W > d + 2$, which is true by the assumption $\#W > a = (d + 2)/2$. If $m \notin W$ and $1 \in W$ the argument is exactly symmetric to the case $m \in W$ and $1 \notin W$. If $m \notin W$ and $1 \notin W$, we have $p_w = p - 1 = \#W - 1$ and $p_w > d - \#W + 1$ is equivalent to $2\,\#W > d + 2$, which is true by the assumption $\#W > a = (d + 2)/2$. This finishes the proof of Lemma 3.1. □

We now further assume that d is an even integer with $d \geq 4$, the case $d = 2$ is discussed in Subsection 3.1. We set $R = k[x_1, \ldots, x_m, z]$, where we put degree 1 for all variables. We consider the ideals $I = (I_{a,1,m-1}, I_{a,2,m})$ and $J = (I_{a-1,2,m-1}, zI_{a-2,3,m-2})$ of R. (When we need to be more precise we will also use the notations $I_{d,m}$ for I and $J_{d,m}$ for J.) It is clear that $I \subset (I_{a-1,2,m-1})$, hence $I \subset J$. Moreover, using Lemma 3.1, $R/I \cong k[\Delta(d, m)][z]$ and $R/J \cong k[\Delta(d - 2, m - 1)][x_1, x_m]$. Consequently, both rings R/I and R/J are Gorenstein by [1, Corollary 5.6.5], and $\dim R/J = \dim R/I - 1$.

The proof of the following key lemma will be given in Subsection 3.3.

Lemma 3.2. *There exists unique* $\phi \in \mathrm{Hom}_{R/I}(J, R/I)$ *such that* $\phi(v) = 0$ *for all* $v \in I_{a-1,2,m-1}$ *and* $\phi(zw) = wx_1x_m$ *for all* $w \in I_{a-2,3,m-2}$. *Moreover, the* R/I*-module*

$\mathrm{Hom}_{R/I}(J, R/I)$ *is generated by the set* $\{i, \phi\}$, *where* $i\colon J \to R/I$ *denotes the inclusion homomorphism.*

Taking into account Lemma 3.2, the Kustin–Miller unprojection ring S of the pair $J \subset R/I$ is equal to

$$S = \frac{(R/I)[T]}{(Tu - \phi(u) \mid u \in J)}.$$

We extend the grading of R to a grading of S by putting the degree of the new variable T equal to 1. By Lemma 3.2 S is a graded k-algebra. Our main result for the case d even is the following theorem.

Theorem 3.3. *The element* $z \in S$ *is* S*-regular, and there is an isomorphism of graded* k*-algebras*

$$S/(z) \cong k[\Delta(d, m+1)].$$

Proof. Denote by $Q \subset R[T]$ the ideal

$$Q = (I, z) + (Tu - \phi(u) \mid u \in J) \subset R[T].$$

By the definition of S we have $S/(z) \cong R[T]/Q$. By the definition of ϕ we have $Q = (I_{a,1,m}, TI_{a-1,2,m-1}, z)$. Hence, Lemma 3.1 implies that $S/(z) \cong k[\Delta(d, m+1)]$. As a consequence, $\dim S/(z) = \dim S - 1$, and since by [10, Theorem 1.5] S is Gorenstein, hence Cohen–Macaulay, we get that z is S-regular. □

EXAMPLE 3.4. Assume $d = 4$ and $m = 6$. We have

$$I = (x_2x_4x_6, x_1x_3x_5), \quad J = (x_2x_4, x_2x_5, x_3x_5, zx_3, zx_4)$$

and

$$S = k[x_1, \ldots, x_6, T, z]/(I, Tx_2x_4, Tx_2x_5, Tx_3x_5, x_3(zT - x_1x_6), x_4(zT - x_1x_6)).$$

3.1. The case $d = 2$ and $d + 1 < m$. Assume $d = 2$ and $d + 1 < m$. It is clear that $\Delta(d, m)$ is just the (unique) triangulation of the 1-sphere S^1 having m vertices. Hence $\Delta(d, m+1)$ is a stellar subdivision of $\Delta(d, m)$, and the results of [3] apply.

In more detail, set $R = k[x_1, \ldots, x_m, z]$, with the degree of all variables equal to 1. Consider the ideals $I = (I_{2,1,m-1}, I_{2,2,m})$ and $J = (I_{1,2,m-1}, z)$ of R. (When we need to be more precise we will also use the notations $I_{2,m}$ for I and $J_{2,m}$ for J.) Clearly $k[\Delta(d, m)][z] \cong R/I$. Moreover, we have that $I \subset J$, that $J \subset R/I$ is a codimension 1 ideal of R/I with R/J Gorenstein, and that if we denote by S the Kustin–Miller unprojection ring of the pair $J \subset R/I$ we have $S/(z) \cong k[\Delta(d, m+1)]$. Moreover, arguing as in the proof of Theorem 3.3 we get that z is an S-regular element.

3.2. The case d is even and $m = d + 1$. Assume $d \geq 2$ is even and $m = d + 1$. We have that

$$k[\Delta(d, m)] \cong k[x_1, \ldots, x_m] \Big/ \left(\prod_{i=1}^{d+1} x_i \right)$$

and

$$k[\Delta(d, m+1)] \cong k[x_1, \ldots, x_{m+1}] \Big/ \left(\prod_{i=0}^{d/2} x_{2i+1}, \prod_{i=1}^{(d/2)+1} x_{2i} \right).$$

We set $R = k[x_1, \ldots, x_m, z]$, with the degree of all variables equal to 1. Consider the ideals $I = \left(\prod_{i=1}^{d+1} x_i\right)$ and $J = \left(\prod_{i=1}^{d/2} x_{2i}, z \prod_{i=1}^{(d/2)-1} x_{2i+1}\right)$ of R. (When we need to be more precise we will also use the notations $I_{d,m}$ for I and $J_{d,m}$ for J.) We have $I \subset J$, that $J \subset R/I$ is a codimension 1 ideal of R/I with R/J Gorenstein, and that if we denote by S the Kustin–Miller unprojection ring of the pair $J \subset R/I$ we have $S/(z) \cong k[\Delta(d, m+1)]$. Moreover, arguing as in the proof of Theorem 3.3 we get that z is an S-regular element.

3.3. Proof of Lemma 3.2. We start the proof of Lemma 3.2. Recall that $I = (I_{a,1,m-1}, I_{a,2,m})$ and $J = (I_{a-1,2,m-1}, zI_{a-2,3,m-2})$. Since J is a codimension 1 ideal of R/I and R/I is Gorenstein, hence Cohen–Macaulay, there exists $b \in J$ which is R/I-regular. Write $b = b_1 + zb_2$, with $b_1 \in I^e_{a-1,2,m-1}$ and $b_2 \in I^e_{a-2,3,m-2}$, where I^e_* denotes the ideal of R/I generated by I_*. Consider the element

$$s_0 = \frac{b_2 x_1 x_m}{b} \in K(R/I),$$

where $K(R/I)$ denotes the total quotient ring of R/I, that is the localization of R/I with respect to the multiplicatively closed subset of regular elements of R/I, cf. [6, p. 60]. We need the following lemma.

Lemma 3.5. (a) *We have that $x_1 x_m v w = 0$ (equality in R/I) for all $v \in I_{a-1,2,m-1}$ and $w \in I_{a-2,3,m-2}$.*
(b) *We have $s_0 z w = w x_1 x_m$ (equality in $K(R/I)$) for all $w \in I_{a-2,3,m-2}$.*

Proof. Proof of (a). It is enough to show that $x_1 x_m x_V x_W = 0$ in $k[\Delta(d, m)]$, whenever x_V is a generating monomial of $I_{a-1,2,m-1}$ and x_W is a generating monomial of $I_{a-2,3,m-2}$, with $V \subset \{2, \ldots, m-1\}$ and $W \subset \{3, \ldots, m-2\}$. Consider the set $A = \{1, m\} \cup V \cup W$. If $2 \notin V$ it is clear that $x_1 x_V = 0$ and, similarly, if $m - 1 \notin V$ we have $x_m x_V = 0$.

Hence for the rest of the proof we can assume that $2 \in V$ and $m - 1 \in V$. Denote by $A_1 = \{1, \ldots, p\}$ the initial segment of A, and by A_2 the final segment of A. Since

$2, m-1 \notin W$, we necessarily have that all odd elements of $A_1 \setminus \{1\}$ are in $W \setminus V$, and all even elements of A_1 are in $V \setminus W$. If the largest element p of A_1 is not in V, the monomial with support $(V \setminus A_1) \cup \{1, 3, \ldots, p\}$ is in I, hence $x_1 x_V x_W = 0$. By a similar argument, if the smallest element of A_2 is not in V we get $x_m x_V x_W = 0$. So we can assume that both the largest element of A_1 and the smallest element of A_2 are in V. By the above discussion, this implies that $\#(A_1 \cap V) = \#(A_1 \cap W) + 1$ and $\#(A_2 \cap V) = \#(A_2 \cap W) + 1$, hence $\#W_a = \#V_a + 1$, where we set $V_a = V \setminus (A_1 \cup A_2)$ and $W_a = W \setminus (A_1 \cup A_2)$. Hence there exists a contiguous subset of $V_a \cup W_a$, say $A_3 = \{i, i+1, \ldots, j\}$, which starts with an element of $W \setminus V$ then either stops or continuous with an element of $V \setminus W$ and finally finishes with an element of $W \setminus V$. The monomial with support in $(V \setminus A_3) \cup \{i, i+2, \ldots, j\}$ is in I, hence we get $x_V x_W = 0$ which finishes the proof of part (a) of Lemma 3.5.

We now prove part (b) of the lemma. It is enough to show that $(b_1 + z b_2) w x_1 x_m = z w (b_2 x_1 x_m)$, for all $w \in W$. For that it is enough to show $x_1 x_m b_1 w = 0$, which follows from part (a). □

Using Lemma 3.5, multiplication by s_0, which a priori is only an R/I-homomorphism $R/I \to K(R/I)$, maps J inside R/I, so defines an R/I-homomorphism $\phi\colon J \to R/I$. By the same Lemma 3.5, we have that $\phi(v) = 0$, for all $v \in I_{a-1,2,m-1}$, and $\phi(zw) = w x_1 x_m$, for all $w \in I_{a-2,3,m-2}$. Since an R/I-homomorphism is uniquely determined by its values on a generating set, the uniqueness of ϕ stated in Lemma 3.2 follows.

We will now prove the part of Lemma 3.2 stating that the R/I-module $\mathrm{Hom}_{R/I}(J, R/I)$ is generated by the set $\{i, \phi\}$. By the arguments contained in the proof of [1, Theorem 5.6.2], we have isomorphisms

$$\omega_{k[\Delta(d,m)]} \cong k[\Delta(d, m)](0), \quad \omega_{k[\Delta(d-2,m-1)]} \cong k[\Delta(d-2, m-1)](0),$$

of graded k-algebras, where ω_R denotes the canonical R-module. Consequently, since $R/I \cong k[\Delta(d, m)][z]$, $R/J \cong k[\Delta(d-2, m-1)][x_1, x_m]$ we get

$$\omega_{R/I} \cong (R/I)(-1) \quad \text{and} \quad \omega_{R/J} \cong (R/J)(-2). \tag{3.1}$$

Combining (3.1) with the short exact sequence ([10, p. 563])

$$0 \to \omega_{R/I} \to \mathrm{Hom}_{R/I}(J, \omega_{R/I}) \to \omega_{R/J} \to 0,$$

we get the short exact sequence

$$0 \to R/I \to \mathrm{Hom}_{R/I}(J, R/I) \to (R/J)(-1) \to 0.$$

As a consequence, $\mathrm{Hom}_{R/I}(J, R/I)$ is generated as an R/I-module by the subset $\{i, \psi\}$, whenever $\psi \in \mathrm{Hom}_{R/I}(J, R/I)$ has homogeneous degree 1 and is not contained in the R/I-submodule of $\mathrm{Hom}_{R/I}(J, R/I)$ generated by the inclusion homomorphism i.

Hence, to prove $\mathrm{Hom}_{R/I}(J, R/I) = (i, \phi)$ is enough to show that there is no $c \in R/I$ with $\phi = ci$. Assume such c exists. Let $w \in I_{a-2,3,m-2}$ be a fixed monomial generator. We then have $czw = \phi(zw) = wx_1x_m$ (equality in R/I), and since R/I is a polynomial ring with respect to z we get $wx_1x_m = 0$, which is impossible, since $I = (I_{a,1,m-1}, I_{a,2,m})$. Hence $\mathrm{Hom}_{R/I}(J, R/I) = (i, \phi)$, which finishes the proof of Lemma 3.2.

4. The main theorem for d odd

Assume k is a fixed field, and d, m two integers with d odd and $5 \leq d < m-1$, the cases $d = 3$ and $m = d+1$ are discussed in Subsections 4.1 and 4.2 respectively. We set $a = (d+1)/2$. Combining Proposition 3.1 with [1, Exercise 5.2.18] we get the following proposition.

Proposition 4.1. *We have*

$$k[\Delta(d, m)] \cong k[x_1, \ldots, x_m]/(I_{a,2,m-1}, x_1x_m I_{a-1,3,m-2}).$$

REMARK 4.2. By Proposition 4.1 and [1, Exercise 5.2.18], for $d \geq 5$ odd the ideal defining $k[\Delta(d, m)]$ is related to the ideal defining $k[\Delta(d-1, m-1)]$. We will use this in what follows to reduce questions for d odd to the easier case d even. A similar remark also applies when $d = 3$.

We set $R = k[x_1, \ldots, x_m, z_1, z_2]$, where we put degree 1 for all variables. Consider the ideals $I = (I_{a,2,m-1}, x_1x_m I_{a-1,3,m-2})$ and $J = (I_{a-1,2,m-2}, z_1z_2 I_{a-2,3,m-3})$ of R. It is clear that $I \subset (I_{a-1,2,m-2})$, hence $I \subset J$. By Proposition 4.1 we have that $R/I \cong k[\Delta(d,m)][z_1, z_2]$ and $R/J \cong k[\Delta(d-2, m-1)][x_1, x_{m-1}, x_m]$. Consequently, both rings R/I and R/J are Gorenstein by [1, Corollary 5.6.5], and $\dim R/J = \dim R/I - 1$. The following lemma is the analogue of Lemma 3.2 for the case d odd.

Lemma 4.3. *There exists unique $\phi \in \mathrm{Hom}_{R/I}(J, R/I)$ such that $\phi(v) = 0$ for all $v \in I_{a-1,2,m-2}$ and $\phi(z_1z_2w) = x_1x_{m-1}x_m w$ for all $w \in I_{a-2,3,m-3}$. Moreover, the R/I-module $\mathrm{Hom}_{R/I}(J, R/I)$ is generated by the set $\{i, \phi\}$, where $i\colon J \to R/I$ denotes the inclusion homomorphism.*

Proof. Taking into account Proposition 4.1 and Remark 4.2, Lemma 4.3 follows by the same arguments as Lemma 3.2. □

Taking into account Lemma 4.3, the Kustin–Miller unprojection ring S of the pair $J \subset R/I$ is equal to

$$S = \frac{(R/I)[T]}{(Tu - \phi(u) \mid u \in J)}.$$

We extend the grading of R to a grading of S by putting the degree of the new variable T equal to 1. Lemma 4.3 tells us that S is a graded k-algebra. Our main result for the case d odd is the following theorem.

Theorem 4.4. *The sequence $z_1, z_2 \in S$ is S-regular, and there is an isomorphism of graded k-algebras*

$$S/(z_1, z_2) \cong k[\Delta(d, m+1)].$$

Proof. Denote by $Q \subset R[T]$ the ideal

$$Q = (I, z_1, z_2) + (Tu - \phi(u) \mid u \in J) \subset R[T].$$

By the definition of S we have $S/(z_1, z_2) \cong R[T]/Q$.

Denote by $g\colon R[T] \to R[x_{m+1}]$ the k-algebra isomorphism which is uniquely specified by $g(z_i) = z_i$ for $i = 1, 2$, $g(x_i) = x_i$ for $1 \le i \le m-1$, $g(x_m) = x_{m+1}$ and $g(T) = x_m$. It is easy to see that $g(Q) = (I_{d,m+1}, z_1, z_2)$. Since g is an isomorphism, we have using Proposition 4.1 that

$$R[T]/Q \cong R[x_{m+1}]/(I_{d,m+1}, z_1, z_2) \cong k[\Delta(d, m+1)],$$

hence $S/(z_1, z_2) \cong k[\Delta(d, m+1)]$. As a consequence, $\dim S/(z_1, z_2) = \dim S - 2$, and since by [10, Theorem 1.5] S is Gorenstein, hence Cohen–Macaulay, we get that z_1, z_2 is an S-regular sequence. □

4.1. The case $d = 3$ and $d + 1 < m$. Assume $d = 3$ and $d + 1 < m$. Combining [1, p. 229, Exercise 5.2.18] with the discussion of Subsection 3.1 we have the following picture. Set $R = k[x_1, \ldots, x_m, z_1, z_2]$, where we put degree 1 for all variables. Consider the ideals $I = (I_{2,2,m-1}, x_1 x_m I_{1,3,m-2})$ and $J = (I_{1,2,m-2}, z_1 z_2)$ of R. Then $k[\Delta(d, m)][z_1, z_2] \cong R/I$. Moreover, we have $I \subset J$, that $J \subset R/I$ is a codimension 1 ideal of R/I with R/J Gorenstein, and that if we denote by S the Kustin–Miller unprojection ring of the pair $J \subset R/I$ then z_1, z_2 is an S-regular sequence and $S/(z_1, z_2) \cong k[\Delta(d, m+1)]$.

4.2. The case d is odd and $m = d + 1$. Assume $d \ge 3$ is odd and $m = d + 1$. We have

$$k[\Delta(d, m)] \cong k[x_1, \ldots, x_m] \Big/ \left(\prod_{i=1}^{d+1} x_i \right)$$

and

$$k[\Delta(d, m+1)] \cong k[x_1, \ldots, x_{m+1}] \Big/ \left(\prod_{i=0}^{(d+1)/2} x_{2i+1}, \prod_{i=1}^{(d+1)/2} x_{2i} \right).$$

Set $R = k[x_1, \ldots, x_m, z_1, z_2]$, where we put degree 1 for all variables. Consider the ideals $I = \left(\prod_{i=1}^{d+1} x_i\right)$ and $J = \left(\prod_{i=1}^{(d+1)/2} x_{2i}, z_1 z_2 \prod_{i=1}^{(d-1)/2} x_{2i+1}\right)$ of R. We have $I \subset J$, that $J \subset R/I$ is a codimension 1 ideal of R/I with R/J Gorenstein, and that if we denote by S the Kustin–Miller unprojection ring of the pair $J \subset R/I$ then z_1, z_2 is an S-regular sequence and $S/(z_1, z_2) \cong k[\Delta(d, m+1)]$.

5. Combinatorial interpretation of our construction

We fix $d \geq 2$ even and $m \geq d+1$, and we will give a combinatorial interpretation of the constructions of Section 3. We introduce the notation $R_{(m)} = k[x_1, \ldots, x_m, z]$. Consider the ideals $I_{d,m}$ and $J_{d,m}$ of $R_{(m)}$ as defined in Section 3 if $d \geq 4$ and $m \geq d+2$, as defined in Subsection 3.1 if $d = 2$ and $m \geq d+2$, and as defined in Subsection 3.2 if $d \geq 2$ and $m = d+1$.

Note that $I_{d,m}$ is the Stanley–Reisner ideal of $\Delta(d,m)$. We will inductively identify $J_{d,m}$. We set $P_{d,m} = I_{d,m} : (x_1 x_m)$, then

$$P_{d,m} = I_{\mathrm{star}_{\Delta(d,m)}(\{1,m\})} + (x_i \mid i \text{is not a vertex of } \mathrm{star}_{\Delta(d,m)}(\{1,m\})).$$

It is clear that the ideal $P_{d,m}$ of $R_{(m)}$ is monomial, and that no minimal monomial generator of it involves the variables x_1, x_m and z. We denote by $\hat{P}_{d,m}$ the ideal of $k[x_2, \ldots, x_{m-1}, z]$ which has the same minimal monomial generating set.

If $d = 2$ we have $J_{d,m} = (P_{d,m}, z)$. Assume now $d \geq 4$. It is easy to see that the ideal $\hat{P}_{d,m}$ is equal to the image of the ideal $I_{d-2,m-2}$ of $R_{(m-2)}$ under the k-algebra isomorphism $R_{(m-2)} \to k[x_2, \ldots x_{m-1}, z]$ that sends z to z and x_i to x_{i+1} for $1 \leq i \leq m-2$, hence $\hat{P}_{d,m}$ is the Stanley–Reisner ideal of a simplicial complex isomorphic to $\Delta(d-2, m-2)$. The unprojection constructions described in Section 3 and Subsections 3.1, 3.2 allow us to pass from the ideal $I_{d-2,m-2}$ of $R_{(m-2)}$ to the ideal $I_{d-2,m-1}$ of $R_{(m-1)}$, which is the Stanley–Reisner ideal of $\Delta(d-2, m-1)$. Denote by $Q_{d,m} \subset k[x_2, \ldots, x_m, z]$ the image of the ideal $I_{d-2,m-1}$ under the k-algebra isomorphism $R_{(m-1)} \to k[x_2, \ldots, x_m, z]$ that sends z to x_m, x_i to x_{i+1} for $1 \leq i \leq m-2$, and x_{m-1} to z. It is then easy to see that $J_{d,m}$ is the ideal of $R_{(m)}$ generated by the image of $Q_{d,m}$ under the inclusion of k-algebras $k[x_2, \ldots, x_m, z] \to R_{(m)}$. In particular, $R_{(m)}/(J_{d,m}, x_1, x_m) \cong k[\Delta(d-2, m-1)]$, as already observed above.

Assume now $d \geq 3$ is odd and $m \geq d+1$. Consider the ideal J as defined in Section 4. Using Remark 4.2, a similar combinatorial interpretation exists for J in terms of the $\Delta(d-2, m-2)$ related to the star of the face $\{1, m\}$ of $\Delta(d, m)$ when $d \geq 5$, and an analogous statement when $d = 3$. We leave the precise formulations to the reader.

6. The minimal resolution of cyclic polytopes

Combining the results of Sections 3 and 4, we have that for $d \geq 4$ and $d+1 < m$, the Stanley–Reisner ring $k[\Delta(d, m+1)]$ can be constructed from the Stanley–Reisner rings $k[\Delta(d, m)]$ and $k[\Delta(d-2, m-1)]$ using Kustin–Miller unprojection. Moreover, we showed that a similar statement is true also for the cases $d = 2, 3$ and $m = d+1$. Using the Kustin–Miller complex construction discussed in Subsection 2.3, we can inductively build a graded free resolution of S, hence using Proposition 6.3 below of $k[\Delta(d, m+1)]$, starting from the minimal graded free resolutions of $k[\Delta(d, m)]$ and $k[\Delta(d-2, m-1)]$. The following theorem, which will be proven in Subsection 6.1, tells us that in this way we get a minimal resolution. Subsection 6.2 contains examples demonstrating the theorem and a link to related computer algebra code.

Theorem 6.1. *For $d \geq 4$ and $d+1 < m$, the graded free resolution of $k[\Delta(d, m+1)]$ obtained from the minimal graded free resolutions of $k[\Delta(d,m)]$ and $k[\Delta(d-2, m-1)]$ using the Kustin–Miller complex construction is minimal. For $d = 2$ or 3 and $d+1 < m$, the graded free resolution of $k[\Delta(d, m+1)]$ obtained from the minimal graded free resolution of $k[\Delta(d, m)]$ and the appropriate Koszul complex (see* Subsections 3.1 *and* 4.1*) using the Kustin–Miller complex construction is also minimal.*

We remark that in the proof of Theorem 6.1 we do not use the calculation of the graded Betti numbers of $k[\Delta(d,m)]$ obtained by Schenzel [12] for even d, and by Terai and Hibi [13] for odd d. Not only that, but in Proposition 6.6 we recover their results, without using Hochster's formula or Alexander duality.

6.1. Proof of Theorem 6.1. For the proof of Theorem 6.1 we will need the following combinatorial discussion.

Assume $d \geq 3$ is odd, $d+1 < m$ and $1 \leq i \leq m-d-1$. We set

$$\eta(d, m, i) = \binom{m-[d/2]-2}{[d/2]+i}\binom{[d/2]+i-1}{[d/2]},$$

compare [13, p. 291]. We also set $\eta(d, m, 0) = \eta(d, m, m-d) = 0$.

Proposition 6.2. *We have, for $1 \leq i \leq m-d$,*

$$\eta(d, m+1, i) = \eta(d, m, i) + \eta(d, m, i-1) + \eta(d-2, m-1, i). \tag{6.1}$$

(*By our conventions, for $i = 1$ the equality becomes $\eta(d,m+1,1) = \eta(d,m,1)+\eta(d-2, m-1, 1)$, while for $i = m-d$ it becomes $\eta(d, m+1, m-d) = \eta(d-2, m-1, m-d)+\eta(d, m, m-d-1)$.*)

Proof. Assume first $2 \le i \le m-d-1$. We will use twice the Pascal triangle identity $\binom{k}{d} = \binom{k-1}{d} + \binom{k-1}{d-1}$. We have

$$\begin{aligned}
&\eta(d, m+1, i)\\
&= \binom{m+1-[d/2]-2}{[d/2]+i}\binom{[d/2]+i-1}{[d/2]}\\
&= \left(\binom{m-[d/2]-2}{[d/2]+i} + \binom{m-[d/2]-2}{[d/2]+i-1}\right)\binom{[d/2]+i-1}{[d/2]}\\
&= \binom{m-[d/2]-2}{[d/2]+i}\binom{[d/2]+i-1}{[d/2]} + \binom{m-[d/2]-2}{[d/2]+i-1}\binom{[d/2]+i-1}{[d/2]}\\
&= \eta(d, m, i) + \binom{m-[d/2]-2}{[d/2]+i-1}\left(\binom{[d/2]+i-2}{[d/2]} + \binom{[d/2]+i-2}{[d/2]-1}\right)\\
&= \eta(d, m, i) + \eta(d, m, i-1) + \eta(d-2, m-1, i).
\end{aligned}$$

The special cases $i = 1$ and $i = m-d$ are proven by the same argument. □

For the proof of Theorem 6.1 we will also need the following general propositions, the first of which is well-known.

Proposition 6.3 ([1, Proposition 1.1.5]). *Assume $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k with the degrees of all variables positive, and $I \subset R$ a homogeneous ideal. Moreover, assume that x_n is R/I-regular. Denote by cF the minimal graded free resolution of R/I as R-module. We then have that $cF \otimes_R R/(x_n)$ is the minimal graded free resolution of $R/(I, x_n)$ as $k[x_1, \ldots, x_{n-1}]$-module, where we used the natural isomorphisms $R \otimes_R R/(x_n) \cong R/(x_n) \cong k[x_1, \ldots, x_{n-1}]$.*

The following proposition is an immediate consequence of Equation (2.1).

Proposition 6.4. *Assume k is a field and $R_1 = k[x_1, \ldots, x_n]$, $R_2 = k[y_1, \ldots, y_n]$ are two polynomial rings with the degrees of all variables positive. Assume $I_1 \subset R_1$ is a monomial ideal, and denote by I_2 the ideal of R_2 generated by the image of I_1 under the k-algebra homomorphism $R_1 \to R_2$, $x_i \mapsto y_i$, for $1 \le i \le n$. Obviously I_2 is a homogeneous ideal of R_2. We claim that for all $i \ge 0$ we have $b_i(R_2/I_2) = b_i(R_1/I_1)$ (of course the graded Betti numbers b_{ij} of R_2/I_2 and R_1/I_1 may differ).*

Proposition 6.5. *Assume k is a field, $R_1 = k[x_1, \ldots, x_n, T]$ and $R_2 = k[y_1, \ldots, y_n, T_1, T_2]$ are two polynomial rings with the degrees of all variables positive,* $\deg x_i = \deg y_i$, *for $1 \le i \le n$, and* $\deg T = \deg T_1 + \deg T_2$. *Assume $I_1 \subset R_1$ is a homogeneous ideal, and denote by $I_2 \subset R_2$ the ideal generated by the image of I_1 under the graded k-algebra homomorphism $\phi\colon R_1 \to R_2$ specified by $\phi(x_i) = y_i$, for $1 \le i \le t$, and $\phi(T) = T_1T_2$. Denote by cF_1 the minimal graded free resolution of R_1/I_1 as*

R_1-module. Then I_2 is a homogeneous ideal R_2, and the complex $cF_1 \otimes_{R_1} R_2$ is a minimal graded free resolution of R_2/I_2 as R_2-module. In particular, the corresponding graded Betti numbers b_{ij} of R_1/I_1 and R_2/I_2 are equal.

Proof. It is clear that I_2 is a homogeneous ideal of R_2. By [6, Theorem 18.16] ϕ is flat. As a consequence, [6, Proposition 6.1] implies that the natural map $I_1 \otimes_{R_1} R_2 \to I_2$ is an isomorphism of graded R_2-modules. By flatness, tensoring the minimal graded free resolution of I_1 as R_1-module with R_2 we get the minimal graded free resolution of I_2 as R_2-module, and Proposition 6.5 follows. □

Theorem 6.1 will follow from the following more precise statement. Notice that, as we already mentioned before, the statements about the graded Betti numbers have been proven before by different arguments in [12, 13], but we do not need to use their results.

Proposition 6.6. *Assume $d \geq 2$ and $d + 1 < m$. Set $b_{ij} = b_{ij}(k[\Delta(d, m)])$. Then the statement of* Theorem 6.1 *is true for (d, m). Moreover, we have that if d is even then $b_{ij} = 1$ for $(i, j) \in \{(0, 0), (m - d, m)\}$,*

$$b_{i,d/2+i} = \eta(d + 1, m + 1, i) + \eta(d + 1, m + 1, m - d - i),$$

for $1 \leq i \leq m - d - 1$, and $b_{ij} = 0$ otherwise. If d is odd, then $b_{ij} = 1$ for $(i, j) \in \{(0, 0), (m - d, m)\}$,

$$b_{i,[d/2]+i} = \eta(d, m, i), \quad b_{i,[d/2]+i+1} = \eta(d, m, m - d - i),$$

for $1 \leq i \leq m - d - 1$, and $b_{ij} = 0$ otherwise.

Proof. We use induction on d and m. If $d \geq 2$ and $m = d + 2$ then $k[\Delta(d, m)]$ is a codimension 2 complete intersection and everything is clear.

The next step, is to notice that, for $d = 2$ and $m \geq 3$, Proposition 6.6 follows from [3, Proposition 5.7], since $\Delta(2, m)$ is equal to $\Delta P_2(m)$ defined in [3, Section 5].

Now assume that d is even with $d \geq 4$ and $d + 3 \leq m$, and, by the inductive hypothesis, Proposition 6.6 holds for the values $(d - 2, m - 1)$ and (d, m). An easy computation, taking into account Proposition 6.2, shows that the Kustin–Miller complex construction resolving $k[\Delta(d, m + 1)]$ has the conjectured graded Betti numbers. Since no degree 0 morphisms appear it is necessarily minimal. This finishes the proof for d even.

Assume now $d \geq 3$ is odd. Combining [1, Exercise 5.2.18] with Propositions 6.4 and 6.5 we get that, for $0 \leq i \leq m - d$,

$$b_i(k[\Delta(d, m)]) = b_i(k[\Delta(d - 1, m - 1)]). \tag{6.2}$$

(Of course the graded Betti numbers b_{ij} can, and in fact are, different for $k[\Delta(d, m)]$ and $k[\Delta(d-1, m-1)]$.) So we can reduce the case d odd to the case $d-1$, by doing an almost identical induction on (d, m) as in the case $(d-1, m-1)$, noticing that the Kustin–Miller complex construction for $k[\Delta(d, m+1)]$ has to be minimal, since we proved that the one for $k[\Delta(d-1, m)]$ is minimal and the corresponding numbers $b_i = \sum_j b_{ij}$ are equal by Equation (6.2). This finishes the proof of Proposition 6.6. □

6.2. Examples and implementation. In this subsection we demonstrate the construction of the cyclic polytope resolution with a sequence of two examples. First we carry out the Kustin–Miller complex construction described in Subsection 2.3 for the step passing from the codimension 4 complete intersection $J_{2,5}$ and the Pfaffian $I_{2,5}$ to the codimension 4 ideal $I_{2,6}$. In the second step we pass from $J_{4,7}$ and the Pfaffian $I_{4,7}$ to $I_{4,8}$, using that $J_{4,7}$ is equal to $I_{2,6}$ after a change of variables. At the end of the subsection we give a link to computer algebra code where we implement our constructions.

Using the notation of Subsection 2.3, we will explicitly compute for each step the auxiliary data α_i, β_i, h_i, u and hence the differentials f_i from the input data a_i and b_i. The ideals $I_{2,5}$ and $I_{4,7}$ are Gorenstein codimension 3, hence Pfaffian, and we will fix below a certain resolution for each of them. In addition, we will also fix below a certain Koszul complex resolving $J_{2,5} = (z, x_2, \ldots, x_4)$.

Assume $q \geq 3$ is an odd integer and M is a skew-symmetric $q \times q$ matrix with entries in a commutative ring. For $1 \leq i \leq q$, we denote by $\mathrm{pf}_i M$ the Pfaffian ([1, Section 3.4]) of the submatrix of M obtained by deleting the i-th row and column of M. The main property of $\mathrm{pf}_i M$ is that its square is the determinant of the corresponding submatrix.

We will use the notation $R_{(m)} = k[x_1, \ldots, x_m, z]$ introduced in Section 5. For $d \geq 2$ even, we denote by M_d the $(d+3) \times (d+3)$ skew-symmetric matrix with entries in $R_{(d+3)}$ whose (i, j) entry for $i \leq j$ is zero except that for $1 \leq i \leq d+2$ we have $(M_d)_{i,i+1} = x_i$ and that $(M_d)_{1,d+3} = -x_{d+3}$. It is an easy calculation that

$$I_{d,d+3} = (\mathrm{pf}_i(M_d) \mid 1 \leq i \leq d+3).$$

In addition, according to the Buchsbaum–Eisenbud theorem [2], the minimal graded free resolution of $R_{(d+3)}/I_{d,d+3}$ is given by

$$0 \to R_{(d+3)} \xrightarrow{v_d^t} R_{(d+3)}^{d+3} \xrightarrow{M_d} R_{(d+3)}^{d+3} \xrightarrow{v_d} R_{(d+3)} \tag{6.3}$$

where v_d denote the $1 \times (d+3)$ matrix with $(1, i)$ entry equal to $(-1)^i \mathrm{pf}_i(M_d)$ and v_d^t denotes the transpose of v_d.

We set $R = R_{(5)}$ and fix the following Koszul complex resolution of $R/J_{2,5}$

$$0 \to R \xrightarrow{a_4} R^4 \xrightarrow{a_3} R^6 \xrightarrow{a_2} R^4 \xrightarrow{a_1} R \tag{6.4}$$

where

$$a_1 = \begin{pmatrix} z & x_3 & x_4 & x_2 \end{pmatrix}, \quad a_2 = \begin{pmatrix} x_3 & x_4 & x_2 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & x_2 & -x_4 \\ 0 & -z & 0 & -x_2 & 0 & x_3 \\ 0 & 0 & -z & x_4 & -x_3 & 0 \end{pmatrix},$$

$$a_3 = \begin{pmatrix} 0 & -x_2 & x_4 & z & 0 & 0 \\ x_2 & 0 & -x_3 & 0 & z & 0 \\ -x_4 & x_3 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & x_3 & x_4 & x_2 \end{pmatrix}^t, \quad a_4 = \begin{pmatrix} x_3 \\ x_4 \\ x_2 \\ -z \end{pmatrix}.$$

We now discuss the Kustin–Miller complex construction for the step passing from $(I_{2,5}, J_{2,5})$ to $I_{2,6}$, which corresponds to the unprojection of $J_{2,5} \subset R/I_{2,5}$. We will use as input for the Kustin–Miller complex construction the resolution (6.4) of $R/J_{2,5}$ and the case $d = 2$ of (6.3), which is a resolution of $R/I_{2,5}$. Performing the computations we obtain, in the notation of Subsection 2.3, the complex C_S specified by $h_1 = h_2 = 0$, $u = -1$ and the maps

$$\alpha_1 \colon R^5 \to R^4, \quad \sum_{i=1}^{5} c_i e_i \mapsto x_1(c_5 e_2 + c_3 e_3) + x_4 c_1 e_4 + x_5(c_2 e_2 + c_4 e_4),$$

$$\alpha_2 \colon R^5 \to R^6, \quad \sum_{i=1}^{5} c_i e_i \mapsto x_1(c_2 e_4 + c_4 e_6) + x_5 c_3 e_5,$$

$$\alpha_3 \colon R \to R^4, \quad e_1 \mapsto x_1 x_5 e_4$$

and

$$\beta_1 \colon R^4 \to R, \quad \sum_{i=1}^{4} c_i e_i \mapsto -x_1 x_5 c_1 e_1,$$

$$\beta_2 \colon R^6 \to R^5, \quad \sum_{i=1}^{6} c_i e_i \mapsto -x_1(c_1 e_2 + c_3 e_4) - x_5 c_2 e_3,$$

$$\beta_3 \colon R^4 \to R^5, \quad \sum_{i=1}^{4} c_i e_i \mapsto -x_1(c_2 e_3 + c_1 e_5) - x_4 c_3 e_1 - x_5(c_1 e_2 + c_3 e_4),$$

where $(e_i)_{1 \le i \le q}$ denotes the canonical basis of R^q as R-module. Substituting x_6 for T and 0 for z in the differential maps of C_S we get the minimal graded free resolution of $R_{(6)}/I_{2,6}$. Moreover, substituting z for x_1 in the differential maps of the resolution of $R_{(6)}/I_{2,6}$ just constructed we get the minimal graded free resolution of $R_{(7)}/J_{4,7}$.

We now set $R = R_{(7)}$ and discuss the Kustin–Miller complex construction for the step passing from $(I_{4,7}, J_{4,7})$ to $I_{4,8}$, which corresponds to the unprojection of $J_{4,7} \subset$

$R/I_{4,7}$. We will use as input for the Kustin–Miller complex construction the resolution of $R/J_{4,7}$ constructed above and the case $d = 4$ of (6.3), which is a resolution of $R/I_{4,7}$. Performing the computations we obtain, in the notation of Subsection 2.3, the complex C_S specified by $h_1 = h_2 = 0$, $u = -1$ and the maps

$$\alpha_1\colon R^7 \to R^9, \quad \sum_{i=1}^{7} c_i e_i \mapsto x_1(c_7e_2 + c_5e_7 + c_3e_8) + x_6c_1e_1 + x_7(c_6e_1 + c_2e_2 + c_4e_4),$$

$$\alpha_2\colon R^7 \to R^{16}, \quad \sum_{i=1}^{7} c_i e_i \mapsto x_7(c_3e_3 + c_5e_5) - x_1(c_2e_9 + c_4e_{11} - c_2e_{12} + c_6e_{13}),$$

$$\alpha_3\colon R \to R^9, \quad e_1 \mapsto x_1x_7(x_5e_4 - x_4e_7 - x_3e_9)$$

and

$$\beta_1\colon R^9 \to R, \quad \sum_{i=1}^{9} c_i e_i \mapsto x_1x_7(-c_3x_4 - c_5x_3 + c_6x_5),$$

$$\beta_2\colon R^{16} \to R^7, \quad \sum_{i=1}^{16} c_i e_i \mapsto -x_1(c_1e_2 + c_6e_2 + c_8e_4 - c_2e_6) - x_7(c_{14}e_3 + c_{16}e_5),$$

$$\beta_3\colon R^9 \to R^7, \quad \sum_{i=1}^{9} c_i e_i \mapsto -x_6c_5e_1 - x_7(c_6e_2 + c_8e_4 + c_5e_6) + x_1(c_2e_3 + c_1e_5 - c_6e_7).$$

Substituting x_8 for T and 0 for z in the differential maps of C_S we get the minimal graded free resolution of $R_{(8)}/I_{4,8}$.

Under the link [4], a related package for the computer algebra system Macaulay2 [5] is available. Applying the ideas of the present paper, it constructs the resolution of the ideal $I_{d,m}$ for d even and $m \geq d + 1$ starting from Koszul complexes and the skew-symmetric Buchsbaum–Eisenbud resolution (6.3) of $I_{d,d+3}$. The functions in the package provide the user with the option to output all the intermediate data a_i, b_i, α_i, β_i, h_i, u, f_i in addition to the final resolution.

ACKNOWLEDGEMENTS. The authors wish to thank Christos Athanasiadis for useful discussions and suggestions, and an anonymous referee for suggestions that improved the presentation of the material.

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