Title	Construction of the fundamental solution for degenerate parabolic systems and its application to construction of a parametrix of □_b
Author(s)	Iwasaki, Chisato
Citation	Osaka Journal of Mathematics. 1984, 21(4), p. 931-954
Version Type	VoR
URL	https://doi.org/10.18910/6732
rights	
Note	

Iwasaki, C.
Osaka J. Math.
21 (1984), 931-954

CONSTRUCTION OF THE FUNDAMENTAL SOLUTION FOR DEGENERATE PARABOLIC SYSTEMS AND ITS APPLICATION TO CONSTRUCTION OF A PARAMETRIX OF $\square_b$

Dedicated to the memory of Professor Hitoshi Kumano-go

CHISATO IWASAKI

(Received December 27, 1983)

Introduction. This paper intends to study the fundamental solution $E(t)$ of a degenerate parabolic system of pseudo-differential operators:

$$(0.1)\qquad \begin{cases} \left[\dfrac{d}{dt}+p(x, D)\right] E(t)=0\,, \quad t>0\,, \quad x\in \boldsymbol{R}^n\,, \\ E(0)=I\,, \end{cases}$$

where $k\times k$ matrix $p(x, \xi)$ has the following expansion:

$$(0.2)\qquad \begin{aligned} & p(x, \xi)=p_m(x, \xi)+p_{m-1}(x, \xi)+p_{m-2}(x, \xi)\,, \\ & p_{m-j}\in S^{m-j}_{1,0} \quad (j=0, 1, 2)\,, \\ & p_{m-j}(x, \lambda\xi)=\lambda^{m-j}\, p_{m-j}(x, \xi) \ \ \lambda>0,\ \xi\neq 0 \quad (j=0, 1) \end{aligned}$$

and $\qquad m>1$.

Our aim is to find $E(t)$ in some class of pseudo-differential operators. We adopt the Weyl symbol for pseudo-differential operators in this paper. The main theorem of this paper is that one can construct the fundamental solution $E(t)$ in the class $S^0_{1/2,1/2}$ of pseudo-differential operators with parameter t provided the symbol (0.2) satisfies the following Condition (A):

Condition (A).

$$\text{(A)–(i)} \quad p_m(x, \xi)=q_m(x, \xi)\, I\,,$$

where q_m $(\in S^m_{1,0})$ is a non-negative scalar symbol.

$$\text{(A)–(ii)} \quad \min_{1\leqslant j\leqslant k} (\mathrm{Re}\,\mu_j(x, \xi))+\tilde{\mathrm{tr}}\, A/2\geqslant c\,|\xi|^{m-1}$$

for some positive constant c on the characteristic set $\Sigma=\{(x, \xi)\in \boldsymbol{R}^n\times\boldsymbol{R}^n; q_m(x, \xi)=0\}$, where $\{\mu_j(x, \xi)\}_{j=1}^k$ are the eigenvalues of $p_{m-1}(x, \xi)$ and $\tilde{\mathrm{tr}}\, A$ is the sum of all positive eigenvalues of A:

$$A=iJH_{q_m}\,. \tag{0.3}$$

Here

$$J=\begin{pmatrix} 0, & I \\ -I, & 0\end{pmatrix} \tag{0.4}$$

and

$$H_q=\begin{pmatrix}\partial_{xx}\, q, & \partial_{x\xi}\, q \\ \partial_{\xi x}\, q, & \partial_{\xi\xi}\, q\end{pmatrix} \quad \text{(the Hessian matrix of } q). \tag{0.5}$$

For a single equation in C. Iwasaki and N. Iwasaki [7] $E(t)$ has been constructed in class $S^0_{1/2,1/2}$ under Condition (A), which is equivalent to the condition that the following inequality holds for some positive constant ε (Melin [9]).

$$\mathrm{Re}(p(x, D)\, u, u)\geqslant \varepsilon\, \|\, u\,\|^2_{(m-1)/2}-C\,\|\, u\,\|_0^2 \quad \text{for} \quad u\in C_0^\infty(\boldsymbol{R}^n)\,. \tag{0.6}$$

Similar results are found in Menikoff and Sjöstrand [10] and Sjöstrand [12]. However for the degenerate systems, namely, when some of the eigenvalues of the principal symbol attain zero, a necessary and sufficient condition in order that (0.6) holds is not known. Although the principal symbol is assumed to have a simple form in our case, our result will turn out to be valid when we apply it to $\Box_b$.

We intend to construct directly the symbol of $E(t)$ having the form $e^\phi f$. The function ϕ is expressible of an explicite function of the principal symbol p_m, its derivatives of the first order, the fundamental matrix A and the subprincipal symbol near the characteristic set Σ with the aid of symbol calculus of pseudo-differential operators. The meaning of (A)–(ii) will be made clear through our construction of ϕ. We obtain $E(t)$ by following the discussion given in [7] carefully.

The exact form of $E(t)$ is available to obtain the asymptotic behavior of $\sum_{j=1}^{\infty}\exp\,(-t\lambda_j)$ as t tends to zero, where $\{\lambda_j\}_{j=1}^\infty$ are the eigenvalues of $p(x, D)$, if $p(x, D)$ is a self-adjoint operator on a bundle over a compact manifold and has exactly double characteristics. As an application of this theorem we get an explicite construction of a parametrix for $\Box_b$ under the condition $Y(q)$ (See [2], Definition 4.1 also) for the Levi form, which will be shown to be equivalent to (A)–(ii). Also we get the asymptotic behavior of the eigenvalues of $\Box_b$.

Folland and Stein [3] and Boutet de Monvel [1] constructed a parametrix for $\Box_b$ under more restrictive conditions. We no longer assume the nondegeneracy of the Levi form, that is, the characteristic Σ is symplectic as they did. On the other hand Rothschild and Tartakoff [11] obtained a parametrix of integral form under $Y(q)$. But its kernel is not given in an exact form and their results are not available to the study of the asymptotic behavior of eigenvalues.

The plan of this paper is as follows. In Section 1 we state the theorems of this paper. Section 2 is devoted to the calculus of pseudo-differential operators with the Weyl symbols and to the construction of the fundamental solution for (0.1). In Section 3 we apply the main theorem to an operator on a manifold. Finally in Section 4 we apply theorems obtained to $\Box_b$.

1. Main results

We say that a C^∞-function $p(x, \xi)$ defined on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ belongs to $S^m_{\rho,\delta} = S^m_{\rho,\delta}(\boldsymbol{R}^n)$ $(0 \leqslant \delta \leqslant \rho \leqslant 1,\ \delta < 1)$ if for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $p^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi \partial^\beta_x p(x, \xi)$ and $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. For $p(x, \xi) \in S^m_{\rho,\delta}$ we define the semi norms $|p|^{(m)}_l$ $(l=0, 1, 2, \cdots)$ by

$$|p|^{(m)}_l = \max_{|\alpha|+|\beta| \leqslant l} \sup_{(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n} \{|p^{(\alpha)}_{(\beta)}(x, \xi)| \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|}\}. \tag{1.1}$$

$S^m_{\rho,\delta}$ is a Fréchet space with the system of semi-norms (1.1).

We employ the Weyl symbol for pseudo-differential operators in this paper, that is, a symbol $p(x, \xi) \in S^m_{\rho,\delta}$ defines an operator as

$$p(x, D)\, u(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n \times \boldsymbol{R}^n} e^{i(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi\right) u(y)\, dy d\xi \quad \text{for} \quad u \in \mathcal{S}(\boldsymbol{R}^n). \tag{1.2}$$

DEFINITION 1.1. We say that $\{p_\varepsilon(x, \xi)\}_{0<\varepsilon<1}$ converges to $p_0(x, \xi)$ as $\varepsilon \to 0$ weakly in $S^m_{\rho,\delta}$ if $\{p_\varepsilon\}_{0<\varepsilon<1}$ is a bounded set in the Fréchet space $S^m_{\rho,\delta}$ and if $p_\varepsilon(x, \xi)$ converges to $p_0(x, \xi)$ as $\varepsilon \to 0$ uniformly on any compact set of $\boldsymbol{R}^n \times \boldsymbol{R}^n$. We denote by $\mathcal{W}-\mathcal{E}^0_t(S^m_{\rho,\delta})$ the set of all functions of t with values in $S^m_{\rho,\delta}$ which are continuous in t with respect to this topology.

The main theorem of this paper is the following

Theorem 1. *Let $p(x, \xi)$ of* (0.2) *satisfy Condition* (A). *Then a fundamental solution $E(t)$ of* (0.1) *is constructed as a matrix whose elements are pseudo-differential operators belonging to $\mathcal{W}-\mathcal{E}^0_t(S^0_{1/2,1/2})$. This is also the unique* *fundamental solution in* $\mathcal{W}-\mathcal{E}_t^0(\bigcup_m S_{\rho,\delta}^m)$ *for any* $0\leqslant\delta\leqslant\rho\leqslant 1$ *and* $\delta<1$. *Moreover* $E(t)$ *belongs to* $S^{-\infty}$ *if* t *is positive and its symbol* $\sigma(E(t))$ *has the following asymptotic expansion for any* N

$$\sigma(E(t))=\sum_{j=0}^{N}(\exp\phi)f_j+g_N,$$

where $f_0=I$, $(\exp\phi)f_j$ *are matrices whose elements belong to* $\mathcal{W}-\mathcal{E}_t^0(S_{1/2,1/2}^{-\varepsilon j})$ *and* g_N *is one with elements belonging to* $\mathcal{W}-\mathcal{E}_t^0(S_{1/2,1/2}^{-\varepsilon(N+1)})$ $(0<\varepsilon<1/6)$. *Here the function* ϕ *is defined by*

$$(1.3)\qquad \phi=\tilde{\psi}\,\phi_1+(1-\tilde{\psi})\,\phi_2,$$

where

$$\tilde{\psi}=\psi^1\,\psi^2$$
$$\psi^1=\psi\,(q_m\langle\xi\rangle^{1-m-2\varepsilon})$$
$$\psi^2=\psi\,(t\langle\xi\rangle^{m-1-\delta})$$

$\psi(s)\in C^\infty([0,\infty))$ *such that* $\psi=1\ (s\leqslant 1)$, $\psi=0\ (s\geqslant 2)$

$\psi'(s)<0\ (1<s<2)$ *and* $|\psi^{(n)}|\leqslant C_{n,\tau}(1-\psi)^\tau\ (0<\tau<1)$

$0<12\delta<1-6\varepsilon<1$

$$(1.4)\qquad \begin{aligned}\phi_1&=-\{q_m t+\langle\nabla q_m t,\,F(At/2)\,J\nabla q_m t\rangle/4\\&\qquad+\mathrm{tr}\,[\log\{\cosh(At/2)\}]/2\}\,I-p_{m-1}\,t,\\F(\lambda)&=(i\lambda)^{-1}(1-\lambda^{-1}\tanh\lambda)\end{aligned}$$

and

$$\phi_2=-\{q_m t+\langle\xi\rangle^{m-1}t\}\,I.$$

REMARK. The operator $\int_0^T E(t)dt$ $(T>0)$ is a parametrix of $p(x, D)$ of class $S_{1/2,1/2}^{1-m}$ since ϕ of (1.3) satisfies

$$\|\exp\phi\|\leqslant c_0\exp(-c'\langle\xi\rangle^{m-1}t)$$

with some positive constant c_0 and c' (See (2.8)). The case that $p_m\equiv 0$ implies the results for parabolic systems of order $m-1$.

If $p(x, \xi)$ is a quadratic polynomial with respect to (x, ξ), then $\exp\phi_1$ is the symbol of $E(t)$. We have

Corollary. *Let* $p=\langle X, HX\rangle/2$ *with* $X=\begin{pmatrix}x\\ \xi\end{pmatrix}$ *for some constant matrix* $H\geqslant 0$. *Then*

$$\sigma(E(t))=\exp\{-i\langle X, J\tanh(iJHt/2)X\rangle\}\,[\det\{\cosh(iJHt/2)\}]^{-1/2}.$$

EXAMPLE 1.

$$P=\sum_{j=1}^{k}(D_{x_j}^2+x_j^2 D_{y_j}^2)+\sum_{j=1}^{l} D_{z_j}^2 .\quad \text{Then}$$

$$\sigma(E(t))=\prod_{j=1}^{k}(\cosh|\eta_j|\, t)^{-1}\exp\{-\sum_{j=1}^{k}(\xi_j^2+x_j^2\eta_j^2)|\eta_j|^{-1}\tanh(|\eta_j|\, t) -\sum_{j=1}^{l}\zeta_j^2 t\} .$$

EXAMPLE 2. Let $\boldsymbol{H}_l$ be the Heisenberg group with l strongly pseudo convex CR structure. Then $\Box_b$ on $\Gamma(\Lambda^{0,q})$ $(q\neq 0, l)$ is

$$\Box_b(\sum_J \phi_J\, d\bar{\omega}^J)=\sum_J\left\{-\frac{1}{2}\sum_{j=1}^{l}(Z_j\bar{Z}_j+\bar{Z}_j Z_j)+i(l-2q)\, T\right\}\phi_J\, d\bar{\omega}^J ,$$

where $Z_j=\dfrac{\partial}{\partial z_j}+i\bar{z}_j\dfrac{\partial}{\partial t}$, $T=\dfrac{\partial}{\partial t}$ and ω^j is the dual base for Z_j.

$$\sigma(\Box_b)(x, y, t, \xi, \eta, \tau)=\{|v_1\tau-Jv_2/2|^2-(l-2q)\,\tau\}\, I ,$$

where $v_1=\begin{pmatrix} x \\ y\end{pmatrix}$ and $v_2=\begin{pmatrix}\xi\\ \eta\end{pmatrix}$.

The fundamental solution $E(s)$ of $\dfrac{d}{ds}+\Box_b$ is given by

$$\sigma(E(s))=\{2/(1+e^{-2|\tau|s})\}^l\exp\{-\tanh(\tau s)|v_1\tau-Jv_2/2|^2/\tau -l|\tau|s+(l-2q)\,\tau s\}\, I .$$

We apply Theorem 1 to a formally self-adjoint operator P on sections of a bundle $\boldsymbol{E}$ of rank k over a compact manifold M of dimension n under the additional assumption (B). As for the definition of pseudo-differential operators on a manifold we use that of Hörmander [6] and Treves [13] which will be illustrated in Section 3.

Condition (B). q_m vanishes exactly to the second order on the characteristic set Σ, that is, $q_m(X)\geqslant C(X)\, d(X, \Sigma)^2$, where $C(X)>0$ and $d(\cdot, \cdot)$ is a distance on T^*M.

Under Condition (B) the characteristic set $\Sigma=\{q_m=0\}$ consists of smooth conic submanifolds of T^*M. Let Σ^i be submanifolds of Σ such that codim $\Sigma^i=d_i$. We put $\Sigma^0=\bigvee_{d_i=d}\Sigma^i$, where $d=\min\{d_i\}$.

Theorem 2. *Let a system of pseudo-differential operators P satisfy Condition* (A) *in any local chart and trivializations of $\boldsymbol{E}$. Then there exists the fundamental solution $E(t)$ of $\dfrac{d}{dt}+P$ in the class $\mathcal{W}-\mathcal{E}_t^0\,(S_{1/2,1/2}^0(M, \boldsymbol{E}))$ of pseudo-differential operators* (*See Definition* 3.2). *Suppose further that P is formally self-adjoint with respect to the inner product associated with some volume of M*

and some Hermitian form on $\boldsymbol{E}$, *and satisfies* (B). *Then we have the following assertions.*

(i) *P is self-adjoint and has only discrete spectrum* $\{\lambda_j\}_{j=1}^{\infty}$.

(ii) $\sum_{j=1}^{\infty} \exp(-t\lambda_j)$ *has the following asymptotic behavior as t tends to zero.*

$$\begin{aligned}\sum_{j=1}^{\infty} \exp(-t\lambda_j) &\sim (C_1+o(1))\, t^{-n/m} \text{ if } n-md/2<0\,,\\ &\sim (C_2 \log(1/t)+O(1))\, t^{-n/m} \text{ if } n-md/2=0\,,\\ &\sim (C_3+o(1))\, t^{-(n-d/2)/(m-1)} \text{ if } n-md/2>0\,,\end{aligned}$$

where

$$\begin{aligned}C_1 &= (2\pi)^{-n}\, k \int_{T^*M} \exp(-q_m(x,\xi))\, dxd\xi\\ C_2 &= (2\pi)^{-(n-d/2)}\, (k/m) \int_{\Sigma^0} h \exp(-h)\, d\Sigma^0\,,\\ C_3 &= (2\pi)^{-(n-d/2)} \int_{\Sigma^0} [\det\{(A/2)^{-1}\sinh(A/2)\}]^{-1/2} \times \operatorname{tr}\{\exp(-p_{m-1})\}\, d\Sigma^0,\end{aligned}$$

where h is an arbitrary positive function on Σ^0 which is homogeneous of degree $m-1$ with respect to ξ, and where $d\Sigma^0$ is a density on Σ^0 induced by q_m and $dxd\xi$ as follows ([7]). *If (u, v) is a system of local coordinates such that $\Sigma^0=\{u=0\}$, we define $d\Sigma^0=(\det H_{uu})^{-1/2}\,\Phi dv$, where $\Phi dudv=dxd\xi$ and H_{uu} is the Hessian matrix of q_m with respect to the variable u.*

REMARK. It will be shown later that Condition (A) is independent of the choise of a local coordinate system.

We apply Theorem 2 to the operator $\Box_b$ on a compact CR-manifold whose definitions will be stated in Section 4 for the sake of convenience.

Theorem 3. *Let M be a compact CR-manifold of dimension $2l+1$ which satisfies the condition $Y(q)$. Then a parametrix Q for $\Box_b$ on $\Gamma(\Lambda^{p,q})$ is constructed as a system of pseudo-differential operators of class $S^{-1}_{1/2,1/2}(M, \Lambda^{p,q})$. Moreover $\bar{\partial}_b Q$ and $\vartheta_b Q$ belong to $S^{-1/2}_{1/2,1/2}(M, \Lambda^{p,q})$. We also get the following asymptotic behavior of $\sum_{j=1}^{\infty} \exp(-t\lambda_j)$, where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of $\Box_b$.*

$$\sum_{j=1}^{\infty} \exp(-t\lambda_j) = (2\pi t)^{-l-1} \binom{l}{p} \int_M c_0\, dM + o(t^{-l-1})\,, \tag{1.5}$$

where dM stands for the natural volume on M defined by the Hermitian metric and c_0 is defined by

$$c_0 = \int_{-\infty}^{\infty} \prod_{j=1}^{l} \{\nu_j\tau/2 \sinh(\nu_j\tau/2)\} \sum_{|J|=q} \exp\{(\sum_{j\notin\{J\}} \nu_j - \sum_{j\in\{J\}} \nu_j)\,\tau/2\}\, d\tau \tag{1.6}$$

$$=\sum_{|J|=q}\{\tilde{\zeta}_l(l+1, (\sum_{j=1}^{l}|\nu_j|-\sum_{j\in\{J\}}\nu_j+\sum_{j\notin\{J\}}\nu_j)/2, \tilde{\nu})$$

$$+\tilde{\zeta}_l(l+1, (\sum_{j=1}^{l}|\nu_j|+\sum_{j\in\{J\}}\nu_j-\sum_{j\notin\{J\}}\nu_j)/2, \tilde{\nu})\},$$

where $\{\nu_j\}_{j=1}^{l}$ *are eigenvalues of the Levi form* L, $\tilde{\nu}=(|\nu_1|, \cdots, |\nu_l|)$ *and* $\tilde{\zeta}_r(s, a, \mu)$ *is given as follows.*

$$\tilde{\zeta}_r(s, a, \mu)=\Gamma(s)\prod_{j=1}^{r}\mu_j\sum_{n_1\cdots n_r=0}^{\infty}(a+\sum_{j=1}^{r}\mu_j n_j)^{-s}$$

$$=\int_0^\infty\prod_{j=1}^{r}\mu_j(1-\exp(-\mu_j x))^{-1}x^{s-1}e^{-ax}dx.$$

$$\mu=(\mu_1, \cdots, \mu_r)\ (\mu_j>0),\ \mathrm{Re}\ s>1\ \textit{and}\ \mathrm{Re}\ a>0.$$

Then

$$\tilde{\zeta}_r(s, a, \mu)=\lim_{\varepsilon\to+0}\tilde{\zeta}_{r+1}(s+1, a, (\mu, \varepsilon)).$$

So we can define $\tilde{\zeta}_r(s, a, \mu)$ *for* $\mu=(\mu_1, \cdots, \mu_r)$ $(\mu_j\geq 0)$, $\mathrm{Re}\ s>r\geq 1$ *and* $\mathrm{Re}\ a>0$. *It is clear that*

$$\tilde{\zeta}_1(s, a, 1)=\Gamma(s)\,\zeta(s, a)$$

where $\zeta(s, a)$ *is Hurwitz zeta function.*

Corollary. *Under the assumption of Theorem* 3, *we get the estimate*

$$\|\phi\|_s\leq C(\|\Box_b\phi\|_{s-1}+\|\phi\|_{s'})\quad \phi\in\Gamma(\Lambda^{p,q})\qquad \textit{for any } s>s'.$$

2. Fundamental solution for degenerate parabolic systems on R^n

Some basic theorems for pseudo-differential operators of the Weyl symbols are stated below. Their proofs are found in the appendix of [7]. At first we give a relation between the ordinary type of pseudo-differential operators

$$p^o(x, D)\,u(x)=(2\pi)^{-n}\int_{R^n\times R^n}e^{i(x-y)\cdot\xi}\,p(x, \xi)\,u(y)\,dyd\xi$$

and those of Weyl symbols.

Theorem 2.1. *If a Weyl symbol* $p(x, \xi)$ *and an ordinary symbol* $q(x, \xi)$ *give the same pseudo-differential operator, that is,* $p(x, D)=q^o(x, D)$, *then they are transformed to each other by the following relations*:

$$q(x, \xi)=(2\pi)^{-n}\int_{R^n\times R^n}e^{-iz\cdot\zeta}\,p(x+z/2, \xi+\zeta)\,dzd\zeta, \tag{2.1}$$

$$(2.2) \qquad p(x, \xi) = (2\pi)^{-n} \int_{R^n \times R^n} e^{iz\cdot\zeta} q(x+z/2, \xi+\zeta)\, dz d\zeta\,.$$

Corollary. *In the above theorem if $p(x, \xi)$ and $q(x, \xi)$ have homogeneous expansions $p \sim \sum_{j=0}^{\infty} p_{m-j}$, $q \sim \sum_{j=0}^{\infty} q_{m-j}$, then $p_m = q_m$ (the principal symbol) and $p_{m-1} = (q_{m-1} + \frac{i}{2} \sum_{j=1}^{n} \partial_{x_j} \partial_{\xi_j} q_m) =$ (the subprincipal symbol of $q^0(x, D)$).*

Remark. We may use the following form of the same operator instead of (1.2).

$$(2.3) \quad p(x, D)\, u(x) = (2\pi)^{-2n} \int_{R^n \times R^n \times R^n} e^{i(x-y)\cdot\xi + iy\cdot\eta}\, p\left(y, \frac{\xi+\eta}{2}\right) \hat{u}(\eta)\, d\eta dy d\xi$$
$$\text{for} \quad u \in \mathcal{S}(R^n), \quad \left(\hat{u}(\eta) = \int_{R^n} e^{-ix\cdot\eta} u(x)\, dx\right).$$

We use the same notation $S^m_{\rho,\delta}$ to denote the set of pseudo-differential operators whose symbols belong to $S^m_{\rho,\delta}$.

Notation 2.2. We denote the symbol of multi-product $p_1(x, D) \cdots p_\nu(x, D)$ of pseudo-differential operators $p_j(x, D)$ with symbol $p_j(x, \xi)$ by $(p_1 \circ \cdots \circ p_\nu)(x, \xi)$. We use the notation $\sigma(P)$ to denote the symbol of P. So $\sigma(p(x, D)) = p(x, \xi)$.

Theorem 2.3.

$$(p_1 \circ \cdots \circ p_\nu)(x, \xi) = (2\pi)^{-n\nu}\, 2^n \int \cdots \int \exp\{i \sum_{j=1}^{\nu} \eta_j (y_j - y_{j-1})\}$$
$$\times \prod_{j=1}^{\nu} p_j(x + y_j/2 + y_{j+1}/2, \xi + \eta_j)\, dy_1 \cdots dy_\nu\, d\eta_1 \cdots d\eta_\nu\,,$$

where $\eta_{\nu+1} = -\eta_1$, $y_{\nu+1} = -y_1$. Moreover if p_j belongs to $S^{m(j)}_{\rho,\delta}$ $(j=1, \cdots, \nu)$ then $p = p_1 \circ \cdots \circ p_\nu \in S^m_{\rho,\delta}$ $(m = \sum_{j=1}^{\nu} m(j))$ satisfies the following estimate for any l

$$|p|_l^{(m)} \leqslant C^\nu \prod_{j=1}^{\nu} |p_j|_{l+l_0}^{(m(j))}\,,$$

where C and l_0 are independent of ν.

Theorem 2.4. *Let p_j belong to $S^{m(j)}_{\rho(j),\delta(j)}$ $(j=1, 2)$ and $\delta(j) < 1$, $\rho(j) \geqslant \delta(k)$ $(j \neq k)$. Then for any N, we get the expansion*

$$p_1 \circ p_2 = \sum_{k=0}^{N-1} (2i)^{-k} (k!)^{-1} \sigma_k(p_1, p_2) + q_N\,,$$

where

$$(2.4) \qquad \sigma_k(p_1, p_2) = \sum_{|\alpha|+|\beta|=k} (-1)^{\beta} \frac{k!}{\alpha!\,\beta!} p_{1(\beta)}^{(\alpha)} p_{2(\alpha)}^{(\beta)},$$

$q_N \in S^{m-\varepsilon N}_{\rho,\delta}$, $m=m(1)+m(2)$, $\rho=\min(\rho(1), \rho(2))$, $\delta=\max(\delta(1), \delta(2))$ *and* $\varepsilon=\min\{\rho(1)-\delta(2), \rho(2)-\delta(1)\}\geqslant 0$. *Moreover there exists constants* l_0 *and* C *for any* l *such that*

$$|q_N|_l^{(m-\varepsilon N)} \leqslant C \sum_{|\alpha|+|\beta|=N} |p_{1(\beta)}^{(\alpha)}|_{l+l_0}^{(m(1)+\delta(1)|\beta|-\rho(1)|\alpha|)} \times |p_{2(\alpha)}^{(\beta)}|_{l+l_0}^{(m(2)+\delta(2)|\alpha|-\rho(2)|\beta|)}.$$

Theorem 2.5. *Let* $p \in S^m_{\rho,\delta}$. *Then its adjoint is a pseudo-differential operator with symbol* $p^*(x, \xi)$.

We get some properties for $\sigma_k(p, q)$ of (2.4) in case that p and q are arbitrary smooth functions on $\boldsymbol{R}^n \times \boldsymbol{R}^n$.

DEFINITION 2.6. (i) ∇p means a vector $\begin{pmatrix}\nabla_x p\\ \nabla_\xi p\end{pmatrix} = {}^t(\partial_1 p, \cdots, \partial_{x_n} p, \partial_{\xi_1} p, \cdots, \partial_{\xi_n} p)$ for $p \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^n)$, where superscript t stands for transpose.

(ii) J is a transformation on $\boldsymbol{C}^n \times \boldsymbol{C}^n$ defined by $J\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}v\\ -u\end{pmatrix}$, (See (0.4)).

(iii) H_p is the Hessian matrix of p (See (0.5)).

(iv) $\langle a, b\rangle = \sum_{j=1}^{2n} a_j b_j$ for a pair of vectors $a={}^t(a_1, \cdots, a_{2n})$ and $b={}^t(b_1, \cdots, b_{2n})$.

For scalar valued smooth functions p, q, ϕ, we have

Proposition 2.7. (i) $\sigma_k(p, q)=(-1)^k \sigma_k(q, p) \quad (k=0, 1, 2, \cdots)$
(ii) $\sigma_1(p, q)=\langle J\nabla p, \nabla q\rangle$
(iii) $\sigma_2(p, q)=-\mathrm{tr}\,(JH_p JH_q)$
(iv) $\sigma_1(p, \exp\phi)=\sigma_1(p, \phi)\exp\phi$
(v) $\sigma_2(p, \exp\phi)=\sigma_2(p, \phi)\exp\phi+\langle J\nabla\phi, H_p J\nabla\phi\rangle\exp\phi$.

Proof of Theorem 1. We obtain the fundamental solution $E(t)$ by applying the same method as that of [7] for a single operator. The uniqueness of $E(t)$ will be shown in Theorem 2.8. The shape of the phase function ϕ_1 of (1.4) near $\{t=0\}\times\Sigma$ is important. So we sketch how to construct ϕ_1 and f_j $(j\geqslant 1)$.

If we assume that $\exp\phi$ belongs to $\mathcal{W}-\mathcal{E}^0_t(S^0_{1/2,1/2})$, we get by Theorem 2.4

$$\frac{d}{dt}\exp\phi + p\circ\exp\phi \equiv \frac{d}{dt}\exp\phi + \sum_{k=0}^{2} (2i)^{-k}(k!)^{-1}\sigma_k(p_m, \exp\phi) + p_{m-1}\exp\phi \qquad \mathrm{mod}\ S^{m-3/2}_{1/2,1/2}.$$

Off the characteristics we take $\phi=-p_m t$. Near $\{t=0\}\times\Sigma$, $\phi=\phi_1$ should satisfy the equation approximately

$$\begin{cases} \dfrac{d}{dt}\,\phi_1+\sum_{k=0}^{2}(2i)^{-k}\,(k!)^{-1}\,\sigma_k(p_m, \exp\phi_1)\exp(-\phi_1)+p_{m-1}=0\,, \\ \phi_1|_{t=0}=0\,. \end{cases}$$

Put $\phi_1=-p_{m-1}\,t+\tilde{\phi}_1\,I$ for some function $\tilde{\phi}_1$. Then $\tilde{\phi}_1$ should satisfy the following equation by Proposition 2.7 if we neglect derivatives of p_{m-1}

$$\text{(2.5)}\qquad \begin{cases} \dfrac{d}{dt}\,\tilde{\phi}_1+q_m+i\langle\nabla q_m, J\nabla\tilde{\phi}_1\rangle/2+\mathrm{tr}\,(JH_{q_m}\,JH_{\tilde{\phi}_1})/8 \\ \qquad\qquad +\langle\nabla\tilde{\phi}_1, JH_{q_m}\,J\nabla\tilde{\phi}_1\rangle/8=0\,, \\ \tilde{\phi}_1|_{t=0}=0\,. \end{cases}$$

We get the following equation for $X=iJH_{\tilde{\phi}_1}$ taking derivatives of (2.5) and neglecting the terms which include the derivation of $\tilde{\phi}_1$, p_m of more than second order.

$$\begin{cases} \dfrac{d}{dt}\,X+A-AX^2/4=0\,,\ A=iJH_{q_m}\ ((0.3)) \\ X|_{t=0}=0\,. \end{cases}$$

Thus we get $X=-2\tanh(At/2)$ and

$$\phi_1=-\{q_m\,t+\langle\nabla q_m\,t, F(At/2)\,J\nabla q_m\,t\rangle/4 \\ +2^{-1}\,\mathrm{tr}\,[\log\{\cosh(At/2)\}]\}\,I-p_{m-1}\,t\,.$$

In our case $p_{m-1}(x, \xi)$ of ϕ_1 is a matrix. So we use the following estimate found in Chapter II of Gel'fand and Shilov [4].

$$\|\exp(-tp_{m-1}(x, \xi))\|\leqslant e^{-t\Lambda}\sum_{j=1}^{k-1}(2t\,\|p_{m-1}(x, \xi)\|)^j\quad t\geqslant 0\,,$$

where $\Lambda=\min_{1\leqslant j\leqslant k}(\mathrm{Re}\,\mu_j)$, $\{\mu_j\}_{j=1}^{k}$ are the eigenvalues of $p_{m-1}(x, \xi)$. Then there exist constant $c_{\alpha,\beta}$ and $d_{\alpha,\beta}$ for any α, β such that

$$\text{(2.6)}\qquad \|\partial_\xi^\alpha\partial_x^\beta(\exp(-tp_{m-1}(x, \xi)))\|\leqslant c_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\,(1+t\langle\xi\rangle^{m-1})^{d_{\alpha,\beta}}\,e^{-t\Lambda}$$

Noting

$$[\det\{\cosh(At/2)\}]^{-1/2}\leqslant C\exp(-\widetilde{\mathrm{tr}}A\,t/2)\,,$$

we have

$$\|\partial_\xi^\alpha\,\partial_x^\beta\,(\exp\phi_1^0)\|\leqslant c'_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\,(1+t\langle\xi\rangle^{m-1})^{d_{\alpha,\beta}}\exp(-(\Lambda+\widetilde{\mathrm{tr}}A/2)\,t)$$

for $\phi_1^0=-2^{-1}\,\mathrm{tr}\,[\log\{\cosh(At/2)\}]\,I-p_{m-1}\,t$.

Hence we have

$$\text{(2.7)} \qquad \| \exp \phi_1^0 \| \leqslant C \exp(-c' |\xi|^{m-1} t)$$

for $c'<c$ with some constant C according to (A)–(ii). By (2.7) and the same method of Proposition 1.25 of [7] the function ϕ of (1.3) satisfies

$$\text{(2.8)} \qquad \| \exp \phi \| \leqslant c_0 \exp(-\phi_0),$$

where

$$\phi_0 = \Phi_2 \tilde{\psi} + q_m\, t(1-\psi^1) + c't\langle\xi\rangle^{m-1} \geqslant c't\langle\xi\rangle^{m-1},$$

with $\Phi_2(x, \xi) = q_m((x, \xi)+h)\, t$, $h = h_0(At/2)\, J\nabla q_m t$
and $h_0(\lambda) = 2^{-1} F(\lambda)\{1+(\lambda^{-1}\tanh\lambda)^{-1/2}\}^{-1}$.
To find f_j ($j=1, 2, \cdots, N$) we must seek the solution of the equation

$$\text{(2.9)} \qquad \frac{d}{dt} f + \sum_{j=1}^{2} (2i)^{-j} (j!)^{-1} \exp(-\phi) \{\sigma_j(p_m, (\exp\phi) f) - \sigma_j(p_m, \exp\phi) f\} = g$$

for some given matrix g. We can apply the same method as in Lemma 1.15 of [7] to solve (2.9) approximately. Noting the estimate of g obtained by (2.6), we have

$$\text{(2.10)} \qquad \| \partial_\xi^\alpha \partial_x^\beta ((\exp\phi) f_j) \| \leqslant C_{j,\alpha,\beta} \langle\xi\rangle^{-\varepsilon j + (|\beta|-|\alpha|)/2} \exp(-\phi_0/2)$$

by the same reason as that of Proposition 1.28 of [7]. It is clear that $(\exp\phi) f_j$ belongs to $S^{-\varepsilon j}_{1/2,1/2}$ by (2.10).

Once an approximate solution $E_N(t)$ has been obtained, a fundamental solution $E(t)$ is constructed by solving the equation

$$\text{(2.11)} \qquad E(t) + \int_0^t E(t-s)\, G_N(s)\, ds = E_N(t),$$

where $E_N(t) = \sum_{j=0}^{N} (\exp\phi) f_j(t, x, D)$ and $G_N(t) = \left(\frac{d}{dt} + P\right) E_N(t)$. To get the solution of (2.11) we use the estimate of symbols of multi-product given in Theorem 2.3. Q.E.D.

Theorem 2.8. *Under Condition* (A), $E(t)$ *is the unique fundamental solution in* $\mathcal{W}-\mathcal{E}_t^0(S^\infty_{\rho,\delta})$ *in any finite interval* $[0, T]$.

Proof. By the same method as in the proof of Theorem 2.2 of [7], we can choose a constant $c>0$ such that

$$\mathrm{Re}\,(p(x, D)\, u, u) + c(u, u) \geqslant 0 \qquad u \in (\mathcal{S}(\boldsymbol{R}^n))^k.$$

Now consider the Cauchy problem

$$\begin{cases} \left\{\dfrac{d}{dt}+p(x, D)\right\} E(t) = 0\,, & t>0\,, \\ E(0) = 0\,. \end{cases}$$

Then $E_c(t)=e^{-ct}E(t)$ satisfies

$$\begin{cases} \left\{\dfrac{d}{dt}+p(x, D)+c\right\} E_c(t) = 0\,, & t>0\,, \\ E_c(0) = 0\,. \end{cases}$$

The inequality

$$\frac{d}{dt}(E_c(t, s)\, u, E_c(t, s)\, u) = -2 \operatorname{Re}((P+c)\, E_c(t, s)\, u, E_c(t, s)\, u)\leqslant 0$$

implies

$$\| E_c(t, s)\, u \| \leqslant \| E_c(s, s)\, u \| = 0\,.$$

Therefore we get conclusion of the theorem. Q.E.D.

Under the same assumption (A) for $p(x, \xi)$, we get

Theorem 2.9. *$p^*(x, \xi)$ also satisfies Condition (A) and we can construct a fundamental solution $V(t)\in\mathcal{W}-\mathcal{E}_t^0\,(S^0_{1/2,1/2})$ of*

$$(2.12)\qquad \begin{cases} \left[\dfrac{d}{dt}+p^*(x, D)\right] V(t) = 0\,, & t>0\,, \\ V(0) = I\,. \end{cases}$$

Moreover

$$(2.13)\qquad E^*(t) = V(t)$$

and

$$(2.14)\qquad \frac{d}{dt}E(t)+E(t)\,p(x, D) = 0\,, \qquad t>0\,.$$

Proof. Let $0<r<t$ be any number. For any f and $g\in(\mathcal{S}(\boldsymbol{R}^n))^k$ we have

$$\frac{\partial}{\partial r}(E(r)\, f, V(t-r)\, g)$$
$$= -(P\, E(r)\, f, V(t-r)\, g)+(E(r)\, f, P^*V(t-r)\, g) = 0\,.$$

Integrating it in r from 0 to t, we obtain

(2.15) $$(E(0)f,\ V(t)g) = (E(t)f,\ V(0)g)\,.$$

From (2.15) we have (2.13). Taking the adjoint of (2.12), we get (2.14) by (2.13). Q.E.D.

Corollary. *Under Condition* (A), *we can construct a parametrix*

$$Q = \int_0^T E(t)\,dt \in S^{1-m}_{1/2,1/2}\ (T>0) \quad \textit{for } p(x, D)\,.$$

Proof. In view of Theorem 1 Q is a right parametrix, and by Theorem 2.9 Q is a left parametrix also. The function ϕ defined by (1.3) satisfies $\|\exp\phi\| \leqslant c_0 \exp(-c'\langle\xi\rangle^{m-1}t)$. So it is easy to see that Q belongs to $S^{1-m}_{1/2,1/2}$. Q.E.D.

In the rest of this section we assume that

(2.16) $$q_m(x, \xi) = \sum_{j=1}^{l} |z_j(x, \xi)|^2\,,$$

where $z_j(x, \xi)$ belongs to $S^{m/2}_{1,0}$ for each j.

Proposition 2.10. *On the characteristic set* $\Sigma = \{(x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}^n;\ q_m(x, \xi) = 0\}$ *the non zero eigenvalues of A coincide with those of $\mathcal{M}$ including multiplicities, where $\mathcal{M}$ is the symmetric matrix defined by*

(2.17) $$\mathcal{M} = \begin{pmatrix} {}^tC, & D \\ -D, & -C \end{pmatrix}.$$

Here C and D are $l\times l$ matrices whose elements are given by

$$c_{j,k} = i\langle J\nabla z_j,\ \nabla \bar{z}_k\rangle$$

and

$$d_{j,k} = i\langle J\nabla z_j,\ \nabla z_k\rangle\,.$$

Proof. The Hesse matrix of q_m is equal to

$$\sum_{j=1}^{l} \{\nabla\bar{z}_j^t(\nabla z_j) + \nabla z_j^t(\nabla\bar{z}_j)\}$$

on Σ when (2.16) holds. Let X be the $2n\times 2l$ matrix defined by

(2.18) $$X = (\nabla z_1, \cdots, \nabla z_l, \nabla\bar{z}_1, \cdots, \nabla\bar{z}_l)\,.$$

Then $A = iJXX^*$ on Σ. On the other hand we have $\mathcal{M} = iX^*JX$ by (2.17) and (2.18). Then

(2.19) $$X^*A = \mathcal{M}X^*$$

and

(2.20) $$AJX = JX\mathcal{M}\,.$$

The conclusion of the proposition follows from (2.19) and (2.20) without difficulty.
Q.E.D.

Theorem 2.11. *We assume* (2.16) *and Condition* (A). *Then the parametrix $Q=q(x, D)$ obtained in Corollary of Theorem* 2.9 *satisfies*

(2.21) $$z_j \circ q,\quad \bar{z}_j \circ q \in S^{(1-m)/2}_{1/2,1/2} \quad (j=1, \cdots, l)$$

and

(2.22) $$z^{\alpha}\, \bar{z}^{\beta} \circ q \in S^{m(|\alpha|+|\beta|-2)/2}_{1/2,1/2} \quad (|\alpha|+|\beta|\geqslant 2)\,,$$

where $z^{\alpha}\, \bar{z}^{\beta} = z_1^{\alpha_1} \cdots z_l^{\alpha_l}\, \bar{z}_1^{\beta_1} \cdots \bar{z}_l^{\beta_l}\, I$.

Proof. We have

$$z^{\alpha} \circ q = \sum_{j=0}^{2} r_j^{\alpha}\,,$$

where $r_j^{\alpha} = (2i)^{-j}\, (j!)^{-1}\, \sigma_j(z^{\alpha}, q) \in S^{1-m-j+m|\alpha|/2}_{1/2,1/2}$ $(j=0, 1)$ and $r_2^{\alpha} \in S^{m|\alpha|/2-m}_{1/2,1/2}$ since Q belongs to $S^{1-m}_{1/2,1/2}$ and z^{α} belongs to $S^{m|\alpha|/2}_{1,0}$. Then it is sufficient to show

$$z_j\, q \in S^{(1-m)/2}_{1/2,1/2}\,,\quad z^{\alpha}\, q \in S^{m(|\alpha|-2)/2}_{1/2,1/2} \quad (|\alpha|=2)$$

and

$$\sigma_1(z^{\alpha}, q) \in S^{m(|\alpha|-2)/2}_{1/2,1/2} \quad (|\alpha|=2)\,.$$

By $q=\int_0^T e(t)\, dt$, for concluding (2.21) and (2.22) it suffices to show

(2.23) $$|z_j \exp(-\phi_0/2)| \leqslant C\langle\xi\rangle^{(m-1)/2} \exp(-\phi_0/4)\,,$$

(2.24) $$|z_i z_j \exp(-\phi_0/2)| \leqslant C\langle\xi\rangle^{m-1} \exp(-\phi_0/4)$$

and

(2.25) $$\|\, z_i\, \sigma_1(z_j I, (\exp \phi)\, f_k)\, \| \leqslant C\langle\xi\rangle^{m-1} \exp(-\phi_0/4)$$

by (2.8) and (2.10). Proposition 1.23 of [7] and the Taylor expansion lead to

$$z_j(x, \xi) = z_j((x, \xi)+h) + z_j'(x, \xi)\,,$$

where $z_j'(x, \xi)$ satisfies

(2.26) $$|z_j'(x, \xi)| \leqslant C\, \{(\Phi_2)^{1/2} + \langle\xi\rangle^{(m-1)/2}\, (\Phi_2 + t\langle\xi\rangle^{m-4/3})\, (1+t\langle\xi\rangle^{m-1})^d\}$$

for some $d \geqslant 0$.

(2.23) and (2.24) are proved by (2.8) and (2.26). (2.25) is clear by (2.10) and (2.24) E.Q.D.

3. Fundamental solution for a degenerate parabolic operator on bundles

In this section we consider pseudo-differential operators on sections of vector bundles over a compact manifold M. The definitions of pseudo-differential operators on a manifold are given, for example, in Hörmander [6], Kumano-go [8] and Treves [13]. We use, in this paper, the definition given by Hörmander and Treves. We state it here for the sake of completeness. In this section we always assume $\rho+\delta\geqslant 1$ and $\delta<1$.

For an open set Ω in $\boldsymbol{R}^n$, $S^m_{\rho,\delta}(\Omega)$ is the set of $C^\infty(\Omega\times\boldsymbol{R}^n)$-functions such that for any compact set $K\subset\Omega$ and multi-indeces α, β, there exists a constant $C_{K,\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x, \xi)|\leqslant C_{K,\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|},\ x\in K,\ \xi\in\boldsymbol{R}^n\,.$$

DEFINITION 3.1. A linear operator L: $C^\infty_0(M)\rightarrow C^\infty(M)$ belongs to $S^m_{\rho,\delta}(M)$ if the kernel of L is smooth off the diagonal in $M\times M$ and if for any local chart θ of M with χ: $\theta\rightarrow\Omega$ a diffeomorphism onto an open set Ω of $\boldsymbol{R}^n$, the mapping of $C^\infty_0(\Omega)$ into $C^\infty(\Omega)$ given by $u\rightarrow L(u\circ\chi)\circ\chi^{-1}$ belongs to $S^m_{\rho,\delta}(\Omega)$. Elements of $S^m_{\rho,\delta}(M)$ are called pseudo-differential operators on M.

DEFINITION 3.2. Let $\boldsymbol{E}$, $\boldsymbol{F}$ be vector bundles over a compact manifold M and let $\Gamma(\boldsymbol{E})$ be the set of sections of $\boldsymbol{E}$. We say a linear operator P: $\Gamma(\boldsymbol{E})\rightarrow\Gamma(\boldsymbol{F})$ is a pseudo-differential operator of class $S^m_{\rho,\delta}(M;\ \boldsymbol{E})$ if the kernel of P is smooth off the diagonal in $M\times M$ and if for any local chart θ and any pair of local basis $e_1, \cdots, e_\mu$, and $f_1, \cdots, f_\nu$ of $\boldsymbol{E}$ and $\boldsymbol{F}$ over θ respectively, there exists $P_{i,j}\in S^m_{\rho,\delta}(\Omega)$ such that

$$(3.1)\qquad (Pv)_i=\sum_{j=1}^{\mu}P_{i,j}\,v_j \text{ in } \Omega,\quad 1\leqslant i\leqslant\nu\,,$$

where $u=\sum_{j=1}^{\mu}(v_j\circ\chi)\,e_j$ and $Pu=\sum_{i=1}^{\nu}((Pv)_i\circ\chi)\,f_i$.

DEFINITION 3.3 (properly supported). A distribution $U\in\mathcal{D}'(\Omega\times\Omega)$ is said to be properly supported if supp U has a compact intersection with $K\times\Omega$ and with $\Omega\times K$ for any compact $K\subset\Omega$. A pseudo-differential operator is said to be properly supported if its kernel is properly supported.

Theorem 3.4. *Let Ω and Ω' be open sets in $\boldsymbol{R}^n$ and let ϕ: $\Omega\rightarrow\Omega'$ be a diffeomorphism. Suppose P is a matrix consisting of property supported operators in $S^m_{\rho,\delta}(\Omega')$. Then we get a matrix $\tilde{P}$ consisting of elements of $S^m_{\rho,\delta}(\Omega)$ such that*

$$\tilde{p}(y, D_y)(u\circ\phi)(y) = [(p(x, D_x)\, u(x))\circ\phi](y) \qquad \textit{for} \quad u\in C_0^\infty(\Omega')\,.$$

$$(3.2) \quad \tilde{p}(y, \eta) = (2\pi)^{-n}\iint e^{iz\cdot\zeta}\, p(\phi(y+z/2)/2+\phi(y-z/2)/2,\ {}^t\Psi(y, z)^{-1}(\zeta+\eta)) \times |\Psi(y, z)|^{-1}\left|\frac{\partial}{\partial y}\,\phi(y-z/2)\right| dzd\zeta\,,$$

where $\Psi(y, z)$ *is a matrix valued smooth function such that*

$$\phi(y+z/2)-\phi(y-z/2) = \Psi(y, z)\, z\,.$$

Corollary. *Pseudo-differential operators on sections of the bundles over a manifold are well-defined. If P has a homogeneous expansion* $p\sim\sum_{j=0}^{\infty} p_{m-j}$, *then* $\tilde{p}$ *has also a homogeneous expansion* $\tilde{p}\sim\sum_{j=0}^{\infty}\tilde{p}_{m-j}$ *which satisfy*

$$(3.3) \qquad \text{(i)} \quad \tilde{p}_m(y, \eta) = p_m(\phi(y), (\partial_y\phi)^{-1}\eta)$$

$$(3.4) \qquad \text{(ii)} \quad \tilde{p}_{m-1}(y, \eta) = p_{m-1}(\phi(y), (\partial_y\,\phi)^{-1}\eta) \quad \textit{if}\ \tilde{p}_m(y, \eta) = \nabla\tilde{p}_m(y, \eta) = 0\,.$$

REMARK. On manifolds it is natural to use (2.3) as a definition of pseudo-differential operators instead of (1.2). Condition (A) is independent of the choise of local coordinates by (3.3) and (3.4).

Proof of Theorem 2. Take a finite covering of M by a local chart $(\theta_\kappa, \chi_\kappa)_{\kappa\in K}$. Using the local coordinates, we get systems of pseudo-differential operators $P^\kappa=(P^\kappa_{i,j})$ satisfying Condition (A) in $\Omega_\kappa=\chi_\kappa(\theta_\kappa)$. We may assume that $p^\kappa\in S^m_{1,0}(\Omega_\kappa)$ are extended to $\tilde{p}^\kappa\in S^m_{1,0}(\boldsymbol{R}^n)$ satisfying Condition (A) of Theorem 1. According to Theorem 1, we can construct a fundamental solution $\tilde{E}^\kappa(t)\in S^0_{1/2,1/2}(\boldsymbol{R}^n)$ of

$$(3.5) \qquad \begin{cases} \left(\dfrac{d}{dt}+\tilde{P}^\kappa\right)\tilde{E}^\kappa(t) = 0\,, & t>0\,,\\ \tilde{E}^\kappa(0) = I\,. \end{cases}$$

For $\tilde{E}^\kappa(t)=(\tilde{E}^\kappa_{i,j}(t))_{i,j=1,\cdots,k}$ we define operators $E^\kappa_{i,j}(t)$ on θ_κ such that

$$(3.6) \qquad \tilde{E}^\kappa_{i,j}(t)\, v = E^\kappa_{i,j}(t)\,(v\circ\chi_\kappa)\circ\chi_\kappa^{-1} \qquad \text{for} \quad v\in C_0^\infty(\chi_\kappa(\theta_\kappa))\,.$$

Choose $\{\phi_\kappa\}_{\kappa\in K}$ a partition of unity subordinate to θ_κ and another function $\psi_\kappa\in C_0^\infty(\theta_\kappa)$ such that $\phi_\kappa\psi_\kappa=\phi_\kappa$ $(\kappa\in K)$. Put

$$(3.7) \qquad \tilde{E}(t)u = \sum_{\kappa\in K}\sum_{i=1}^{k}\sum_{j=1}^{k}\psi_\kappa\, E^\kappa_{i,j}(t)\,(\phi_\kappa\, u)_j\, e^\kappa_i\,,$$

where $\phi_\kappa u = \sum_{j=1}^{k} (\phi_\kappa u)_j e_j^\kappa$. Then $\tilde{E}(t)$ belongs to $S^0_{1/2,1/2}(M, \boldsymbol{E})$ and by (3.5) $\sim$(3.7) it is clear that

$$\begin{cases} \left(\dfrac{d}{dt}+P\right)\tilde{E}(t) = K(t) \quad t>0\,, \\ \tilde{E}(0) = I\,, \end{cases}$$

where $K(t)$ is a smoothing operator.
We assume that the fundamental solution $E(t)$ is of the form

$$E(t) = \tilde{E}(t)+\int_0^t \tilde{E}(t-s)\,\Phi(s)\,ds\,.$$

Then $\Phi(t)$ must satisfy the following integral equation

$$K(t)+\Phi(t)+\int_0^t K(t-s)\,\Phi(s)\,ds = 0\,. \tag{3.8}$$

$\Phi(t)=\sum_{j=1}^{\infty}\Phi^j(t)$ satisfies (3.8), where $\Phi^j(t)$ is defined by

$$\Phi^0(t) = -K(t)$$
$$\Phi^j(t) = -\int_0^t K(t-s)\,\Phi^{j-1}(s)\,ds\,.$$

It is clear that $\Phi(t)$ is a smoothing operator. Uniqueness is shown by the same method as that of Theorem 2.8. The proofs of the assertions (i), (ii) for a formally self-adjoint system satisfying Condition (B) are omitted since they are obtained by applying the method of Section 4 of [7] to a system of pseudo-differential operators on M with diagonal principal symbol instead of pseudo-differential operators on M. Q.E.D.

4. Proof of Theorem 3

M is a CR-manifold of dimension $2l+1$ i.e. a real orientable C^∞-manifold with a subbundle S of complex tangent bundle $\boldsymbol{C}TM$ satisfying the following conditions:

(i) $\dim_C S = l$

(ii) $S\cap\bar{S} = \{0\}$

(iii) $[\Gamma(S), \Gamma(S)]\subset\Gamma(S)$,

where $\Gamma(S)$ stands for the space of C^∞ cross sections of S. We fix F a complexification of the line bundle of TM such that

$$\boldsymbol{C}TM = S\oplus\bar{S}\oplus F\,.$$

We denote $(\Lambda^p S^*)\otimes(\Lambda^q \bar{S}^*)$ by $\Lambda^{p,q}$. The operator $\bar{\partial}_b$: $\Gamma(\Lambda^{p,q})\to\Gamma(\Lambda^{p,q+1})$ is defined as

$$\langle\bar{\partial}_b\phi, (Z_1\wedge\cdots\wedge Z_p)\otimes(W_1\wedge\cdots\wedge W_{q+1})\rangle$$
$$=\sum_{j=1}^{q+1}(-1)^{j-1}W_j\langle\phi, (Z_1\wedge\cdots\wedge Z_p)\otimes(W_1\wedge\cdots\wedge\widehat{W}_j\wedge\cdots\wedge W_{q+1})\rangle$$
$$+\sum_{i<j}(-1)^{i+j}\langle\phi, (Z_1\wedge\cdots\wedge Z_q)\otimes([W_i, W_j]\wedge W_1\cdots\wedge\widehat{W}_i\wedge\cdots\widehat{W}_j\wedge\cdots\wedge W_{q+1}\rangle$$
$$\text{for}\quad Z_1, \cdots, Z_p\in\Gamma(S),\ W_1, \cdots, W_{q+1}\in\Gamma(\bar{S}).$$

Then $\bar{\partial}_b$ forms a complex. Let $L_1, \cdots, L_l$ be a local basis for sections of S and choose a non zero local section T of $TM\cap F$. Then $L_1, \cdots, L_l, \bar{L}_1, \cdots, \bar{L}_l$ and T span $\boldsymbol{C}TM$. The Levi form $L=(L_{i,j})$ defined by

$$i\,[L_i, \bar{L}_j]=L_{i,j}\,T \mod (L_1, \cdots, L_l, \bar{L}_1, \cdots, \bar{L}_l) \tag{4.1}$$

is hermitian on $\boldsymbol{C}^l$. If we introduce a Hermitian metric on M such that S, $\bar{S}$ and F are mutually orthogonal, then we can define the formal adjoint operator ϑ_b: $\Gamma(\Lambda^{p,q})\to\Gamma(\Lambda^{p,q-1})$ of $\bar{\partial}_b$ and the Laplacian $\Box_b=\bar{\partial}_b\vartheta_b+\vartheta_b\bar{\partial}_b$: $\Gamma(\Lambda^{p,q})\to\Gamma(\Lambda^{p,q})$. Following Folland and Kohn [2] we introduce the condition $Y(q)$ for the Levi form.

DEFINITION 4.1. We say that M satisfies $Y(q)$ if

$$\max(\mu_+, \mu_-)\geqslant\max(q+1, l+1-q)$$

or

$$\min(\mu_+, \mu_-)\geqslant\min(q+1, l+1-q)$$

at each point of M, where μ_+ (μ_-) are numbers of the positive (negative) eigenvalues of the Levi form L, respectively.

REMARK. The condition $Y(q)$ is independent of the choise of $L_1, \cdots, L_l$, T (See [2]). The Hermitian metric which we use is arbitrary as far as S, $\bar{S}$ and F are mutually orthogonal. Hence, we need not choose Hermitian metrices bearing a relationship to the Levi form as Folland–Stein [3] and Greiner–Stein [5] did.

Now we will calculate the symbol of self-adjoint operator $\Box_b$ in a local chart to apply Theorem 2. Let $L_1, \cdots, L_l$ and T be an orthonormal basis over an open set U of M. Let $\omega^1, \cdots, \omega^l, \omega^{l+1}$ be its dual basis. We can write by (4.1) with the Levi form $L=(L_{i,j})$

$$i\,[L_i, \bar{L}_j]=L_{i,j}\,T+\sum_{k=1}^{l}a_{i,j}^k\,L_k+\sum_{k=1}^{l}b_{i,j}^k\,\bar{L}_k \quad \text{on } U.$$

For simplicity we consider $\Box_b$ on $\Gamma(\Lambda^{0,q})$. No difference appears in other cases.

For $\phi=\sum_{|J|=q}\phi_J\,\bar{\omega}^J\in\Gamma(\Lambda^{0,q})$, where $\bar{\omega}^J=\bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_q}$, $J=(j_1,\cdots,j_q)$, $j_1<\cdots<j_q$ ($j_k\in\{1,\cdots,l\}$), we have

$$\bar{\partial}_b\,\phi=\sum_{j=1}^{l}\bar{L}_j\,\phi_J\,\bar{\omega}^j\wedge\bar{\omega}^J+\mathcal{E}(\phi)\,,$$

$$\vartheta_b\,\phi=-\sum_{j=1}^{j}L_j\,\phi_J\,\bar{\omega}^j\lrcorner\bar{\omega}^J+\mathcal{E}(\phi)\,,$$

where $\mathcal{E}(\phi)$ means linear combinations of ϕ_J with coefficients of smooth functions and

$$\begin{aligned}\bar{\omega}^j\lrcorner\bar{\omega}^J&=(-1)^{k-1}\,\bar{\omega}^{\tilde{J}}\text{ for }J=(j_1,\cdots,j_k,\cdots j_q)\ j_k=j\text{ and }\tilde{J}=(j_1,\cdots,\hat{j_k},\cdots j_q)\\&=0\text{ if }j\notin J\,.\end{aligned}$$

Then we have

$$\begin{aligned}\Box_b\,\phi&=(\bar{\partial}_b\,\vartheta_b+\vartheta_b\,\bar{\partial}_b)\,\phi\\&=-\sum_{|J|=q}\Big(\sum_{m\notin\{J\}}L_m\,\bar{L}_m+\sum_{m\in\{J\}}\bar{L}_m\,L_m\Big)\,\phi_J\,\bar{\omega}^J\\&\quad-\sum_{|J|=q}\Big(\sum_{j\neq m}L_j\,\bar{L}_m-\bar{L}_m\,L_j\Big)\,\phi_J\,\bar{\omega}^j\lrcorner\,(\bar{\omega}^m\wedge\bar{\omega}^J)\\&\quad+\mathcal{E}(L\phi,\,\bar{L}\phi)\,,\end{aligned}\tag{4.2}$$

where $\mathcal{E}(L\phi,\,\bar{L}\phi)$ means linear combinations of $L_j\phi_J$, $\bar{L}_j\phi_J$ and ϕ_J with coefficients of smooth functions.

Put $k=l!/(l-q)!\,q!$ and $n=2l+1$. For local basis $(\bar{\omega}^J)_J$, the symbol of $\Box_b$ is given by

$$\sigma(\Box_b)=p_2+p_1+p_0\,,$$

where p_j ($j=0,1,2$) is a $k\times k$ matrix homogeneous of order j in ξ such that

Proposition 4.2. (i) $p_2=\sum_{j=1}^{l}|z_j(x,\xi)|^2\,I=q_2\,I,$

where

$$z_j(x,\xi)=\sum_{m=1}^{n}\alpha_j^m(x)\,\xi_m$$

when

$$L_j=\sum_{m=1}^{n}\alpha_j^m\,\frac{\partial}{\partial x_m}\,.$$

(ii) $\widetilde{\mathrm{tr}}\,(iJH_{q_2})=\sum_{j=1}^{l}|\mu_j(x,\xi)|$ *on the characteristic* $\Sigma=\{(x,\xi);\ z_j(x,\xi)=0,$

$j=1, \cdots, l\}$, *where* $\{\mu_j(x, \xi)\}$ *are the eigenvalues of the matrix* $C(x, \xi)=(c_{i,j}(x, \xi))$ *defined by*

$$c_{i,j}(x, \xi) = i\langle J\nabla z_i(x, \xi), \nabla\bar{z}_j(x, \xi)\rangle . \tag{4.3}$$

(iii) *On* Σ, $p_1(x, \xi)$ *is a Hermitian matrix whose eigenvalues are* $(\mu_J(x, \xi))_J$, *where*

$$\mu_J(x, \xi) = \{\sum_{j\in\{J\}} \mu_j(x, \xi) - \sum_{j\notin\{J\}} \mu_j(x, \xi)\}/2 . \tag{4.4}$$

Proof. We can find smooth functions $\{d_j(x)\}_{j=1}^l$ such that $\sigma(L_j)=iz_j+d_j$ and $\sigma(\bar{L}_j)=i\bar{z}_j+\bar{d}_j$. By Theorem 2.4 we get

$$\begin{aligned} &\sigma(\sum_{m\notin\{J\}} L_m \bar{L}_m + \sum_{m\in\{J\}} \bar{L}_m L_m) \\ &\equiv -\sum_{m=1}^{l} |z_m|^2 - \{\sum_{m\notin\{J\}} \sigma_1(z_m, \bar{z}_m) + \sum_{m\in\{J\}} \sigma_1(\bar{z}_m, z_m)\}/2i \\ &\quad + i\sum_{m=1}^{l} (\bar{d}_m z_m + d_m \bar{z}_m) \\ &\equiv -\{\sum_{m=1}^{l} |z_m|^2 + \sum_{m=1}^{l} \sigma_1(z_m, \bar{z}_m)/2i - \sum_{m\in\{J\}} \sigma_1(z_m, \bar{z}_m)/i\} \\ &\quad + i\sum_{m=1}^{l} (\bar{d}_m z_m + d_m \bar{z}_m) \qquad \bmod S^0_{1,0} \end{aligned} \tag{4.5}$$

$$\begin{aligned} \sigma(L_j \bar{L}_m - \bar{L}_m L_j) &= \sigma([L_j, \bar{L}_m]) \\ &\equiv i\,\sigma_1(z_j, \bar{z}_m) = i\langle J\nabla z_j, \nabla\bar{z}_m\rangle = c_{j,m} \qquad \bmod S^0_{1,0} . \end{aligned} \tag{4.6}$$

So (i) is clear by (4.2) and (4.5). By the property $[\Gamma(S), \Gamma(S)]\subset\Gamma(S)$ we have

$$\sigma([L_i, L_j]) = i\langle J\nabla z_i, \nabla z_j\rangle = 0 \quad \text{on} \quad \Sigma . \tag{4.7}$$

Proposition 2.10 means on Σ

$$\begin{aligned} &\widetilde{\operatorname{tr}}\,(iJH_{q_2}) \\ &= \Big\{\text{the sum of absolute values of eigenvalues} \\ &\qquad\qquad \text{of } \mathcal{M} = \begin{pmatrix} {}^tC, & 0 \\ 0, & -C \end{pmatrix} \Big\}/2 \\ &= \sum_{j=1}^{l} |\mu_j| . \end{aligned}$$

Thus (ii) is proved. By (4.5), (4.6) and $\sigma_1(a, b)=\langle J\nabla a, \nabla b\rangle$ we have on Σ

$$p_1(x, \xi)\,(\sum_{|J|=q}\phi_J\,\varpi^J)$$

(4.8)
$$=\sum_{|J|=q}\{-\sum_{m=1}^{l}c_{m,m}(x, \xi)/2+\sum_{m\in\{J\}}c_{m,m}(x, \xi)\}\;\phi_J\,\varpi^J + \sum_{|K|=|J|=q}(q_1(x, \xi))_{K,J}\;\phi_J\,\varpi^K\,,$$

where $q_1(x, \xi)$ is given by

(4.9)
$$\sum_{|K|=q}(q_1(x, \xi))_{K,J}\,\varpi^K = -\sum_{\substack{j\neq m\\ j\cup\{K\}=m\cup\{J\}}}c_{j,m}\,\varpi^j \lrcorner (\varpi^m\wedge\varpi^J)\,.$$

q_1 is a Hermitian matrix because C is a Hermitian matrix. Choose a unitary matrix U such that

(4.10)
$$UCU^{-1}=\begin{pmatrix}\mu_1 & & 0\\ & \mu_2 & \\ 0 & \ddots & \\ & & \mu_l\end{pmatrix}.$$

Let $\bar{\eta}={}^t(\bar{\eta}^1, \cdots, \bar{\eta}^l)$ be defined as $\bar{\eta}=U\varpi$, $\varpi={}^t(\varpi^1, \cdots, \varpi^l)$. Then we have the following lemma. So the proof is complete since $\{\bar{\eta}^J\}_J$ are linearly independent. Q.E.D.

Lemma 4.3. *$p_1(x, \xi)\,\bar{\eta}^J=\mu_J(x, \xi)\,\bar{\eta}^J$ for any J.*

Proof. We assume the following equality

(4.11)
$$p_1(x, \xi) = -\frac{1}{2}\,(\text{tr } C)\;I+p_1'(x, \xi)\,,$$

where

(4.12)
$$p_1'(\phi_J\,\varpi^J) = \sum_{j=1}^{q}\phi_J(\varpi^{j_1}\wedge\cdots\wedge\varpi^{j_{i-1}}\wedge C\varpi^{j_i}\wedge\varpi^{j_{i+1}}\wedge\cdots\wedge\varpi^{j_q})$$
$$\text{for } J=(j_1, \cdots, j_q)\quad (j_1<j_2<\cdots<j_q)\,,$$
$$C\,\varpi^j = \sum_{m=1}^{l}c_{j,m}\,\varpi^m\,.$$

Then we also prove the formula (4.12) for any ϖ^J ($J=(j_1, \cdots, j_q)\; j_k\in\{1, \cdots, l\}$), because $\varpi^J=\varepsilon_K^J\,\varpi^K$ $K=(k_1, \cdots, k_q)$ $k_1<\cdots<k_q$

$$\begin{aligned}p_1'(\varpi^J) &= \varepsilon_K^J\,p_1'(\varpi^K)\\ &= \varepsilon_K^J\sum_{i=1}^{q}\varpi^{k_1}\wedge\cdots\wedge\varpi^{k_{i-1}}\wedge C\,\varpi^{k_i}\wedge\varpi^{k_{i+1}}\wedge\cdots\wedge\varpi^{k_q}\\ &= \sum_{i=1}^{q}\varpi^{j_1}\wedge\cdots\wedge\varpi^{j_{\sigma(i)-1}}\wedge C\varpi^{j_{\sigma(i)}}\wedge\varpi^{j_{\sigma(i)+1}}\wedge\cdots\wedge\varpi^{j_q}\\ &= \sum_{i=1}^{q}\varpi^{j_1}\wedge\cdots\wedge\varpi^{i_{i-1}}\wedge C\,\varpi^{j_i}\wedge\cdots\wedge\varpi^{j_q}\,,\end{aligned}$$

where σ is the permutation such that $j_{\sigma(i)}=k_i$. By linearlity we can prove (4.12) for any vector of Λ^q. So we can prove by (4.10)

$$p_1'(\bar{\eta}^J) = (\sum_{j\in\{J\}} \mu_j)\, \bar{\eta}^J\,. \tag{4.13}$$

The conclusion follows from (4.11) and (4.13). (4.12) is justified as follows. Noting that if $J=(j_1, \cdots, j_q)$ and $j=j_i$, $\bar{\omega}^j \lrcorner (\bar{\omega}^m \wedge \bar{\omega}^J)=(-1)\ \bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_{i-1}}\wedge \bar{\omega}^m\wedge\bar{\omega}^{j_{i+1}}\wedge\cdots\wedge\bar{\omega}^{j_q}$, we get by (4.8) and (4.9)

$$\begin{aligned}
p_1'(\phi_J\, \omega^J) &= \phi_J \,\{\sum_{i=1}^q \sum_{m=j_i} c_{j_i,m}\, \bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_{i-1}}\wedge\bar{\omega}^m\wedge\bar{\omega}^{j_{i+1}}\wedge\cdots\wedge\bar{\omega}^{j_q} \\
&\quad - \sum_{j\neq m} c_{j,m}\, \bar{\omega}^j \lrcorner (\bar{\omega}^m\wedge\bar{\omega}^J)\} \\
&= \phi_J \,\{\sum_{i=1}^q \sum_{m=j_i} c_{j_i,m}\, \bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_{i-1}}\wedge\bar{\omega}^m\wedge\bar{\omega}^{j_{i+1}}\wedge\cdots\wedge\bar{\omega}^{j_q} \\
&\quad + \sum_{i=1}^q \sum_{m\neq j_i} c_{j_i,m}\, \bar{\omega}^{j_1}\wedge\cdots\wedge\bar{\omega}^{j_{i-1}}\wedge\bar{\omega}^m\wedge\bar{\omega}^{j_{i+1}}\wedge\cdots\wedge\bar{\omega}^{j_q}\} \\
&= \phi_J\, \{\sum_{i=1}^q \bar{\omega}^{j_1}\wedge\cdots\wedge C\bar{\omega}^{j_i}\wedge\cdots\wedge\bar{\omega}^{j_q}\}\,. \qquad \text{Q.E.D.}
\end{aligned}$$

Before we prove Theorem 3, we give a remark for (1.6).

Remark. Of course the expression (1.6) is independent of the choise of an orthonomal basis $\{L_j\}_{j=1}^l$ of S and T. In fact for another orthonormal basis $\{L_j'\}_{j=1}^l$, $C'=BCB^*$ holds with a unitary matrix B, where $C=(c_{j,m})=\sigma([L_j, \bar{L}_m])$ and $C'=(c'_{j,m})=\sigma([L'_j, \bar{L}'_m])$. So $\{\mu_j\}_{j=1}^l$ are invariant. We also see that the integrand is an even function of τ. Thus (1.6) is independent of choise of T.

Proof of Theorem 3. We have only to check Condition (A) of Theorem 1. By Proposition 4.2 $q_2=\sum_{j=1}^l |z_j|^2\geqslant 0$ and (A)–(ii) is equivalent to the following inequality.

$$2\,\mu_J(x, \xi)+\sum_{j=1}^l |\mu_j(x, \xi)| \geqslant c|\xi| \tag{4.14}$$

for any J and for some positive constant c on $\Sigma=\{(x, \xi);\, z_j=0, j=1, \cdots, l\}$.

Set $\sigma(T)=i\tau$. Then τ is a real-valued function of (x, ξ). By (4.1) and (4.6) we have

$$c_{i,j} = L_{i,j}\, \tau \text{ on } \Sigma\,. \tag{4.15}$$

It is clear by (4.15) that

$$\mu_j(x, \xi) = \nu_j(x)\, \tau(x, \xi) \quad j = 1, \cdots, l\,.$$

(4.14) is equivalent to the inequality

$$\sum_{j\in\{J\}} \nu_j(x)\,\tau - \sum_{j\notin\{J\}} \nu_j(x)\,\tau + \sum_{j=1}^{l} |\nu_j(x)|\,|\tau| \geqslant c|\tau|,\ \tau\in\boldsymbol{R} \quad \text{for any } J.$$

This is equivalent to the inequalities

$$\sum_{j\in\{J\}} (|\nu_j|+\nu_j) + \sum_{j\notin\{J\}} (|\nu_j|-\nu_j) \geqslant c$$

(4.16) and

$$\sum_{j\in\{J\}} (|\nu_j|-\nu_j) + \sum_{j\notin\{J\}} (|\nu_j|+\nu_j) \geqslant c \quad \text{for any } J.$$

It is easily shown that (4.16) is equivalent to $Y(q)$. We also use Theorem 2 for our operator $\Box_b$. Take $n=2l+1$, $m=2$ and $d=2l$. Then we have

$$\sum_{j=1}^{\infty} \exp(-\lambda_j t)$$

$$\underset{t\downarrow 0}{\approx} (2t)^{-l-1} \int_{\Sigma^0} [\det\{(A/2)^{-1} \sinh (A/2)\}]^{-1/2} \operatorname{tr}(e^{-p_1})\, d\Sigma^0, \tag{4.17}$$

$$\text{with } A = iJH_{q_2}.$$

By Proposition 2.10, the eigenvalues of A are $\{\mu_j, -\mu_j\}_{j=1}^{l}$ and zero and those of p_1 are $\{\mu_J\}_J$ by Proposition 4.2. The integrand of (4.17) is

$$\prod_{j=1}^{l} \mu_j/2\, (\sinh (\mu_j/2))^{-1} \Big(\sum_{|J|=q} e^{-\mu_J}\Big).$$

Take $u_j=(z_j+\bar{z}_j)/2$, $u_{j+l}=(z_j-\bar{z}_j)/2i$ $(j=1, \cdots, l)$ and $v=(\tau, x)$. Then we have

$$\Sigma = \{u_1 = \cdots u_{2l} = 0\},\ p_2 = \sum_{j=1}^{2l} u_j^2/2,\ H_{uu} = I$$

$$dxd\xi = \Phi dudv \text{ with } \Phi = \left| \frac{\partial u}{\partial \xi}, \frac{\partial \tau}{\partial \xi} \right|^{-1}.$$

By the assumption that L_j, T are orthonormal with respect to the Hermitian metric, $(U_j)_{j=1}^{2l+1}$ $(U_j=(L_j+\bar{L}_j)/2,\ U_{l+j}=(L_j-\bar{L}_j)/2i,\ j=1, \cdots, l,\ U_{2l+1}=T)$ are mutually orthonormal with respect to the Riemannian metric g. This means

$$\sum_{m,k=1}^{n} a_j^k\, g_{k,m}\, a_i^m = \delta_{j,i}, \tag{4.18}$$

where $g_{i,j}=g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and $U_j=\sum_{k=1}^{n} a_j^k \frac{\partial}{\partial x_k}$ $(j=1, \cdots, 2l+1)$. By (4.18)

$$|\det a| = |\det G|^{-1/2},$$

where $G=(g_{i,j})$. Then $\Phi=|\det a|^{-1}=|\det G|^{1/2}$. So we have

$$d\Sigma = \Phi dv = |\det G|^{1/2}\, dx d\tau = dM d\tau\,.$$

$\bar{\partial}_b Q$, $\vartheta_b Q \in S^{-1/2}_{1/2,1/2}(M, \Lambda^{p,q})$ are shown as an application of Theorem 2.11.

Q.E.D.

References

[1] L. Boutet de Monvel: *Hypoelliptic operators with double characteristics and related pseudodifferential operators,* Comm. Pure Appl. Math. **27** (1974), 585–639.

[2] G.B. Folland and J.J. Kohn: The Neumann problem for the Cauchy-Riemann complex, Ann. of Math. Studies 75 (1972).

[3] G.B. Folland and E.M. Stein: *Estimates for the $\bar{\partial}_b$-complex and Analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[4] I.M. Gel'fand and G.E. Shilov: Generalized functions III, Academic Press, 1967.

[5] P. Greiner and E.M. Stein: Estimates for the $\bar{\partial}$-Neumann problem, Math. Notes No. 19, Princeton Univ. Press, 1977.

[6] L. Hörmander: *Pseudo-differential operators and hypoelliptic equations,* Proc. Symposium on Singular Integrals, Amer. Math. Soc. **19** (1967), 138–183.

[7] C. Iwasaki and N. Iwasaki: *Parametrix for a degerenate parabolic equation and its application to the asymptotic behavior of spectral function for stationary problems,* Publ. Res. Inst. Math. Sci. **17** (1981), 557–655.

[8] H. Kumano-go: Pseudo-differential operators, Iwanami Shoten, 1974; transl: MIT Press, 1981.

[9] A. Melin: *Lower bounds for pseudo-differential operators,* Ark. Mat. **9** (1971), 117–140.

[10] A. Menikoff and J. Sjöstrand: *On the eigenvalues of a class of hypoelliptic operators* II, Lecture Note in Math. 755 (1979), 201–247.

[11] L.P. Rothschild and D. Tartakoff: *Parametrices with C^∞ error for $\square_b$ and operators of Hörmander type,* Partial differential equations and geometry, (Proc. Conf., Park City, Utah, 1977), Lecture Note in Pure and Appl. Math. 48 (1979), 255–271.

[12] J. Sjöstrand: *On the eigenvalues of a class of hypoelliptic operators,* Ann. Inst. Fourier **30** (1980), 109–169.

[13] F. Treves: Introduction to pseudodifferential and Fourier integral operators I, II, Plenum, 1980.

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan