KUPKA-REEB PHENOMENA AND UNIVERSAL UNFOLDINGES OF CERTAIN FOLIATION SINGULARITIES

Dedicated to Professor Yozo Matsushima on his 60th birthday

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If $\tilde{\omega}$ is an integrable 1-form, under certain circumstances, $\tilde{\omega}$ is given as the pull back of a 1-form ω on a lower dimensional space by a submersion, that is, $\tilde{\omega}$ is a trivial unfolding of ω (Kupka-Reeb phenomenon). Especially, if we have an integrable 1-form ω which is a universal unfolding of some other 1-form, then every unfolding of ω is trivial. Thus we obtain "stable" singularities as universal unfoldings.

In this note, we construct universal unfoldings of some complex foliation singularities as an application of the versality theorem proved in [5]. For generalities on unfolding theory of complex analytic foliations, we refer to [4] and [5]. We briefly discuss universal unfoldings in section 1. In section 2, we consider the form $\omega=(\alpha x+\beta y)ydx-(\gamma x+\delta y)xdy$ on $C^2=\{(x,y)\}$ and show that, under some condition on α, β, γ and δ, we can construct a universal unfolding $\tilde{\omega}$ of ω (Theorem 2.1). As a foliation singularity, $\tilde{\omega}$ turns out to be a simple one (Remark 2.2). This fact can be used, for example, to find the solutions of the differential equation $\omega=0$ and its "perturbations". In section 3, we take up the form $\varpi=x_1\cdots x_n\sum_{i=1}^{n}a_i\frac{dx_i}{x_i}$ studied by Cerveau and Neto in [1]. They proved, among others, that every unfolding of (a form whose $n-1$ st jet is equal to) ϖ is trivial, provided that $a_i\neq a_j\neq 0$. We give (Theorem 3.2) an alternative proof of this using the versality theorem in [5]. When $n=3$ and two of the a_i's are the same, we show that some unfolding of ϖ is identical with one of the universal unfoldings constructed in section 2. We also indicate how to "stabilize" ϖ in general when two or more of the a_i's are the same (Proposition 3.8).

1. Universal unfoldings. Let $F=(\omega)$ be a codim 1 local foliation at the origin 0 in C^n ([5] section 1).

DEFINITION 1.1. An unfolding $\mathscr{F}$ of F is universal if it is versal and if the infinitesimal unfolding map of $\mathscr{F}$ ([5] section 1) is injective.

DEFINITION 1.2. An unfolding $\mathscr{F}$ of F is trivial if there is a local holomorphic submersion $\Phi : (C^n \times C^m, \, 0) \to (C^n, \, 0)$ such that $\mathscr{F}$ is generated by the pull back $\Phi^* \omega$ of ω by Φ, where C^m is the parameter space of $\mathscr{F}$.

Proposition 1.3. *Let F be a local foliation at 0 in C^n. If $\mathscr{F} = (\tilde{\omega})$ is a universal unfolding of F, then every unfolding of $\mathscr{F}$ is trivial.*

Proof. Let $x = (x_1, \cdots, x_n)$ be a coordinate system on C^n and let $C^m = \{t = (t_1, \cdots, t_m)\}$ be the parameter space of $\mathscr{F}$. Thus $\mathscr{F}$ is a local foliation at the origin in $C^n \times C^m$. Let $\mathscr{F}'$ be an arbitrary unfolding of $\mathscr{F}$ with parameter space $C^l = \{s = (s_1, \cdots, s_l)\}$. Then we may think of $\mathscr{F}'$ as an unfolding of F with parameter space $C^m \times C^l$. By the universality of $\mathscr{F}'$ there are map germs Φ and φ such that (i) the diagram

$$
\begin{CD}
(C^n \times C^m \times C^l, 0) @>\Phi>> (C^n \times C^m, 0) \\
@V\pi'VV @VV\pi V \\
(C^m \times C^l, 0) @>\varphi>> (C^m, 0),
\end{CD}
$$

where π' and π are canonical projections, is commutative, (ii) for each (t, s), the restriction of Φ to $\pi'^{-1}(t, s)$ is a biholomorphic map into $\pi^{-1}(\varphi(t, s))$ and (iii) $\mathscr{F}'$ is generated by $\Phi^* \tilde{\omega}$. In order to prove the proposition, it suffices to show that Φ is a submersion. We may write $\Phi(x, t, s) = (\psi(x, t, s), \varphi(t, s))$, where ψ is a local map $(C^n \times C^m \times C^l, 0) \to (C^n, 0)$. By the above property (ii), ψ is a submersion. Consider the diagram

$$
\begin{array}{ccc}
T_{C^m \times C^l, 0} & \underset{\iota}{\overset{d\varphi}{\rightleftarrows}} & T_{C^m, 0} \\
\rho' \searrow & & \swarrow \rho \\
& U(F), &
\end{array}
$$

where ρ and ρ' are the infinitesimal unfolding maps ([4] (2.9), (4.3), [5] section 1) of $\mathscr{F}$ and $\mathscr{F}'$ respectively, $d\varphi$ is the differential of φ and ι is the natural inclusion. Since $\mathscr{F}'$ is an unfolding of $\mathscr{F}$, we have $\rho' \circ \iota = \rho$. Also, by the naturality of infinitesimal unfolding maps, we have $\rho' = \rho \circ d\varphi$. Hence $\rho = \rho \circ d\varphi \circ \iota$. Since $\mathscr{F}$ is a universal unfolding, ρ is injective. Therefore, $d\varphi$ must be surjective and φ is a submersion,   Q.E.D.

REMARKS 1.4. $1°$. For Proposition 1.3, the codimension of F need not be one.

$2°$. For the universal unfoldings given in this note, we could use [5] (4.1) Corollary instead of Proposition 1.3.

2. Universal unfoldings of some singularities on C^2. We consider the 1-form

$$\omega = (\alpha x + \beta y)y\,dx - (\gamma x + \delta y)x\,dy$$

on $C^2 = \{(x, y)\}$ with α, β, γ and δ complex numbers. We assume that the set of zeros of ω consists only of the origin 0, that is, we assume that

$$\beta \neq 0,\ \gamma \neq 0 \text{ and } D = \alpha\delta - \beta\gamma \neq 0 .$$

Let $F = (\omega)$ be the codim 1 local foliation at 0 in C^2 generated by the germ of ω at 0 ([5] section 1). We set

$$A = \gamma\{D + \beta(\alpha - \gamma)\},\quad B = \beta\{D + \gamma(\delta - \beta)\} \text{ and } C = \alpha\beta - \gamma\delta .$$

Then it is not difficult to show that F is a Haefliger foliation ([4] (1.10) Definition), that is, ω admits an integrating factor, if and only if $A = B = 0$.

We assume that F is non-Haefliger hereafter. Furthermore, we consider only the following three cases:

(I) $\qquad\qquad A = 0,\ B \neq 0,$

(II) $\qquad\qquad A \neq 0,\ B = 0,$

(III) $\qquad\qquad A \neq 0,\ B \neq 0,\ C = 0.$

In the case (I), we may set $\alpha = 2ak$, $\beta = a$, $\gamma = bk$, $\delta = b - a$. Then we have $D = -a(2a - b)k$ and $B = -a(a + b)(2a - b)k$. The constants a, b and k are arbitrary as long as $abk \neq 0$, $a \neq -b$ and $2a \neq b$.

In the case (II), we may set $\alpha = a - b$, $\beta = ak$, $\gamma = b$ and $\delta = 2bk$. Then we have $D = -b(2b - a)k$ and $A = -b(b + a)(2b - a)k$. The constants a, b and k are arbitrary as long as $abk \neq 0$, $b \neq -a$ and $2b \neq a$.

In the case (III), we may set $\alpha = ak$, $\beta = b$, $\gamma = a$ and $\delta = bk$. Then we have $D = ab(k^2 - 1)$, $A = a^2b(k - 1)(k + 2)$ and $B = ab^2(k - 1)(k + 2)$. The constants a, b and k are arbitrary as long as $ab \neq 0$, $k^2 \neq 1$ and $k \neq -2$.

Theorem 2.1. *If one of the conditions* (I), (II) *and* (III) *is satisfied, then there is a universal unfolding $\mathcal{F} = (\tilde{\omega})$ of the foliation $F = (\omega)$. $\mathcal{F}$ is a (codim 1, local) foliation at 0 in $C^4 = \{(x, y, s, t)\}$. In each case we may choose the following as a generator $\tilde{\omega}$ of $\mathcal{F}$:*

(I) $\quad \tilde{\omega} = (2akx + ay + as)y\,dx - \{bkx^2 + (b - a)xy + bxs + bt\}\,dy + axy\,ds + ay\,dt .$

(II) $\quad \tilde{\omega} = \{(a - b)xy + aky^2 - ays - at\}\,dx - (bx + 2bky - bs)xy\,dy + bxy\,ds + bx\,dt .$

(III) $\quad \tilde{\omega} = \{akxy + by^2 - ys - (k + 1)at\}\,dx - \{ax^2 + bkxy + xs - (k + 1)bt\}\,dy$
$\qquad\qquad + (k + 1)(xy - t)\,ds + (ax - by + s)\,dt .$

Proof.   Let $\mathcal{O}$ be the ring of germs of holomorphic functions at 0 in $\boldsymbol{C}^2$ and let Ω be the $\mathcal{O}$-module of germs of holomorphic 1-forms.   We set $\Omega_F = \Omega/F$.   If we denote by f and g the germs of the functions $(\alpha x + \beta y)y$ and $-(\gamma x + \delta y)x$, respectively, we have ([4] (4.5), [5] section 1) $\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) = \mathcal{O}/(f, g)$, where (f, g) is the ideal generated by f and g.   For any element h in $\mathcal{O}$, we denote by $[h]$ the class of h in $\mathcal{O}/(f, g)$.   In our case, since $\beta \neq 0$ and $\gamma \neq 0$, we have

$$\mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) = \boldsymbol{C}^4$$

and we may take $[1]$, $[x]$, $[y]$ and $[xy]$ as basis elements.   Next we determine the set $U(F)$ of equivalence classes of first order unfoldings of F, which is given by ([4] (6.1) Theorem, (6.8) Remark)

$$U(F) = \{[h] \in \mathrm{Ext}^1_{\mathcal{O}}(\Omega_F, \mathcal{O}) \mid h\,d\omega = \eta \wedge \omega \text{ for some } \eta \in \Omega\} .$$

First we have

$$d\omega = -\{(\alpha+2\gamma)x + (\delta+2\beta)y\}\,dx \wedge dy .$$

If $\alpha+2\gamma = \delta+2\beta = 0$, then $A = B = 0$.   Hence by our assumption, $d\omega \neq 0$. Since the coefficients of ω are homogeneous polynomials of degree 2, if an element of the form $\lambda_1[1] + \lambda_2[x] + \lambda_3[y] + \lambda_4[xy]$ is in $U(F)$, then we must have $\lambda_1 = 0$.   Now the element $[xy]$ is in $U(F)$, since if we set $h_1 = Dxy$, we have

$$h_1\,d\omega = \eta_1 \wedge \omega$$

with $\eta_1 = \{D + \beta(\alpha-\gamma)\}\,ydx + \{D + \gamma(\delta-\beta)\}\,xdy$.   We now look for elements of the form $h = \lambda x + \mu y$ such that $[h] \in U(F)$.   It is not difficult to see that the equation $h\,d\omega = \eta \wedge \omega$ has a solution for η if and only if

$$\begin{vmatrix} \gamma & 0 & (\alpha+2\gamma)\lambda \\ \delta & \alpha & (\delta+2\beta)\lambda + (\alpha+2\gamma)\mu \\ 0 & \beta & (\delta+2\beta)\mu \end{vmatrix} = 0$$

or equivalently $A\mu + B\lambda = 0$.   Thus if we set $h_2 = Ax - By$, then

$$h_2\,d\omega = \eta_2 \wedge \omega$$

with $\eta_2 = \left(2 + \dfrac{\alpha}{\gamma}\right)Adx - \left(2 + \dfrac{\delta}{\beta}\right)Bdy$.   Hence we see that $U(F) = \boldsymbol{C}^2$ and we may take $[h_1] = [Dxy]$ and $[h_2] = [Ax - By]$ as basis elements.   The element h_1 determines a first order unfolding

$$\tilde{\omega}_1 = \omega + \omega_1^{(1)}s + h_1 ds$$

with $\omega_1^{(1)} = dh_1 - \eta_1 = \beta(\gamma-\alpha)ydx + \gamma(\beta-\delta)xdy$, where s is a parameter.   The

second order (in s) term in $d\tilde{\omega}_1 \wedge \tilde{\omega}_1$ is

$$d\omega_1^{(1)} \wedge \omega_1^{(1)}s^2 + (h_1 d\omega_1^{(1)} - dh_1 \wedge \omega_1^{(1)})sds .$$

We have $d\omega_1^{(1)} \wedge \omega_1^{(1)} = 0$, since $\omega_1^{(1)}$ is a 1-form on C^2. Using $d\omega_1^{(1)} = Cdx \wedge dy$, we have $h_1 d\omega_1^{(1)} - dh_1 \wedge \omega_1^{(1)} = (DC - DC)xydx \wedge dy = 0$. Hence $\tilde{\omega}_1$ satisfies the integrability condition $d\tilde{\omega}_1 \wedge \tilde{\omega}_1 = 0$. Thus $\tilde{\omega}_1$ is actually an unfolding of ω. The element h_2 determines a first order unfolding

$$\tilde{\omega}_2 = \omega + \omega_2^{(1)}t + h_2 dt$$

with $\omega_2^{(1)} = dh_2 - \eta_2 = -\left(1 + \dfrac{\alpha}{\gamma}\right)Adx + \left(1 + \dfrac{\delta}{\beta}\right)Bdy$,

where t is a parameter. Since $d\omega_2^{(1)} = 0$, the second order term in $d\tilde{\omega}_2 \wedge \tilde{\omega}_2$ is $-dh_2 \wedge \omega_2^{(1)} \wedge tdt = \dfrac{1}{\beta\gamma}ABCdx \wedge dy \wedge tdt$. Hence under the condition (I), (II) or (III), this vanishes. Thus $\tilde{\omega}_2$ is actually an unfolding of ω.

(I) $A = 0$, $B \neq 0$. In this case, we have $h_1 = Dxy = -a(2a-b)kxy$, $\omega_1^{(1)} = -(2a-b)k(aydx - bxdy)$, $h_2 = -By = aCy$ and $\omega_2^{(1)} = -bCdy$. We set $h_1' = axy$, $\omega_1^{(1)'} = aydx - bxdy$, $h_2' = ay$ and $\omega_2^{(1)'} = -bdy$. Then clearly

$$\tilde{\omega}_1' = \omega + \omega_1^{(1)'}s + h_1'ds \quad \text{and} \quad \tilde{\omega}_2' = \omega + \omega_2^{(1)'}t + h_2'dt$$

are unfoldings of ω. We combine $\tilde{\omega}_1'$ and $\tilde{\omega}_2'$ to obtain a form on $C^4 = \{(x, y, s, t)\}$:

$$\tilde{\omega} = \omega + \omega_1^{(1)'}s + \omega_2^{(1)'}t + h_1'ds + h_2'dt .$$

We now show that $\tilde{\omega}$ satisfies the integrability condition and is thus an unfolding of ω. Noting that $d\omega_2^{(1)'} = 0$, we have

$$d\tilde{\omega} = d\omega + d\omega_1^{(1)'}s - (\omega_1^{(1)'} - dh_1') \wedge ds - (\omega_2^{(1)'} - dh_2') \wedge dt .$$

For our purpose, it suffices to show that the terms in $d\tilde{\omega} \wedge \tilde{\omega}$ involving sdt, tds or $ds \wedge dt$ vanish. First, the coefficient of sdt is

$$(\omega_2^{(1)'} - dh_2') \wedge \omega_1^{(1)'} + h_2'd\omega_1^{(1)'} = a(a+b)ydx \wedge dy - a(a+b)ydx \wedge dy = 0 .$$

The coefficient of tds is

$$(\omega_1^{(1)'} - dh_1') \wedge \omega_2^{(1)'} = b(a+b)xdy \wedge dy = 0 .$$

Finally the coefficient of $ds \wedge dt$ is

$$h_1'(\omega_2^{(1)'} - dh_2') - h_2'(\omega_1^{(1)'} - dh_1') = -a(a+b)xydy + a(a+b)xydy = 0 .$$

Therefore, $\tilde{\omega}$ is integrable. Let $\mathcal{F} = (\tilde{\omega})$ be the unfolding of $F = (\omega)$ generated

by $\tilde{\omega}$. The infinitesimal unfolding map of $\mathscr{F}$ sends the tangent vectors $\dfrac{\partial}{\partial s}$ and

$\dfrac{\partial}{\partial t}$ of the parameter space $C^2 = \{(s, t)\}$ to the classes $[h'_1]$ and $[h'_2]$ in $U(F)$.   Since

$[h'_1]$ and $[h'_2]$ form a basis of $U(F)$, by the versality theorem in [5], $\mathscr{F}$ is a universal unfolding of F.

(II) $A \neq 0$, $B = 0$.   Similar to the case (I).

(III) $A \neq 0$, $B \neq 0$, $C = 0$.   In this case, we have $h_1 = Dxy = ab(k^2-1)xy$, $\omega_1^{(1)} = -ab(k-1)d(xy)$, $h_2 = ab(k-1)(k+2)(ax-by)$ and $\omega_2^{(1)} = -ab(k^2-1)(k+2)(adx-bdy)$.   We set $h''_1 = (k+1)xy$, $\omega_1^{(1)''} = -d(xy)$, $h''_2 = ax-by$ and $\omega_2^{(1)''} = -(k+1)(adx-bdy)$.   Then clearly

$$\tilde{\omega}''_1 = \omega + \omega_1^{(1)''}s + h''_1 ds \quad \text{and} \quad \tilde{\omega}''_2 = \omega + \omega_2^{(1)''}t + h''_2 dt$$

are unfoldings of ω.   We construct a universal unfolding of ω by combining $\tilde{\omega}''_1$ and $\tilde{\omega}''_2$.   If we simply add them as in the case (I), we do not get an integrable form.   However, again by a straightforward computation, it can be shown that the form

$$\tilde{\omega} = \omega + \omega_1^{(1)''}s + \omega_2^{(1)''}t + h''_1 ds + h''_2 dt - (k+1)tds + sdt$$

on $C^4 = \{(x, y, s, t)\}$ is integrable (see also Remark 2.2 below).   Thus $\mathscr{F} = (\tilde{\omega})$ is an unfolding of F.   Moreover, since $[h'_1]$ and $[h'_2]$ form a basis of $U(F)$, by the versality theorem in [5], $\mathscr{F}$ is a universal unfolding of F,   Q.E.D.

REMARK 2.2.   For each $\tilde{\omega}$ in Theorem 2.1, we have $d\tilde{\omega}(0) \neq 0$.   Hence there must be a coordinate system on C^4 in terms of which the form $\tilde{\omega}$ involves only two variables (Kupka-Reeb phenomenon [3], [1] p. 2).   In fact, in each case, such a coordinate system (x', y', s', t') is given as follows:

(I)   If $x' = x$, $y' = y$, $s' = s$ and $t' = kx^2 + xy + xs + t$, then

$\tilde{\omega} = ay'dt' - bt'dy'$.

(II)   If $x' = x$, $y' = y$, $s' = s$ and $t' = -xy - ky^2 + ys + t$, then

$\tilde{\omega} = bx'dt' - at'dx'$.

(III)   If $x' = x$, $y' = y$, $s' = ax + by + s$ and $t' = -xy + t$, then

$\tilde{\omega} = s'dt' - (k+1)t'ds'$.

From this we can readily find the singular set and the leaves of the foliation $\mathscr{F} = (\tilde{\omega})$.   The leaves of $\mathscr{F}$ are given, in terms of the old system (x, y, s, t), by

(2.3)
$$\begin{cases} (\text{ I })\quad cy^b = (kx^2 + xy + xs + t)^a , \\ (\text{ II })\quad cx^a = (-xy - ky^2 + ys + t)^b , \\ (\text{III})\quad c(xy - t) = (ax - by + s)^{k+1} , \end{cases}$$

where c is an arbitrary constant. Also, if we consider, for each fixed (s, t), the foliation $F_{s,t} = (\omega_{s,t})$ on $\mathbf{C}^2 = \{(x, y)\}$ generated by

(I) $\omega_{s,t} = (2akx + ay + as)ydx - \{bkx^2 + (b-a)xy + bxs + bt\}dy$,

(II) $\omega_{s,t} = \{(a-b)xy + aky^2 - ays - at\}dx - (bx + 2bky - bs)xdy$,

(III) $\omega_{s,t} = \{akxy + by^2 - ys - (k+1)at\}dx - \{ax^2 + bkxy + xs - (k+1)bt\}dy$,

then (2.3) also gives the leaves of $F_{s,t}$, or solutions of the differential equation $\omega_{s,t} = 0$.

EXAMPLES 2.4. $1°$. If $\alpha = 0$, $\beta = 1$, $\gamma = 1$ and $\delta = 0$, then $D = -1, A = B = -2$ and $C = 0$. Hence (III) is satisfied and $a = b = 1$, $k = 0$. Thus

$$\tilde{\omega} = (y^2 - ys - t)dx - (x^2 + xs - t)dy + (xy - t)ds + (x - y + s)dt$$

is a universal unfolding of $\omega = y^2 dx - x^2 dy$.

$2°$. If $\alpha = 2$, $\beta = 1$, $\gamma = 1$ and $\delta = 2$, then $D = 3$, $A = B = 4$ and $C = 0$. Hence (III) is satisfied and $a = b = 1$, $k = 2$. Thus

$$\tilde{\omega} = (2xy + y^2 - ys - 3t)dx - (x^2 + 2xy + xs - 3t)dy + 3(xy - t)ds + (x - y + s)dt$$

is a universal unfolding of $\omega = (2x + y)ydx - (x + 2y)xdy$.

$3°$. If $\alpha = 2$, $\beta = -1$, $\gamma = 1$ and $\delta = 0$, then $D = 1$, $A = 0$, $B = -2$ and $C = -2$. Hence (I) is satisfied and $a = b = k = -1$. Thus

$$\tilde{\omega} = (2x - y - s)ydx - (x^2 - xs - t)dy - xyds - ydt$$

is a universal unfolding of $\omega = (2x - y)ydx - x^2 dy$.

Examples $2°$ and $3°$ give universal unfoldings of singularities of Dumortier [2] p. 95.

3. Singularity of Cerveau and Neto. Consider the integrable 1-form

$$(3.1) \qquad \bar{\omega} = x_1 \cdots x_n \sum_{i=1}^{n} a_i \frac{dx_i}{x_i}$$

on $\mathbf{C}^n = \{(x_1, \cdots, x_n)\}$. First we give an alternative proof, which uses the versality theorem in [5], of the following result of Cerveau and Neto [1].

Theorem 3.2. *Let $\bar{F}$ be the foliation generated by $\bar{\omega}$ in (3.1). If $a_i \neq a_j \neq 0$, then the set $U(\bar{F})$ of first order unfoldings of $\bar{F}$ is zero. Thus $\bar{F}$ is a universal unfolding of $\bar{F}$ itself and every unfolding of $\bar{F}$ is trivial.*

Proof. Let $\mathcal{O}$ be the ring of germs of holomorphic functions at 0 in $\mathbf{C}^n$ and let Ω be the $\mathcal{O}$-module of germs of holomorphic 1-forms. We set $\Omega_{\bar{F}} = \Omega/\bar{F}$. Also we denote by f_i the germ of the function $x_1 \cdots \hat{x}_i \cdots x_n$ (omit x_i)

at 0. Then we have

(3.3) $Ext_{\mathcal{O}}^1(\Omega_{\bar{F}}, \mathcal{O}) = \mathcal{O}/(f_1, \cdots, f_n) \,,$

where $(f_1, \cdots, f_n)$ is the ideal generated by $f_1, \cdots, f_n$. Now we find the set $U(\bar{F})$, which is given by

$$U(\bar{F}) = \{[h] \in Ext_{\mathcal{O}}^1(\Omega_{\bar{F}}, \mathcal{O}) \,|\, hd\bar{\omega} = \eta \wedge \bar{\omega} \quad \text{for some } \eta \in \Omega\} \,.$$

We have $d\bar{\omega} = x_1 \cdots x_n \sum_{i<j} (a_j - a_i) \dfrac{dx_i \wedge dx_j}{x_i x_j}$ and, if we write $\eta = \sum_{i=1}^{n} g_i dx_i$, $\eta \wedge \bar{\omega} =$

$x_1 \cdots x_n \sum_{i<j} \left(\dfrac{a_j g_i}{x_j} - \dfrac{a_i g_j}{x_i} \right) dx_i \wedge dx_j$. Hence $hd\bar{\omega} = \eta \wedge \bar{\omega}$ is equivalent to

(3.4) $(a_j - a_i)h = a_j x_i g_i - a_i x_j g_j \quad$ for all i, j with $1 \le i < j \le n$.

By (3.3), we may assume that each monomial in h involves at most $n-2$ different x_i's. Thus, if $a_i \neq a_j \neq 0$, (3.4) is satisfied only when $h=0$. Therefore $U(\bar{F}) = 0$. By the versality theorem in [5], $\bar{F}$ is a universal unfolding of $\bar{F}$ itself and every unfolding of $\bar{F}$ is trivial ([5] (4.1) Corollary),   Q.E.D.

Now we consider the case where two or more of the a_i's are the same. First we assume that $n=3$ and $a_1 \neq a_2 = a_3$. We may set $a_2 = a_3 = 1$. Also we set $a_1 = \lambda$. Thus

(3.5) $\bar{\omega} = \lambda x_2 x_3 dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3 \,,$

where $\lambda \neq 1$. We also impose a technical condition $\lambda \neq -2$. The following proposition, which is a direct consequence of Theorem 2.1, shows that we can "stabilize" $\bar{\omega}$ in (3.5) by unfolding it suitably.

Proposition 3.6. *Let $\tilde{\omega}$ be the 1-form on $C^4 = \{(x_1, x_2, x_3, t)\}$ given by*

$$\tilde{\omega} = \lambda(x_2 x_3 + t)dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3 + x_1 dt, \quad \lambda \neq 0, 1, -2 \,.$$

Then $\tilde{\omega}$ is integrable; $d\tilde{\omega} \wedge \tilde{\omega} = 0$, and the foliation $\mathcal{F} = (\tilde{\omega})$ can be viewed as a universal unfolding of a certain foliation on C^2. Thus every unfolding of $\mathcal{F}$ is trivial.

Proof. Consider the 1-form

$$\omega = (-2x+y)ydx - \{\lambda x - (\lambda+1)y\}xdy$$

on $C^2 = \{(x, y)\}$. Then we have $D = \lambda+2$, $A=0$, $B = -(\lambda-1)(\lambda+2)$. Thus by Theorem 2.1 (I),

$$\tilde{\omega} = (-2x+y+s)ydx - \{\lambda x^2 - (\lambda+1)xy - \lambda xs - \lambda t\}dy + xyds + ydt$$

is a universal unfolding of ω. By the coordinate transformation $x_1 = y$, $x_2 = x$, $x_3 = -x+y+s$, $t = t$ of C^4, $\tilde{\omega}$ becomes the one in the statement,   Q.E.D.

REMARK 3.7.   Let F be the foliation in $\boldsymbol{C}^3 = \{(x_1, \; x_2, \; x_3)\}$ generated by ϖ in (3.5). If we set, for each integer $p \geq 1$, $h_p = x_1^p$, then $[h_p]$ is in $U(\bar{F})$. In fact, h_p determines an actual unfolding

$$\tilde{\omega}_p = \{\lambda x_2 x_3 + (\lambda + p - 1)x_1^{p-1}t\}\,dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3 + x_1^p dt$$

of ϖ. Moreover, if $p \neq q$, then $[h_p] \neq [h_q]$ in $U(\bar{F})$. Hence the unfoldings $\tilde{\omega}_p$ and $\tilde{\omega}_q$ are not equivalent. However, the above proposition shows that essentially it suffices if we unfold ϖ to $\tilde{\omega}_1$ which is the unfolding determined by $h_1 = x_1$. Thus if we consider the form

$$\tilde{\tilde{\omega}} = \{\lambda x_2 x_3 + \lambda t + (\lambda + 1)x_1 s\}\,dx_1 + x_1 x_3 dx_2 + x_1 x_2 dx_3 + x_1 dt + x_1^2 ds$$

on $\boldsymbol{C}^5 = \{(x_1, \; x_2, \; x_3, \; t, \; s)\}$, which is readily checked to be integrable, as an unfolding of ϖ, it contains the two independent unfoldings determined by $h_1 = x_1$ and $h_2 = x_1^2$. However, as an unfolding of $\tilde{\omega}$ in Proposition 3.6, it is trivial.

More generally consider the form ϖ in (3.1) and assume that $a_1 = \lambda_1, \cdots,$ $a_m = \lambda_m$, $a_{m+1} = \cdots = a_n = 1$, $m \geq 1$, $\lambda_i \neq \lambda_j \neq 0, 1$. Thus

$$\varpi = x_1 \cdots x_n \sum_{i=1}^m \lambda_i \frac{dx_i}{x_i} + x_1 \cdots x_n \sum_{i=m+1}^n \frac{dx_i}{x_i}.$$

Let $\bar{F}$ be the foliation generated by ϖ. If $h = x_1 \cdots x_m$, then it is not difficult to show that $[h]$ is in $U(\bar{F})$ and is not obstructed. In fact, the following proposition shows that we can "stabilize" ϖ if we unfold ϖ to the unfolding $\tilde{\omega}$ determined by $h = x_1 \cdots x_m$.

Proposition 3.8.   *Let $\tilde{\omega}$ be the 1-form on $\boldsymbol{C}^{n+1} = \{(x_1, \cdots, x_n, t)\}$ given by*

$$\tilde{\omega} = x_1 \cdots x_m(x_{m+1} \cdots x_n + t)\sum_{i=1}^m \lambda_i \frac{dx_i}{x_i} + x_1 \cdots x_n \sum_{i=m+1}^n \frac{dx_i}{x_i} + x_i \cdots x_m dt,$$

$$\lambda_i \neq \lambda_j \neq 0, \; 1 \, .$$

Then $\tilde{\omega}$ is integrable. If $\mathcal{F} = (\tilde{\omega})$ is the foliation generated by $\tilde{\omega}$, then $U(\mathcal{F}) = 0$. Thus $\mathcal{F}$ is a universal unfolding of $\mathcal{F}$ itself and every unfolding of $\mathcal{F}$ is trivial.

Proof. We introduce a new coordinate system $(y_1, \cdots, y_{n+1})$ on $\boldsymbol{C}^{n+1}$ by $y_1 = x_1, \cdots, y_m = x_m$, $y_{m+1} = x_{m+1} \cdots x_n + t$, $y_{m+2} = x_{m+1}, \cdots, y_{n+1} = x_n$. Then $\tilde{\omega}$ becomes

$$\tilde{\omega} = y_1 \cdots y_{m+1} \sum_{i=1}^{m+1} \lambda_i \frac{dy_i}{y_i},$$

where we set $\lambda_{m+1} = 1$. The proposition is then proved by a similar argument as in the proof of Theorem 3.2.

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