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## ELEMENTARY INTERSECTION NUMBERS ON PUNCTURED SPHERES

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### Introduction

According to Thurston, for any analytically finite Riemann surface  $\mathcal{R}$ , the set  $\overline{\mathcal{G}}(\mathcal{R})$  of all projective geodesic laminations in  $\mathcal{R}$  can be made into a topological space homeomorphic to a sphere of dimension depending on the topology of  $\mathcal{R}$ . Understanding the space  $\overline{\mathcal{G}}(\mathcal{R})$  is important for various approaches to the Teichmüller space and the mapping class group of  $\mathcal{R}$ . The space  $\overline{\mathcal{G}}(\mathcal{R})$  was then investigated by several authors from many different points of view. See [3–10], [12, 13, 15], and references there in.

In this paper, we consider the space  $\overline{\mathcal{G}}_n = \overline{\mathcal{G}}(\Sigma_n)$  for any integer  $n \geq 4$ , where  $\Sigma_n$  is an  $n$ -punctured sphere endowed with a hyperbolic metric. Note that  $\overline{\mathcal{G}}_n$  is homeomorphic to a sphere of dimension  $2n - 7$ .

This work was an attempt to generalize the projective coordinates defined in [3, 4] to an arbitrary  $\overline{\mathcal{G}}_n$ . This work and that of Keen, Parker and Series [10] are essentially based on cutting sequence technique developed by Birman and Series [2], and complement the works of Masur and Minsky [12, 13].

Let  $\mathcal{G}_n$  be the set of all simple closed geodesics on  $\Sigma_n$ . For  $n = 4$  or  $5$ , the author has defined a set of projective coordinates for  $\mathcal{G}_n$  so that the completion of these coordinates parametrize  $\overline{\mathcal{G}}_n$ , (see [3, 4]). The coordinates of each  $\gamma \in \mathcal{G}_n$  are geometric intersection numbers of  $\gamma$  with  $2(n - 3)$  fixed geodesics in  $\mathcal{G}_n$ , and read off directly from the topology of  $\gamma$ . Moreover, these coordinates have three remarkable applications. First, the geometric intersection number of any two geodesics in  $\mathcal{G}_n$  can be formulated explicitly in terms of the corresponding coordinates. Secondly, the coordinates of each  $\gamma \in \mathcal{G}_n$  determine a canonical expression of  $\gamma$  as a word in a given set of generators for the fundamental group  $\pi_1(\Sigma_n)$ . Finally, the coordinates of each  $\gamma \in \mathcal{G}_n$  are related to trace polynomials of the transformations corresponding to  $\gamma$  in a family of regular  $B$ -groups uniformizing  $\Sigma_n$ .

For an arbitrary  $n \geq 5$ , following [3, 4], we shall choose  $n - 3$  fixed triples  $(\gamma_j^1, \gamma_j^2, \gamma_j^3)$  of geodesics in  $\mathcal{G}_n$  ( $1 \leq j \leq n - 3$ ), and compute the geometric intersec-

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tion numbers  $i(\gamma, \gamma_j^k)$ , called the *elementary intersection numbers* of  $\gamma$ . The elementary intersection numbers of  $\gamma$  will determine a set of parameters for  $\gamma$ .

The geodesics  $\gamma_j^k$  are defined explicitly in §1.2. They are chosen intuitively as described below. First, we line up the punctures of  $\Sigma_n$ , say  $\zeta_1, \dots, \zeta_n$ . For every  $j$ , the geodesic  $\gamma_j^1$  is chosen to separate  $\zeta_1, \dots, \zeta_{j+1}$  from  $\zeta_{j+2}, \dots, \zeta_n$ . These geodesics  $\gamma_j^1$  determine  $n-3$  subsurfaces of  $\Sigma_n$  each of which is homeomorphic to a four punctured sphere. Two of them are isometric to spheres with three punctures and one hole, denoted by  $\Sigma_4^{(1)}$  and  $\Sigma_4^{(n-3)}$ , and the others are isometric to spheres with two punctures and two holes, denoted by  $\Sigma_4^{(j)}$ ,  $2 \leq j \leq n-4$ . More explicitly,  $\Sigma_4^{(1)}$  is the subsurface containing  $\zeta_1, \zeta_2$  and  $\zeta_3$  with the boundary geodesic  $\gamma_2^1$ ;  $\Sigma_4^{(n-3)}$  is the subsurface containing  $\zeta_{n-2}, \zeta_{n-1}$  and  $\zeta_n$  with the boundary geodesic  $\gamma_{n-4}^1$ ;  $\Sigma_4^{(j)}$  is the subsurface bounded by  $\gamma_{j-1}^1$  and  $\gamma_{j+1}^1$  for  $2 \leq j \leq n-4$ . For every  $j$ , we choose  $\gamma_j^2$  so that  $\gamma_j^1$  and  $\gamma_j^2$  form a marking of a four punctured sphere as  $\gamma_\infty$  and  $\gamma_0$  given in [3]. The geodesic  $\gamma_j^3$  plays the role of  $\gamma_1$  given [3] which is obtained from  $\gamma_j^2$  by a half-twist along  $\gamma_j^1$ .

The main work of this paper is to find formulas for computing elementary intersection numbers so that the formulas agree with that given in [4] when  $n = 5$ . These formulas will be called *elementary intersection formulas*. To derive these formulas, we introduce  $2(n-3)$  integers for each  $\gamma \in \mathcal{G}_n$ , denoted by  $I_j(\gamma)$  and  $N_j(\gamma)$  for  $1 \leq j \leq n-3$ , (see §2.1 and §2.4). These integers are defined analogously to the projective coordinates given in [3, 4]. For  $\gamma \in \mathcal{G}_n$ , every  $I_j(\gamma)$  is defined to be  $(1/2)i(\gamma, \gamma_j^1)$ , and the sign of every  $N_j(\gamma)$  is determined by the symmetry of a fundamental domain for a Fuchsian representation of  $\pi_1(\Sigma_n)$  acting on the upper half plane. With these integer valued functions  $I_j$  and  $N_j$ , we prove in §2.5 the elementary intersection formulas (Theorem 2.10) by applying induction to the number  $n$  of punctures. In this paper, we develop a new idea that makes the induction work for  $n \geq 5$ , (cf. Remark 2.3).

As an application of elementary intersection formulas, at the end of the paper, we construct a continuous map  $\Psi$  from  $\overline{\mathcal{G}}_n$  into a sphere  $\Delta_n \subset \mathbb{R}^{3(n-3)}$  of dimension  $2n-7$  whose restriction to  $\mathcal{G}_n$  is written explicitly in terms of  $I_j$  and  $N_j$ .

It would be very interesting to derive a geometric intersection formula as given in [3, Theorem 2.6] and [4, Theorem 3.1] for any two geodesics in  $\mathcal{G}_n$ . With the formula, one proves easily the injectivity of  $\Psi$ . To prove that the integers  $I_j(\gamma)$  and  $N_j(\gamma)$  form a set of projective coordinates for  $\gamma \in \mathcal{G}_n$ , one also need the surjectivity of the map  $\Psi: \overline{\mathcal{G}}_n \rightarrow \Delta_n$ . This will follow if  $\Psi(\mathcal{G}_n)$  is dense in  $\Delta_n$ . For the proof, one may consider  $\pi_1$ -train tracks introduced by Birman and Series [1], (cf. [3, 4]). The work will appear elsewhere.

## 1. Preliminaries

**1.1. The space of complete simple geodesics.** For any integer  $n \geq 4$ , a loop on  $\Sigma_n$  with no self intersections will be called a *simple loop*. An *essential simple loop*

on  $\Sigma_n$  is a simple loop which is neither homotopically trivial nor homotopic to a simple closed curve around to a puncture of  $\Sigma_n$ . A finite union of mutually disjoint essential simple loops on  $\Sigma_n$  will be called a *multiple simple loop*. The set of all free homotopy classes of non-oriented essential simple loops on  $\Sigma_n$  is denoted by  $\mathcal{G}_n$ , while the set of all free homotopy classes of non-oriented multiple simple loops is denoted by  $\mathcal{GL}_n$ . Obviously,  $\mathcal{G}_n \subset \mathcal{GL}_n$ .

In general, we shall use  $[\alpha]$  for the free homotopy class represented by a curve  $\alpha$  lying on  $\Sigma_n$ . Every element of  $\mathcal{G}_n$  contains a unique geodesic  $\gamma$  on  $\Sigma_n$ . By abuse of notation, we shall also use  $\gamma$  for the free homotopy class containing  $\gamma$ .

We shall write every element of  $\mathcal{GL}_n$  as an integral combination of elements of  $\mathcal{G}_n$ . For every integer  $m > 1$ , we use  $\mathcal{Z}_+^m$  for the set of  $m$ -tuples  $(k_1, \dots, k_m)$  of integers  $k_j \geq 0$  with  $\sum_{j=1}^m k_j > 0$ , and  $\Lambda_n^m$  for the set of  $m$ -tuples  $(\gamma_1, \dots, \gamma_m)$  of mutually disjoint geodesics in  $\mathcal{G}_n$ .

Let  $\alpha$  be an arbitrary multiple simple loop on  $\Sigma_n$ . All connected components of  $\alpha$  fall into at most  $n - 3$  distinct free homotopy classes. There exist  $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$  and  $(\gamma_1, \dots, \gamma_{n-3}) \in \Lambda_n^{n-3}$  such that, for every  $j$ ,  $\alpha$  has exactly  $k_j$  connected components freely homotopic to  $\gamma_j$ . We shall write:

$$[\alpha] = k_1 \gamma_1 \oplus \dots \oplus k_{n-3} \gamma_{n-3} = \bigoplus_{j=1}^{n-3} k_j \gamma_j.$$

Let  $[\mathcal{G}_n, \mathbb{R}_+]$  be the set of all functions from  $\mathcal{G}_n$  into the set  $\mathbb{R}_+$  of all non-negative real numbers. We provide  $\mathcal{G}_n$  with the discrete topology, and provide  $[\mathcal{G}_n, \mathbb{R}_+]$  with the compact-open topology.

Two elements  $f$  and  $g$  of  $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$  are called *projectively equivalent* if there is a positive number  $t$  such that  $f = tg$ . Let  $P[\mathcal{G}_n, \mathbb{R}_+]$  be the set of all projective equivalence classes in  $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$  provided with the quotient topology. Let  $\pi_n$  be the quotient map of  $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$  onto  $P[\mathcal{G}_n, \mathbb{R}_+]$ .

Following [5], we embed  $\mathcal{GL}_n$  into  $[\mathcal{G}_n, \mathbb{R}_+]$  by using geometric intersection numbers of elements of  $\mathcal{GL}_n$ . For any two curves  $\alpha_1$  and  $\alpha_2$  on  $\Sigma_n$ , let  $\#(\alpha_1 \cap \alpha_2)$  denote the cardinality of the intersection  $\alpha_1 \cap \alpha_2$ . The *geometric intersection number*  $i([\alpha_1], [\alpha_2])$  of  $[\alpha_1]$  with  $[\alpha_2]$  is defined by

$$i([\alpha_1], [\alpha_2]) = \min\{\#(\alpha'_1 \cap \alpha'_2) : [\alpha'_j] = [\alpha_j] \text{ for } j = 1, 2\}.$$

It follows immediately from the definition that, for any curve  $\beta$  on  $\Sigma_n$ ,

$$i\left(\bigoplus_{j=1}^{n-3} k_j \gamma_j, [\beta]\right) = \sum_{j=1}^{n-3} k_j \cdot i(\gamma_j, [\beta]).$$

Every  $\alpha \in \mathcal{GL}_n$  induces a function  $\mathcal{I}_\alpha^{(n)}: \mathcal{G}_n \longrightarrow \mathbb{R}_+$  given by

$$\mathcal{I}_\alpha^{(n)}(\gamma) = i(\alpha, \gamma) \quad \text{for all } \gamma \in \mathcal{G}_n.$$

Let  $\mathcal{I}^{(n)}: \mathcal{GL}_n \longrightarrow [\mathcal{G}_n, \mathbb{R}_+]$  be defined by

$$\mathcal{I}^{(n)}(\alpha) = \mathcal{I}_\alpha^{(n)} \quad \text{for all } \alpha \in \mathcal{GL}_n.$$

When there is no risk of confusion, we shall simply write  $\pi_n$  as  $\pi$ , write  $\mathcal{I}_\alpha^{(n)}$  as  $\mathcal{I}_\alpha$ , and write  $\mathcal{I}^{(n)}$  as  $\mathcal{I}$ .

It is well known that the composition  $\pi\mathcal{I}$  is injective [5, Exposé 3], and that  $\overline{\pi\mathcal{I}(\mathcal{GL}_n)} = \overline{\pi\mathcal{I}(\mathcal{G}_n)}$  [5, Exposé 4, Theorem 4], where  $\overline{\pi\mathcal{I}(\mathcal{GL}_n)}$  and  $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$  denote the closures of  $\pi\mathcal{I}(\mathcal{GL}_n)$  and  $\pi\mathcal{I}(\mathcal{G}_n)$  in  $P[\mathcal{G}_n, \mathbb{R}_+]$ , respectively. These results are original due to Thurston [15].

Note that an element  $\mathcal{L}$  of  $P[\mathcal{G}_n, \mathbb{R}_+]$  is in  $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$  if and only if for any  $l$  in  $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$  with  $\pi(l) = \mathcal{L}$  there exist a sequence  $\{t_k\}_{k=1}^\infty$  of positive numbers and a sequence  $\{\gamma_k\}_{k=1}^\infty$  of geodesics in  $\mathcal{G}_n$  such that the sequence  $\{t_k \mathcal{I}_{\gamma_k}\}_{k=1}^\infty$  converges to  $l$ . A sequence  $\{l_k\}_{k=1}^\infty$  in  $[\mathcal{G}_n, \mathbb{R}_+]$  is called *convergent* to  $l \in [\mathcal{G}_n, \mathbb{R}_+]$  if for every  $\gamma \in \mathcal{G}_n$  the sequence  $\{l_k(\gamma)\}_{k=1}^\infty$  converges in  $\mathbb{R}$  to  $l(\gamma)$ .

**1.2. Cyclic reduced words.** It is well known that every free homotopy class in  $\mathcal{G}_n$  corresponds to a unique conjugacy class in the fundamental group of  $\Sigma_n$ . Now, we consider a Fuchsian representation  $G_n$  of the fundamental group of  $\Sigma_n$  acting on the upper half plane  $\mathcal{U} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ , and find a representative for each conjugacy class in  $G_n$  by using Birman and Series' cutting sequence technique [2].

Let  $G_n$  be the subgroup of  $PSL(2, \mathbb{R})$  generated by the following transformations

$$S_1 = \begin{pmatrix} 1 & 2(n-2) \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad T_{n-j-2} = \begin{pmatrix} 2j+1 & 2j(j+1) \\ 2 & 2j+1 \end{pmatrix},$$

where  $1 \leq j \leq n-3$  are integers.

For every integer  $j$  with  $1 \leq j \leq n-3$ , let  $C_j$  be the isometric circle of  $T_j$ , and  $C'_j$  be the isometric circle of  $T_j^{-1}$ . Let  $C_{n-2}$  be the isometric circle of  $S_2$ , and  $C'_{n-2}$  be the isometric circle of  $S_2^{-1}$ . Let

$$C'_0 = \{z \in \mathbb{C} : \text{Re } z = -(n-2)\} \quad \text{and} \quad C_0 = \{z \in \mathbb{C} : \text{Re } z = n-2\}.$$

Note that  $S_1(C'_0) = C_0$ , and that the polygon  $\mathcal{D}_n \subset \mathcal{U}$  bounded by  $C_j$  and  $C'_j$ ,  $0 \leq j \leq n-2$ , is a fundamental domain for  $G_n$  acting on  $\mathcal{U}$ .

For simplicity, we shall schematically draw  $\mathcal{D}_n$  as a rectangular region. See Fig. 1 for  $n = 4, 5, 6$ , where the points on the boundary of  $\mathcal{D}_n$  marked by “ $\times$ ” correspond to punctures of  $\Sigma_n$ .

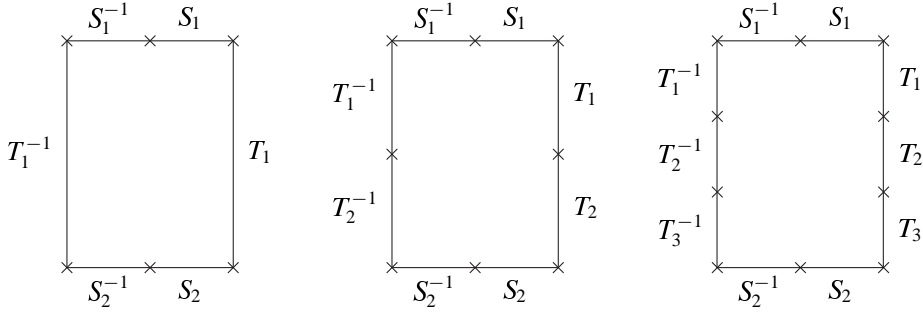


Fig. 1. The fundamental domain  $\mathcal{D}_n$  for  $n = 4, 5, 6$ .

Let  $\Gamma_n$  denote the set of all side pairings of  $\mathcal{D}_n$ , i.e.,

$$\Gamma_n = \{S_1, S_1^{-1}, S_2, S_2^{-1}, T_j, T_j^{-1} : j = 1, \dots, n-3\}.$$

For every  $X \in \Gamma_n$ , we label the common side  $s$  of  $\mathcal{D}_n$  and  $X(\mathcal{D}_n)$  by  $X^{-1}$  on the side inside  $\mathcal{D}_n$  and by  $X$  on the side inside  $X(\mathcal{D}_n)$ , (cf. Fig. 1). This side  $s$  will be called the  $X$ -side of  $\mathcal{D}_n$ .

For every  $g \in G_n$ , the image  $g(\mathcal{D}_n)$  will be called a  $G_n$ -translate of  $\mathcal{D}_n$ . We transport the above side labelling to all  $G_n$ -translates of  $\mathcal{D}_n$ .

For any closed curve  $\gamma$  in  $\Sigma_n$ , let  $\tilde{\gamma}$  be a lift of  $\gamma$  to  $\mathcal{U}$  which starts on a side of a  $G_n$ -translate of  $\mathcal{D}_n$  and projects to  $\gamma$  bijectively, except the endpoints of  $\tilde{\gamma}$ . Let  $z_0 \in \mathcal{U}$  be an endpoint of  $\tilde{\gamma}$ , and we orient  $\tilde{\gamma}$  so that its initial point is  $z_0$ . The arc  $\tilde{\gamma}$  cuts in order the  $G_n$ -translates  $g_0(\mathcal{D}_n), g_1(\mathcal{D}_n), \dots, g_k(\mathcal{D}_n)$  of  $\mathcal{D}_n$ . For every integer  $j$  with  $1 \leq j \leq k$ , let  $X_j \in \Gamma_n$  be the label of the common side of  $g_{j-1}(\mathcal{D}_n)$  and  $g_j(\mathcal{D}_n)$ , interior to  $g_j(\mathcal{D}_n)$ . Then  $X_j = g_{j-1}^{-1} \circ g_j$  for every  $j$ , and  $\gamma$  is represented by

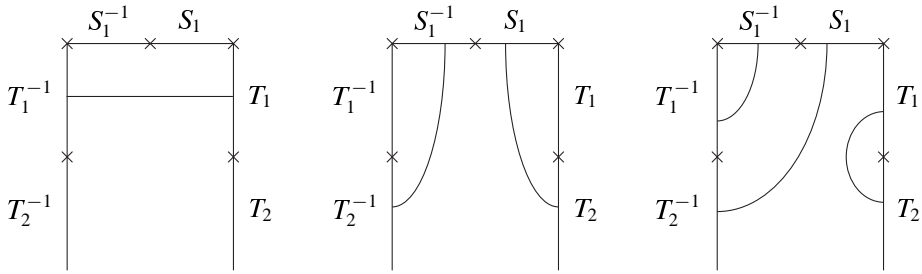
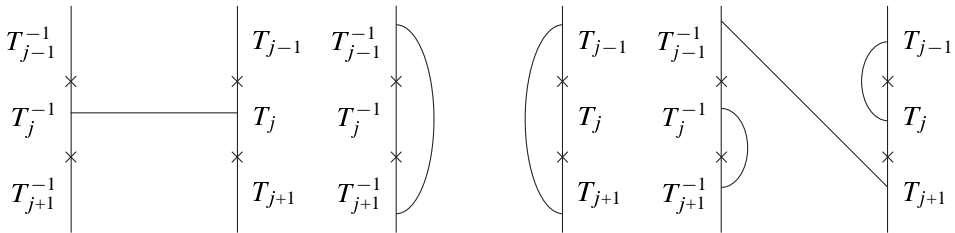
$$(g_0^{-1} \circ g_1) \circ (g_1^{-1} \circ g_2) \circ \dots \circ (g_{k-1}^{-1} \circ g_k) = X_1 \circ X_2 \circ \dots \circ X_k = \prod_{j=1}^k X_j.$$

Such an expression is called a  $\Gamma_n$ -word representing  $\gamma$ . See [4, §1.2] for a full discussion.

A  $\Gamma_n$ -word  $\prod_{j=1}^k X_j$  will be called *reduced* if  $X_j \neq X_{j+1}^{-1}$  for  $1 \leq j \leq k-1$ . It is called *cyclically reduced* if in addition  $X_1 \neq X_k^{-1}$ .

Let  $\gamma$  be a simple loop on  $\Sigma_n$  represented by a  $\Gamma_n$ -word given above. For every  $1 \leq j \leq k$ , let  $l_j$  be the image of the intersection of  $\tilde{\gamma}$  with  $g_j(\overline{\mathcal{D}_n})$  mapped by  $g_j^{-1}$ , where  $\overline{\mathcal{D}_n}$  is the relative closure of  $\mathcal{D}_n$  in  $\mathcal{U}$ . Note that each  $l_j$  is a simple arc in  $\overline{\mathcal{D}_n}$  connecting the  $X_j^{-1}$ -side to the  $X_{j+1}$ -side, where  $X_{k+1} = X_1$ . Each  $l_j$  will be called a *strand* of  $\gamma$ .

Let  $\alpha$  be a multiple simple loop on  $\Sigma_n$ . A strand of a connected component of  $\alpha$  will be also called a *strand* of  $\alpha$ .

Fig. 2. From the left to the right :  $\gamma_1^1, \gamma_1^2, \gamma_1^3$ .Fig. 3. From the left to the right :  $\gamma_j^1, \gamma_j^2, \gamma_j^3, 2 \leq j \leq n-4$ .

A loop on  $\Sigma_n$  is called *reduced* if it is represented by a cyclically reduced  $\Gamma_n$ -word. A multiple simple loop  $\alpha$  on  $\Sigma_n$  is called *reduced* if every connected component of  $\alpha$  is reduced. Note that a simple loop or a multiple simple loop on  $\Sigma_n$  is reduced if and only if every strand of the loop connects two different sides of  $\mathcal{D}_n$ . It is easy to see that every simple closed geodesic on  $\Sigma_n$  is a reduced loop. Thus every free homotopy class of multiple simple loops on  $\Sigma_n$  contains a reduced one.

If  $\gamma \in \mathcal{G}_n$  is a geodesic represented by a reduced  $\Gamma_n$ -word  $W$ , then  $\gamma$  is also represented by an arbitrary cyclic permutation of  $W$ . If  $\gamma'$  is a geodesic which has the same underlying set with  $\gamma$  but opposite orientation, then  $\gamma'$  is represented by  $W^{-1}$ . Because we are only interested in non-oriented simple loops, we shall identify all reduced  $\Gamma_n$ -words which are cyclic permutations of  $W$  or cyclic permutations of  $W^{-1}$ , and call any one of them a *cyclic reduced  $\Gamma_n$ -word* representing  $\gamma$ . Every cyclic reduced  $\Gamma_n$ -word is cyclically reduced.

As examples, let  $\gamma_j^k \in \mathcal{G}_n$  be the geodesics given in Fig. 2, Fig. 3 and Fig. 4, where  $j$  and  $k$  are integers with  $1 \leq j \leq n-3$  and  $1 \leq k \leq 3$ . See introduction for a geometric interpretation of  $\gamma_j^k$ . Each  $\gamma_j^k$  is represented by a cyclic reduced  $\Gamma_n$ -word  $W_j^k$  as given below:

- (i)  $W_1^1 = T_1, W_1^2 = S_1 T_2^{-1}, W_1^3 = S_1 T_1^{-1} T_2$ ;
- (ii)  $W_j^1 = T_j, W_j^2 = T_{j-1} T_{j+1}^{-1}, W_j^3 = T_{j+1} T_j^{-1} T_{j-1}$  for  $2 \leq j \leq n-4$ ;
- (iii)  $W_{n-3}^1 = T_{n-3}, W_{n-3}^2 = S_2 T_{n-4}^{-1}$  and  $W_{n-3}^3 = S_2 T_{n-3}^{-1} T_{n-4}$ .

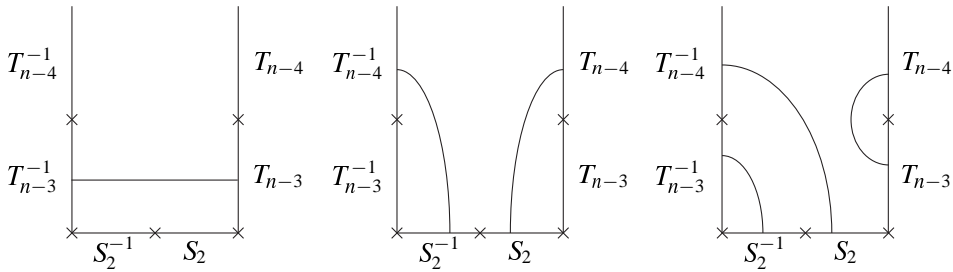


Fig. 4. From the left to the right :  $\gamma_{n-3}^1$ ,  $\gamma_{n-3}^2$ ,  $\gamma_{n-3}^3$ .

**1.3. Subwords and admissible subarcs.** Let  $\hat{\mathcal{G}}_n = \mathcal{G}_n - \{\gamma_j^1 : 1 \leq j \leq n-3\}$ . Let  $\gamma \in \hat{\mathcal{G}}_n$  be a geodesic represented by a cyclic reduced  $\Gamma_n$ -word  $W = \prod_{j=1}^k X_j$ . Note that  $k > 1$ . For any two integers  $1 \leq j \leq k$  and  $1 \leq l \leq k$ , the reduced  $\Gamma_n$ -word

$$(1) \quad W' = X_j \cdots X_{j+l-1}$$

will be called a *subword* of  $W$ , where  $X_{j+i} = X_{j+i-k}$  whenever  $1 \leq i \leq l$  and  $i+j > k$ .

We shall relate  $W'$  to  $\gamma$  geometrically. For every  $i$ , let  $l_i$  be the strand of  $\gamma$  connecting the  $X_{i-1}^{-1}$ -side to the  $X_i$ -side, where  $X_{i-1} = X_k$  if  $i = 1$ . Assume that  $1 \leq l < k$ , i.e.,  $W' \neq W$ . We think that  $W'$  “represents” a subarc  $\gamma'$  of  $\gamma$ . We choose  $\gamma'$  to be the projection to  $\Sigma_n$  of the union  $\cup_{i=j}^{j+l-1} l_i$ . Each of the arcs  $l_1, \dots, l_{j+l-1}$  is called a strand of  $\gamma'$ .

This arc  $\gamma'$  has two distinct endpoints. One of the two endpoints is the projection of the endpoint of  $l_j$  on the  $X_{j-1}^{-1}$ -side, and the other one is the projection of the endpoint of  $l_{j+l-1}$  on the  $X_{j+l-1}$ -side. The word given in (1) is not clear enough to indicate the endpoint on the  $X_{j-1}^{-1}$ -side. To distinguish it from cyclic reduced words representing simple closed geodesics, we write the reduced  $\Gamma_n$ -word representing  $\gamma'$  as

$$(2) \quad \vec{X}_{j-1} W' = \vec{X}_{j-1} X_j \cdots X_{j+l-1},$$

where  $\vec{X}_{j-1}$  is to indicate that  $\vec{X}_{j-1} W'$  is not cyclic, and to indicate that one of the endpoints of  $\gamma'$  is the projection of a point on the  $X_{j-1}^{-1}$ -side.

A subarc of a geodesic  $\gamma \in \mathcal{G}_n$  will be called *admissible* if either it is  $\gamma$  itself, or is represented by a reduced  $\Gamma_n$ -word as given in (2).

**REMARK 1.1.** For  $\varepsilon = \pm 1$ ,  $X \in \Gamma_n$ ,  $X_1, X_2 \in \Gamma_n - \{X^{\pm 1}\}$ , and an integer  $k > 1$ , we shall write

$$X_1 \underbrace{X^\varepsilon \cdots X^\varepsilon}_{k \text{ times}} X_2 = X_1 X^{k\varepsilon} X_2.$$

Let  $\gamma \in \hat{\mathcal{G}}_n$  be a geodesic represented by a cyclic reduced  $\Gamma_n$ -word  $W(\gamma)$ . By the



same reasoning as that in [3, §3], there are no admissible subarcs of  $\gamma \in \mathcal{G}_n$  represented by any one of the following words:

$$\begin{array}{cccc} \vec{S}_1^\varepsilon S_1^\varepsilon, & \vec{S}_2^\varepsilon S_2^\varepsilon, & \vec{T}_1^\delta S_1^\varepsilon T_1^\delta, & \vec{T}_{n-3}^\delta S_2^\varepsilon T_{n-3}^\delta, \\ \vec{S}_1^\varepsilon T_1^k S_1^\delta, & \vec{S}_2^\varepsilon T_{n-3}^k S_2^\delta, & \vec{T}_j^\varepsilon T_{j+1}^\delta T_j^\varepsilon, & \vec{T}_{j+1}^\varepsilon T_j^\delta T_{j+1}^\varepsilon, \end{array}$$

where  $\varepsilon, \delta \in \{1, -1\}$ ,  $k \neq 0$  is an integer, and  $1 \leq j \leq n-4$ . Therefore, none of the words  $S_1^\varepsilon S_1^\varepsilon$ ,  $S_2^\varepsilon S_2^\varepsilon$ ,  $T_1^\delta S_1^\varepsilon T_1^\delta$ ,  $T_{n-3}^\delta S_2^\varepsilon T_{n-3}^\delta$ ,  $S_1^\varepsilon T_1^k S_1^\delta$ ,  $S_2^\varepsilon T_{n-3}^k S_2^\delta$ ,  $T_j^\varepsilon T_{j+1}^\delta T_j^\varepsilon$  and  $T_{j+1}^\varepsilon T_j^\delta T_{j+1}^\varepsilon$  is a subword of  $W(\gamma)$ .

**1.4. The free homotopy relative to  $\partial\mathcal{D}_n$ .** To be able to relate the geometric intersection number of two geodesics in  $\mathcal{G}_n$  to the intersection of their admissible subarcs, we shall define the *free homotopy relative to  $\partial\mathcal{D}_n$*  on a family of curves on  $\Sigma_n$  which contains all admissible subarcs of geodesics in  $\mathcal{G}_n$ .

The union of a finite number of mutually disjoint simple curves on  $\Sigma_n$  will be called a *multiple simple curve*. Let  $\mathcal{A}$  be the family of all multiple simple curves  $\beta$  on  $\Sigma_n$  satisfying the following three properties.

- (i)  $\beta$  lifts to a finite number of mutually disjoint simple arcs in  $\mathcal{D}_n$ , called the *strands* of  $\beta$ .
- (ii) Except the endpoints, each strand of  $\beta$  lies in the interior of  $\mathcal{D}_n$ .
- (iii) Each strand of  $\beta$  connects two different sides of  $\mathcal{D}_n$ .

Note that  $\mathcal{A}$  contains all reduced multiple simple loops on  $\Sigma_n$ , and contains all admissible subarcs of geodesics in  $\mathcal{G}_n$ .

Two multiple simple curves  $\beta_1$  and  $\beta_2$  in  $\mathcal{A}$  will be called *freely homotopic relative to  $\partial\mathcal{D}_n$* , written by  $\beta_1 \sim \beta_2$  (rel.  $\partial\mathcal{D}_n$ ), if for any two distinct  $X, X' \in \Gamma_n$

$$\begin{aligned} & \#(\text{strands of } \beta_1 \text{ connecting the } X\text{-side and the } X'\text{-side}) \\ &= \#(\text{strands of } \beta_2 \text{ connecting the } X\text{-side and the } X'\text{-side}). \end{aligned}$$

Note that two reduced multiple simple loops on  $\Sigma_n$  are freely homotopic if and only if they are freely homotopic relative to  $\partial\mathcal{D}_n$ . For  $\beta \in \mathcal{A}$ , let

$$[\beta]_{\partial\mathcal{D}_n} = \{\beta' \in \mathcal{A} : \beta' \sim \beta \text{ (rel. } \partial\mathcal{D}_n)\},$$

and we shall call a strand of  $\beta$  a *strand* of  $[\beta]_{\partial\mathcal{D}_n}$ .

Now, we may define the *strands* of a free homotopy class  $\alpha \in \mathcal{GL}_n$  as follows. Write  $\alpha = \bigoplus_{j=1}^m k_j \gamma_j$ , where  $(\gamma_1, \dots, \gamma_m) \in \Lambda_n^m$ , and  $m, k_1, \dots, k_m$  are positive integers with  $m \leq n-3$ . A strand of some  $\gamma_j$  is called a *strand* of  $\alpha$ . Similarly, an admissible subarc of some  $\gamma_j$  is called an *admissible subarc* of  $\alpha$ .

For  $\beta_1, \beta_2 \in \mathcal{A}$ , we define

$$i([\beta_1]_{\partial\mathcal{D}_n}, [\beta_2]_{\partial\mathcal{D}_n}) = \min\{\#(\beta'_1 \cap \beta'_2) : \beta'_1 \sim \beta_1 \text{ and } \beta'_2 \sim \beta_2 \text{ (rel. } \partial\mathcal{D}_n)\}.$$

where  $\#(\beta'_1 \cap \beta'_2)$  denotes the cardinality of the intersection  $\beta'_1 \cap \beta'_2$ . For simplicity, from now on we shall write

$$i([\beta_1]_{\partial \mathcal{D}_n}, [\beta_2]_{\partial \mathcal{D}_n}) = i([\beta_1], [\beta_2])_{\partial \mathcal{D}_n}$$

for  $\beta_1, \beta_2 \in \mathcal{A}$ . Note that if  $\beta_1$  and  $\beta_2$  are reduced multiple simple loops, then  $i([\beta_1], [\beta_2])_{\partial \mathcal{D}_n} = i([\beta_1], [\beta_2])$ .

**1.5. Four automorphisms of  $\mathcal{GL}_n$ .** We have set up a very symmetric fundamental domain  $\mathcal{D}_n$  for  $G_n$ . As we did in [4], by use of the symmetry of  $\mathcal{D}_n$ , we may cut down our discussion to fewer cases by introducing four automorphisms  $\Theta_1, \Theta_2, T_1$  and  $T_2$  of  $G_n$  defined by

$$\begin{aligned} \Theta_1(X) &= X^{-1} \text{ for } X \in \{S_1, S_2, T_j : 1 \leq j \leq n-3\}; \\ \Theta_2(S_1) &= S_2, \quad \Theta_2(S_2) = S_1, \text{ and } \Theta_2(T_j) = T_{n-j-2} \text{ for } 1 \leq j \leq n-3. \\ T_1(S_1) &= S_1^{-1}T_1 \text{ and } T_1(X) = X \text{ for } X \in \{S_2, T_j : 1 \leq j \leq n-3\}; \\ T_2(S_2) &= S_2^{-1}T_{n-3} \text{ and } T_2(X) = X \text{ for } X \in \{S_1, T_j : 1 \leq j \leq n-3\}. \end{aligned}$$

It follows from Nielsen's isomorphism theorem ([14] or [11, Theorem V.H.1]) that for  $j = 1$  or  $2$ , each of the automorphisms  $\Theta_j$  and  $T_j$  induces a homeomorphism of  $\Sigma_n$  onto itself, still denoted by  $\Theta_j$  and  $T_j$ .

Let  $\varphi$  be any one of the four homeomorphisms  $\Theta_1, \Theta_2, T_1$  and  $T_2$ . The action of  $\varphi$  on  $\mathcal{GL}_n$  is defined as follows. For every geodesic  $\gamma \in \mathcal{G}_n$ , let  $\varphi(\gamma)$  denote the free homotopy class containing the homeomorphic image of  $\gamma$  under  $\varphi$ . As before, let  $\varphi(\gamma)$  also denote the geodesic in the free homotopy class  $\varphi(\gamma)$ . Thus  $\varphi$  extends naturally to  $\mathcal{GL}_n$  such that

$$\varphi \left( \bigoplus_{j=1}^{n-3} k_j \gamma_j \right) = \bigoplus_{j=1}^{n-3} k_j \varphi(\gamma_j),$$

where  $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$  and  $(\gamma_1, \dots, \gamma_{n-3}) \in \Lambda_n^{n-3}$ .

Note that if  $\gamma \in \mathcal{G}_n$  is represented by a cyclic reduced  $\Gamma_n$ -word  $W$ , then  $\varphi(\gamma)$  is represented by  $\varphi(W)$ .

## 2. Elementary Intersection Numbers

In this section, we generalize elementary intersection numbers of elements of  $\mathcal{GL}_5$  [4, §2.1] to elements of  $\mathcal{GL}_n$ , and prove the elementary intersection formulas.

**2.1. The integer valued functions  $I_j$ .** Let  $\gamma_j^k \in \mathcal{G}_n$  be the geodesics given in §1.2. For any  $\alpha \in \mathcal{GL}_n$ , the geometric intersection numbers  $i(\alpha, \gamma_j^k)$  are called the *elementary intersection numbers* of  $\alpha$ .

Note that if  $\beta_1$  and  $\beta_2$  are two simple closed curves on a 2-sphere, and if they intersect transversally at every point of intersection, then  $\#(\beta_1 \cap \beta_2)$  is an even integer. Thus  $i(\alpha_1, \alpha_2)$  is an even integer for any two  $\alpha_1, \alpha_2 \in \mathcal{GL}_n$ . We shall write

$$I_j(\alpha) = \frac{i(\alpha, \gamma_j^1)}{2}$$

for  $\alpha \in \mathcal{GL}_n$ , and for  $1 \leq j \leq n-3$ . Note that if  $\gamma \in \mathcal{G}_n$  is represented by a cyclic reduced  $\Gamma_n$ -word  $W(\gamma) = W$ , then

$$\begin{aligned} I_1(\gamma) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_1\text{-side}) \\ &= \text{the total number of the letters } S_1 \text{ and } S_1^{-1} \text{ appearing in } W; \\ I_{n-3}(\gamma) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_2\text{-side}) \\ &= \text{the total number of the letters } S_2 \text{ and } S_2^{-1} \text{ appearing in } W. \end{aligned}$$

Thus for  $\alpha \in \mathcal{GL}_n$  we have

$$\begin{aligned} I_1(\alpha) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_1\text{-side}); \\ I_{n-3}(\alpha) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_2\text{-side}). \end{aligned}$$

Since  $\Theta_1(\gamma_j^1) = \gamma_j^1$  and  $\Theta_2(\gamma_j^1) = \gamma_{n-j-2}^1$ , the following proposition is an immediate consequence of the definition.

**Proposition 2.1.** *If  $\alpha \in \mathcal{GL}_n$ , then*

$$I_j(\alpha) = I_j(\Theta_1(\alpha)) \quad \text{and} \quad I_j(\alpha) = I_{n-j-2}(\Theta_2(\alpha)) \quad \text{for } 1 \leq j \leq n-3.$$

By an argument similar to the one in the proof of [4, Proposition 2.2 (i), (ii)], we obtain:

**Proposition 2.2.** *Let  $\alpha \in \mathcal{GL}_n$ . For any integer  $m$ ,  $\mathcal{T}_1^m(\alpha) = \alpha$  when  $I_1(\alpha) = 0$ , while  $\mathcal{T}_2^m(\alpha) = \alpha$  if  $I_{n-3}(\alpha) = 0$ .*

**Proposition 2.3.** *If  $\alpha \in \mathcal{GL}_n$ , and if  $m$  is an integer, then*

$$I_1(\alpha) = I_1(\mathcal{T}_j^m(\alpha)) \quad \text{and} \quad I_{n-3}(\alpha) = I_{n-3}(\mathcal{T}_j^m(\alpha)) \quad \text{for } j = 1, 2.$$

Proof. Since  $\gamma_1^1$  and  $\gamma_{n-3}^1$  are invariant under each  $\mathcal{T}_j$ , the proof is straightforward.  $\square$

**Proposition 2.4.** *If  $\alpha \in \mathcal{GL}_n$ , and if  $j, k$  and  $m$  are integers with  $1 \leq k \leq 3$ , then*

$$i(\alpha, \gamma_j^k) = i(\mathcal{T}_1^m(\alpha), \gamma_j^k) \quad \text{for } 1 < j \leq n-3, \text{ and}$$

$$i(\alpha, \gamma_j^k) = i(\mathcal{T}_2^m(\alpha), \gamma_j^k) \quad \text{for } 1 \leq j < n-3.$$

Proof. By Proposition 2.2, we have  $\mathcal{T}_1(\gamma_j^k) = \gamma_j^k$  for  $1 < j \leq n-3$ , and  $\mathcal{T}_2(\gamma_j^k) = \gamma_j^k$  for  $1 \leq j < n-3$ . The proof is complete.  $\square$

**2.2. Cyclic semi-reduced words.** To compute elementary intersection numbers, we associate to geodesics in  $\mathcal{G}_n$  *cyclic semi-reduced*  $\Gamma_n$ -words, which are defined analogously to those in [4, §2.2].

Let  $\gamma \in \mathcal{G}_n$  with  $I_{n-3}(\gamma) > 0$ . Assume that  $\gamma$  is represented by a cyclic reduced  $\Gamma_n$ -word  $W(\gamma)$ . If  $S_2^\varepsilon X$  or  $XS_2^\varepsilon$  is a subword of  $W(\gamma)$  with  $\varepsilon = \pm 1$  and  $X \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$ , we shall write

$$S_2^\varepsilon X = S_2^\varepsilon T_{n-3}^0 X \quad \text{and} \quad XS_2^\varepsilon = XT_{n-3}^0 S_2^\varepsilon.$$

Similarly, for a geodesic  $\gamma \in \mathcal{G}_n$  with  $I_1(\gamma) > 0$ , if  $X \in \Gamma_n - \{S_1^{\pm 1}, T_1^{\pm 1}\}$ , and if  $S_1^\varepsilon X$  or  $XS_1^\varepsilon$  is a subword of  $W(\gamma)$ , then we write  $S_1^\varepsilon X = S_1^\varepsilon T_1^0 X$  and  $XS_1^\varepsilon = XT_1^0 S_1^\varepsilon$ . The resulting cyclic  $\Gamma_n$ -word is called a *semi-reduced*, still denoted by  $W(\gamma)$ .

As in [4, §2.5], we shall write cyclic semi-reduced  $\Gamma_n$ -words in two canonical forms. First, we subdivide  $\mathcal{GL}_n$  into four classes.

Note that every geodesic in  $\mathcal{G}_n$  can not simultaneously have a strand joining the  $S_2^\varepsilon$ -side to the  $T_{n-3}$ -side and a strand joining the  $S_2^\varepsilon$ -side to the  $T_{n-3}^{-1}$ -side for  $\varepsilon = 1$  or  $-1$ , (see Remark 1.1).

Let  $\mathcal{GL}_n^+(T_{n-3})$  be the set of elements of  $\mathcal{GL}_n$  which have no strands connecting the  $T_{n-3}$ -side to the  $S_2^\varepsilon$ -side for  $\varepsilon = \pm 1$ . Let

$$\mathcal{GL}_n^-(T_{n-3}) = \Theta_1(\mathcal{GL}_n^+(T_{n-3})),$$

and let

$$\mathcal{GL}_n^+(T_1) = \Theta_2(\mathcal{GL}_n^+(T_{n-3})) \quad \text{and} \quad \mathcal{GL}_n^-(T_1) = \Theta_2(\mathcal{GL}_n^-(T_{n-3})).$$

Consequently,  $\mathcal{GL}_n^-(T_1) = \Theta_1(\mathcal{GL}_n^+(T_1))$ . We remark that  $\alpha \in \mathcal{GL}_n^+(T_1)$  if and only if  $\alpha$  has no strands connecting the  $T_1$ -side to the  $S_1^\varepsilon$ -side for  $\varepsilon = \pm 1$ . The set  $\mathcal{G}_n$  is then subdivided into four subclasses as:

$$\begin{aligned} \mathcal{G}_n^+(T_1) &= \mathcal{G}_n \cap \mathcal{GL}_n^+(T_1) \quad \text{and} \quad \mathcal{G}_n^-(T_1) = \Theta_1(\mathcal{G}_n^+(T_1)), \\ \mathcal{G}_n^+(T_{n-3}) &= \mathcal{G}_n \cap \mathcal{GL}_n^+(T_{n-3}) \quad \text{and} \quad \mathcal{G}_n^-(T_{n-3}) = \Theta_1(\mathcal{G}_n^+(T_{n-3})). \end{aligned}$$

Now, by the same reasoning as in [4, §2.5], every  $\gamma \in \mathcal{G}_n$  with  $I_1(\gamma) > 0$  or  $I_{n-3}(\gamma) > 0$  is represented by a cyclic semi-reduced  $\Gamma_n$ -word  $W$  as given below.

First, assume that  $I_{n-3}(\gamma) = m > 0$ . There exist  $m$  triples  $(\varepsilon_j, p_j, q_j)$  of integers with  $\varepsilon_j = \pm 1$ ,  $p_j \geq 0$  and  $q_j \geq 0$ , and there exist  $m$  reduced  $\Gamma_n$ -words  $W_j = \prod_{i=1}^{\nu_j} X_{ji}$

with  $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$ , and  $X_{ji} \in \Gamma_n - \{S_2^{\pm 1}\}$  when  $1 < i < \nu_j$  such that

$$(3) \quad \gamma \in \mathcal{G}_n^-(T_{n-3}) \implies W = \prod_{j=1}^m T_{n-3}^{-p_j} S_2^{\varepsilon_j} T_{n-3}^{q_j} W_j;$$

$$(4) \quad \gamma \in \mathcal{G}_n^+(T_{n-3}) \implies W = \prod_{j=1}^m T_{n-3}^{p_j} S_2^{\varepsilon_j} T_{n-3}^{-q_j} W_j.$$

If  $I_1(\gamma) = m > 0$ , by considering  $\Theta_2(\gamma)$ , then  $\gamma$  is represented by

$$W = \prod_{j=1}^m T_1^{-p_j} S_1^{\varepsilon_j} T_1^{q_j} W_j,$$

where  $(\varepsilon_j, p_j, q_j)$  are integers with  $\varepsilon_j = \pm 1$  and  $p_j q_j \geq 0$ , and where  $W_j = \prod_{i=1}^{\nu_j} X_{ji}$  are reduced  $\Gamma_n$ -words with  $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{S_1^{\pm 1}, T_1^{\pm 1}\}$ , and  $X_{ji} \in \Gamma_n - \{S_1^{\pm 1}\}$  when  $1 < i < \nu_j$ . Moreover,  $\gamma \in \mathcal{G}_n^-(T_1)$  if and only if  $p_j \geq 0$  and  $q_j \geq 0$  for all  $j$ , while  $\gamma \in \mathcal{G}_n^+(T_1)$  if and only if  $p_j \leq 0$  and  $q_j \leq 0$  for all  $j$ .

We remark that any word given above is reduced if each  $p_j q_j > 0$ .

**2.3. Essential blocks and puncture-like blocks.** We shall compute elementary intersection numbers by applying mathematical induction to the number  $n$  of punctures. From now on, we assume that  $n \geq 5$ .

To be able to apply mathematical induction to  $n$ , we first embed  $\mathcal{GL}_{n-1}$  into  $\mathcal{GL}_n$ . Let  $\Phi_n: G_{n-1} \longrightarrow G_n$  be the monomorphism defined by

$$\Phi_n(S_1) = S_1, \quad \Phi_n(S_2) = T_{n-3} \quad \text{and} \quad \Phi_n(T_j) = T_j \quad \text{for } 1 \leq j \leq n-4.$$

The monomorphism  $\Phi_n$  induces an injective map of  $\mathcal{G}_{n-1}$  into  $\mathcal{G}_n$ , also denoted by  $\Phi_n$ . If  $\gamma \in \mathcal{G}_{n-1}$  is represented by a cyclic reduced (or semi-reduced)  $\Gamma_{n-1}$ -word  $W$ , then  $\Phi_n(\gamma)$  is represented by  $\Phi_n(W)$ .

Let  $\mathcal{G}_n^{(n-1)}$  be the image of  $\mathcal{G}_{n-1}$  mapped by  $\Phi_n$ , and let  $\mathcal{GL}_n^{(n-1)}$  be the set of all elements of  $\mathcal{GL}_n$  of the form

$$\bigoplus_{j=1}^{n-4} k_j \gamma_j,$$

where  $(k_1, \dots, k_{n-4}) \in \mathbb{Z}_+^{n-4}$ , and  $\gamma_j \in \mathcal{G}_n^{(n-1)}$  are mutually disjoint geodesics. Note that if  $\gamma \in \mathcal{GL}_n^{(n-1)}$ , then  $2I_{n-3}(\gamma) = i(\gamma, \gamma_{n-3}^1) = 0$ .

The geodesic  $\gamma_{n-3}^1$  divides  $\Sigma_n$  into two connected components. One of them is a sphere with  $n-2$  punctures and one hole, denoted by  $\Sigma_n^{(n-1)}$ , and the other one is a sphere with two punctures and one hole, denoted by  $\Sigma_n^{(3)}$ . Note that the punctures of  $\Sigma_n^{(3)}$  correspond to the fixed points of the transformations  $S_2$  and  $S_2 T_{n-3}^{-1}$ . Also note

that  $\Sigma_n^{(n-1)}$  is homeomorphic to  $\Sigma_{n-1}$ , and  $\Sigma_n^{(3)}$  is homeomorphic to a 3-punctured sphere. It follows from the definition that every  $\gamma \in \mathcal{GL}_n^{(n-1)}$  contains a representative lying on  $\Sigma_n^{(n-1)}$ . Thus we can do an induction after we relate free homotopy classes in  $\mathcal{GL}_n$  to free homotopy classes in  $\mathcal{GL}_n^{(n-1)}$ .

To relate free homotopy classes in  $\mathcal{GL}_n$  to that in  $\mathcal{GL}_n^{(n-1)}$ , we consider the set  $\mathcal{GL}_n^0$  of all free homotopy classes in  $\mathcal{GL}_n$  which have no strands connecting the  $S_2^\varepsilon$ -side to the  $X$ -side, where  $\varepsilon = \pm 1$ , and where  $X \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$ . Let  $\mathcal{G}_n^0 = \mathcal{G}_n \cap \mathcal{GL}_n^0$ .

It follows immediately from the definition that if  $\gamma \in \mathcal{GL}_n$  with  $I_{n-3}(\gamma) = 0$ , then  $\gamma \in \mathcal{GL}_n^0$ . In particular,  $\mathcal{GL}_n^{(n-1)} \subset \mathcal{GL}_n^0$ .

Let  $\gamma \in \mathcal{G}_n$  with  $I_{n-3}(\gamma) = m > 0$ . Then  $\gamma \in \mathcal{G}_n^-(T_{n-3}) \cap \mathcal{G}_n^0$  if and only if it is represented by a cyclic reduced  $\Gamma_n$ -word as given in (3), while  $\gamma \in \mathcal{G}_n^+(T_{n-3}) \cap \mathcal{G}_n^0$  if and only if it is represented by a cyclic reduced  $\Gamma_n$ -word as given in (4) with  $p_j > 0$  and  $q_j > 0$  for all  $j$ .

The admissible subarcs of every  $\gamma \in \mathcal{GL}_n^0$  fall into two classes. One contains admissible subarcs of  $\gamma$  which are freely homotopic relative to  $\partial\mathcal{D}_n$  to simple curves lying on  $\Sigma_n^{(n-1)}$ . The other class contains admissible subarcs of  $\gamma$  which are freely homotopic relative to  $\partial\mathcal{D}_n$  to simple curves lying on  $\Sigma_n^{(3)}$ . We shall relate  $\gamma$  to free homotopy classes in  $\mathcal{GL}_n^{(n-1)}$  by relating the admissible subarcs of  $\gamma$  in the first class to elements of  $\mathcal{GL}_n^{(n-1)}$ .

Any  $\gamma \in \mathcal{GL}_n$  can be related to an element of  $\mathcal{GL}_n^0$  as follows.

**Proposition 2.5.** *Let  $\gamma \in \mathcal{GL}_n$ .*

- (i) *If  $\gamma \in \mathcal{GL}_n^+(T_{n-3})$ , then  $T_2^{-2}(\gamma) \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ .*
- (ii) *If  $\gamma \in \mathcal{GL}_n^-(T_{n-3})$ , then  $T_2^2(\gamma) \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^-(T_{n-3})$ .*

*Proof.* It suffices to prove (i) for  $\gamma \in \mathcal{G}_n^+(T_{n-3})$ . There is nothing to prove if  $I_{n-3}(\gamma) = 0$ . If  $I_{n-3}(\gamma) = m > 0$ , then  $\gamma$  is represented by the cyclic semi-reduced  $\Gamma_n$ -word given in (4). Now, the assertion follows since  $T_2^{-2}(W) = \prod_{j=1}^m T_{n-3}^{p_j+1} S_2^{\varepsilon_j} T_{n-3}^{-q_j-1} W_j$ .  $\square$

With Proposition 2.5, we may restrict our attention to the subclass  $\mathcal{GL}_n^0$  of  $\mathcal{GL}_n$ .

Before continuing our discussion, we choose once for all an orientation for the  $X$ -side of  $\mathcal{D}_n$ , where  $X \in \{T_{n-3}, T_{n-3}^{-1}, T_{n-4}, T_{n-4}^{-1}\}$ . Note that  $T_{n-4}T_{n-3}^{-1}$  is parabolic since the trace of  $T_{n-4}T_{n-3}^{-1}$  is  $-2$ . Let  $\zeta$  be the fixed point of the transformation  $T_{n-4}T_{n-3}^{-1}$ . For  $X = T_{n-3}$  or  $T_{n-4}$ , if  $P_1$  and  $P_2$  are two points lying on the  $X$ -side, and if  $P_1$  lies between  $\zeta$  and  $P_2$ , then we write  $P_1 \prec P_2$ . If  $Q_1$  and  $Q_2$  are two points lying on the  $X^{-1}$ -side, we write  $Q_1 \prec Q_2$  whenever  $X(Q_1) \prec X(Q_2)$ .

**Proposition 2.6.** *Let  $\gamma \in \mathcal{G}_n^+(T_{n-3})$  with  $I_{n-3}(\gamma) = m > 0$ , and let  $\gamma$  be represented by the cyclic reduced  $\Gamma_n$ -word given in (4).*

*If  $\gamma$  has a strand joining the  $T_{n-3}$ -side to the  $T_{n-4}$ -side, and has a strand joining*

the  $T_{n-3}$ -side to the  $T_{n-4}^{-1}$ -side, then  $W_j = T_{n-4}$  or  $W_j = T_{n-4}^{-1}$  for some  $j$ .

Proof. Let  $P_1 \prec \cdots \prec P_k$  be the points where the strands of  $\gamma$  meet the  $T_{n-4}$ -side. For every integer  $l$  with  $1 \leq l \leq k$ , let  $P'_l$  be the point on the  $T_{n-4}^{-1}$ -side identified with  $P_l$  by the transformation  $T_{n-4}$ . Let  $l$  be the strand of  $\gamma$  with an endpoint at  $P_1$ , and let  $l'$  be the strand of  $\gamma$  with an endpoint at  $P'_1$ .

By assumption,  $l$  must connect the  $T_{n-3}$ -side and the  $T_{n-4}$ -side, and  $l'$  must connect the  $T_{n-3}$ -side and the  $T_{n-4}^{-1}$ -side. The union  $l \cup l'$  projects to an admissible subarc  $\gamma'$  of  $\gamma$  represented by  $\vec{T}_{n-3}^{-1} T_{n-4} T_{n-3}$  or  $\vec{T}_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$ .

Assume that  $\gamma'$  is represented by  $\vec{T}_{n-3}^{-1} T_{n-4} T_{n-3}$ . Let  $Q$  be the endpoint of  $l$  on the  $T_{n-3}$ -side. We orient  $\gamma'$  so that the projection of  $Q$  to  $\Sigma_n$  is the initial point of  $\gamma'$ . Since  $\gamma \in \mathcal{G}_n^0$ , then the subword  $W' = T_{n-3}^{-1} T_{n-4} T_{n-3}$  of  $W$  must be followed by a subword of the form  $T_{n-3}^p S_2^\varepsilon T_{n-3}^{-q}$  for some integers  $\varepsilon = \pm 1$ ,  $p \geq 0$  and  $q > 0$ .

On the other hand, consider the subarc  $\gamma''$  of  $\gamma$  which has the same underlying set with  $\gamma'$  but with the opposite orientation. Then  $\gamma''$  is represented by  $\vec{T}_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$ . Thus  $T_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$  is a subword of  $W^{-1}$  and is followed by a subword of the form  $T_{n-3}^{p'} S_2^{\varepsilon'} T_{n-3}^{-q'}$  for some integers  $\varepsilon' = \pm 1$ ,  $p' \geq 0$  and  $q' > 0$ . We conclude that

$$T_{n-3}^{q'} S_2^{-\varepsilon'} T_{n-3}^{-p'} \cdot W' \cdot T_{n-3}^p S_2^\varepsilon T_{n-3}^{-q} = T_{n-3}^{q'} S_2^{-\varepsilon'} T_{n-3}^{-p'-1} T_{n-4} T_{n-3}^{p+1} S_2^\varepsilon T_{n-3}^{-q}$$

is a subword of  $W$ . This proves that  $W_j = T_{n-4}$  for some  $j$ .

Similarly, if  $\gamma'$  is represented by  $\vec{T}_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$ , then there is an integer  $j$  such that  $W_j = T_{n-4}^{-1}$ .  $\square$

**Blocks of simple closed geodesics.** Let  $\gamma$  be given in Proposition 2.6. For every integer  $j$  with  $1 \leq j \leq m = I_{n-3}(\gamma)$ , let  $\gamma_j$  be the admissible subarc of  $\gamma$  represented by  $\vec{T}_{n-3}^{-1} W_j T_{n-3}$ . Every  $\gamma_j$  will be called a *block* of  $\gamma$ .

Let  $l_0^{(j)}$  be the strand of  $\gamma_j$  joining the  $T_{n-3}$ -side to the  $X_{j1}$ -side with  $P$  the endpoint on the  $T_{n-3}$ -side, and let  $l_1^{(j)}$  be the strand of  $\gamma_j$  joining the  $X_{j\nu_j}^{-1}$ -side to the  $T_{n-3}$ -side with  $Q$  the endpoint on the  $X_{j\nu_j}^{-1}$ -side. Let  $P'$  be the point on the  $T_{n-3}^{-1}$ -side which is identified with  $P$  by the transformation  $T_{n-3}$ .

Now, we replace  $l_1^{(j)}$  by a simple arc  $\tilde{l}_1^{(j)}$  joining  $Q$  to  $P'$  so that  $\tilde{l}_1^{(j)}$  is disjoint from all strands of  $\gamma_j$  except possibly  $l_1^{(j)}$ . Let  $\mathcal{L}_j$  be the union of all strands of  $\gamma_j$  other than  $l_1^{(j)}$ . The union  $\mathcal{L}_j \cup \tilde{l}_1^{(j)}$  projects to a simple closed curve  $\tilde{\gamma}_j$  on  $\Sigma_n$ . See the proof of [4, Theorem 5.3].

If  $W_j = T_{n-4}$  or  $T_{n-4}^{-1}$ , then  $\tilde{\gamma}_j$  is a simple loop around the puncture corresponding to the fixed point of  $T_{n-3} T_{n-4}^{-1}$ . In this case, we shall call  $\gamma_j$  a *puncture-like block* of  $\gamma$ .

We call  $\gamma_j$  an *essential block* of  $\gamma$  if  $\gamma_j$  is not a puncture-like block. Thus  $\gamma_j$  is an essential block if and only if  $\tilde{\gamma}_j \in \mathcal{G}_n^{(n-1)}$ .

Next, let  $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^-(T_{n-3})$  with  $I_{n-3}(\gamma) > 0$ . An admissible subarc  $\gamma'$  of  $\gamma$  is

called a *puncture-like block* if  $\Theta_1(\gamma')$  is a puncture-like block of  $\Theta_1(\gamma)$ , and is called an *essential block* if  $\Theta_1(\gamma')$  is an essential block of  $\Theta_1(\gamma)$ . By Proposition 2.6,  $\gamma'$  is a puncture-like block of  $\gamma$  if and only if it is represented by  $\vec{T}_{n-3}T_{n-4}^\varepsilon T_{n-3}^{-1}$  with  $\varepsilon = \pm 1$ .

**Blocks of free homotopy classes in  $\mathcal{GL}_n^0$ .** For  $\gamma \in \mathcal{GL}_n^0$  with  $I_{n-3}(\gamma) > 0$ , there are positive integers  $k_1, \dots, k_m$ , and mutually disjoint geodesics  $\beta_1, \dots, \beta_m$  in  $\mathcal{G}_n^0$  such that

$$\gamma = \bigoplus_{i=1}^m k_i \beta_i,$$

where  $m$  is a positive integer with  $m \leq n-3$ . An admissible subarc  $\gamma'$  of  $\gamma$  is called a *block* of  $\gamma$  if it is either a connected component of  $\gamma$  with  $I_{n-3}(\gamma') = 0$ , or is a block of some  $\beta_i$ . A block  $\gamma'$  of  $\gamma$  is called *puncture-like* if it is a puncture-like block of some  $\beta_i$ , and is called *essential* if it is not a puncture-like block. Note that if  $\gamma'$  is a connected component of  $\gamma$  with  $I_{n-3}(\gamma') = 0$ , then  $\gamma' \in \mathcal{G}_n^{(n-1)}$ . Such an essential block will be called of the *second kind*. An essential block of  $\gamma$  will be called of the *first kind* if it is not of the second kind.

**REMARK 2.1.** It follows from Proposition 2.6 that if  $\gamma \in \mathcal{GL}_n^+(T_{n-3})$  has a strand joining the  $T_{n-3}$ -side to the  $T_{n-4}$ -side, and has a strand joining the  $T_{n-3}$ -side to the  $T_{n-4}^{-1}$ -side, then  $\gamma$  has a puncture-like block. Similarly, if  $\gamma \in \mathcal{GL}_n^-(T_{n-3})$  has a strand joining the  $T_{n-3}^{-1}$ -side to the  $T_{n-4}$ -side, and has a strand joining the  $T_{n-3}^{-1}$ -side to the  $T_{n-4}^{-1}$ -side, then  $\gamma$  has a puncture-like block.

**REMARK 2.2.** Let  $\gamma \in \mathcal{GL}_n^0$  with  $I_{n-3}(\gamma) > 0$ . If  $\gamma$  has no essential blocks, then  $I_1(\gamma) = 0$  and  $I_{n-3}(\Theta_2(\gamma)) = 0$ . Note that  $\Theta_2(\gamma) \in \mathcal{GL}_n^{(n-1)}$ . Thus the elementary intersection numbers of  $\gamma$  will be obtained from that of  $\Theta_2(\gamma)$  by applying induction to  $n$ . Therefore, we shall only consider the case where  $\gamma$  has essential blocks.

The following theorem plays an important role in the sequel.

**Theorem 2.7.** Let  $\gamma \in \mathcal{GL}_n^0$  with  $I_{n-3}(\gamma) > 0$ . If  $\gamma$  has essential blocks, then there is an  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$  such that

$$i(\alpha_\gamma, \gamma_{n-4}^1) = i(\gamma, \gamma_{n-4}^1) \quad \text{and} \quad i(\alpha_\gamma, \gamma_j^k) = i(\gamma, \gamma_j^k)$$

for  $1 \leq j < n-4$  and  $1 \leq k \leq 3$ . Furthermore,  $\alpha_\gamma$  can be chosen so that  $\Theta_1(\alpha_{\Theta_1(\gamma)}) = \alpha_\gamma$ .



Since for all  $j, k$ ,

$$i(\Theta_1(\alpha_{\Theta_1(\gamma)}), \gamma_j^k) = i(\alpha_{\Theta_1(\gamma)}, \Theta_1(\gamma_j^k)) = i(\Theta_1(\gamma), \Theta_1(\gamma_j^k)) = i(\gamma, \gamma_j^k),$$

$\alpha_\gamma$  can be chosen so that  $\Theta_1(\alpha_{\Theta_1(\gamma)}) = \alpha_\gamma$  since for all  $j, k$ . Thus, we may assume that  $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ .

First, we prove Theorem 2.7 for  $\gamma$  which has no puncture-like blocks. Let  $\mathcal{E}$  be the set of all essential blocks of  $\gamma$ , and for every  $\gamma' \in \mathcal{E}$  let

$$t(\gamma') = \text{the number of strands of } \gamma' \text{ meeting the } T_{n-3}\text{-side.}$$

If  $\gamma$  has no essential blocks of the first kind, then any essential block of  $\gamma$  serves as  $\alpha_\gamma$ .

Now, we assume that  $\gamma$  has exactly  $e > 0$  essential blocks of the first kind, say  $\gamma_1, \dots, \gamma_e$ . Let  $\tilde{\gamma}_j$  be the geodesic in  $\mathcal{G}_n^{(n-1)}$  corresponding to  $\gamma_j$  (see the definition of blocks), and let  $t_j$  be the number of strands of  $\tilde{\gamma}_j$  meeting the  $T_{n-3}$ -side. Note that  $t(\gamma_j) = t_j + 1$ , and the strands of  $\gamma_j$  meet the  $T_{n-3}^{-1}$ -side in exactly  $t_j - 1$  points. Then the strands of  $\cup_{j=1}^e \gamma_j$  meet the  $T_{n-3}^{-1}$ -side in exactly  $t_0 = \sum_{j=1}^e (t_j - 1)$  points, and meet the  $T_{n-3}$ -side in exactly  $t_0 + 2e$  points.

We consider the disjoint union  $\mathcal{L}$  of strands of all essential blocks of  $\gamma$ . Let  $Q_1 \prec Q_2 \prec \dots \prec Q_q$  be the points where  $\mathcal{L}$  meets the  $T_{n-3}$ -side, where  $q$  is an integer with  $q \geq t_0 + 2e$ .

CLAIM 1. For every integer  $j$  with  $q - 2e + 1 \leq j \leq q$ , the point  $Q_j$  is an endpoint of a strand  $L_j$  of  $\cup_{j=1}^e \gamma_j$ .

We shall show that Claim 1 implies Theorem 2.7 when  $\gamma$  has no puncture-like blocks. For every integer  $j$  with  $q - e + 1 \leq j \leq q$ , let  $P_j$  be the endpoint of  $L_j$  other than  $Q_j$ , and let  $Q'_{j-e}$  be the point lying on the  $T_{n-3}^{-1}$ -side which is identified with  $Q_{j-e}$  by the transformation  $T_{n-3}$ . There are mutually disjoint simple arcs  $L'_j$ ,  $q - e + 1 \leq j \leq q$ , in  $\mathcal{D}_n$  satisfying the following two properties:

- (i) Each  $L'_j$  connects  $P_j$  to  $Q'_{j-e}$ .
- (ii) Each  $L'_j$  is disjoint from the strands of any essential block of  $\gamma$  except possibly the strands  $L_{q-e+1}, \dots, L_q$ .

The set  $\mathcal{L}' = (\mathcal{L} - \cup_{j=q-e+1}^q L_j) \cup (\cup_{j=q-e+1}^q L'_j)$  projects to a multiple simple loop  $\alpha_\gamma$  in  $\mathcal{GL}_n^{(n-1)}$ , and the free homotopy class represented by  $\alpha_\gamma$ , still denoted by  $\alpha_\gamma$ , satisfies the required conditions since  $i(\alpha_\gamma, \gamma_{n-4}^1) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n}$  and  $i(\alpha_\gamma, \gamma_j^k) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_j^k)_{\partial \mathcal{D}_n}$  for  $1 \leq j < n - 4$  and  $1 \leq k \leq 3$ .

Proof of Claim 1. There is nothing to prove if  $\gamma$  has no essential blocks  $\gamma'$  of the second kind with  $t(\gamma') > 0$ . Assume that  $\gamma$  has exactly  $p > 0$  essential blocks  $\gamma_{e+1}, \dots, \gamma_{e+p}$  of the second kind with  $t(\gamma_{e+j}) > 0$ ,  $1 \leq j \leq p$ .

For every  $j$  with  $1 \leq j \leq e$ , the block  $\gamma_j$  is represented by a reduced  $\Gamma_n$ -word  $\vec{T}_{n-3}^{-1} W_j T_{n-3}$ , where  $W_j \neq T_{n-4}^{\pm 1}$  is of the form  $W_j = \prod_{i=1}^{\nu_j} X_{ji}$  with  $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$  and  $X_{ji} \in \Gamma_n - \{S_2^{\pm 1}\}$  for  $1 < i < \nu_j$ . Let

$l_j^{(1)}$  be the strand of  $\gamma_j$  joining the  $T_{n-3}$ -side to the  $X_{j1}$ -side,

$l_j^{(2)}$  be the strand of  $\gamma_j$  joining the  $X_{j\nu_j}^{-1}$ -side to the  $T_{n-3}$ -side,

$Q_{jk}$  be the endpoint of  $l_j^{(k)}$  on the  $T_{n-3}$ -side for  $k = 1, 2$ , and

$Q'_{jk}$  be the point on the  $T_{n-3}^{-1}$ -side identified with  $Q_{jk}$  by the transformation  $T_{n-3}$ .

By the definition of  $\gamma_j$ , the point  $Q'_{jk}$  is an endpoint of a strand  $L_j^{(k)}$  of  $\gamma$  joining the  $T_{n-3}^{-1}$ -side to the  $X$ -side with  $X \in \{T_{n-3}, S_2^{\pm 1}\}$ .

Suppose that there is an integer  $m$  with  $q-2e+1 \leq m \leq q$  such that  $Q_m$  is an endpoint of a strand of  $\cup_{j=1}^p \gamma_{e+j}$ . Then there is a  $Q_{jk}$  such that  $Q_{jk} \prec Q_m$ . Let  $Q'_m$  be the point on the  $T_{n-3}^{-1}$ -side identified with  $Q_m$  by the transformation  $T_{n-3}$ . It follows from the definition of  $\gamma_{e+j}$  that  $Q'_m$  is an endpoint of a strand  $L$  of  $\gamma$  joining the  $T_{n-3}^{-1}$ -side to some  $X$ -side with  $X \in \Gamma_n - \{T_{n-3}, S_2^{\pm 1}\}$ . Since  $Q_{jk} \prec Q_m$ , then  $Q'_{jk} \prec Q'_m$ , and thus  $L_j^{(k)}$  must intersect  $L$  transversally. This contradiction completes the proof of the claim.  $\square$

In the following, we prove Theorem 2.7 for  $\gamma$  which has puncture-like blocks. For this case, we need the following two lemmas.

**Lemma 2.8.** *If  $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$  with  $I_{n-3}(\gamma) > 0$ , and if  $\gamma$  has a puncture-like block, then every essential block of  $\gamma$  has no strands meeting the  $T_{n-3}^{-1}$ -side.*

*Proof.* Let  $\gamma_0$  be a puncture-like block of  $\gamma$ . There is a strand  $l_0$  of  $\gamma_0$  connecting the  $T_{n-4}$ -side and the  $T_{n-3}$ -side. Let  $Q_0$  be the endpoint of  $l_0$  on the  $T_{n-3}$ -side. We may choose  $\gamma_0$  so that  $Q_0 \prec Q$  whenever  $Q$  is an endpoint of a strand of  $\gamma$  on the  $T_{n-3}$ -side. Let  $Q'_0$  be the point on the  $T_{n-3}^{-1}$ -side which is identified with  $Q_0$  by the transformation  $T_{n-3}$ . Note that  $Q'_0$  is an endpoint of a strand  $L_0$  of  $\gamma$  joining the  $T_{n-3}^{-1}$ -side to the  $X$ -side with  $X \in \{T_{n-3}, S_2, S_2^{-1}\}$ . Also note that if  $Q'$  is an endpoint of a strand of  $\gamma$  on the  $T_{n-3}^{-1}$ -side, then  $Q'_0 \prec Q'$  by the definition of  $Q_0$ .

Now, suppose that there is an essential block  $\gamma'$  of  $\gamma$  such that  $\gamma'$  has a strand  $l'$  meeting the  $T_{n-3}^{-1}$ -side at a point  $Q'$ . Since  $Q'_0 \prec Q'$ , and since  $\gamma'$  has no strands joining the  $T_{n-3}^{-1}$ -side to the  $X$ -side with  $X \in \{T_{n-3}, S_2, S_2^{-1}\}$ , then  $l'$  must intersect  $L_0$  transversally. This is a contradiction.  $\square$

**Lemma 2.9.** *Let  $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$  with  $I_{n-3}(\gamma) > 0$ , and for an arbitrary block  $\gamma'$  of  $\gamma$ , let  $t(\gamma')$  be the number of strands of  $\gamma'$  meeting the  $T_{n-3}$ -side. If  $\gamma$  has*

puncture-like blocks, then

$$t(\gamma') = \begin{cases} 2 & \text{if } \gamma' \text{ is a puncture-like block,} \\ 2 & \text{if } \gamma' \text{ is an essential block of the first kind,} \\ 0 & \text{if } \gamma' \text{ is an essential block of the second kind.} \end{cases}$$

*Proof.* It follows immediately from the definition that  $t(\gamma') = 2$  whenever  $\gamma'$  is a puncture-like block of  $\gamma$ .

Let  $\gamma'$  be an essential block of the second kind, i.e.,  $\gamma'$  is a simple closed geodesic in  $\mathcal{GL}_n^{(n-1)}$ . If  $\gamma'$  has a strand meeting the  $T_{n-3}$ -side, then  $\gamma'$  must have a strand meeting the  $T_{n-3}^{-1}$ -side. This contradicts to Lemma 2.8. Therefore,  $t(\gamma') = 0$ .

If  $\gamma'$  is an essential block of the first kind, then  $\gamma'$  is represented by a reduced  $\Gamma_n$ -word  $\bar{T}_{n-3}^{-1}WT_{n-3}$ , where  $W \neq T_{n-4}^{\pm 1}$  is of the form  $W = \prod_{j=1}^m X_j$  with  $X_1, X_m \in \Gamma_n - \{T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$ , and  $X_j \in \Gamma_n - \{S_2^{\pm 1}\}$  for  $1 < j < m$ . There is a strand  $l_0$  of  $\gamma'$  joining the  $T_{n-3}$ -side to the  $X_1$ -side, and there is another strand  $l_1$  of  $\gamma'$  joining the  $X_m$ -side to the  $T_{n-3}$ -side. Thus  $t(\gamma') \geq 2$ .

Suppose that  $t(\gamma') > 2$ . There is a  $k \in \{2, \dots, m-1\}$  such that  $X_k = T_{n-3}$  or  $X_k = T_{n-3}^{-1}$ . If  $X_k = T_{n-3}$ , then  $\gamma'$  has a strand joining the  $T_{n-3}^{-1}$ -side to the  $X_{k+1}$ -side. This is a contradiction to Lemma 2.8. If  $X_k = T_{n-3}^{-1}$ , then  $\gamma'$  has a strand joining the  $X_{k-1}$ -side to the  $T_{n-3}^{-1}$ -side. This is a contradiction to Lemma 2.8 again. Therefore,  $t(\gamma') = 2$ .  $\square$

Now, we complete the proof of Theorem 2.7 as follows. Let  $\gamma_1, \dots, \gamma_e$  be all the first kind essential blocks of  $\gamma$ , and assume that  $\gamma$  has exactly  $p > 0$  puncture-like blocks, say  $\gamma_{e+1}, \dots, \gamma_{e+p}$ . Note that  $t(\gamma_j) = 2$  for all  $j$  by Lemma 2.9.

Let  $Q_1 \prec \dots \prec Q_k$  be the points where the strands of  $\gamma$  meet the  $T_{n-3}$ -side. Note that  $k \geq 2p + 2e$ . Since  $\gamma \in \mathcal{GL}_n^0$ , and since  $t(\gamma') = 0$  whenever  $\gamma'$  is an essential block of  $\gamma$  of the second kind, then  $Q_1, \dots, Q_{2p+2e}$  are endpoints of strands of  $\cup_{j=1}^{p+e} \gamma_j$ , and, for  $2p + 2e + 1 \leq j \leq k$ , each  $Q_j$  is an endpoint of a strand of  $\gamma$  connecting the  $T_{n-3}$ -side and the  $T_{n-3}^{-1}$ -side whenever  $2p + 2e + 1 \leq j \leq k$ .

CLAIM 2.  $Q_{p+1}, \dots, Q_{p+2e}$  are the points where the strands of  $\cup_{j=1}^e \gamma_j$  meet the  $T_{n-3}$ -side.

Now, for every integer  $j$  with  $1 \leq j \leq e$ , let  $L_j$  be the strand of  $\cup_{j=1}^e \gamma_j$  with  $Q_{p+e+j}$  an endpoint, let  $P_j$  be the other endpoint of  $L_j$ . Let  $Q'_{p+j}$  be the point on the  $T_{n-3}^{-1}$ -side which is identified with  $Q_{p+j}$  by the transformation  $T_{n-3}$ .

There are  $e$  mutually disjoint simple arcs  $L'_j$  in  $\mathcal{D}_n$  connecting  $P_j$  to  $Q'_{p+j}$  for every  $j$  such that every  $L'_j$  is disjoint from the strands of any essential block of  $\gamma$  except possibly the strands  $L_1, \dots, L_e$ . As before, let  $\mathcal{E}$  be the set of all essential blocks of  $\mathcal{L}' = (\mathcal{L} - \cup_{j=1}^e L_j) \cup (\cup_{j=1}^e L'_j)$  projects to  $\Sigma_n$  a multiple simple loop  $\alpha_\gamma$

in  $\mathcal{GL}_n^{(n-1)}$ . Let  $\alpha_\gamma$  also denote the corresponding free homotopy class. Note that if  $\gamma'$  is a puncture-like block of  $\gamma$ , then  $i(\gamma', \gamma_{n-4}^1)_{\partial\mathcal{D}_n} = 0 = i(\gamma', \gamma_j^k)_{\partial\mathcal{D}_n}$  for  $1 \leq j < n-4$  and  $1 \leq k \leq 3$ . This completes the proof of Theorem 2.7.

**Proof of Claim 2.** It suffices to prove that if  $Q$  is the endpoint of a strand of  $\cup_{j=1}^e \gamma_j$  lying on the  $T_{n-3}$ -side, then  $Q_j \prec Q \prec Q_{p+2e+j}$  for all  $j$  with  $1 \leq j \leq p$ .

Let  $\gamma'$  be the essential block of  $\gamma$  of the first kind such that  $Q$  is one of the two points where the strands of  $\gamma'$  meet the  $T_{n-3}$ -side, and let  $L$  be the strand of  $\gamma'$  with  $Q$  as an endpoint.

If  $Q \in \{Q_1, \dots, Q_p\}$ , then there is an integer  $m$  with  $p < m \leq 2p + 2e$  such that  $Q_m$  is the endpoint of a strand  $l$  of  $\cup_{j=1}^p \gamma_{e+j}$  connecting the  $T_{n-3}$ -side to the  $T_{n-4}$ -side. Thus the other endpoint  $P$  of  $L$  must lie on the  $T_{n-4}$ -side with  $P \prec P_m$ , where  $P_m$  is the endpoint of  $l$  other than  $Q_m$ . Let  $P'$  and  $P'_m$  be the points lying on the  $T_{n-4}^{-1}$ -side which are identified with  $P$  and  $P_m$  respectively by the transformation  $T_{n-4}$ . Let  $L'$  be the strand of  $\gamma'$  with  $P'$  as an endpoint. Since  $P' \prec P'_m$ , then  $L'$  must connect the  $T_{n-4}$ -side to the  $T_{n-3}$ -side. This implies that  $\gamma'$  is a puncture-like block of  $\gamma$ , which is a contradiction. Therefore,  $Q_j \prec Q$  for all  $j$  with  $1 \leq j \leq p$ .

By a similar argument, one proves that  $Q \prec Q_{p+2e+j}$  for  $1 \leq j \leq p$ .  $\square$

**2.4. The integer valued functions  $N_j$ .** To formulate elementary intersection numbers, in addition to the integer valued functions  $I_j$  defined in §2.1, we shall need other  $n-3$  integer valued functions  $N_j$ ,  $1 \leq j \leq n-3$ . These functions  $N_j$  are analogues of the integer valued functions  $N_T$  and  $N_S$  defined in [4].

We shall define an integer valued function  $N_j^{(n)}$  on  $\mathcal{GL}_n$  for any given integer  $j > 0$  with  $j \leq n-3$  so that

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\gamma))$$

whenever  $\gamma \in \mathcal{GL}_n^{(n-1)}$  and  $j \leq n-4$ , where  $\Phi_n$  is defined in §2.3. This means that  $N_j^{(n-1)}$  can be regarded as the restriction of  $N_j^{(n)}$  to  $\mathcal{GL}_n^{(n-1)}$  whenever  $1 \leq j \leq n-4$ . Thus  $N_j^{(n)}$  can be simply written as  $N_j$ . Furthermore, this allows us to define  $N_j$  inductively by using Theorem 2.7.

First, we define the functions  $N_1^{(n)}$  and  $N_{n-3}^{(n)}$ . If  $\gamma = \bigoplus_{j=1}^{n-3} k_j \gamma_j^1$  with  $(k_1, \dots, k_{n-3}) \in \mathbb{Z}_+^{n-3}$ , we define

$$N_j^{(n)}(\gamma) = k_j = \#(\text{strands of } \gamma \text{ connecting the } T_j\text{-side and the } T_j^{-1}\text{-side}),$$

for  $j = 1$  or  $n-3$ .

Now, we define  $N_1^{(n)}(\gamma)$  and  $N_{n-3}^{(n)}(\gamma)$  for  $\gamma \in \widehat{\mathcal{GL}}_n$ , where

$$\widehat{\mathcal{GL}}_n = \mathcal{GL}_n - \left\{ \bigoplus_{j=1}^{n-3} k_j \gamma_j^1 : (k_1, \dots, k_{n-3}) \in \mathbb{Z}_+^{n-3} \right\}.$$

If  $\gamma \in \mathcal{GL}_n^+(T_1)$ , let

$$N_1^{(n)}(\gamma) = \#(\text{strands of } \gamma \text{ joining the } T_1^{-1}\text{-side to the } S_1^\varepsilon\text{-side}) \\ + \#(\text{strands of } \gamma \text{ joining the } T_1\text{-side to the } T_1^{-1}\text{-side}),$$

where  $\varepsilon = \pm 1$ . If  $\gamma \in \mathcal{GL}_n^+(T_{n-3})$ , let

$$N_{n-3}^{(n)}(\gamma) = \#(\text{strands of } \gamma \text{ joining the } T_{n-3}^{-1}\text{-side to the } S_2^\varepsilon\text{-side}) \\ + \#(\text{strands of } \gamma \text{ joining the } T_{n-3}\text{-side to the } T_{n-3}^{-1}\text{-side}).$$

For  $j = 1$  or  $n - 3$ , and for  $\gamma \in \mathcal{GL}_n^-(T_j) \cap \widehat{\mathcal{GL}}_n$ , let

$$N_j^{(n)}(\gamma) = -N_j^{(n)}(\Theta_1(\gamma)).$$

It is clear that the definition of  $N_1^{(n)}$  is independent of  $n$  since  $n \geq 5$ . Thus  $N_1^{(n)}$  will be simply written as  $N_1$ .

REMARK 2.3. For  $n = 5$ , let  $N_T$  and  $N_S$  be the integer valued functions defined in [4], and let  $N_1$  and  $N_2 = N_{n-3}^{(n)}$  be the integer valued functions defined above. Then for  $\gamma \in \mathcal{GL}_5$  we have

$$N_1(\gamma) = N_T(\gamma) \quad \text{and} \quad N_2(\gamma) = -N_S(\gamma).$$

Note that the geodesic  $\gamma_{23}$  defined in [4] and the geodesic  $\gamma_2^3$  defined in this article are images of each other under  $\Theta_1$ . Thus, the following equations are also valid for  $\gamma \in \mathcal{GL}_5$  (see [4, Corollary 3.4]):

$$\begin{aligned} i(\gamma, \gamma_1^2) &= 2|N_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma) \\ i(\gamma, \gamma_1^3) &= 2|N_1(\gamma) - I_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma) \\ i(\gamma, \gamma_2^2) &= 2|N_2(\gamma)| + |I_1(\gamma) - I_2(\gamma)| + I_1(\gamma) - I_2(\gamma) \\ i(\gamma, \gamma_2^3) &= 2|N_2(\gamma) - I_2(\gamma)| + |I_1(\gamma) - I_2(\gamma)| + I_1(\gamma) - I_2(\gamma) \end{aligned}$$

In §2.5, we shall prove similar formulas for elementary intersection numbers of  $\gamma \in \mathcal{GL}_n$  for an arbitrary integer  $n \geq 5$ .

For integers  $n$  and  $j$  with  $1 < j \leq n - 4$ , the integer valued functions  $N_j^{(n)}$  on  $\mathcal{GL}_n$  are defined as follows. We first define  $N_j^{(n)}(\gamma)$  for  $\gamma \in \mathcal{GL}_n^0$ .

(i) If  $I_{n-3}(\gamma) = 0$ , then there exist  $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$  and mutually disjoint geodesics  $\gamma_1, \dots, \gamma_{n-4}$  in  $\mathcal{GL}_n^{(n-1)}$  such that

$$(5) \quad \gamma = \bigoplus_{i=1}^{n-4} k_i \gamma_i \oplus k_{n-3} \gamma_{n-3}^1.$$

Let

$$(6) \quad \alpha_\gamma = \bigoplus_{i=1}^{n-4} k_i \gamma_i,$$

and we define

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\alpha_\gamma)).$$

In particular, if  $\gamma \in \mathcal{GL}_n^{(n-1)}$ , then  $k_{n-3} = 0$ ,  $\alpha_\gamma = \gamma$ , and

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\gamma)).$$

(ii) If  $I_{n-3}(\gamma) > 0$ , and if  $\gamma$  has essential blocks, we define

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\alpha_\gamma)),$$

where  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$  is given in Theorem 2.7.

(iii) If  $I_{n-3}(\gamma) > 0$ , and if  $\gamma$  has no essential blocks, we define

$$N_j^{(n)}(\gamma) = 0.$$

From (i), we know that  $N_j^{(n-1)}$  is the restriction of  $N_j^{(n)}$  to  $\mathcal{GL}_n^{(n-1)} \equiv \mathcal{GL}_{n-1}$  for any two integers  $j$  and  $n$  with  $1 < j \leq n-4$ . Note that  $N_{n-4}^{(n-1)} = N_{\nu-3}^{(\nu)}$ , where  $\nu = n-1$ . From now on, we shall write  $N_j^{(n)}$  as  $N_j$  for  $1 \leq j \leq n-3$ .

Now, for an arbitrary  $\gamma \in \mathcal{GL}_n$  and for an arbitrary integer  $j$  with  $1 < j \leq n-4$ , we define

$$N_j(\gamma) = \begin{cases} N_j(\mathcal{T}_2^{-2}(\gamma)) & \text{if } \gamma \in \mathcal{GL}_n^+(T_{n-3}), \\ N_j(\mathcal{T}_2^2(\gamma)) & \text{if } \gamma \in \mathcal{GL}_n^-(T_{n-3}). \end{cases}$$

To prove that  $N_j$  is well-defined, we have to show that

$$N_j(\gamma) = \begin{cases} N_j(\mathcal{T}_2^{-2}(\gamma)) & \text{for all } \gamma \in \mathcal{GL}_n^+(T_{n-3}) \cap \mathcal{GL}_n^0, \\ N_j(\mathcal{T}_2^2(\gamma)) & \text{for all } \gamma \in \mathcal{GL}_n^-(T_{n-3}) \cap \mathcal{GL}_n^0. \end{cases}$$

Without loss of generality, we may assume that  $\gamma \in \mathcal{G}_n^0$ . There is nothing to prove if  $I_{n-3}(\gamma) = 0$  since in this case  $\mathcal{T}_2(\gamma) = \gamma$ . Assume that  $\gamma \in \mathcal{G}_n^-(T_{n-3})$  with  $I_{n-3}(\gamma) = m > 0$ . Then  $\gamma$  is represented by a cyclic reduced  $\Gamma_n$ -word as given in (3), say  $W = \prod_{i=1}^m T_{n-3}^{-p_i} S_2^{\varepsilon_i} T_{n-3}^{q_i} W_i$  with  $p_i > 0$  and  $q_i > 0$  for all  $i$ . Since

$$\mathcal{T}_2^2(W) = \prod_{i=1}^m T_{n-3}^{-p_i-1} S_2^{\varepsilon_i} T_{n-3}^{q_i+1} W_i,$$

$\gamma$  has essential blocks if and only if  $T_2^2(\gamma) = \tilde{\gamma}$  has essential blocks. Thus  $N_j(\gamma) = 0 = N_j(\tilde{\gamma})$  whenever  $\gamma$  has no essential blocks. When  $\gamma$  has essential blocks,  $\alpha_\gamma$  is completely determined by the subwords  $T_{n-3}W_iT_{n-3}^{-1}$ ,  $1 \leq i \leq m$ , and so is  $\alpha_{\tilde{\gamma}}$ . This proves that  $N_j(\gamma) = N_j(\tilde{\gamma})$  since  $\alpha_\gamma = \alpha_{\tilde{\gamma}}$ .

If  $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^+(T_{n-3})$ , then  $\gamma$  is represented by a cyclic reduced  $\Gamma_n$ -word as given in (4). A similar argument as above, one proves easily that  $N_j(\gamma) = N_j(T_2^{-2}(\gamma))$ . Therefore,  $N_j$  is well-defined.

Note that since  $N_{n-4}^{(n)} \equiv N_{\nu-3}^{(\nu)}$  with  $\nu = n - 1$ , from the definition of  $N_{n-3}$ , we may interpretate  $N_{n-4}$  geometrically. This gives  $N_j$  a geometric interpretation for every integer  $j$  with  $1 < j \leq n - 4$ . From Proposition 2.5, we assume that  $\gamma \in \mathcal{GL}_n^0$ .

Let  $\mathcal{GL}_n^+(T_{n-4})$  be the set of all  $\gamma$  in  $\mathcal{GL}_n^0$  which satisfy either one of the following two conditions:

- (i) If  $I_{n-3}(\gamma) = 0$ , then  $\gamma$  has no strands connecting the  $T_{n-4}$ -side to the  $T_{n-3}^\varepsilon$ -side, where  $\varepsilon = \pm 1$ .
- (ii) If  $I_{n-3}(\gamma) > 0$ , then every essential block of  $\gamma$  has no strands connecting the  $T_{n-4}$ -side to the  $T_{n-3}^\varepsilon$ -side, where  $\varepsilon = \pm 1$ .

Let  $\mathcal{GL}_n^-(T_{n-4}) = \Theta_1(\mathcal{GL}_n^+(T_{n-4}))$ . If  $\gamma = \bigoplus_{j=1}^{n-3} k_j \gamma_j^1$  with  $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$ , then

$$N_{n-4}^{(n)}(\gamma) = k_{n-4} = \#(\text{strands of } \gamma \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}).$$

Let  $\varepsilon = \pm 1$ . If  $\gamma \in \mathcal{GL}_n^+(T_{n-4}) \cap \widehat{\mathcal{GL}}_n$  with  $I_{n-3}(\gamma) = 0$ , then

$$\begin{aligned} N_{n-4}^{(n)}(\gamma) &= \#(\text{strands of } \gamma \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } T_{n-3}^\varepsilon\text{-side to the } T_{n-4}^{-1}\text{-side}). \end{aligned}$$

If  $\gamma \in \mathcal{GL}_n^+(T_{n-4}) \cap \widehat{\mathcal{GL}}_n$  with  $I_{n-3}(\gamma) > 0$ , then

$$(7) \quad N_{n-4}^{(n)}(\gamma) = \sum_{\gamma' \in \mathcal{E}} N_{n-4}^{(n)}(\gamma'),$$

where  $\mathcal{E}$  is the set of all essential blocks of  $\gamma$ , and where

$$\begin{aligned} N_{n-4}^{(n)}(\gamma') &= \#(\text{strands of } \gamma' \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma' \text{ joining the } T_{n-3}^\varepsilon\text{-side to the } T_{n-4}^{-1}\text{-side}) \end{aligned}$$

for  $\gamma' \in \mathcal{E}$ . When  $\mathcal{E}$  is empty, the integer on the right of (7) is defined to be zero. If  $\gamma \in \mathcal{GL}_n^-(T_{n-4}) \cap \widehat{\mathcal{GL}}_n$ , then  $N_{n-4}^{(n)}(\gamma) = -N_{n-4}^{(n)}(\Theta_1(\gamma))$ .

**2.5. Elementary intersection formulas.** This subsection is devoted to proving the main theorem:

**Theorem 2.10** (Elementary intersection formulas). *For an arbitrary integer  $n \geq 6$ , if  $\gamma \in \mathcal{GL}_n$ , then*

$$\begin{aligned} i(\gamma, \gamma_1^2) &= 2|N_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma), \\ i(\gamma, \gamma_1^3) &= 2|N_1(\gamma) - I_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma), \\ i(\gamma, \gamma_{n-3}^2) &= 2|N_{n-3}(\gamma)| + |I_{n-4}(\gamma) - I_{n-3}(\gamma)| + I_{n-4}(\gamma) - I_{n-3}(\gamma), \\ i(\gamma, \gamma_{n-3}^3) &= 2|N_{n-3}(\gamma) - I_{n-3}(\gamma)| + |I_{n-4}(\gamma) - I_{n-3}(\gamma)| + I_{n-4}(\gamma) - I_{n-3}(\gamma), \end{aligned}$$

and for every integer  $j$  with  $1 < j < n - 3$

$$\begin{aligned} i(\gamma, \gamma_j^2) &= 2|N_j(\gamma)| + |I_{j-1}(\gamma) - I_j(\gamma)| + I_{j-1}(\gamma) - I_j(\gamma) \\ &\quad + |I_{j+1}(\gamma) - I_j(\gamma)| + I_{j+1}(\gamma) - I_j(\gamma), \\ i(\gamma, \gamma_j^3) &= 2|N_j(\gamma) - I_j(\gamma)| + |I_{j-1}(\gamma) - I_j(\gamma)| + I_{j-1}(\gamma) - I_j(\gamma) \\ &\quad + |I_{j+1}(\gamma) - I_j(\gamma)| + I_{j+1}(\gamma) - I_j(\gamma). \end{aligned}$$

For the proof of Theorem 2.10, we need the following two immediate consequences of the definition of  $N_j$ .

**Lemma 2.11.** *If  $\gamma \in \mathcal{GL}_n$ , then  $N_1(\gamma) = N_{n-3}(\Theta_2(\gamma))$ .*

**Lemma 2.12.** *If  $(k_1, \dots, k_{n-3}) \in \mathbb{Z}_+^{n-3}$  and  $(\gamma_1, \dots, \gamma_{n-3}) \in \Lambda_n^{n-3}$ , then*

$$N_j \left( \bigoplus_{i=1}^{n-3} k_i \gamma_i \right) = \sum_{j=i}^{n-3} k_i N_j(\gamma_j) \quad \text{for every integer } j \text{ with } 1 \leq j \leq n-3.$$

For  $k = 2$  or  $3$ , the elementary intersection numbers  $i(\gamma, \gamma_1^k)$  and  $i(\gamma, \gamma_{n-3}^k)$  are related as follows:

$$i(\gamma, \gamma_{n-3}^k) = i(\Theta_2(\gamma), \Theta_2(\gamma_{n-3}^k)) = i(\Theta_2(\gamma), \gamma_1^k).$$

From Proposition 2.1, we obtain  $I_1(\Theta_2(\gamma)) = I_{n-3}(\gamma)$  and  $I_2(\Theta_2(\gamma)) = I_{n-4}(\gamma)$ . Now, by Lemma 2.11, the elementary intersection formulas for  $i(\gamma, \gamma_{n-3}^2)$  and  $i(\gamma, \gamma_{n-3}^3)$  follow immediately from those for  $i(\gamma, \gamma_1^2)$  and  $i(\gamma, \gamma_1^3)$ .

On the other hand,  $i(\gamma, \gamma_1^3) = i(\mathcal{T}_1(\gamma), \gamma_1^2)$  since  $\gamma_1^3 = \mathcal{T}_1^{-1}(\gamma_1^2)$ . Thus, by Proposition 2.3, one derives easily the elementary intersection formula for  $i(\gamma, \gamma_1^3)$  from that for  $i(\gamma, \gamma_1^2)$  if

$$N_1(\mathcal{T}_1(\gamma)) = N_1(\gamma) - I_1(\gamma).$$

By use of the word given in (3), one proves easily the following more general results by a similar argument as that in [4, Proposition 2.8].



**Lemma 2.13.** *Let  $\gamma \in \mathcal{GL}_n$ , and let  $\nu$  be an arbitrary integer. Then*

$$\begin{aligned} N_1(\mathcal{T}_2^\nu(\gamma)) &= N_1(\gamma), & N_1(\mathcal{T}_1^\nu(\gamma)) &= N_1(\gamma) - \nu I_1(\gamma), \\ N_{n-3}(\mathcal{T}_1^\nu(\gamma)) &= N_{n-3}(\gamma), & N_{n-3}(\mathcal{T}_2^\nu(\gamma)) &= N_{n-3}(\gamma) - \nu I_{n-3}(\gamma). \end{aligned}$$

For the proof of Theorem 2.10, it remains to prove the elementary intersection formulas for  $i(\gamma, \gamma_1^2)$ ,  $i(\gamma, \gamma_j^2)$  and  $i(\gamma, \gamma_j^3)$  for  $1 < j < n - 3$ .

First, we prove the elementary intersection formula for  $i(\gamma, \gamma_1^2)$  by applying induction to  $n$  for  $n \geq 5$ . For the case of  $n = 5$ , the assertion is proved in [4, Corollary 3.4]. Assume that  $n > 5$ , and that the equation holds for  $\gamma \in \mathcal{GL}_n^{(n-1)}$ .

Now, let  $\gamma \in \mathcal{GL}_n$ . If  $I_{n-3}(\gamma) = 0$ , write  $\gamma$  as given in (5), and let  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$  be given in (6). By the definition,  $N_1(\gamma) = N_1(\alpha_\gamma)$ . Since  $i(\gamma_{n-3}^1, \beta) = 0$  for  $\beta \in \mathcal{GL}_n^{(n-1)}$ , then  $I_j(\gamma) = I_j(\alpha_\gamma)$  for  $j = 1, 2$ . The assertion follows for the case since  $i(\gamma, \gamma_1^2) = i(\alpha_\gamma, \gamma_1^2)$ .

Assume that  $I_{n-3}(\gamma) > 0$ . Since  $i(\gamma, \gamma_1^2) = i(\Theta_1(\gamma), \gamma_1^2)$ , we may assume that  $\gamma \in \mathcal{GL}_n^+(T_{n-3})$ . Moreover, by considering  $\mathcal{T}_2^{-2}(\gamma)$ , from Proposition 2.4, Proposition 2.5 and Lemma 2.13 we may assume that  $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ .

If  $\gamma$  has no essential blocks, we have  $I_1(\gamma) = I_2(\gamma) = 0 = i(\gamma, \gamma_1^2)$ . By the definition of  $N_1$ , we have  $N_1(\gamma) = 0$  since  $I_1(\gamma) = 0$ . Now, the intersection formula for  $i(\gamma, \gamma_1^2)$  holds trivially in this case.

If  $\gamma$  has essential blocks, then  $I_j(\alpha_\gamma) = I_j(\gamma)$  for  $j = 1, 2$ , and  $i(\alpha_\gamma, \gamma_1^2) = i(\gamma, \gamma_1^2)$ , where  $\alpha_\gamma$  is given in Theorem 2.7. Note that  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$  and  $N_1(\gamma) = N_1(\alpha_\gamma)$ . The proof of the intersection formula for  $i(\gamma, \gamma_1^2)$  is then completed by induction hypothesis.

In the rest of this subsection, we prove the intersection formulas for  $i(\gamma, \gamma_j^2)$  and  $i(\gamma, \gamma_j^3)$  with  $1 < j < n - 3$ , by applying induction to  $n \geq 6$ . If  $n = 6$ , then the formulas are exactly the same as given below.

**Lemma 2.14.** *If  $n \geq 6$ , and if  $\gamma \in \mathcal{GL}_n$ , then*

$$\begin{aligned} i(\gamma, \gamma_{n-4}^2) &= 2|N_{n-4}(\gamma)| + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \\ i(\gamma, \gamma_{n-4}^3) &= 2|N_{n-4}(\gamma) - I_{n-4}(\gamma)| \\ &\quad + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \end{aligned}$$

With Lemma 2.14, we first complete induction step as follows. Assume that  $n > 6$ . From Lemma 2.14, we may assume that  $1 < j < n - 4$ . If  $I_{n-3}(\gamma) = 0$ , then we write  $\gamma$  and  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ , respectively, as in (5) and (6). Since  $N_j(\alpha_\gamma) = N_j(\gamma)$  and  $I_k(\alpha_\gamma) = I_k(\gamma)$  for  $0 < j - 1 \leq k \leq j < n - 4$ , the assertions hold for this case by induction hypothesis.

Assume that  $I_{n-3}(\gamma) > 0$ . If  $\gamma$  has no essential blocks, then  $I_j(\gamma) = 0 = N_j(\gamma)$  for  $1 < j < n-3$ , and  $i(\gamma, \gamma_j^k) = 0$  for  $1 < j < n-4$  and for  $k = 2, 3$ . If  $\gamma$  has essential blocks, we may assume that  $\gamma \in \mathcal{GL}_n^0$ . Let  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$  be given in Theorem 2.7. By the induction hypothesis again, the proof is complete.

For the proof of Lemma 2.14, we need:

**Lemma 2.15.** *If  $\gamma \in \mathcal{GL}_n^0$  with  $I_{n-3}(\gamma) > 0$ , then  $\gamma$  has exactly*

$$\frac{|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma)}{2}$$

*puncture-like blocks.*

**Proof.** Without loss of generality, we assume that  $\gamma \in \mathcal{GL}_n^+(T_{n-3})$ . Let  $\mathcal{E}$  denote the set of all essential blocks of  $\gamma$ . If  $\gamma'$  is a puncture-like block of  $\gamma$ , then  $i(\gamma', \gamma_{n-4}^1) = 0$  and  $2I_{n-4}(\gamma) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n}$ .

Let  $I_{n-3}(\gamma) = m$ , and let  $p \geq 0$  be the number of puncture-like blocks of  $\gamma$ . Then  $\gamma$  has exactly  $e = m - p$  essential blocks of the first kind. If  $p = 0$ , then  $2I_{n-4}(\gamma) \geq \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} \geq 2m = 2I_{n-3}(\gamma)$ , and  $|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 0 = 2p$ .

Now, assume that  $p > 0$ . It follows from Lemma 2.9 that

$$i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = \begin{cases} 2 & \text{if } \gamma' \text{ is an essential block of } \gamma \text{ of the first kind,} \\ 0 & \text{if } \gamma' \text{ is an essential block of } \gamma \text{ of the second kind.} \end{cases}$$

If  $p = m$ , then  $\gamma$  has no essential blocks of the first kind, and

$$2I_{n-4}(\gamma) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = 0.$$

Thus  $|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 2m = 2p$ .

If  $0 < p < m$ , let  $\gamma_1, \dots, \gamma_e$  be the essential blocks of  $\gamma$  of the first kind. Then  $2I_{n-4}(\gamma) = \sum_{j=1}^e i(\gamma_j, \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = 2e = 2I_{n-3}(\gamma) - 2p$ , and

$$2p = 2\{I_{n-3}(\gamma) - I_{n-4}(\gamma)\} = |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma). \quad \square$$

**Proof of Lemma 2.14.** It suffices to prove the lemma for  $\gamma \in \mathcal{G}_n$ . We shall prove the lemma for  $\gamma \in \mathcal{G}_n^+(T_{n-3})$ . By a similar argument, one proves the lemma for  $\gamma \in \mathcal{G}_n^-(T_{n-3})$ .

If  $\gamma \in \mathcal{G}_n^+(T_{n-3})$ , then  $N_{n-4}(\mathcal{T}^{-2}(\gamma)) = N_{n-4}(\gamma)$  by the definition of  $N_{n-4}$ . Note that  $i(\gamma, \gamma_{n-4}^k) = i(\mathcal{T}_2^{-2}(\gamma), \gamma_{n-4}^k)$  for  $k = 2, 3$ , and that  $2I_j(\gamma) = 2I_j(\mathcal{T}_2^{-2}(\gamma))$  for  $n-5 \leq j \leq n-3$ . By Proposition 2.5, we may assume that  $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^+(T_{n-3})$ .

If  $I_{n-3}(\gamma) = 0$ , then  $\gamma \in \mathcal{G}_n^{(n-1)}$ , and

$$|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 0.$$

By letting  $\nu = n - 1$ , we have

$$\begin{aligned} i(\gamma, \gamma_{n-4}^2) &= 2|N_{\nu-3}(\gamma)| + |I_{\nu-4}(\gamma) - I_{\nu-3}(\gamma)| + I_{\nu-4}(\gamma) - I_{\nu-3}(\gamma) \\ &= 2|N_{n-4}(\gamma)| + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \end{aligned}$$

Similarly, we obtain the intersection formula for  $i(\gamma, \gamma_{n-4}^3)$ .

If  $I_{n-3}(\gamma) = m > 0$ , then  $\gamma$  is represented by a cyclic reduced  $\Gamma_n$ -word  $W$  as given in (3). Note that  $p_j > 0$  and  $q_j > 0$  for  $1 \leq j \leq m$ . For every  $j$ , let  $\gamma_j$  be the block of  $\gamma$  represented by  $\vec{T}_{n-3}^{-1} W_j T_{n-3}$ , and let  $\beta(\gamma_j)$  be the admissible subarc of  $\gamma$  represented by

$$\vec{T}_{n-3} T_{n-3}^{p_j-1} S_2^{\varepsilon_j} T_{n-3}^{-q_j} W_j T_{n-3}.$$

Note that every  $\gamma_j$  is a subarc of  $\beta(\gamma_j)$ , and that  $i(\beta(\gamma_j), \gamma_{n-4}^k) = 2$  for  $k = 2$  or  $3$  whenever  $\gamma_j$  is puncture-like. Let  $\mathcal{E}$  be the set of all essential blocks of  $\gamma$ . From Lemma 2.15, we have, for  $k = 2$  or  $3$ ,

$$i(\gamma, \gamma_{n-4}^k) = |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) + \sum_{\gamma_j \in \mathcal{E}} i(\beta(\gamma_j), \gamma_{n-4}^k)_{\partial \mathcal{D}_n}.$$

If  $\gamma$  has no essential blocks, then the lemma holds trivially for  $\gamma$  since  $I_{n-3}(\gamma) = I_{n-4}(\gamma) = N_{n-4}(\gamma) = 0$ .

Now, assume that  $\mathcal{E}$  is not empty. Note that every essential block of  $\gamma$  is of the first kind since  $\gamma \in \mathcal{G}_n$ . Let  $\mathcal{L}$  be the union of all strands of  $\gamma$  which connect the  $T_{n-3}^{-1}$ -side to the  $X$ -side with  $X \in \{T_{n-3}, S_2, S_2^{-1}\}$ .

For  $k = 2$  or  $3$ , each  $\gamma_{n-4}^k$  has a unique strand  $l_k$  meeting the  $T_{n-3}^{-1}$ -side. Let  $Q'_k$  be the endpoint of  $l_k$  lying on the  $T_{n-3}^{-1}$ -side, and let  $Q_k$  be the point on the  $T_{n-3}$ -side which is identified with  $Q'_k$  by the transformation  $T_{n-3}^{-1}$ .

Since  $i(\gamma_{n-4}^k, \gamma_{n-3}^1) = 0$ , we may assume that  $l_k$  is disjoint from  $\mathcal{L}$ . This implies that  $Q'_k \prec Q'$  whenever  $Q'$  is an endpoint of some strand in  $\mathcal{L}$  meeting the  $T_{n-3}^{-1}$ -side, and that  $Q_k \prec Q$  whenever  $Q$  is the endpoint of some strand of  $\gamma$  lying on the  $T_{n-3}$ -side. Thus, we have

$$\sum_{\gamma_j \in \mathcal{E}} i(\beta(\gamma_j), \gamma_{n-4}^k)_{\partial \mathcal{D}_n} = \sum_{\gamma_j \in \mathcal{E}} i(\gamma_j, \gamma_{n-4}^k)_{\partial \mathcal{D}_n} = i(\alpha_\gamma, \gamma_{n-4}^k),$$

where  $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$  is given in Theorem 2.7. By letting  $\nu = n - 1$ , we obtain

$$\begin{aligned} i(\alpha_\gamma, \gamma_{n-4}^2) &= 2|N_{\nu-3}(\alpha_\gamma)| + |I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| \\ &\quad + I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma) \\ i(\alpha_\gamma, \gamma_{n-4}^3) &= 2|N_{\nu-3}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| + |I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| \\ &\quad + I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma) \end{aligned}$$

The proof of Lemma 2.14 is complete.  $\square$

### 3. A Mapping of $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ into a Sphere

In this section, we construct a continuous mapping  $\Psi$  from  $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$  into  $\mathbb{R}^{3(n-3)}$  whose image set is a sphere of dimension  $2n-7$ . The mapping  $\Psi$  will be constructed in a similar way as that given in [4] for the case of  $n=5$ . We shall first define the restriction of  $\Psi$  on  $\mathcal{GL}_n$  homogeneously, and extend it to  $\pi^{-1}\pi\mathcal{I}(\mathcal{G}_n)$ . Note that  $\overline{\pi\mathcal{I}(\mathcal{G}_n)} = \overline{\pi\mathcal{I}(\mathcal{GL}_n)}$ . By a continuity argument as in [4, §4.3], one proves that  $\Psi$  extends continuously to  $\pi^{-1}\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ . Since the restriction  $\pi$  to  $\pi^{-1}\overline{\pi\mathcal{I}(\mathcal{G}_n)}$  is a quotient map, the required continuous mapping  $\Psi$  is then obtained.

For the definition of  $\Psi$  on  $\mathcal{GL}_n$ , we first construct a function  $\psi_0$  from  $\mathcal{GL}_n$  into  $\mathbb{R}^{3(n-3)}$  whose values are written in terms of elementary intersection numbers. For every  $\gamma \in \mathcal{GL}_n$ , we write

$$\psi_0(\gamma) = (x_1^1(\gamma), x_1^2(\gamma), x_1^3(\gamma), \dots, x_{n-3}^1(\gamma), x_{n-3}^2(\gamma), x_{n-3}^3(\gamma)),$$

where  $x_j^k(\gamma) = i(\gamma, \gamma_j^k)/\lambda(\gamma)$  for  $1 \leq j \leq n-3$  and for  $1 \leq k \leq 3$ , and where  $\lambda(\gamma) = \sum_{j=1}^{n-3} \sum_{k=1}^3 i(\gamma, \gamma_j^k)$ . Note that the image of  $\psi_0$  lies in

$$\Pi' = \Pi \cap \left\{ (t_1, t_2, \dots, t_{3(n-3)}) \in \mathbb{R}^{3(n-3)} : 1 - 2 \sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}| > 0 \right\},$$

where  $\Pi = \{(t_1, t_2, \dots, t_{3(n-3)}) \in \mathbb{R}^{3(n-3)} : \sum_{j=1}^{3(n-3)} t_j = 1\}$ . For later use, we define the function  $f: \mathbb{R}^{3(n-3)} \rightarrow \mathbb{R}$  by

$$f(t_1, t_2, \dots, t_{3(n-3)}) = 1 - 2 \sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}|.$$

Following [4], we define the mapping  $\Psi: \mathcal{GL}_n \rightarrow \mathbb{R}^{3(n-3)}$  by

$$\Psi(\gamma) = (\xi_1^1(\gamma), \xi_1^2(\gamma), \xi_1^3(\gamma), \dots, \xi_{n-3}^1(\gamma), \xi_{n-3}^2(\gamma), \xi_{n-3}^3(\gamma)),$$

where for every  $1 \leq j \leq n-3$

$$\xi_j^1(\gamma) = \frac{2I_j(\gamma)}{\rho(\gamma)}, \quad \xi_j^2(\gamma) = \frac{2|N_j(\gamma)|}{\rho(\gamma)} \quad \text{and} \quad \xi_j^3(\gamma) = \frac{2|N_j(\gamma) - I_j(\gamma)|}{\rho(\gamma)},$$

and  $\rho(\gamma) = 2 \sum_{j=1}^{n-3} \{I_j(\gamma) + |N_j(\gamma)| + |N_j(\gamma) - I_j(\gamma)|\}$ . It is easy to see that  $\Psi(\gamma) \in \Delta_n = \mathcal{C}^{n-3} \cap \Pi$  for every  $\gamma \in \mathcal{GL}_n$ , where  $\mathcal{C}$  is the set of points  $(t_1, t_2, t_3) \in \mathbb{R}_+^3$  satisfying:

$$t_2 + t_3 = t_1, \quad t_1 + t_3 = t_2, \quad \text{or} \quad t_1 + t_2 = t_3.$$

A similar argument to that given in [4, §4.2] proves that  $\Delta_n$  is homeomorphic to a sphere of dimension  $2n - 7$ .

We shall prove that there is a homeomorphism  $\psi_1$  of  $\Pi'$  onto  $\Pi$  so that  $\Psi = \psi_1 \circ \psi_0$ . Then we obtain:

**Theorem 3.1.** *The function  $\Psi$  extends to  $\overline{\pi\mathcal{I}(\mathcal{G}_n)} = \overline{\pi\mathcal{I}(\mathcal{GL}_n)}$  as a continuous mapping into a sphere of dimension  $2n - 7$ .*

It remains to construct the mapping  $\psi_1$ . For  $\gamma \in \mathcal{GL}_n$ , let

$$\nu(\gamma) = 1 - \frac{4}{\lambda(\gamma)} \sum_{j=1}^{n-4} |I_j(\gamma) - I_{j+1}(\gamma)| = 1 - 2 \sum_{j=1}^{n-4} |x_j^1(\gamma) - x_{j+1}^1(\gamma)|.$$

A direct computation shows that  $\rho(\gamma) = \lambda(\gamma)\nu(\gamma)$ , and the followings:

$$\begin{aligned} \xi_j^1(\gamma) &= \frac{x_j^1(\gamma)}{\nu(\gamma)} \quad \text{for } 1 \leq j \leq n-3, \\ \xi_1^2(\gamma) &= \frac{x_1^2(\gamma)}{\nu(\gamma)} - \frac{|x_2^1(\gamma) - x_1^1(\gamma)| + \{x_2^1(\gamma) - x_1^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_1^3(\gamma) &= \frac{x_1^3(\gamma)}{\nu(\gamma)} - \frac{|x_2^1(\gamma) - x_1^1(\gamma)| + \{x_2^1(\gamma) - x_1^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_{n-3}^2(\gamma) &= \frac{x_{n-3}^2(\gamma)}{\nu(\gamma)} - \frac{|x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)| + \{x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_{n-3}^3(\gamma) &= \frac{x_{n-3}^3(\gamma)}{\nu(\gamma)} - \frac{|x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)| + \{x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)\}}{2\nu(\gamma)}, \end{aligned}$$

and for  $1 < j < n-3$

$$\begin{aligned} \xi_j^2(\gamma) &= \frac{x_j^2(\gamma)}{\nu(\gamma)} - \frac{|x_{j-1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j-1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)} \\ &\quad - \frac{|x_{j+1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j+1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_j^3(\gamma) &= \frac{x_j^3(\gamma)}{\nu(\gamma)} - \frac{|x_{j-1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j-1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)} \\ &\quad - \frac{|x_{j+1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j+1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)}. \end{aligned}$$

The above equations motivate the function  $\psi_1: \Pi' \longrightarrow \mathbb{R}^{3(n-3)}$  defined by

$\psi_1(r_1, r_2, \dots, r_{3(n-3)}) = (t_1, t_2, \dots, t_{3(n-3)})$ , where

$$t_j = \frac{r_j}{f(r_1, r_2, \dots, r_{3(n-3)})} \quad \text{for } j = 3k - 2 \text{ with } 1 \leq k \leq n - 3$$

$$t_j = \frac{r_j}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_4 - r_1| + (r_4 - r_1)}{2f(r_1, r_2, \dots, r_{3(n-3)})} \quad \text{for } j = 2, 3,$$

$$t_j = \frac{r_j}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_{3n-14} - r_{3n-11}| + (r_{3n-14} - r_{3n-11})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

for  $j = 3n - 10$  or  $3(n - 3)$ , and

$$t_{3k-1} = \frac{r_{3k-1}}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_{3k-5} - r_{3k-2}| + (r_{3k-5} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

$$- \frac{|r_{3k+1} - r_{3k-2}| + (r_{3k+1} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

$$t_{3k} = \frac{r_{3k}}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_{3k-5} - r_{3k-2}| + (r_{3k-5} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

$$- \frac{|r_{3k+1} - r_{3k-2}| + (r_{3k+1} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})} \quad \text{for } 1 < k < n - 3.$$

A direct computation proves that  $\psi_1$  maps  $\Pi'$  into  $\Pi$  by showing that

$$\sum_{j=1}^{3(n-3)} t_j = 1 \quad \text{and} \quad 1 + 2 \sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}| = \frac{1}{f(r_1, r_2, \dots, r_{3(n-3)})}.$$

From the definition of  $\psi_1$ , one proves easily that  $\psi_1$  is indeed a homeomorphism of  $\Pi'$  onto  $\Pi$ .

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