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ELEMENTARY INTERSECTION NUMBERS ON PUNCTURED SPHERES

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Introduction

According to Thurston, for any analytically finite Riemann surface \mathcal{R} , the set $\overline{\mathcal{G}}(\mathcal{R})$ of all projective geodesic laminations in \mathcal{R} can be made into a topological space homeomorphic to a sphere of dimension depending on the topology of \mathcal{R} . Understanding the space $\overline{\mathcal{G}}(\mathcal{R})$ is important for various approaches to the Teichmüller space and the mapping class group of \mathcal{R} . The space $\overline{\mathcal{G}}(\mathcal{R})$ was then investigated by several authors from many different points of view. See [3–10], [12, 13, 15], and references there in.

In this paper, we consider the space $\overline{\mathcal{G}}_n = \overline{\mathcal{G}}(\Sigma_n)$ for any integer $n \geq 4$, where Σ_n is an n -punctured sphere endowed with a hyperbolic metric. Note that $\overline{\mathcal{G}}_n$ is homeomorphic to a sphere of dimension $2n - 7$.

This work was an attempt to generalize the projective coordinates defined in [3, 4] to an arbitrary $\overline{\mathcal{G}}_n$. This work and that of Keen, Parker and Series [10] are essentially based on cutting sequence technique developed by Birman and Series [2], and complement the works of Masur and Minsky [12, 13].

Let \mathcal{G}_n be the set of all simple closed geodesics on Σ_n . For $n = 4$ or 5 , the author has defined a set of projective coordinates for \mathcal{G}_n so that the completion of these coordinates parametrize $\overline{\mathcal{G}}_n$, (see [3, 4]). The coordinates of each $\gamma \in \mathcal{G}_n$ are geometric intersection numbers of γ with $2(n - 3)$ fixed geodesics in \mathcal{G}_n , and read off directly from the topology of γ . Moreover, these coordinates have three remarkable applications. First, the geometric intersection number of any two geodesics in \mathcal{G}_n can be formulated explicitly in terms of the corresponding coordinates. Secondly, the coordinates of each $\gamma \in \mathcal{G}_n$ determine a canonical expression of γ as a word in a given set of generators for the fundamental group $\pi_1(\Sigma_n)$. Finally, the coordinates of each $\gamma \in \mathcal{G}_n$ are related to trace polynomials of the transformations corresponding to γ in a family of regular B -groups uniformizing Σ_n .

For an arbitrary $n \geq 5$, following [3, 4], we shall choose $n - 3$ fixed triples $(\gamma_j^1, \gamma_j^2, \gamma_j^3)$ of geodesics in \mathcal{G}_n ($1 \leq j \leq n - 3$), and compute the geometric intersec-

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tion numbers $i(\gamma, \gamma_j^k)$, called the *elementary intersection numbers* of γ . The elementary intersection numbers of γ will determine a set of parameters for γ .

The geodesics γ_j^k are defined explicitly in §1.2. They are chosen intuitively as described below. First, we line up the punctures of Σ_n , say ζ_1, \dots, ζ_n . For every j , the geodesic γ_j^1 is chosen to separate $\zeta_1, \dots, \zeta_{j+1}$ from $\zeta_{j+2}, \dots, \zeta_n$. These geodesics γ_j^1 determine $n-3$ subsurfaces of Σ_n each of which is homeomorphic to a four punctured sphere. Two of them are isometric to spheres with three punctures and one hole, denoted by $\Sigma_4^{(1)}$ and $\Sigma_4^{(n-3)}$, and the others are isometric to spheres with two punctures and two holes, denoted by $\Sigma_4^{(j)}$, $2 \leq j \leq n-4$. More explicitly, $\Sigma_4^{(1)}$ is the subsurface containing ζ_1, ζ_2 and ζ_3 with the boundary geodesic γ_2^1 ; $\Sigma_4^{(n-3)}$ is the subsurface containing ζ_{n-2}, ζ_{n-1} and ζ_n with the boundary geodesic γ_{n-4}^1 ; $\Sigma_4^{(j)}$ is the subsurface bounded by γ_{j-1}^1 and γ_{j+1}^1 for $2 \leq j \leq n-4$. For every j , we choose γ_j^2 so that γ_j^1 and γ_j^2 form a marking of a four punctured sphere as γ_∞ and γ_0 given in [3]. The geodesic γ_j^3 plays the role of γ_1 given [3] which is obtained from γ_j^2 by a half-twist along γ_j^1 .

The main work of this paper is to find formulas for computing elementary intersection numbers so that the formulas agree with that given in [4] when $n = 5$. These formulas will be called *elementary intersection formulas*. To derive these formulas, we introduce $2(n-3)$ integers for each $\gamma \in \mathcal{G}_n$, denoted by $I_j(\gamma)$ and $N_j(\gamma)$ for $1 \leq j \leq n-3$, (see §2.1 and §2.4). These integers are defined analogously to the projective coordinates given in [3, 4]. For $\gamma \in \mathcal{G}_n$, every $I_j(\gamma)$ is defined to be $(1/2)i(\gamma, \gamma_j^1)$, and the sign of every $N_j(\gamma)$ is determined by the symmetry of a fundamental domain for a Fuchsian representation of $\pi_1(\Sigma_n)$ acting on the upper half plane. With these integer valued functions I_j and N_j , we prove in §2.5 the elementary intersection formulas (Theorem 2.10) by applying induction to the number n of punctures. In this paper, we develop a new idea that makes the induction work for $n \geq 5$, (cf. Remark 2.3).

As an application of elementary intersection formulas, at the end of the paper, we construct a continuous map Ψ from $\overline{\mathcal{G}}_n$ into a sphere $\Delta_n \subset \mathbb{R}^{3(n-3)}$ of dimension $2n-7$ whose restriction to \mathcal{G}_n is written explicitly in terms of I_j and N_j .

It would be very interesting to derive a geometric intersection formula as given in [3, Theorem 2.6] and [4, Theorem 3.1] for any two geodesics in \mathcal{G}_n . With the formula, one proves easily the injectivity of Ψ . To prove that the integers $I_j(\gamma)$ and $N_j(\gamma)$ form a set of projective coordinates for $\gamma \in \mathcal{G}_n$, one also need the surjectivity of the map $\Psi: \overline{\mathcal{G}}_n \rightarrow \Delta_n$. This will follow if $\Psi(\mathcal{G}_n)$ is dense in Δ_n . For the proof, one may consider π_1 -train tracks introduced by Birman and Series [1], (cf. [3, 4]). The work will appear elsewhere.

1. Preliminaries

1.1. The space of complete simple geodesics. For any integer $n \geq 4$, a loop on Σ_n with no self intersections will be called a *simple loop*. An *essential simple loop*

on Σ_n is a simple loop which is neither homotopically trivial nor homotopic to a simple closed curve around to a puncture of Σ_n . A finite union of mutually disjoint essential simple loops on Σ_n will be called a *multiple simple loop*. The set of all free homotopy classes of non-oriented essential simple loops on Σ_n is denoted by \mathcal{G}_n , while the set of all free homotopy classes of non-oriented multiple simple loops is denoted by \mathcal{GL}_n . Obviously, $\mathcal{G}_n \subset \mathcal{GL}_n$.

In general, we shall use $[\alpha]$ for the free homotopy class represented by a curve α lying on Σ_n . Every element of \mathcal{G}_n contains a unique geodesic γ on Σ_n . By abuse of notation, we shall also use γ for the free homotopy class containing γ .

We shall write every element of \mathcal{GL}_n as an integral combination of elements of \mathcal{G}_n . For every integer $m > 1$, we use \mathcal{Z}_+^m for the set of m -tuples (k_1, \dots, k_m) of integers $k_j \geq 0$ with $\sum_{j=1}^m k_j > 0$, and Λ_n^m for the set of m -tuples $(\gamma_1, \dots, \gamma_m)$ of mutually disjoint geodesics in \mathcal{G}_n .

Let α be an arbitrary multiple simple loop on Σ_n . All connected components of α fall into at most $n - 3$ distinct free homotopy classes. There exist $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$ and $(\gamma_1, \dots, \gamma_{n-3}) \in \Lambda_n^{n-3}$ such that, for every j , α has exactly k_j connected components freely homotopic to γ_j . We shall write:

$$[\alpha] = k_1\gamma_1 \oplus \dots \oplus k_{n-3}\gamma_{n-3} = \bigoplus_{j=1}^{n-3} k_j\gamma_j.$$

Let $[\mathcal{G}_n, \mathbb{R}_+]$ be the set of all functions from \mathcal{G}_n into the set \mathbb{R}_+ of all non-negative real numbers. We provide \mathcal{G}_n with the discrete topology, and provide $[\mathcal{G}_n, \mathbb{R}_+]$ with the compact-open topology.

Two elements f and g of $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ are called *projectively equivalent* if there is a positive number t such that $f = tg$. Let $P[\mathcal{G}_n, \mathbb{R}_+]$ be the set of all projective equivalence classes in $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ provided with the quotient topology. Let π_n be the quotient map of $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ onto $P[\mathcal{G}_n, \mathbb{R}_+]$.

Following [5], we embed \mathcal{GL}_n into $[\mathcal{G}_n, \mathbb{R}_+]$ by using geometric intersection numbers of elements of \mathcal{GL}_n . For any two curves α_1 and α_2 on Σ_n , let $\#(\alpha_1 \cap \alpha_2)$ denote the cardinality of the intersection $\alpha_1 \cap \alpha_2$. The *geometric intersection number* $i([\alpha_1], [\alpha_2])$ of $[\alpha_1]$ with $[\alpha_2]$ is defined by

$$i([\alpha_1], [\alpha_2]) = \min\{\#(\alpha'_1 \cap \alpha'_2) : [\alpha'_j] = [\alpha_j] \text{ for } j = 1, 2\}.$$

It follows immediately from the definition that, for any curve β on Σ_n ,

$$i\left(\bigoplus_{j=1}^{n-3} k_j\gamma_j, [\beta]\right) = \sum_{j=1}^{n-3} k_j \cdot i(\gamma_j, [\beta]).$$

Every $\alpha \in \mathcal{GL}_n$ induces a function $\mathcal{I}_\alpha^{(n)}: \mathcal{G}_n \rightarrow \mathbb{R}_+$ given by

$$\mathcal{I}_\alpha^{(n)}(\gamma) = i(\alpha, \gamma) \quad \text{for all } \gamma \in \mathcal{G}_n.$$

Let $\mathcal{I}^{(n)}: \mathcal{GL}_n \rightarrow [\mathcal{G}_n, \mathbb{R}_+]$ be defined by

$$\mathcal{I}^{(n)}(\alpha) = \mathcal{I}_\alpha^{(n)} \quad \text{for all } \alpha \in \mathcal{GL}_n.$$

When there is no risk of confusion, we shall simply write π_n as π , write $\mathcal{I}_\alpha^{(n)}$ as \mathcal{I}_α , and write $\mathcal{I}^{(n)}$ as \mathcal{I} .

It is well known that the composition $\pi\mathcal{I}$ is injective [5, Exposé 3], and that $\overline{\pi\mathcal{I}(\mathcal{GL}_n)} = \overline{\pi\mathcal{I}(\mathcal{G}_n)}$ [5, Exposé 4, Theorem 4], where $\overline{\pi\mathcal{I}(\mathcal{GL}_n)}$ and $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ denote the closures of $\pi\mathcal{I}(\mathcal{GL}_n)$ and $\pi\mathcal{I}(\mathcal{G}_n)$ in $P[\mathcal{G}_n, \mathbb{R}_+]$, respectively. These results are original due to Thurston [15].

Note that an element \mathcal{L} of $P[\mathcal{G}_n, \mathbb{R}_+]$ is in $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ if and only if for any l in $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ with $\pi(l) = \mathcal{L}$ there exist a sequence $\{t_k\}_{k=1}^\infty$ of positive numbers and a sequence $\{\gamma_k\}_{k=1}^\infty$ of geodesics in \mathcal{G}_n such that the sequence $\{t_k\mathcal{I}_{\gamma_k}\}_{k=1}^\infty$ converges to l . A sequence $\{l_k\}_{k=1}^\infty$ in $[\mathcal{G}_n, \mathbb{R}_+]$ is called *convergent* to $l \in [\mathcal{G}_n, \mathbb{R}_+]$ if for every $\gamma \in \mathcal{G}_n$ the sequence $\{l_k(\gamma)\}_{k=1}^\infty$ converges in \mathbb{R} to $l(\gamma)$.

1.2. Cyclic reduced words. It is well known that every free homotopy class in \mathcal{G}_n corresponds to a unique conjugacy class in the fundamental group of Σ_n . Now, we consider a Fuchsian representation G_n of the fundamental group of Σ_n acting on the upper half plane $\mathcal{U} = \{z \in \mathbb{C} : \text{Im } z > 0\}$, and find a representative for each conjugacy class in G_n by using Birman and Series' cutting sequence technique [2].

Let G_n be the subgroup of $PSL(2, \mathbb{R})$ generated by the following transformations

$$S_1 = \begin{pmatrix} 1 & 2(n-2) \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad T_{n-j-2} = \begin{pmatrix} 2j+1 & 2j(j+1) \\ 2 & 2j+1 \end{pmatrix},$$

where $1 \leq j \leq n-3$ are integers.

For every integer j with $1 \leq j \leq n-3$, let C_j be the isometric circle of T_j , and C'_j be the isometric circle of T_j^{-1} . Let C_{n-2} be the isometric circle of S_2 , and C'_{n-2} be the isometric circle of S_2^{-1} . Let

$$C'_0 = \{z \in \mathbb{C} : \text{Re } z = -(n-2)\} \quad \text{and} \quad C_0 = \{z \in \mathbb{C} : \text{Re } z = n-2\}.$$

Note that $S_1(C'_0) = C_0$, and that the polygon $\mathcal{D}_n \subset \mathcal{U}$ bounded by C_j and C'_j , $0 \leq j \leq n-2$, is a fundamental domain for G_n acting on \mathcal{U} .

For simplicity, we shall schematically draw \mathcal{D}_n as a rectangular region. See Fig. 1 for $n = 4, 5, 6$, where the points on the boundary of \mathcal{D}_n marked by "x" correspond to punctures of Σ_n .

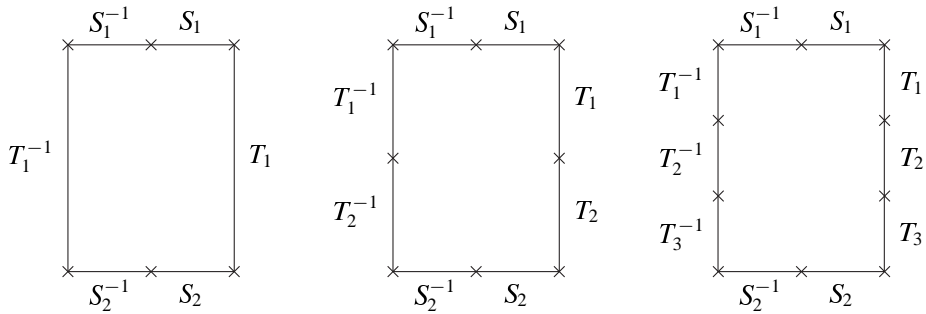


Fig. 1. The fundamental domain \mathcal{D}_n for $n = 4, 5, 6$.

Let Γ_n denote the set of all side pairings of \mathcal{D}_n , i.e.,

$$\Gamma_n = \{S_1, S_1^{-1}, S_2, S_2^{-1}, T_j, T_j^{-1} : j = 1, \dots, n - 3\}.$$

For every $X \in \Gamma_n$, we label the common side s of \mathcal{D}_n and $X(\mathcal{D}_n)$ by X^{-1} on the side inside \mathcal{D}_n and by X on the side inside $X(\mathcal{D}_n)$, (cf. Fig. 1). This side s will be called the X -side of \mathcal{D}_n .

For every $g \in G_n$, the image $g(\mathcal{D}_n)$ will be called a G_n -translate of \mathcal{D}_n . We transport the above side labelling to all G_n -translates of \mathcal{D}_n .

For any closed curve γ in Σ_n , let $\tilde{\gamma}$ be a lift of γ to \mathcal{U} which starts on a side of a G_n -translate of \mathcal{D}_n and projects to γ bijectively, except the endpoints of $\tilde{\gamma}$. Let $z_0 \in \mathcal{U}$ be an endpoint of $\tilde{\gamma}$, and we orient $\tilde{\gamma}$ so that its initial point is z_0 . The arc $\tilde{\gamma}$ cuts in order the G_n -translates $g_0(\mathcal{D}_n), g_1(\mathcal{D}_n), \dots, g_k(\mathcal{D}_n)$ of \mathcal{D}_n . For every integer j with $1 \leq j \leq k$, let $X_j \in \Gamma_n$ be the label of the common side of $g_{j-1}(\mathcal{D}_n)$ and $g_j(\mathcal{D}_n)$, interior to $g_j(\mathcal{D}_n)$. Then $X_j = g_{j-1}^{-1} \circ g_j$ for every j , and γ is represented by

$$(g_0^{-1} \circ g_1) \circ (g_1^{-1} \circ g_2) \circ \dots \circ (g_{k-1}^{-1} \circ g_k) = X_1 \circ X_2 \circ \dots \circ X_k = \prod_{j=1}^k X_j.$$

Such an expression is called a Γ_n -word representing γ . See [4, §1.2] for a full discussion.

A Γ_n -word $\prod_{j=1}^k X_j$ will be called *reduced* if $X_j \neq X_{j+1}^{-1}$ for $1 \leq j \leq k - 1$. It is called *cyclically reduced* if in addition $X_1 \neq X_k^{-1}$.

Let γ be a simple loop on Σ_n represented by a Γ_n -word given above. For every $1 \leq j \leq k$, let l_j be the image of the intersection of $\tilde{\gamma}$ with $g_j(\overline{\mathcal{D}_n})$ mapped by g_j^{-1} , where $\overline{\mathcal{D}_n}$ is the relative closure of \mathcal{D}_n in \mathcal{U} . Note that each l_j is a simple arc in $\overline{\mathcal{D}_n}$ connecting the X_j^{-1} -side to the X_{j+1} -side, where $X_{k+1} = X_1$. Each l_j will be called a *strand* of γ .

Let α be a multiple simple loop on Σ_n . A strand of a connected component of α will be also called a *strand* of α .

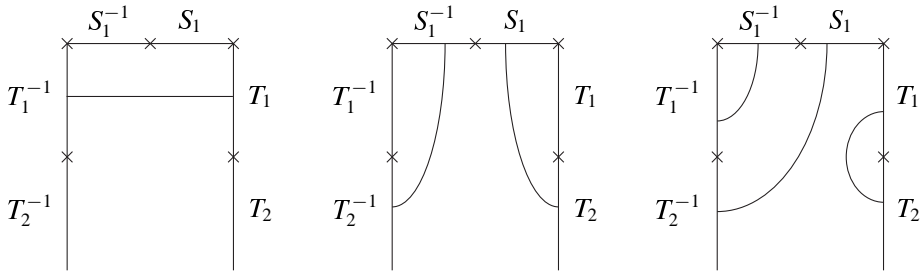


Fig. 2. From the left to the right : $\gamma_1^1, \gamma_1^2, \gamma_1^3$.

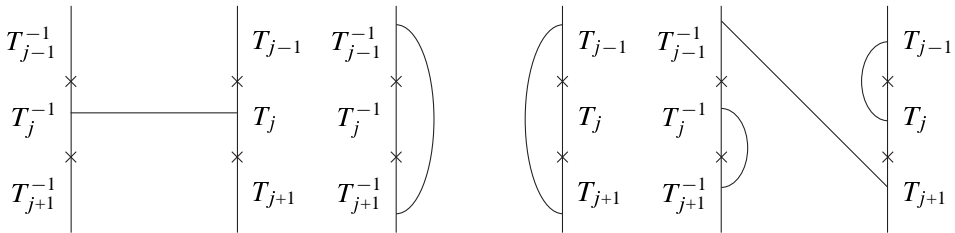


Fig. 3. From the left to the right : $\gamma_j^1, \gamma_j^2, \gamma_j^3, 2 \leq j \leq n - 4$.

A loop on Σ_n is called *reduced* if it is represented by a cyclically reduced Γ_n -word. A multiple simple loop α on Σ_n is called *reduced* if every connected component of α is reduced. Note that a simple loop or a multiple simple loop on Σ_n is reduced if and only if every strand of the loop connects two different sides of \mathcal{D}_n . It is easy to see that every simple closed geodesic on Σ_n is a reduced loop. Thus every free homotopy class of multiple simple loops on Σ_n contains a reduced one.

If $\gamma \in \mathcal{G}_n$ is a geodesic represented by a reduced Γ_n -word W , then γ is also represented by an arbitrary cyclic permutation of W . If γ' is a geodesic which has the same underlying set with γ but opposite orientation, then γ' is represented by W^{-1} . Because we are only interested in non-oriented simple loops, we shall identify all reduced Γ_n -words which are cyclic permutations of W or cyclic permutations of W^{-1} , and call any one of them a *cyclic reduced Γ_n -word* representing γ . Every cyclic reduced Γ_n -word is cyclically reduced.

As examples, let $\gamma_j^k \in \mathcal{G}_n$ be the geodesics given in Fig. 2, Fig. 3 and Fig. 4, where j and k are integers with $1 \leq j \leq n - 3$ and $1 \leq k \leq 3$. See introduction for a geometric interpretation of γ_j^k . Each γ_j^k is represented by a cyclic reduced Γ_n -word W_j^k as given below:

- (i) $W_1^1 = T_1, W_1^2 = S_1 T_2^{-1}, W_1^3 = S_1 T_1^{-1} T_2$;
- (ii) $W_j^1 = T_j, W_j^2 = T_{j-1} T_{j+1}^{-1}, W_j^3 = T_{j+1} T_j^{-1} T_{j-1}$ for $2 \leq j \leq n - 4$;
- (iii) $W_{n-3}^1 = T_{n-3}, W_{n-3}^2 = S_2 T_{n-4}^{-1}$ and $W_{n-3}^3 = S_2 T_{n-3}^{-1} T_{n-4}$.

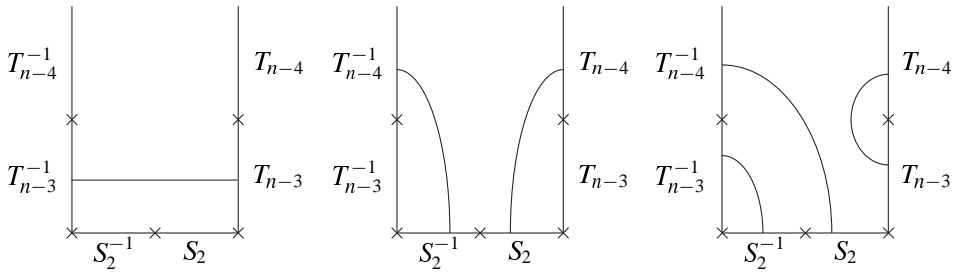


Fig. 4. From the left to the right : $\gamma_{n-3}^1, \gamma_{n-3}^2, \gamma_{n-3}^3$.

1.3. Subwords and admissible subarcs. Let $\hat{\mathcal{G}}_n = \mathcal{G}_n - \{\gamma_j^1 : 1 \leq j \leq n - 3\}$. Let $\gamma \in \hat{\mathcal{G}}_n$ be a geodesic represented by a cyclic reduced Γ_n -word $W = \prod_{j=1}^k X_j$. Note that $k > 1$. For any two integers $1 \leq j \leq k$ and $1 \leq l \leq k$, the reduced Γ_n -word

$$(1) \quad W' = X_j \cdots X_{j+l-1}$$

will be called a *subword* of W , where $X_{j+i} = X_{j+i-k}$ whenever $1 \leq i \leq l$ and $i+j > k$.

We shall relate W' to γ geometrically. For every i , let l_i be the strand of γ connecting the X_{i-1}^{-1} -side to the X_i -side, where $X_{i-1} = X_k$ if $i = 1$. Assume that $1 \leq l < k$, i.e., $W' \neq W$. We think that W' “represents” a subarc γ' of γ . We choose γ' to be the projection to Σ_n of the union $\cup_{i=j}^{j+l-1} l_i$. Each of the arcs l_1, \dots, l_{j+l-1} is called a strand of γ' .

This arc γ' has two distinct endpoints. One of the two endpoints is the projection of the endpoint of l_j on the X_{j-1}^{-1} -side, and the other one is the projection of the endpoint of l_{j+l-1} on the X_{j+l-1} -side. The word given in (1) is not clear enough to indicate the endpoint on the X_{j-1}^{-1} -side. To distinguish it from cyclic reduced words representing simple closed geodesics, we write the reduced Γ_n -word representing γ' as

$$(2) \quad \vec{X}_{j-1} W' = \vec{X}_{j-1} X_j \cdots X_{j+l-1},$$

where \vec{X}_{j-1} is to indicate that $\vec{X}_{j-1} W'$ is not cyclic, and to indicate that one of the endpoints of γ' is the projection of a point on the X_{j-1}^{-1} -side.

A subarc of a geodesic $\gamma \in \mathcal{G}_n$ will be called *admissible* if either it is γ itself, or is represented by a reduced Γ_n -word as given in (2).

REMARK 1.1. For $\varepsilon = \pm 1$, $X \in \Gamma_n$, $X_1, X_2 \in \Gamma_n - \{X^{\pm 1}\}$, and an integer $k > 1$, we shall write

$$X_1 \underbrace{X^\varepsilon \cdots X^\varepsilon}_{k \text{ times}} X_2 = X_1 X^{k\varepsilon} X_2.$$

Let $\gamma \in \hat{\mathcal{G}}_n$ be a geodesic represented by a cyclic reduced Γ_n -word $W(\gamma)$. By the

same reasoning as that in [3, §3], there are no admissible subarcs of $\gamma \in \mathcal{G}_n$ represented by any one of the following words:

$$\begin{array}{cccc} \vec{S}_1^\varepsilon S_1^\varepsilon, & \vec{S}_2^\varepsilon S_2^\varepsilon, & \vec{T}_1^\delta S_1^\varepsilon T_1^\delta, & \vec{T}_{n-3}^\delta S_2^\varepsilon T_{n-3}^\delta, \\ \vec{S}_1^\varepsilon T_1^k S_1^\delta, & \vec{S}_2^\varepsilon T_{n-3}^k S_2^\delta, & \vec{T}_j^\varepsilon T_{j+1}^\delta T_j^\varepsilon, & \vec{T}_{j+1}^\varepsilon T_j^\delta T_{j+1}^\varepsilon, \end{array}$$

where $\varepsilon, \delta \in \{1, -1\}$, $k \neq 0$ is an integer, and $1 \leq j \leq n - 4$. Therefore, none of the words $S_1^\varepsilon S_1^\varepsilon, S_2^\varepsilon S_2^\varepsilon, T_1^\delta S_1^\varepsilon T_1^\delta, T_{n-3}^\delta S_2^\varepsilon T_{n-3}^\delta, S_1^\varepsilon T_1^k S_1^\delta, S_2^\varepsilon T_{n-3}^k S_2^\delta, T_j^\varepsilon T_{j+1}^\delta T_j^\varepsilon$ and $T_{j+1}^\varepsilon T_j^\delta T_{j+1}^\varepsilon$ is a subword of $W(\gamma)$.

1.4. The free homotopy relative to $\partial\mathcal{D}_n$. To be able to relate the geometric intersection number of two geodesics in \mathcal{G}_n to the intersection of their admissible subarcs, we shall define the *free homotopy relative to $\partial\mathcal{D}_n$* on a family of curves on Σ_n which contains all admissible subarcs of geodesics in \mathcal{G}_n .

The union of a finite number of mutually disjoint simple curves on Σ_n will be called a *multiple simple curve*. Let \mathcal{A} be the family of all multiple simple curves β on Σ_n satisfying the following three properties.

- (i) β lifts to a finite number of mutually disjoint simple arcs in \mathcal{D}_n , called the *strands* of β .
- (ii) Except the endpoints, each strand of β lies in the interior of \mathcal{D}_n .
- (iii) Each strand of β connects two different sides of \mathcal{D}_n .

Note that \mathcal{A} contains all reduced multiple simple loops on Σ_n , and contains all admissible subarcs of geodesics in \mathcal{G}_n .

Two multiple simple curves β_1 and β_2 in \mathcal{A} will be called *freely homotopic relative to $\partial\mathcal{D}_n$* , written by $\beta_1 \sim \beta_2$ (rel. $\partial\mathcal{D}_n$), if for any two distinct $X, X' \in \Gamma_n$

$$\begin{aligned} & \#(\text{strands of } \beta_1 \text{ connecting the } X\text{-side and the } X'\text{-side}) \\ & = \#(\text{strands of } \beta_2 \text{ connecting the } X\text{-side and the } X'\text{-side}). \end{aligned}$$

Note that two reduced multiple simple loops on Σ_n are freely homotopic if and only if they are freely homotopic relative to $\partial\mathcal{D}_n$. For $\beta \in \mathcal{A}$, let

$$[\beta]_{\partial\mathcal{D}_n} = \{\beta' \in \mathcal{A} : \beta' \sim \beta \text{ (rel. } \partial\mathcal{D}_n)\},$$

and we shall call a strand of β a *strand* of $[\beta]_{\partial\mathcal{D}_n}$.

Now, we may define the *strands* of a free homotopy class $\alpha \in \mathcal{GL}_n$ as follows. Write $\alpha = \bigoplus_{j=1}^m k_j \gamma_j$, where $(\gamma_1, \dots, \gamma_m) \in \Lambda_n^m$, and m, k_1, \dots, k_m are positive integers with $m \leq n - 3$. A strand of some γ_j is called a *strand* of α . Similarly, an admissible subarc of some γ_j is called an *admissible subarc* of α .

For $\beta_1, \beta_2 \in \mathcal{A}$, we define

$$i([\beta_1]_{\partial\mathcal{D}_n}, [\beta_2]_{\partial\mathcal{D}_n}) = \min\{\#(\beta'_1 \cap \beta'_2) : \beta'_1 \sim \beta_1 \text{ and } \beta'_2 \sim \beta_2 \text{ (rel. } \partial\mathcal{D}_n)\}.$$

where $\#(\beta'_1 \cap \beta'_2)$ denotes the cardinality of the intersection $\beta'_1 \cap \beta'_2$. For simplicity, from now on we shall write

$$i([\beta_1]_{\partial \mathcal{D}_n}, [\beta_2]_{\partial \mathcal{D}_n}) = i([\beta_1], [\beta_2])_{\partial \mathcal{D}_n}$$

for $\beta_1, \beta_2 \in \mathcal{A}$. Note that if β_1 and β_2 are reduced multiple simple loops, then $i([\beta_1], [\beta_2])_{\partial \mathcal{D}_n} = i([\beta_1], [\beta_2])$.

1.5. Four automorphisms of \mathcal{GL}_n . We have set up a very symmetric fundamental domain \mathcal{D}_n for G_n . As we did in [4], by use of the symmetry of \mathcal{D}_n , we may cut down our discussion to fewer cases by introducing four automorphisms $\Theta_1, \Theta_2, \mathcal{T}_1$ and \mathcal{T}_2 of G_n defined by

$$\begin{aligned} \Theta_1(X) &= X^{-1} \text{ for } X \in \{S_1, S_2, T_j : 1 \leq j \leq n-3\}; \\ \Theta_2(S_1) &= S_2, \Theta_2(S_2) = S_1, \text{ and } \Theta_2(T_j) = T_{n-j-2} \text{ for } 1 \leq j \leq n-3. \\ \mathcal{T}_1(S_1) &= S_1^{-1}T_1 \text{ and } \mathcal{T}_1(X) = X \text{ for } X \in \{S_2, T_j : 1 \leq j \leq n-3\}; \\ \mathcal{T}_2(S_2) &= S_2^{-1}T_{n-3} \text{ and } \mathcal{T}_2(X) = X \text{ for } X \in \{S_1, T_j : 1 \leq j \leq n-3\}. \end{aligned}$$

It follows from Nielsen’s isomorphism theorem ([14] or [11, Theorem V.H.1]) that for $j = 1$ or 2 , each of the automorphisms Θ_j and \mathcal{T}_j induces a homeomorphism of Σ_n onto itself, still denoted by Θ_j and \mathcal{T}_j .

Let φ be any one of the four homeomorphisms $\Theta_1, \Theta_2, \mathcal{T}_1$ and \mathcal{T}_2 . The action of φ on \mathcal{GL}_n is defined as follows. For every geodesic $\gamma \in \mathcal{G}_n$, let $\varphi(\gamma)$ denote the free homotopy class containing the homeomorphic image of γ under φ . As before, let $\varphi(\gamma)$ also denote the geodesic in the free homotopy class $\varphi(\gamma)$. Thus φ extends naturally to \mathcal{GL}_n such that

$$\varphi \left(\bigoplus_{j=1}^{n-3} k_j \gamma_j \right) = \bigoplus_{j=1}^{n-3} k_j \varphi(\gamma_j),$$

where $(k_1, \dots, k_{n-3}) \in \mathbb{Z}_+^{n-3}$ and $(\gamma_1, \dots, \gamma_{n-3}) \in \Lambda_n^{n-3}$.

Note that if $\gamma \in \mathcal{G}_n$ is represented by a cyclic reduced Γ_n -word W , then $\varphi(\gamma)$ is represented by $\varphi(W)$.

2. Elementary Intersection Numbers

In this section, we generalize elementary intersection numbers of elements of \mathcal{GL}_5 [4, §2.1] to elements of \mathcal{GL}_n , and prove the elementary intersection formulas.

2.1. The integer valued functions I_j . Let $\gamma_j^k \in \mathcal{G}_n$ be the geodesics given in §1.2. For any $\alpha \in \mathcal{GL}_n$, the geometric intersection numbers $i(\alpha, \gamma_j^k)$ are called the *elementary intersection numbers* of α .

Note that if β_1 and β_2 are two simple closed curves on a 2-sphere, and if they intersect transversally at every point of intersection, then $\#(\beta_1 \cap \beta_2)$ is an even integer. Thus $i(\alpha_1, \alpha_2)$ is an even integer for any two $\alpha_1, \alpha_2 \in \mathcal{GL}_n$. We shall write

$$I_j(\alpha) = \frac{i(\alpha, \gamma_j^1)}{2}$$

for $\alpha \in \mathcal{GL}_n$, and for $1 \leq j \leq n-3$. Note that if $\gamma \in \mathcal{G}_n$ is represented by a cyclic reduced Γ_n -word $W(\gamma) = W$, then

$$\begin{aligned} I_1(\gamma) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_1\text{-side}) \\ &= \text{the total number of the letters } S_1 \text{ and } S_1^{-1} \text{ appearing in } W; \\ I_{n-3}(\gamma) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_2\text{-side}) \\ &= \text{the total number of the letters } S_2 \text{ and } S_2^{-1} \text{ appearing in } W. \end{aligned}$$

Thus for $\alpha \in \mathcal{GL}_n$ we have

$$\begin{aligned} I_1(\alpha) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_1\text{-side}); \\ I_{n-3}(\alpha) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_2\text{-side}). \end{aligned}$$

Since $\Theta_1(\gamma_j^1) = \gamma_j^1$ and $\Theta_2(\gamma_j^1) = \gamma_{n-j-2}^1$, the following proposition is an immediate consequence of the definition.

Proposition 2.1. *If $\alpha \in \mathcal{GL}_n$, then*

$$I_j(\alpha) = I_j(\Theta_1(\alpha)) \quad \text{and} \quad I_j(\alpha) = I_{n-j-2}(\Theta_2(\alpha)) \quad \text{for } 1 \leq j \leq n-3.$$

By an argument similar to the one in the proof of [4, Proposition 2.2 (i), (ii)], we obtain:

Proposition 2.2. *Let $\alpha \in \mathcal{GL}_n$. For any integer m , $\mathcal{T}_1^m(\alpha) = \alpha$ when $I_1(\alpha) = 0$, while $\mathcal{T}_2^m(\alpha) = \alpha$ if $I_{n-3}(\alpha) = 0$.*

Proposition 2.3. *If $\alpha \in \mathcal{GL}_n$, and if m is an integer, then*

$$I_1(\alpha) = I_1(\mathcal{T}_j^m(\alpha)) \quad \text{and} \quad I_{n-3}(\alpha) = I_{n-3}(\mathcal{T}_j^m(\alpha)) \quad \text{for } j = 1, 2.$$

Proof. Since γ_1^1 and γ_{n-3}^1 are invariant under each \mathcal{T}_j , the proof is straightforward. \square

Proposition 2.4. *If $\alpha \in \mathcal{GL}_n$, and if j, k and m are integers with $1 \leq k \leq 3$, then*

$$i(\alpha, \gamma_j^k) = i(\mathcal{T}_1^m(\alpha), \gamma_j^k) \quad \text{for } 1 < j \leq n-3, \text{ and}$$

$$i(\alpha, \gamma_j^k) = i(T_2^m(\alpha), \gamma_j^k) \text{ for } 1 \leq j < n - 3.$$

Proof. By Proposition 2.2, we have $T_1(\gamma_j^k) = \gamma_j^k$ for $1 < j \leq n - 3$, and $T_2(\gamma_j^k) = \gamma_j^k$ for $1 \leq j < n - 3$. The proof is complete. \square

2.2. Cyclic semi-reduced words. To compute elementary intersection numbers, we associate to geodesics in \mathcal{G}_n cyclic semi-reduced Γ_n -words, which are defined analogously to those in [4, §2.2].

Let $\gamma \in \mathcal{G}_n$ with $I_{n-3}(\gamma) > 0$. Assume that γ is represented by a cyclic reduced Γ_n -word $W(\gamma)$. If $S_2^\varepsilon X$ or XS_2^ε is a subword of $W(\gamma)$ with $\varepsilon = \pm 1$ and $X \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$, we shall write

$$S_2^\varepsilon X = S_2^\varepsilon T_{n-3}^0 X \text{ and } XS_2^\varepsilon = XT_{n-3}^0 S_2^\varepsilon.$$

Similarly, for a geodesic $\gamma \in \mathcal{G}_n$ with $I_1(\gamma) > 0$, if $X \in \Gamma_n - \{S_1^{\pm 1}, T_1^{\pm 1}\}$, and if $S_1^\varepsilon X$ or XS_1^ε is a subword of $W(\gamma)$, then we write $S_1^\varepsilon X = S_1^\varepsilon T_1^0 X$ and $XS_1^\varepsilon = XT_1^0 S_1^\varepsilon$. The resulting cyclic Γ_n -word is called a *semi-reduced*, still denoted by $W(\gamma)$.

As in [4, §2.5], we shall write cyclic semi-reduced Γ_n -words in two canonical forms. First, we subdivide \mathcal{GL}_n into four classes.

Note that every geodesic in \mathcal{G}_n can not simultaneously have a strand joining the S_2^ε -side to the T_{n-3} -side and a strand joining the S_2^ε -side to the T_{n-3}^{-1} -side for $\varepsilon = 1$ or -1 , (see Remark 1.1).

Let $\mathcal{GL}_n^+(T_{n-3})$ be the set of elements of \mathcal{GL}_n which have no strands connecting the T_{n-3} -side to the S_2^ε -side for $\varepsilon = \pm 1$. Let

$$\mathcal{GL}_n^-(T_{n-3}) = \Theta_1(\mathcal{GL}_n^+(T_{n-3})),$$

and let

$$\mathcal{GL}_n^+(T_1) = \Theta_2(\mathcal{GL}_n^+(T_{n-3})) \text{ and } \mathcal{GL}_n^-(T_1) = \Theta_2(\mathcal{GL}_n^-(T_{n-3})).$$

Consequently, $\mathcal{GL}_n^-(T_1) = \Theta_1(\mathcal{GL}_n^+(T_1))$. We remark that $\alpha \in \mathcal{GL}_n^+(T_1)$ if and only if α has no strands connecting the T_1 -side to the S_1^ε -side for $\varepsilon = \pm 1$. The set \mathcal{G}_n is then subdivided into four subclasses as:

$$\begin{aligned} \mathcal{G}_n^+(T_1) &= \mathcal{G}_n \cap \mathcal{GL}_n^+(T_1) \text{ and } \mathcal{G}_n^-(T_1) = \Theta_1(\mathcal{G}_n^+(T_1)). \\ \mathcal{G}_n^+(T_{n-3}) &= \mathcal{G}_n \cap \mathcal{GL}_n^+(T_{n-3}) \text{ and } \mathcal{G}_n^-(T_{n-3}) = \Theta_1(\mathcal{G}_n^+(T_{n-3})). \end{aligned}$$

Now, by the same reasoning as in [4, §2.5], every $\gamma \in \mathcal{G}_n$ with $I_1(\gamma) > 0$ or $I_{n-3}(\gamma) > 0$ is represented by a cyclic semi-reduced Γ_n -word W as given below.

First, assume that $I_{n-3}(\gamma) = m > 0$. There exist m triples $(\varepsilon_j, p_j, q_j)$ of integers with $\varepsilon_j = \pm 1$, $p_j \geq 0$ and $q_j \geq 0$, and there exist m reduced Γ_n -words $W_j = \prod_{i=1}^{p_j} X_{ji}$

with $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$, and $X_{ji} \in \Gamma_n - \{S_2^{\pm 1}\}$ when $1 < i < \nu_j$ such that

$$(3) \quad \gamma \in \mathcal{G}_n^-(T_{n-3}) \implies W = \prod_{j=1}^m T_{n-3}^{-p_j} S_2^{\varepsilon_j} T_{n-3}^{q_j} W_j;$$

$$(4) \quad \gamma \in \mathcal{G}_n^+(T_{n-3}) \implies W = \prod_{j=1}^m T_{n-3}^{p_j} S_2^{\varepsilon_j} T_{n-3}^{-q_j} W_j.$$

If $I_1(\gamma) = m > 0$, by considering $\Theta_2(\gamma)$, then γ is represented by

$$W = \prod_{j=1}^m T_1^{-p_j} S_1^{\varepsilon_j} T_1^{q_j} W_j,$$

where $(\varepsilon_j, p_j, q_j)$ are integers with $\varepsilon_j = \pm 1$ and $p_j q_j \geq 0$, and where $W_j = \prod_{i=1}^{\nu_j} X_{ji}$ are reduced Γ_n -words with $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{S_1^{\pm 1}, T_1^{\pm 1}\}$, and $X_{ji} \in \Gamma_n - \{S_1^{\pm 1}\}$ when $1 < i < \nu_j$. Moreover, $\gamma \in \mathcal{G}_n^-(T_1)$ if and only if $p_j \geq 0$ and $q_j \geq 0$ for all j , while $\gamma \in \mathcal{G}_n^+(T_1)$ if and only if $p_j \leq 0$ and $q_j \leq 0$ for all j .

We remark that any word given above is reduced if each $p_j q_j > 0$.

2.3. Essential blocks and puncture-like blocks. We shall compute elementary intersection numbers by applying mathematical induction to the number n of punctures. From now on, we assume that $n \geq 5$.

To be able to apply mathematical induction to n , we first embed \mathcal{GL}_{n-1} into \mathcal{GL}_n . Let $\Phi_n: G_{n-1} \rightarrow G_n$ be the monomorphism defined by

$$\Phi_n(S_1) = S_1, \quad \Phi_n(S_2) = T_{n-3} \quad \text{and} \quad \Phi_n(T_j) = T_j \quad \text{for } 1 \leq j \leq n-4.$$

The monomorphism Φ_n induces an injective map of \mathcal{G}_{n-1} into \mathcal{G}_n , also denoted by Φ_n . If $\gamma \in \mathcal{G}_{n-1}$ is represented by a cyclic reduced (or semi-reduced) Γ_{n-1} -word W , then $\Phi_n(\gamma)$ is represented by $\Phi_n(W)$.

Let $\mathcal{G}_n^{(n-1)}$ be the image of \mathcal{G}_{n-1} mapped by Φ_n , and let $\mathcal{GL}_n^{(n-1)}$ be the set of all elements of \mathcal{GL}_n of the form

$$\bigoplus_{j=1}^{n-4} k_j \gamma_j,$$

where $(k_1, \dots, k_{n-4}) \in \mathbb{Z}_+^{n-4}$, and $\gamma_j \in \mathcal{G}_n^{(n-1)}$ are mutually disjoint geodesics. Note that if $\gamma \in \mathcal{GL}_n^{(n-1)}$, then $2I_{n-3}(\gamma) = i(\gamma, \gamma_{n-3}^1) = 0$.

The geodesic γ_{n-3}^1 divides Σ_n into two connected components. One of them is a sphere with $n-2$ punctures and one hole, denoted by $\Sigma_n^{(n-1)}$, and the other one is a sphere with two punctures and one hole, denoted by $\Sigma_n^{(3)}$. Note that the punctures of $\Sigma_n^{(3)}$ correspond to the fixed points of the transformations S_2 and $S_2 T_{n-3}^{-1}$. Also note

that $\Sigma_n^{(n-1)}$ is homeomorphic to Σ_{n-1} , and $\Sigma_n^{(3)}$ is homeomorphic to a 3-punctured sphere. It follows from the definition that every $\gamma \in \mathcal{GL}_n^{(n-1)}$ contains a representative lying on $\Sigma_n^{(n-1)}$. Thus we can do an induction after we relate free homotopy classes in \mathcal{GL}_n to free homotopy classes in $\mathcal{GL}_n^{(n-1)}$.

To relate free homotopy classes in \mathcal{GL}_n to that in $\mathcal{GL}_n^{(n-1)}$, we consider the set \mathcal{GL}_n^0 of all free homotopy classes in \mathcal{GL}_n which have no strands connecting the S_2^ε -side to the X -side, where $\varepsilon = \pm 1$, and where $X \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$. Let $\mathcal{G}_n^0 = \mathcal{G}_n \cap \mathcal{GL}_n^0$.

It follows immediately from the definition that if $\gamma \in \mathcal{GL}_n$ with $I_{n-3}(\gamma) = 0$, then $\gamma \in \mathcal{GL}_n^0$. In particular, $\mathcal{GL}_n^{(n-1)} \subset \mathcal{GL}_n^0$.

Let $\gamma \in \mathcal{G}_n$ with $I_{n-3}(\gamma) = m > 0$. Then $\gamma \in \mathcal{G}_n^-(T_{n-3}) \cap \mathcal{G}_n^0$ if and only if it is represented by a cyclic reduced Γ_n -word as given in (3), while $\gamma \in \mathcal{G}_n^+(T_{n-3}) \cap \mathcal{G}_n^0$ if and only if it is represented by a cyclic reduced Γ_n -word as given in (4) with $p_j > 0$ and $q_j > 0$ for all j .

The admissible subarcs of every $\gamma \in \mathcal{GL}_n^0$ fall into two classes. One contains admissible subarcs of γ which are freely homotopic relative to $\partial\mathcal{D}_n$ to simple curves lying on $\Sigma_n^{(n-1)}$. The other class contains admissible subarcs of γ which are freely homotopic relative to $\partial\mathcal{D}_n$ to simple curves lying on $\Sigma_n^{(3)}$. We shall relate γ to free homotopy classes in $\mathcal{GL}_n^{(n-1)}$ by relating the admissible subarcs of γ in the first class to elements of $\mathcal{GL}_n^{(n-1)}$.

Any $\gamma \in \mathcal{GL}_n$ can be related to an element of \mathcal{GL}_n^0 as follows.

Proposition 2.5. *Let $\gamma \in \mathcal{GL}_n$.*

- (i) *If $\gamma \in \mathcal{GL}_n^+(T_{n-3})$, then $T_2^{-2}(\gamma) \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$.*
- (ii) *If $\gamma \in \mathcal{GL}_n^-(T_{n-3})$, then $T_2^2(\gamma) \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^-(T_{n-3})$.*

Proof. It suffices to prove (i) for $\gamma \in \mathcal{G}_n^+(T_{n-3})$. There is nothing to prove if $I_{n-3}(\gamma) = 0$. If $I_{n-3}(\gamma) = m > 0$, then γ is represented by the cyclic semi-reduced Γ_n -word given in (4). Now, the assertion follows since $T_2^{-2}(W) = \prod_{j=1}^m T_{n-3}^{p_j+1} S_2^{\varepsilon_j} T_{n-3}^{-q_j-1} W_j$. □

With Proposition 2.5, we may restrict our attention to the subclass \mathcal{GL}_n^0 of \mathcal{GL}_n .

Before continuing our discussion, we choose once for all an orientation for the X -side of \mathcal{D}_n , where $X \in \{T_{n-3}, T_{n-3}^{-1}, T_{n-4}, T_{n-4}^{-1}\}$. Note that $T_{n-4}T_{n-3}^{-1}$ is parabolic since the trace of $T_{n-4}T_{n-3}^{-1}$ is -2 . Let ζ be the fixed point of the transformation $T_{n-4}T_{n-3}^{-1}$. For $X = T_{n-3}$ or T_{n-4} , if P_1 and P_2 are two points lying on the X -side, and if P_1 lies between ζ and P_2 , then we write $P_1 \prec P_2$. If Q_1 and Q_2 are two points lying on the X^{-1} -side, we write $Q_1 \prec Q_2$ whenever $X(Q_1) \prec X(Q_2)$.

Proposition 2.6. *Let $\gamma \in \mathcal{G}_n^+(T_{n-3})$ with $I_{n-3}(\gamma) = m > 0$, and let γ be represented by the cyclic reduced Γ_n -word given in (4).*

If γ has a strand joining the T_{n-3} -side to the T_{n-4} -side, and has a strand joining

the T_{n-3} -side to the T_{n-4}^{-1} -side, then $W_j = T_{n-4}$ or $W_j = T_{n-4}^{-1}$ for some j .

Proof. Let $P_1 \prec \dots \prec P_k$ be the points where the strands of γ meet the T_{n-4} -side. For every integer l with $1 \leq l \leq k$, let P'_l be the point on the T_{n-4}^{-1} -side identified with P_l by the transformation T_{n-4} . Let l be the strand of γ with an endpoint at P_1 , and let l' be the strand of γ with an endpoint at P'_1 .

By assumption, l must connect the T_{n-3} -side and the T_{n-4} -side, and l' must connect the T_{n-3} -side and the T_{n-4}^{-1} -side. The union $l \cup l'$ projects to an admissible subarc γ' of γ represented by $\vec{T}_{n-3}^{-1}T_{n-4}T_{n-3}$ or $\vec{T}_{n-3}^{-1}T_{n-4}^{-1}T_{n-3}$.

Assume that γ' is represented by $\vec{T}_{n-3}^{-1}T_{n-4}T_{n-3}$. Let Q be the endpoint of l on the T_{n-3} -side. We orient γ' so that the projection of Q to Σ_n is the initial point of γ' . Since $\gamma \in \mathcal{G}_n^0$, then the subword $W' = T_{n-3}^{-1}T_{n-4}T_{n-3}$ of W must be followed by a subword of the form $T_{n-3}^p S_2^\varepsilon T_{n-3}^{-q}$ for some integers $\varepsilon = \pm 1$, $p \geq 0$ and $q > 0$.

On the other hand, consider the subarc γ'' of γ which has the same underlying set with γ' but with the opposite orientation. Then γ'' is represented by $\vec{T}_{n-3}^{-1}T_{n-4}^{-1}T_{n-3}$. Thus $T_{n-3}^{-1}T_{n-4}^{-1}T_{n-3}$ is a subword of W^{-1} and is followed by a subword of the form $T_{n-3}^{p'} S_2^{\varepsilon'} T_{n-3}^{-q'}$ for some integers $\varepsilon' = \pm 1$, $p' \geq 0$ and $q' > 0$. We conclude that

$$T_{n-3}^{q'} S_2^{-\varepsilon'} T_{n-3}^{-p'} \cdot W' \cdot T_{n-3}^p S_2^\varepsilon T_{n-3}^{-q} = T_{n-3}^{q'} S_2^{-\varepsilon'} T_{n-3}^{-p'-1} T_{n-4} T_{n-3}^{p+1} S_2^\varepsilon T_{n-3}^{-q}$$

is a subword of W . This proves that $W_j = T_{n-4}$ for some j .

Similarly, if γ' is represented by $\vec{T}_{n-3}^{-1}T_{n-4}^{-1}T_{n-3}$, then there is an integer j such that $W_j = T_{n-4}^{-1}$. □

Blocks of simple closed geodesics. Let γ be given in Proposition 2.6. For every integer j with $1 \leq j \leq m = I_{n-3}(\gamma)$, let γ_j be the admissible subarc of γ represented by $\vec{T}_{n-3}^{-1}W_j T_{n-3}$. Every γ_j will be called a *block* of γ .

Let $l_0^{(j)}$ be the strand of γ_j joining the T_{n-3} -side to the X_{j1} -side with P the endpoint on the T_{n-3} -side, and let $l_1^{(j)}$ be the strand of γ_j joining the $X_{j\nu_j}^{-1}$ -side to the T_{n-3} -side with Q the endpoint on the $X_{j\nu_j}^{-1}$ -side. Let P' be the point on the T_{n-3}^{-1} -side which is identified with P by the transformation T_{n-3} .

Now, we replace $l_1^{(j)}$ by a simple arc $\tilde{l}_1^{(j)}$ joining Q to P' so that $\tilde{l}_1^{(j)}$ is disjoint from all strands of γ_j except possibly $l_1^{(j)}$. Let \mathcal{L}_j be the union of all strands of γ_j other than $l_1^{(j)}$. The union $\mathcal{L}_j \cup \tilde{l}_1^{(j)}$ projects to a simple closed curve $\tilde{\gamma}_j$ on Σ_n . See the proof of [4, Theorem 5.3].

If $W_j = T_{n-4}$ or T_{n-4}^{-1} , then $\tilde{\gamma}_j$ is a simple loop around the puncture corresponding to the fixed point of $T_{n-3}T_{n-4}^{-1}$. In this case, we shall call γ_j a *puncture-like block* of γ .

We call γ_j an *essential block* of γ if γ_j is not a puncture-like block. Thus γ_j is an essential block if and only if $\tilde{\gamma}_j \in \mathcal{G}_n^{(n-1)}$.

Next, let $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^-(T_{n-3})$ with $I_{n-3}(\gamma) > 0$. An admissible subarc γ' of γ is

called a *puncture-like block* if $\Theta_1(\gamma')$ is a puncture-like block of $\Theta_1(\gamma)$, and is called an *essential block* if $\Theta_1(\gamma')$ is an essential block of $\Theta_1(\gamma)$. By Proposition 2.6, γ' is a puncture-like block of γ if and only if it is represented by $\vec{T}_{n-3} T_{n-4}^\varepsilon T_{n-3}^{-1}$ with $\varepsilon = \pm 1$.

Blocks of free homotopy classes in \mathcal{GL}_n^0 . For $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$, there are positive integers k_1, \dots, k_m , and mutually disjoint geodesics β_1, \dots, β_m in \mathcal{G}_n^0 such that

$$\gamma = \bigoplus_{i=1}^m k_i \beta_i,$$

where m is a positive integer with $m \leq n - 3$. An admissible subarc γ' of γ is called a *block* of γ if it is either a connected component of γ with $I_{n-3}(\gamma') = 0$, or is a block of some β_i . A block γ' of γ is called *puncture-like* if it is a puncture-like block of some β_i , and is called *essential* if it is not a puncture-like block. Note that if γ' is a connected component of γ with $I_{n-3}(\gamma') = 0$, then $\gamma' \in \mathcal{G}_n^{(n-1)}$. Such an essential block will be called of the *second kind*. An essential block of γ will be called of the *first kind* if it is not of the second kind.

REMARK 2.1. It follows from Proposition 2.6 that if $\gamma \in \mathcal{GL}_n^+(T_{n-3})$ has a strand joining the T_{n-3} -side to the T_{n-4} -side, and has a strand joining the T_{n-3} -side to the T_{n-4}^{-1} -side, then γ has a puncture-like block. Similarly, if $\gamma \in \mathcal{GL}_n^-(T_{n-3})$ has a strand joining the T_{n-3}^{-1} -side to the T_{n-4} -side, and has a strand joining the T_{n-3}^{-1} -side to the T_{n-4}^{-1} -side, then γ has a puncture-like block.

REMARK 2.2. Let $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$. If γ has no essential blocks, then $I_1(\gamma) = 0$ and $I_{n-3}(\Theta_2(\gamma)) = 0$. Note that $\Theta_2(\gamma) \in \mathcal{GL}_n^{(n-1)}$. Thus the elementary intersection numbers of γ will be obtained from that of $\Theta_2(\gamma)$ by applying induction to n . Therefore, we shall only consider the case where γ has essential blocks.

The following theorem plays an important role in the sequel.

Theorem 2.7. *Let $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$. If γ has essential blocks, then there is an $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ such that*

$$i(\alpha_\gamma, \gamma_{n-4}^1) = i(\gamma, \gamma_{n-4}^1) \quad \text{and} \quad i(\alpha_\gamma, \gamma_j^k) = i(\gamma, \gamma_j^k)$$

for $1 \leq j < n-4$ and $1 \leq k \leq 3$. Furthermore, α_γ can be chosen so that $\Theta_1(\alpha_{\Theta_1(\gamma)}) = \alpha_\gamma$.

Since for all j, k ,

$$i(\Theta_1(\alpha_{\Theta_1(\gamma)}), \gamma_j^k) = i(\alpha_{\Theta_1(\gamma)}, \Theta_1(\gamma_j^k)) = i(\Theta_1(\gamma), \Theta_1(\gamma_j^k)) = i(\gamma, \gamma_j^k),$$

α_γ can be chosen so that $\Theta_1(\alpha_{\Theta_1(\gamma)}) = \alpha_\gamma$ since for all j, k . Thus, we may assume that $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$.

First, we prove Theorem 2.7 for γ which has no puncture-like blocks. Let \mathcal{E} be the set of all essential blocks of γ , and for every $\gamma' \in \mathcal{E}$ let

$$t(\gamma') = \text{the number of strands of } \gamma' \text{ meeting the } T_{n-3}\text{-side.}$$

If γ has no essential blocks of the first kind, then any essential block of γ serves as α_γ .

Now, we assume that γ has exactly $e > 0$ essential blocks of the first kind, say $\gamma_1, \dots, \gamma_e$. Let $\tilde{\gamma}_j$ be the geodesic in $\mathcal{G}_n^{(n-1)}$ corresponding to γ_j (see the definition of blocks), and let t_j be the number of strands of $\tilde{\gamma}_j$ meeting the T_{n-3} -side. Note that $t(\gamma_j) = t_j + 1$, and the strands of γ_j meet the T_{n-3}^{-1} -side in exactly $t_j - 1$ points. Then the strands of $\cup_{j=1}^e \gamma_j$ meet the T_{n-3}^{-1} -side in exactly $t_0 = \sum_{j=1}^e (t_j - 1)$ points, and meet the T_{n-3} -side in exactly $t_0 + 2e$ points.

We consider the disjoint union \mathcal{L} of strands of all essential blocks of γ . Let $Q_1 < Q_2 < \dots < Q_q$ be the points where \mathcal{L} meets the T_{n-3} -side, where q is an integer with $q \geq t_0 + 2e$.

CLAIM 1. For every integer j with $q - 2e + 1 \leq j \leq q$, the point Q_j is an endpoint of a strand L_j of $\cup_{j=1}^e \gamma_j$.

We shall show that Claim 1 implies Theorem 2.7 when γ has no puncture-like blocks. For every integer j with $q - e + 1 \leq j \leq q$, let P_j be the endpoint of L_j other than Q_j , and let Q'_{j-e} be the point lying on the T_{n-3}^{-1} -side which is identified with Q_{j-e} by the transformation T_{n-3} . There are mutually disjoint simple arcs L'_j , $q - e + 1 \leq j \leq q$, in \mathcal{D}_n satisfying the following two properties:

- (i) Each L'_j connects P_j to Q'_{j-e} .
- (ii) Each L'_j is disjoint from the strands of any essential block of γ except possibly the strands L_{q-e+1}, \dots, L_q .

The set $\mathcal{L}' = (\mathcal{L} - \cup_{j=q-e+1}^q L_j) \cup (\cup_{j=q-e+1}^q L'_j)$ projects to a multiple simple loop α_γ in $\mathcal{GL}_n^{(n-1)}$, and the free homotopy class represented by α_γ , still denoted by α_γ , satisfies the required conditions since $i(\alpha_\gamma, \gamma_{n-4}^1) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n}$ and $i(\alpha_\gamma, \gamma_j^k) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_j^k)_{\partial \mathcal{D}_n}$ for $1 \leq j < n - 4$ and $1 \leq k \leq 3$.

Proof of Claim 1. There is nothing to prove if γ has no essential blocks γ' of the second kind with $t(\gamma') > 0$. Assume that γ has exactly $p > 0$ essential blocks $\gamma_{e+1}, \dots, \gamma_{e+p}$ of the second kind with $t(\gamma_{e+j}) > 0$, $1 \leq j \leq p$.

For every j with $1 \leq j \leq e$, the block γ_j is represented by a reduced Γ_n -word $\vec{T}_{n-3}^{-1}W_jT_{n-3}$, where $W_j \neq T_{n-4}^{\pm 1}$ is of the form $W_j = \prod_{i=1}^{\nu_j} X_{ji}$ with $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$ and $X_{ji} \in \Gamma_n - \{S_2^{\pm 1}\}$ for $1 < i < \nu_j$. Let

$l_j^{(1)}$ be the strand of γ_j joining the T_{n-3} -side to the X_{j1} -side,

$l_j^{(2)}$ be the strand of γ_j joining the $X_{j\nu_j}^{-1}$ -side to the T_{n-3} -side,

Q_{jk} be the endpoint of $l_j^{(k)}$ on the T_{n-3} -side for $k = 1, 2$, and

Q'_{jk} be the point on the T_{n-3}^{-1} -side identified with Q_{jk} by the transformation T_{n-3} .

By the definition of γ_j , the point Q'_{jk} is an endpoint of a strand $L_j^{(k)}$ of γ joining the T_{n-3}^{-1} -side to the X -side with $X \in \{T_{n-3}, S_2^{\pm 1}\}$.

Suppose that there is an integer m with $q-2e+1 \leq m \leq q$ such that Q_m is an endpoint of a strand of $\cup_{j=1}^p \gamma_{e+j}$. Then there is a Q_{jk} such that $Q_{jk} \prec Q_m$. Let Q'_m be the point on the T_{n-3}^{-1} -side identified with Q_m by the transformation T_{n-3} . It follows from the definition of γ_{e+j} that Q'_m is an endpoint of a strand L of γ joining the T_{n-3}^{-1} -side to some X -side with $X \in \Gamma_n - \{T_{n-3}, S_2^{\pm 1}\}$. Since $Q_{jk} \prec Q_m$, then $Q'_{jk} \prec Q'_m$, and thus $L_j^{(k)}$ must intersect L transversally. This contradiction completes the proof of the claim. □

In the following, we prove Theorem 2.7 for γ which has puncture-like blocks. For this case, we need the following two lemmas.

Lemma 2.8. *If $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ with $I_{n-3}(\gamma) > 0$, and if γ has a puncture-like block, then every essential block of γ has no strands meeting the T_{n-3}^{-1} -side.*

Proof. Let γ_0 be a puncture-like block of γ . There is a strand l_0 of γ_0 connecting the T_{n-4} -side and the T_{n-3} -side. Let Q_0 be the endpoint of l_0 on the T_{n-3} -side. We may choose γ_0 so that $Q_0 \prec Q$ whenever Q is an endpoint of a strand of γ on the T_{n-3} -side. Let Q'_0 be the point on the T_{n-3}^{-1} -side which is identified with Q_0 by the transformation T_{n-3} . Note that Q'_0 is an endpoint of a strand L_0 of γ joining the T_{n-3}^{-1} -side to the X -side with $X \in \{T_{n-3}, S_2, S_2^{-1}\}$. Also note that if Q' is an endpoint of a strand of γ on the T_{n-3}^{-1} -side, then $Q'_0 \prec Q'$ by the definition of Q_0 .

Now, suppose that there is an essential block γ' of γ such that γ' has a strand l' meeting the T_{n-3}^{-1} -side at a point Q' . Since $Q'_0 \prec Q'$, and since γ' has no strands joining the T_{n-3}^{-1} -side to the X -side with $X \in \{T_{n-3}, S_2, S_2^{-1}\}$, then l' must intersect L_0 transversally. This is a contradiction. □

Lemma 2.9. *Let $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ with $I_{n-3}(\gamma) > 0$, and for an arbitrary block γ' of γ , let $t(\gamma')$ be the number of strands of γ' meeting the T_{n-3} -side. If γ has*

puncture-like blocks, then

$$t(\gamma') = \begin{cases} 2 & \text{if } \gamma' \text{ is a puncture-like block,} \\ 2 & \text{if } \gamma' \text{ is an essential block of the first kind,} \\ 0 & \text{if } \gamma' \text{ is an essential block of the second kind.} \end{cases}$$

Proof. It follows immediately from the definition that $t(\gamma') = 2$ whenever γ' is a puncture-like block of γ .

Let γ' be an essential block of the second kind, i.e., γ' is a simple closed geodesic in $\mathcal{GL}_n^{(n-1)}$. If γ' has a strand meeting the T_{n-3} -side, then γ' must have a strand meeting the T_{n-3}^{-1} -side. This contradicts to Lemma 2.8. Therefore, $t(\gamma') = 0$.

If γ' is an essential block of the first kind, then γ' is represented by a reduced Γ_n -word $\bar{T}_{n-3}^{-1}WT_{n-3}$, where $W \neq T_{n-4}^{\pm 1}$ is of the form $W = \prod_{j=1}^m X_j$ with $X_1, X_m \in \Gamma_n - \{T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$, and $X_j \in \Gamma_n - \{S_2^{\pm 1}\}$ for $1 < j < m$. There is a strand l_0 of γ' joining the T_{n-3} -side to the X_1 -side, and there is another strand l_1 of γ' joining the X_m -side to the T_{n-3} -side. Thus $t(\gamma') \geq 2$.

Suppose that $t(\gamma') > 2$. There is a $k \in \{2, \dots, m-1\}$ such that $X_k = T_{n-3}$ or $X_k = T_{n-3}^{-1}$. If $X_k = T_{n-3}$, then γ' has a strand joining the T_{n-3}^{-1} -side to the X_{k+1} -side. This is a contradiction to Lemma 2.8. If $X_k = T_{n-3}^{-1}$, then γ' has a strand joining the X_{k-1} -side to the T_{n-3}^{-1} -side. This is a contradiction to Lemma 2.8 again. Therefore, $t(\gamma') = 2$. □

Now, we complete the proof of Theorem 2.7 as follows. Let $\gamma_1, \dots, \gamma_e$ be all the first kind essential blocks of γ , and assume that γ has exactly $p > 0$ puncture-like blocks, say $\gamma_{e+1}, \dots, \gamma_{e+p}$. Note that $t(\gamma_j) = 2$ for all j by Lemma 2.9.

Let $Q_1 \prec \dots \prec Q_k$ be the points where the strands of γ meet the T_{n-3} -side. Note that $k \geq 2p + 2e$. Since $\gamma \in \mathcal{GL}_n^0$, and since $t(\gamma') = 0$ whenever γ' is an essential block of γ of the second kind, then Q_1, \dots, Q_{2p+2e} are endpoints of strands of $\cup_{j=1}^{p+e} \gamma_j$, and, for $2p + 2e + 1 \leq j \leq k$, each Q_j is an endpoint of a strand of γ connecting the T_{n-3} -side and the T_{n-3}^{-1} -side whenever $2p + 2e + 1 \leq j \leq k$.

CLAIM 2. Q_{p+1}, \dots, Q_{p+2e} are the points where the strands of $\cup_{j=1}^e \gamma_j$ meet the T_{n-3} -side.

Now, for every integer j with $1 \leq j \leq e$, let L_j be the strand of $\cup_{j=1}^e \gamma_j$ with Q_{p+e+j} an endpoint, let P_j be the other endpoint of L_j . Let Q'_{p+j} be the point on the T_{n-3}^{-1} -side which is identified with Q_{p+j} by the transformation T_{n-3} .

There are e mutually disjoint simple arcs L'_j in \mathcal{D}_n connecting P_j to Q'_{p+j} for every j such that every L'_j is disjoint from the strands of any essential block of γ except possibly the strands L_1, \dots, L_e . As before, let \mathcal{E} be the set of all essential blocks of $\mathcal{L}' = (\mathcal{L} - \cup_{j=1}^e L_j) \cup (\cup_{j=1}^e L'_j)$ projects to Σ_n a multiple simple loop α_γ

in $\mathcal{GL}_n^{(n-1)}$. Let α_γ also denote the corresponding free homotopy class. Note that if γ' is a puncture-like block of γ , then $i(\gamma', \gamma_{n-4}^1)_{\partial\mathcal{D}_n} = 0 = i(\gamma', \gamma_j^k)_{\partial\mathcal{D}_n}$ for $1 \leq j < n-4$ and $1 \leq k \leq 3$. This completes the proof of Theorem 2.7.

Proof of Claim 2. It suffices to prove that if Q is the endpoint of a strand of $\cup_{j=1}^e \gamma_j$ lying on the T_{n-3} -side, then $Q_j \prec Q \prec Q_{p+2e+j}$ for all j with $1 \leq j \leq p$.

Let γ' be the essential block of γ of the first kind such that Q is one of the two points where the strands of γ' meet the T_{n-3} -side, and let L be the strand of γ' with Q as an endpoint.

If $Q \in \{Q_1, \dots, Q_p\}$, then there is an integer m with $p < m \leq 2p + 2e$ such that Q_m is the endpoint of a strand l of $\cup_{j=1}^p \gamma_{e+j}$ connecting the T_{n-3} -side to the T_{n-4} -side. Thus the other endpoint P of l must lie on the T_{n-4} -side with $P \prec P_m$, where P_m is the endpoint of l other than Q_m . Let P' and P'_m be the points lying on the T_{n-4}^{-1} -side which are identified with P and P_m respectively by the transformation T_{n-4} . Let L' be the strand of γ' with P' as an endpoint. Since $P' \prec P'_m$, then L' must connect the T_{n-4} -side to the T_{n-3} -side. This implies that γ' is a puncture-like block of γ , which is a contradiction. Therefore, $Q_j \prec Q$ for all j with $1 \leq j \leq p$.

By a similar argument, one proves that $Q \prec Q_{p+2e+j}$ for $1 \leq j \leq p$. □

2.4. The integer valued functions N_j . To formulate elementary intersection numbers, in addition to the integer valued functions I_j defined in §2.1, we shall need other $n - 3$ integer valued functions N_j , $1 \leq j \leq n - 3$. These functions N_j are analogues of the integer valued functions N_T and N_S defined in [4].

We shall define an integer valued function $N_j^{(n)}$ on \mathcal{GL}_n for any given integer $j > 0$ with $j \leq n - 3$ so that

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\gamma))$$

whenever $\gamma \in \mathcal{GL}_n^{(n-1)}$ and $j \leq n - 4$, where Φ_n is defined in §2.3. This means that $N_j^{(n-1)}$ can be regarded as the restriction of $N_j^{(n)}$ to $\mathcal{GL}_n^{(n-1)}$ whenever $1 \leq j \leq n - 4$. Thus $N_j^{(n)}$ can be simply written as N_j . Furthermore, this allows us to define N_j inductively by using Theorem 2.7.

First, we define the functions $N_1^{(n)}$ and $N_{n-3}^{(n)}$. If $\gamma = \bigoplus_{j=1}^{n-3} k_j \gamma_j^1$ with $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$, we define

$$N_j^{(n)}(\gamma) = k_j = \#(\text{strands of } \gamma \text{ connecting the } T_j\text{-side and the } T_j^{-1}\text{-side}),$$

for $j = 1$ or $n - 3$.

Now, we define $N_1^{(n)}(\gamma)$ and $N_{n-3}^{(n)}(\gamma)$ for $\gamma \in \widehat{\mathcal{GL}}_n$, where

$$\widehat{\mathcal{GL}}_n = \mathcal{GL}_n - \left\{ \bigoplus_{j=1}^{n-3} k_j \gamma_j^1 : (k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3} \right\}.$$

If $\gamma \in \mathcal{GL}_n^+(T_1)$, let

$$N_1^{(n)}(\gamma) = \#(\text{strands of } \gamma \text{ joining the } T_1^{-1}\text{-side to the } S_1^\varepsilon\text{-side}) \\ + \#(\text{strands of } \gamma \text{ joining the } T_1\text{-side to the } T_1^{-1}\text{-side}),$$

where $\varepsilon = \pm 1$. If $\gamma \in \mathcal{GL}_n^+(T_{n-3})$, let

$$N_{n-3}^{(n)}(\gamma) = \#(\text{strands of } \gamma \text{ joining the } T_{n-3}^{-1}\text{-side to the } S_2^\varepsilon\text{-side}) \\ + \#(\text{strands of } \gamma \text{ joining the } T_{n-3}\text{-side to the } T_{n-3}^{-1}\text{-side}).$$

For $j = 1$ or $n - 3$, and for $\gamma \in \mathcal{GL}_n^-(T_j) \cap \widehat{\mathcal{GL}}_n$, let

$$N_j^{(n)}(\gamma) = -N_j^{(n)}(\Theta_1(\gamma)).$$

It is clear that the definition of $N_1^{(n)}$ is independent of n since $n \geq 5$. Thus $N_1^{(n)}$ will be simply written as N_1 .

REMARK 2.3. For $n = 5$, let N_T and N_S be the integer valued functions defined in [4], and let N_1 and $N_2 = N_{n-3}^{(n)}$ be the integer valued functions defined above. Then for $\gamma \in \mathcal{GL}_5$ we have

$$N_1(\gamma) = N_T(\gamma) \quad \text{and} \quad N_2(\gamma) = -N_S(\gamma).$$

Note that the geodesic γ_{23} defined in [4] and the geodesic γ_2^3 defined in this article are imgaes of each other under Θ_1 . Thus, the following equations are also valid for $\gamma \in \mathcal{GL}_5$ (see [4, Corollary 3.4]):

$$i(\gamma, \gamma_1^2) = 2|N_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma) \\ i(\gamma, \gamma_1^3) = 2|N_1(\gamma) - I_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma) \\ i(\gamma, \gamma_2^2) = 2|N_2(\gamma)| + |I_1(\gamma) - I_2(\gamma)| + I_1(\gamma) - I_2(\gamma) \\ i(\gamma, \gamma_2^3) = 2|N_2(\gamma) - I_2(\gamma)| + |I_1(\gamma) - I_2(\gamma)| + I_1(\gamma) - I_2(\gamma)$$

In §2.5, we shall prove similar formulas for elementary intersection numbers of $\gamma \in \mathcal{GL}_n$ for an arbitrary integer $n \geq 5$.

For integers n and j with $1 < j \leq n - 4$, the integer valued functions $N_j^{(n)}$ on \mathcal{GL}_n are defined as follows. We first define $N_j^{(n)}(\gamma)$ for $\gamma \in \mathcal{GL}_n^0$.

(i) If $I_{n-3}(\gamma) = 0$, then there exist $(k_1, \dots, k_{n-3}) \in \mathbb{Z}_+^{n-3}$ and mutually disjoint geodesics $\gamma_1, \dots, \gamma_{n-4}$ in $\mathcal{GL}_n^{(n-1)}$ such that

$$(5) \quad \gamma = \bigoplus_{i=1}^{n-4} k_i \gamma_i \oplus k_{n-3} \gamma_{n-3}^1.$$

Let

$$(6) \quad \alpha_\gamma = \bigoplus_{i=1}^{n-4} k_i \gamma_i,$$

and we define

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\alpha_\gamma)).$$

In particular, if $\gamma \in \mathcal{GL}_n^{(n-1)}$, then $k_{n-3} = 0$, $\alpha_\gamma = \gamma$, and

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\gamma)).$$

(ii) If $I_{n-3}(\gamma) > 0$, and if γ has essential blocks, we define

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}(\Phi_n^{-1}(\alpha_\gamma)),$$

where $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ is given in Theorem 2.7.

(iii) If $I_{n-3}(\gamma) > 0$, and if γ has no essential blocks, we define

$$N_j^{(n)}(\gamma) = 0.$$

From (i), we know that $N_j^{(n-1)}$ is the restriction of $N_j^{(n)}$ to $\mathcal{GL}_n^{(n-1)} \equiv \mathcal{GL}_{n-1}$ for any two integers j and n with $1 < j \leq n - 4$. Note that $N_{n-4}^{(n-1)} = N_{\nu-3}^{(\nu)}$, where $\nu = n - 1$. From now on, we shall write $N_j^{(n)}$ as N_j for $1 \leq j \leq n - 3$.

Now, for an arbitrary $\gamma \in \mathcal{GL}_n$ and for an arbitrary integer j with $1 < j \leq n - 4$, we define

$$N_j(\gamma) = \begin{cases} N_j(\mathcal{T}_2^{-2}(\gamma)) & \text{if } \gamma \in \mathcal{GL}_n^+(T_{n-3}), \\ N_j(\mathcal{T}_2^2(\gamma)) & \text{if } \gamma \in \mathcal{GL}_n^-(T_{n-3}). \end{cases}$$

To prove that N_j is well-defined, we have to show that

$$N_j(\gamma) = \begin{cases} N_j(\mathcal{T}_2^{-2}(\gamma)) & \text{for all } \gamma \in \mathcal{GL}_n^+(T_{n-3}) \cap \mathcal{GL}_n^0, \\ N_j(\mathcal{T}_2^2(\gamma)) & \text{for all } \gamma \in \mathcal{GL}_n^-(T_{n-3}) \cap \mathcal{GL}_n^0. \end{cases}$$

Without loss of generality, we may assume that $\gamma \in \mathcal{G}_n^0$. There is nothing to prove if $I_{n-3}(\gamma) = 0$ since in this case $\mathcal{T}_2(\gamma) = \gamma$. Assume that $\gamma \in \mathcal{G}_n^-(T_{n-3})$ with $I_{n-3}(\gamma) = m > 0$. Then γ is represented by a cyclic reduced Γ_n -word as given in (3), say $W = \prod_{i=1}^m T_{n-3}^{-p_i} S_2^{\varepsilon_i} T_{n-3}^{q_i} W_i$ with $p_i > 0$ and $q_i > 0$ for all i . Since

$$\mathcal{T}_2^2(W) = \prod_{i=1}^m T_{n-3}^{-p_i-1} S_2^{\varepsilon_i} T_{n-3}^{q_i+1} W_i,$$

γ has essential blocks if and only if $\mathcal{T}_2^2(\gamma) = \tilde{\gamma}$ has essential blocks. Thus $N_j(\gamma) = 0 = N_j(\tilde{\gamma})$ whenever γ has no essential blocks. When γ has essential blocks, α_γ is completely determined by the subwords $T_{n-3}W_iT_{n-3}^{-1}$, $1 \leq i \leq m$, and so is $\alpha_{\tilde{\gamma}}$. This proves that $N_j(\gamma) = N_j(\tilde{\gamma})$ since $\alpha_\gamma = \alpha_{\tilde{\gamma}}$.

If $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^+(T_{n-3})$, then γ is represented by a cyclic reduced Γ_n -word as given in (4). A similar argument as above, one proves easily that $N_j(\gamma) = N_j(\mathcal{T}_2^{-2}(\gamma))$. Therefore, N_j is well-defined.

Note that since $N_{n-4}^{(n)} \equiv N_{\nu-3}^{(\nu)}$ with $\nu = n - 1$, from the definition of N_{n-3} , we may interpretate N_{n-4} geometrically. This gives N_j a geometric interpretation for every integer j with $1 < j \leq n - 4$. From Proposition 2.5, we assume that $\gamma \in \mathcal{G}\mathcal{L}_n^0$.

Let $\mathcal{G}\mathcal{L}_n^+(T_{n-4})$ be the set of all γ in $\mathcal{G}\mathcal{L}_n^0$ which satisfy either one of the following two conditions:

- (i) If $I_{n-3}(\gamma) = 0$, then γ has no strands connecting the T_{n-4} -side to the T_{n-3}^ε -side, where $\varepsilon = \pm 1$.
- (ii) If $I_{n-3}(\gamma) > 0$, then every essential block of γ has no strands connecting the T_{n-4} -side to the T_{n-3}^ε -side, where $\varepsilon = \pm 1$.

Let $\mathcal{G}\mathcal{L}_n^-(T_{n-4}) = \Theta_1(\mathcal{G}\mathcal{L}_n^+(T_{n-4}))$. If $\gamma = \bigoplus_{j=1}^{n-3} k_j \gamma_j^1$ with $(k_1, \dots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$, then

$$N_{n-4}^{(n)}(\gamma) = k_{n-4} = \#(\text{strands of } \gamma \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}).$$

Let $\varepsilon = \pm 1$. If $\gamma \in \mathcal{G}\mathcal{L}_n^+(T_{n-4}) \cap \widehat{\mathcal{G}\mathcal{L}}_n$ with $I_{n-3}(\gamma) = 0$, then

$$\begin{aligned} N_{n-4}^{(n)}(\gamma) &= \#(\text{strands of } \gamma \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } T_{n-3}^\varepsilon\text{-side to the } T_{n-4}^{-1}\text{-side}). \end{aligned}$$

If $\gamma \in \mathcal{G}\mathcal{L}_n^+(T_{n-4}) \cap \widehat{\mathcal{G}\mathcal{L}}_n$ with $I_{n-3}(\gamma) > 0$, then

$$(7) \quad N_{n-4}^{(n)}(\gamma) = \sum_{\gamma' \in \mathcal{E}} N_{n-4}^{(n)}(\gamma'),$$

where \mathcal{E} is the set of all essential blocks of γ , and where

$$\begin{aligned} N_{n-4}^{(n)}(\gamma') &= \#(\text{strands of } \gamma' \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma' \text{ joining the } T_{n-3}^\varepsilon\text{-side to the } T_{n-4}^{-1}\text{-side}) \end{aligned}$$

for $\gamma' \in \mathcal{E}$. When \mathcal{E} is empty, the integer on the right of (7) is defined to be zero. If $\gamma \in \mathcal{G}\mathcal{L}_n^-(T_{n-4}) \cap \widehat{\mathcal{G}\mathcal{L}}_n$, then $N_{n-4}^{(n)}(\gamma) = -N_{n-4}^{(n)}(\Theta_1(\gamma))$.

2.5. Elementary intersection formulas. This subsection is devoted to proving the main theorem:

Theorem 2.10 (Elementary intersection formulas). *For an arbitrary integer $n \geq 6$, if $\gamma \in \mathcal{GL}_n$, then*

$$\begin{aligned} i(\gamma, \gamma_1^2) &= 2|N_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma), \\ i(\gamma, \gamma_1^3) &= 2|N_1(\gamma) - I_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma), \\ i(\gamma, \gamma_{n-3}^2) &= 2|N_{n-3}(\gamma)| + |I_{n-4}(\gamma) - I_{n-3}(\gamma)| + I_{n-4}(\gamma) - I_{n-3}(\gamma), \\ i(\gamma, \gamma_{n-3}^3) &= 2|N_{n-3}(\gamma) - I_{n-3}(\gamma)| + |I_{n-4}(\gamma) - I_{n-3}(\gamma)| + I_{n-4}(\gamma) - I_{n-3}(\gamma), \end{aligned}$$

and for every integer j with $1 < j < n - 3$

$$\begin{aligned} i(\gamma, \gamma_j^2) &= 2|N_j(\gamma)| + |I_{j-1}(\gamma) - I_j(\gamma)| + I_{j-1}(\gamma) - I_j(\gamma) \\ &\quad + |I_{j+1}(\gamma) - I_j(\gamma)| + I_{j+1}(\gamma) - I_j(\gamma), \\ i(\gamma, \gamma_j^3) &= 2|N_j(\gamma) - I_j(\gamma)| + |I_{j-1}(\gamma) - I_j(\gamma)| + I_{j-1}(\gamma) - I_j(\gamma) \\ &\quad + |I_{j+1}(\gamma) - I_j(\gamma)| + I_{j+1}(\gamma) - I_j(\gamma). \end{aligned}$$

For the proof of Theorem 2.10, we need the following two immediate consequences of the definition of N_j .

Lemma 2.11. *If $\gamma \in \mathcal{GL}_n$, then $N_1(\gamma) = N_{n-3}(\Theta_2(\gamma))$.*

Lemma 2.12. *If $(k_1, \dots, k_{n-3}) \in \mathbb{Z}_+^{n-3}$ and $(\gamma_1, \dots, \gamma_{n-3}) \in \Lambda_n^{n-3}$, then*

$$N_j \left(\bigoplus_{i=1}^{n-3} k_i \gamma_i \right) = \sum_{j=i}^{n-3} k_i N_j(\gamma_j) \quad \text{for every integer } j \text{ with } 1 \leq j \leq n - 3.$$

For $k = 2$ or 3 , the elementary intersection numbers $i(\gamma, \gamma_1^k)$ and $i(\gamma, \gamma_{n-3}^k)$ are related as follows:

$$i(\gamma, \gamma_{n-3}^k) = i(\Theta_2(\gamma), \Theta_2(\gamma_{n-3}^k)) = i(\Theta_2(\gamma), \gamma_1^k).$$

From Proposition 2.1, we obtain $I_1(\Theta_2(\gamma)) = I_{n-3}(\gamma)$ and $I_2(\Theta_2(\gamma)) = I_{n-4}(\gamma)$. Now, by Lemma 2.11, the elementary intersection formulas for $i(\gamma, \gamma_{n-3}^2)$ and $i(\gamma, \gamma_{n-3}^3)$ follow immediately from those for $i(\gamma, \gamma_1^2)$ and $i(\gamma, \gamma_1^3)$.

On the other hand, $i(\gamma, \gamma_1^3) = i(\mathcal{T}_1(\gamma), \gamma_1^2)$ since $\gamma_1^3 = \mathcal{T}_1^{-1}(\gamma_1^2)$. Thus, by Proposition 2.3, one derives easily the elementary intersection formula for $i(\gamma, \gamma_1^3)$ from that for $i(\gamma, \gamma_1^2)$ if

$$N_1(\mathcal{T}_1(\gamma)) = N_1(\gamma) - I_1(\gamma).$$

By use of the word given in (3), one proves easily the following more general results by a similar argument as that in [4, Proposition 2.8].

Lemma 2.13. *Let $\gamma \in \mathcal{GL}_n$, and let ν be an arbitrary integer. Then*

$$\begin{aligned} N_1(\mathcal{T}_2^\nu(\gamma)) &= N_1(\gamma), & N_1(\mathcal{T}_1^\nu(\gamma)) &= N_1(\gamma) - \nu I_1(\gamma), \\ N_{n-3}(\mathcal{T}_1^\nu(\gamma)) &= N_{n-3}(\gamma), & N_{n-3}(\mathcal{T}_2^\nu(\gamma)) &= N_{n-3}(\gamma) - \nu I_{n-3}(\gamma). \end{aligned}$$

For the proof of Theorem 2.10, it remains to prove the elementary intersection formulas for $i(\gamma, \gamma_1^2)$, $i(\gamma, \gamma_j^2)$ and $i(\gamma, \gamma_j^3)$ for $1 < j < n - 3$.

First, we prove the elementary intersection formula for $i(\gamma, \gamma_1^2)$ by applying induction to n for $n \geq 5$. For the case of $n = 5$, the assertion is proved in [4, Corollary 3.4]. Assume that $n > 5$, and that the equation holds for $\gamma \in \mathcal{GL}_n^{(n-1)}$.

Now, let $\gamma \in \mathcal{GL}_n$. If $I_{n-3}(\gamma) = 0$, write γ as given in (5), and let $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ be given in (6). By the definition, $N_1(\gamma) = N_1(\alpha_\gamma)$. Since $i(\gamma_{n-3}^1, \beta) = 0$ for $\beta \in \mathcal{GL}_n^{(n-1)}$, then $I_j(\gamma) = I_j(\alpha_\gamma)$ for $j = 1, 2$. The assertion follows for the case since $i(\gamma, \gamma_1^2) = i(\alpha_\gamma, \gamma_1^2)$.

Assume that $I_{n-3}(\gamma) > 0$. Since $i(\gamma, \gamma_1^2) = i(\Theta_1(\gamma), \gamma_1^2)$, we may assume that $\gamma \in \mathcal{GL}_n^+(T_{n-3})$. Moreover, by considering $\mathcal{T}_2^{-2}(\gamma)$, from Proposition 2.4, Proposition 2.5 and Lemma 2.13 we may assume that $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$.

If γ has no essential blocks, we have $I_1(\gamma) = I_2(\gamma) = 0 = i(\gamma, \gamma_1^2)$. By the definition of N_1 , we have $N_1(\gamma) = 0$ since $I_1(\gamma) = 0$. Now, the intersection formula for $i(\gamma, \gamma_1^2)$ holds trivially in this case.

If γ has essential blocks, then $I_j(\alpha_\gamma) = I_j(\gamma)$ for $j = 1, 2$, and $i(\alpha_\gamma, \gamma_1^2) = i(\gamma, \gamma_1^2)$, where α_γ is given in Theorem 2.7. Note that $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ and $N_1(\gamma) = N_1(\alpha_\gamma)$. The proof of the intersection formula for $i(\gamma, \gamma_1^2)$ is then completed by induction hypothesis.

In the rest of this subsection, we prove the intersection formulas for $i(\gamma, \gamma_j^2)$ and $i(\gamma, \gamma_j^3)$ with $1 < j < n - 3$, by applying induction to $n \geq 6$. If $n = 6$, then the formulas are exactly the same as given below.

Lemma 2.14. *If $n \geq 6$, and if $\gamma \in \mathcal{GL}_n$, then*

$$\begin{aligned} i(\gamma, \gamma_{n-4}^2) &= 2|N_{n-4}(\gamma)| + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \\ i(\gamma, \gamma_{n-4}^3) &= 2|N_{n-4}(\gamma) - I_{n-4}(\gamma)| \\ &\quad + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \end{aligned}$$

With Lemma 2.14, we first complete induction step as follows. Assume that $n > 6$. From Lemma 2.14, we may assume that $1 < j < n - 4$. If $I_{n-3}(\gamma) = 0$, then we write γ and $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$, respectively, as in (5) and (6). Since $N_j(\alpha_\gamma) = N_j(\gamma)$ and $I_k(\alpha_\gamma) = I_k(\gamma)$ for $0 < j - 1 \leq k \leq j < n - 4$, the assertions hold for this case by induction hypothesis.

Assume that $I_{n-3}(\gamma) > 0$. If γ has no essential blocks, then $I_j(\gamma) = 0 = N_j(\gamma)$ for $1 < j < n - 3$, and $i(\gamma, \gamma_j^k) = 0$ for $1 < j < n - 4$ and for $k = 2, 3$. If γ has essential blocks, we may assume that $\gamma \in \mathcal{GL}_n^0$. Let $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ be given in Theorem 2.7. By the induction hypothesis again, the proof is complete.

For the proof of Lemma 2.14, we need:

Lemma 2.15. *If $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$, then γ has exactly*

$$\frac{|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma)}{2}$$

puncture-like blocks.

Proof. Without loss of generality, we assume that $\gamma \in \mathcal{GL}_n^+(T_{n-3})$. Let \mathcal{E} denote the set of all essential blocks of γ . If γ' is a puncture-like block of γ , then $i(\gamma', \gamma_{n-4}^1) = 0$ and $2I_{n-4}(\gamma) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n}$.

Let $I_{n-3}(\gamma) = m$, and let $p \geq 0$ be the number of puncture-like blocks of γ . Then γ has exactly $e = m - p$ essential blocks of the first kind. If $p = 0$, then $2I_{n-4}(\gamma) \geq \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} \geq 2m = 2I_{n-3}(\gamma)$, and $|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 0 = 2p$.

Now, assume that $p > 0$. It follows from Lemma 2.9 that

$$i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = \begin{cases} 2 & \text{if } \gamma' \text{ is an essential block of } \gamma \text{ of the first kind,} \\ 0 & \text{if } \gamma' \text{ is an essential block of } \gamma \text{ of the second kind.} \end{cases}$$

If $p = m$, then γ has no essential blocks of the first kind, and

$$2I_{n-4}(\gamma) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = 0.$$

Thus $|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 2m = 2p$.

If $0 < p < m$, let $\gamma_1, \dots, \gamma_e$ be the essential blocks of γ of the first kind. Then $2I_{n-4}(\gamma) = \sum_{j=1}^e i(\gamma_j, \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = 2e = 2I_{n-3}(\gamma) - 2p$, and

$$2p = 2\{I_{n-3}(\gamma) - I_{n-4}(\gamma)\} = |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma). \quad \square$$

Proof of Lemma 2.14. It suffices to prove the lemma for $\gamma \in \mathcal{G}_n$. We shall prove the lemma for $\gamma \in \mathcal{G}_n^+(T_{n-3})$. By a similar argument, one proves the lemma for $\gamma \in \mathcal{G}_n^-(T_{n-3})$.

If $\gamma \in \mathcal{G}_n^+(T_{n-3})$, then $N_{n-4}(\mathcal{T}^{-2}(\gamma)) = N_{n-4}(\gamma)$ by the definition of N_{n-4} . Note that $i(\gamma, \gamma_{n-4}^k) = i(\mathcal{T}_2^{-2}(\gamma), \gamma_{n-4}^k)$ for $k = 2, 3$, and that $2I_j(\gamma) = 2I_j(\mathcal{T}_2^{-2}(\gamma))$ for $n - 5 \leq j \leq n - 3$. By Proposition 2.5, we may assume that $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^+(T_{n-3})$.

If $I_{n-3}(\gamma) = 0$, then $\gamma \in \mathcal{G}_n^{(n-1)}$, and

$$|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 0.$$

By letting $\nu = n - 1$, we have

$$\begin{aligned} i(\gamma, \gamma_{n-4}^2) &= 2|N_{\nu-3}(\gamma)| + |I_{\nu-4}(\gamma) - I_{\nu-3}(\gamma)| + I_{\nu-4}(\gamma) - I_{\nu-3}(\gamma) \\ &= 2|N_{n-4}(\gamma)| + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \end{aligned}$$

Similarly, we obtain the intersection formula for $i(\gamma, \gamma_{n-4}^3)$.

If $I_{n-3}(\gamma) = m > 0$, then γ is represented by a cyclic reduced Γ_n -word W as given in (3). Note that $p_j > 0$ and $q_j > 0$ for $1 \leq j \leq m$. For every j , let γ_j be the block of γ represented by $\vec{T}_{n-3}^{-1}W_jT_{n-3}$, and let $\beta(\gamma_j)$ be the admissible subarc of γ represented by

$$\vec{T}_{n-3}T_{n-3}^{p_j-1}S_2^{\epsilon_j}T_{n-3}^{-q_j}W_jT_{n-3}.$$

Note that every γ_j is a subarc of $\beta(\gamma_j)$, and that $i(\beta(\gamma_j), \gamma_{n-4}^k) = 2$ for $k = 2$ or 3 whenever γ_j is puncture-like. Let \mathcal{E} be the set of all essential blocks of γ . From Lemma 2.15, we have, for $k = 2$ or 3 ,

$$i(\gamma, \gamma_{n-4}^k) = |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) + \sum_{\gamma_j \in \mathcal{E}} i(\beta(\gamma_j), \gamma_{n-4}^k)_{\partial \mathcal{D}_n}.$$

If γ has no essential blocks, then the lemma holds trivially for γ since $I_{n-3}(\gamma) = I_{n-4}(\gamma) = N_{n-4}(\gamma) = 0$.

Now, assume that \mathcal{E} is not empty. Note that every essential block of γ is of the first kind since $\gamma \in \mathcal{G}_n$. Let \mathcal{L} be the union of all strands of γ which connect the T_{n-3}^{-1} -side to the X -side with $X \in \{T_{n-3}, S_2, S_2^{-1}\}$.

For $k = 2$ or 3 , each γ_{n-4}^k has a unique strand l_k meeting the T_{n-3}^{-1} -side. Let Q'_k be the endpoint of l_k lying on the T_{n-3}^{-1} -side, and let Q_k be the point on the T_{n-3} -side which is identified with Q'_k by the transformation T_{n-3}^{-1} .

Since $i(\gamma_{n-4}^k, \gamma_{n-3}^1) = 0$, we may assume that l_k is disjoint from \mathcal{L} . This implies that $Q'_k \prec Q'$ whenever Q' is an endpoint of some strand in \mathcal{L} meeting the T_{n-3}^{-1} -side, and that $Q_k \prec Q$ whenever Q is the endpoint of some strand of γ lying on the T_{n-3} -side. Thus, we have

$$\sum_{\gamma_j \in \mathcal{E}} i(\beta(\gamma_j), \gamma_{n-4}^k)_{\partial \mathcal{D}_n} = \sum_{\gamma_j \in \mathcal{E}} i(\gamma_j, \gamma_{n-4}^k)_{\partial \mathcal{D}_n} = i(\alpha_\gamma, \gamma_{n-4}^k),$$

where $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ is given in Theorem 2.7. By letting $\nu = n - 1$, we obtain

$$\begin{aligned} i(\alpha_\gamma, \gamma_{n-4}^2) &= 2|N_{\nu-3}(\alpha_\gamma)| + |I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| \\ &\quad + I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma) \\ i(\alpha_\gamma, \gamma_{n-4}^3) &= 2|N_{\nu-3}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| + |I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| \\ &\quad + I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma) \end{aligned}$$

The proof of Lemma 2.14 is complete. □

3. A Mapping of $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ into a Sphere

In this section, we construct a continuous mapping Ψ from $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ into $\mathbb{R}^{3(n-3)}$ whose image set is a sphere of dimension $2n - 7$. The mapping Ψ will be constructed in a similar way as that given in [4] for the case of $n = 5$. We shall first define the restriction of Ψ on \mathcal{GL}_n homogeneously, and extend it to $\pi^{-1}\pi\mathcal{I}(\mathcal{G}_n)$. Note that $\overline{\pi\mathcal{I}(\mathcal{G}_n)} = \overline{\pi\mathcal{I}(\mathcal{GL}_n)}$. By a continuity argument as in [4, §4.3], one proves that Ψ extends continuously to $\pi^{-1}\overline{\pi\mathcal{I}(\mathcal{G}_n)}$. Since the restriction π to $\pi^{-1}\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ is a quotient map, the required continuous mapping Ψ is then obtained.

For the definition of Ψ on \mathcal{GL}_n , we first construct a function ψ_0 from \mathcal{GL}_n into $\mathbb{R}^{3(n-3)}$ whose values are written in terms of elementary intersection numbers. For every $\gamma \in \mathcal{GL}_n$, we write

$$\psi_0(\gamma) = (x_1^1(\gamma), x_1^2(\gamma), x_1^3(\gamma), \dots, x_{n-3}^1(\gamma), x_{n-3}^2(\gamma), x_{n-3}^3(\gamma)),$$

where $x_j^k(\gamma) = i(\gamma, \gamma_j^k)/\lambda(\gamma)$ for $1 \leq j \leq n - 3$ and for $1 \leq k \leq 3$, and where $\lambda(\gamma) = \sum_{j=1}^{n-3} \sum_{k=1}^3 i(\gamma, \gamma_j^k)$. Note that the image of ψ_0 lies in

$$\Pi' = \Pi \cap \left\{ (t_1, t_2, \dots, t_{3(n-3)}) \in \mathbb{R}^{3(n-3)} : 1 - 2 \sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}| > 0 \right\},$$

where $\Pi = \{(t_1, t_2, \dots, t_{3(n-3)}) \in \mathbb{R}^{3(n-3)} : \sum_{j=1}^{3(n-3)} t_j = 1\}$. For later use, we define the function $f: \mathbb{R}^{3(n-3)} \rightarrow \mathbb{R}$ by

$$f(t_1, t_2, \dots, t_{3(n-3)}) = 1 - 2 \sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}|.$$

Following [4], we define the mapping $\Psi: \mathcal{GL}_n \rightarrow \mathbb{R}^{3(n-3)}$ by

$$\Psi(\gamma) = (\xi_1^1(\gamma), \xi_1^2(\gamma), \xi_1^3(\gamma), \dots, \xi_{n-3}^1(\gamma), \xi_{n-3}^2(\gamma), \xi_{n-3}^3(\gamma)),$$

where for every $1 \leq j \leq n - 3$

$$\xi_j^1(\gamma) = \frac{2I_j(\gamma)}{\rho(\gamma)}, \quad \xi_j^2(\gamma) = \frac{2|N_j(\gamma)|}{\rho(\gamma)} \quad \text{and} \quad \xi_j^3(\gamma) = \frac{2|N_j(\gamma) - I_j(\gamma)|}{\rho(\gamma)},$$

and $\rho(\gamma) = 2 \sum_{j=1}^{n-3} \{I_j(\gamma) + |N_j(\gamma)| + |N_j(\gamma) - I_j(\gamma)|\}$. It is easy to see that $\Psi(\gamma) \in \Delta_n = \mathcal{C}^{n-3} \cap \Pi$ for every $\gamma \in \mathcal{GL}_n$, where \mathcal{C} is the set of points $(t_1, t_2, t_3) \in \mathbb{R}_+^3$ satisfying:

$$t_2 + t_3 = t_1, \quad t_1 + t_3 = t_2, \quad \text{or} \quad t_1 + t_2 = t_3.$$

A similar argument to that given in [4, §4.2] proves that Δ_n is homeomorphic to a sphere of dimension $2n - 7$.

We shall prove that there is a homeomorphism ψ_1 of Π' onto Π so that $\Psi = \psi_1 \circ \psi_0$. Then we obtain:

Theorem 3.1. *The function Ψ extends to $\overline{\pi\mathcal{I}(\mathcal{G}_n)} = \overline{\pi\mathcal{I}(\mathcal{GL}_n)}$ as a continuous mapping into a sphere of dimension $2n - 7$.*

It remains to construct the mapping ψ_1 . For $\gamma \in \mathcal{GL}_n$, let

$$\nu(\gamma) = 1 - \frac{4}{\lambda(\gamma)} \sum_{j=1}^{n-4} |I_j(\gamma) - I_{j+1}(\gamma)| = 1 - 2 \sum_{j=1}^{n-4} |x_j^1(\gamma) - x_{j+1}^1(\gamma)|.$$

A direct computation shows that $\rho(\gamma) = \lambda(\gamma)\nu(\gamma)$, and the followings:

$$\begin{aligned} \xi_j^1(\gamma) &= \frac{x_j^1(\gamma)}{\nu(\gamma)} \quad \text{for } 1 \leq j \leq n-3, \\ \xi_1^2(\gamma) &= \frac{x_1^2(\gamma)}{\nu(\gamma)} - \frac{|x_2^1(\gamma) - x_1^1(\gamma)| + \{x_2^1(\gamma) - x_1^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_1^3(\gamma) &= \frac{x_1^3(\gamma)}{\nu(\gamma)} - \frac{|x_2^1(\gamma) - x_1^1(\gamma)| + \{x_2^1(\gamma) - x_1^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_{n-3}^2(\gamma) &= \frac{x_{n-3}^2(\gamma)}{\nu(\gamma)} - \frac{|x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)| + \{x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_{n-3}^3(\gamma) &= \frac{x_{n-3}^3(\gamma)}{\nu(\gamma)} - \frac{|x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)| + \{x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)\}}{2\nu(\gamma)}, \end{aligned}$$

and for $1 < j < n-3$

$$\begin{aligned} \xi_j^2(\gamma) &= \frac{x_j^2(\gamma)}{\nu(\gamma)} - \frac{|x_{j-1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j-1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)} \\ &\quad - \frac{|x_{j+1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j+1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)}, \\ \xi_j^3(\gamma) &= \frac{x_j^3(\gamma)}{\nu(\gamma)} - \frac{|x_{j-1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j-1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)} \\ &\quad - \frac{|x_{j+1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j+1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)}. \end{aligned}$$

The above equations motivate the function $\psi_1: \Pi' \rightarrow \mathbb{R}^{3(n-3)}$ defined by

$\psi_1(r_1, r_2, \dots, r_{3(n-3)}) = (t_1, t_2, \dots, t_{3(n-3)})$, where

$$t_j = \frac{r_j}{f(r_1, r_2, \dots, r_{3(n-3)})} \quad \text{for } j = 3k - 2 \text{ with } 1 \leq k \leq n - 3$$

$$t_j = \frac{r_j}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_4 - r_1| + (r_4 - r_1)}{2f(r_1, r_2, \dots, r_{3(n-3)})} \quad \text{for } j = 2, 3,$$

$$t_j = \frac{r_j}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_{3n-14} - r_{3n-11}| + (r_{3n-14} - r_{3n-11})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

for $j = 3n - 10$ or $3(n - 3)$, and

$$t_{3k-1} = \frac{r_{3k-1}}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_{3k-5} - r_{3k-2}| + (r_{3k-5} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

$$- \frac{|r_{3k+1} - r_{3k-2}| + (r_{3k+1} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

$$t_{3k} = \frac{r_{3k}}{f(r_1, r_2, \dots, r_{3(n-3)})} - \frac{|r_{3k-5} - r_{3k-2}| + (r_{3k-5} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})}$$

$$- \frac{|r_{3k+1} - r_{3k-2}| + (r_{3k+1} - r_{3k-2})}{2f(r_1, r_2, \dots, r_{3(n-3)})} \quad \text{for } 1 < k < n - 3.$$

A direct computation proves that ψ_1 maps Π' into Π by showing that

$$\sum_{j=1}^{3(n-3)} t_j = 1 \quad \text{and} \quad 1 + 2 \sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}| = \frac{1}{f(r_1, r_2, \dots, r_{3(n-3)})}.$$

From the definition of ψ_1 , one proves easily that ψ_1 is indeed a homeomorphism of Π' onto Π .

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