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Linear-Order on a Group

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Let us discuss here under what condition a group admits a linear-order.¹⁾ Related ideas to my previous paper²⁾ will be adopted.

Preliminaries. A partial-order on a group G is wholly determined by such a subset g of G — we shall call it a (partial-) *ordering set* briefly — that satisfies the following two conditions :

- 1) g is an invariant sub-semigroup with 1,
- 2) g cannot contain an element ($\neq 1$) together with its inverse.

A linear-ordering set is therefore characterized by one more additional condition :

- 3) It contains either x or x^{-1} for any x of G .

For a finite subset $\{x_1, \dots, x_n\}$ of G and an invariant sub-semigroup g of G the invariant sub-semigroup generated by x_1, \dots, x_n and g shall be denoted by $g(x_1, \dots, x_n)$. Especially the invariant sub-semigroup generated by $\{x_1, \dots, x_n\}$ alone is (x_1, \dots, x_n) .

Theorem. *The following three conditions are mutually equivalent :*

- (I) G admits a linear-order.
- (II) For any finite subset $\{x_1, \dots, x_n\}$ of G the intersection of all possible $2^n g(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$, where $\varepsilon_i = \pm 1$, is equal to 1.
- (III) For any element a of G there exists an ordering set g_a containing a and having the property :

- (*) If $xy (\neq 1)$ belongs to g_a , then either x or y belongs to g_a .

Such an ordering set in (III) will be called (*)-ordering set.

Proof. We shall divide this into three parts :

(I) \rightarrow (II). By a linear-order on G every element x of G attains a sign $\varepsilon^0 = \pm 1$ in such a way that x^{ε^0} is ≥ 1 with respect to this order. Then obviously all elements of $(x_1^{\varepsilon_1^0}, \dots, x_n^{\varepsilon_n^0})$ are ≥ 1 , and all elements of $(x_1^{-\varepsilon_1^0}, \dots, x_n^{-\varepsilon_n^0})$ are ≤ 1 . Therefore the intersection of these two sets is already equal to 1.

1) Cf. K. Iwasawa, On linearly ordered groups. Journ. of Math. Soc. of Japan, 1 (1948).

Also, P. Lorenzen, Ueber halbgeordnete Gruppen. Math. Zeits. 52 (1949).

2) M. Ohnishi, On linearization of ordered groups. Osaka Math. Journ. 2 (1950).

(II)→(III). Let us consider the family of all subsets g of G which have the properties :

(1) g is an invariant sub-semigroup with 1,

(2) For any finite subset $\{x_1, \dots, x_n\}$ of G the intersection of all possible $2^n g(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$, where $\varepsilon_i = \pm 1$, does not contain a .

This family being not empty by our assumption (II), and these above properties being clearly of finite character,³⁾ Zorn's lemma ascertains that there exists a maximal set g' with respect to them. Then g' contains either x or x^{-1} for any x of G . In fact, if neither $x(\neq 1)$ nor x^{-1} belongs to g' , then the invariant sub-semigroups $g'(x)$ and $g'(x^{-1})$ contain the maximal set g' properly, hence they cannot satisfy the above property (2); in other words there exist some finite subsets $\{y_1, \dots, y_k\}$ and $\{z_1, \dots, z_s\}$, and the intersection of all $g'(x, y_1^{\pm 1}, \dots, y_k^{\pm 1})$ and that of all $g'((x^{-1}, z_1^{\pm 1}, \dots, z_s^{\pm 1}))$ contain the element a .

Consequently the intersection of all $g'(x^{\pm 1}, y_1^{\pm 1}, \dots, y_k^{\pm 1}, z_1^{\pm 1}, \dots, z_s^{\pm 1})$ contains a , which contradicts the above property (2) of g' . Thus g' containing either x or x^{-1} for any x , its complement $g_a = G - g'$ proves, as is easily seen, to be the desired $(*)$ -ordering set containing the element a .

Finally (III)→(I). Again considering the family of all $(*)$ -ordering sets, there also exists a maximal $(*)$ -ordering set g_0 by applying of Zorn's lemma. We must show that g_0 is really a *linear-ordering* set. Let us now assume that g_0 contains neither a nor a^{-1} for some $a(\neq 1)$ of G . (III) assures us that we can find a $(*)$ -ordering set g_a containing a .

The set $g_0 + g_a - g_0^{-1}$ (here $+$ and $-$ are the usual set-operations, and g_0^{-1} is the set composed of the inverses of g_0), or what is the same, $g_0 + \{x \in g_a; x \notin g_0, x^{-1} \notin g_0\}$, obviously contains a , and by rather easy computations we know that this set is also a $(*)$ -ordering set, which contradicts the maximal property of g_0 , hence g_0 is a linear-ordering set, and G admits a linear order.

These three parts complete the proof of our theorem.

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3) Cf. J. Tukey, *Convergence and Uniformity in Topology*, Princeton Univ. Press, (1940), pp. 7-8.