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Linear-Order on a Group

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Let us discuss here under what condition a group admits a linear-order.[1] Related ideas to my previous paper[2] will be adopted.

Preliminaries. A partial-order on a group G is wholly determined by such a subset $\mathfrak{g}$ of G — we shall call it a (partial-) *ordering set* briefly — that satisfies the following two conditions:

1) $\mathfrak{g}$ is an invariant sub-semigroup with 1,

2) $\mathfrak{g}$ cannot contain an element $(\neq 1)$ together with its inverse.

A linear-ordering set is therefore characterized by one more additional condition:

3) It contains either x or x^{-1} for any x of G.

For a finite subset $\{x_1, \dots, x_n\}$ of G and an invariant sub-semigroup $\mathfrak{g}$ of G the invariant sub-semigroup generated by $x_1, \dots, x_n$ and $\mathfrak{g}$ shall be denoted by $\mathfrak{g}(x_1, \dots, x_n)$. Especially the invariant sub-semigroup generated by $\{x_1, \dots, x_n\}$ alone is $(x_1, \dots, x_n)$.

Theorem. *The following three conditions are mutually equivalent:*

(I) *G admits a linear-order.*

(II) *For any finite subset $\{x_1, \dots, x_n\}$ of G the intersection of all possible 2^n $\mathfrak{g}(x_1^{\varepsilon_1}, \dots x_n^{\varepsilon_n})$, where $\varepsilon_i = \pm 1$, is equal to 1.*

(III) *For any element a of G there exists an ordering set $\mathfrak{g}_a$ containing a and having the property:*

($*$) *If $xy (\neq 1)$ belongs to $\mathfrak{g}_a$, then either x or y belongs to $\mathfrak{g}_a$.*

Such an ordering set in (III) will be called ($*$)-ordering set.

Proof. We shall divide this into three parts:

(I)→(II). By a linear-order on G every element x of G attains a sign $\varepsilon^0 = \pm 1$ in such a way that x^{ε_0} is $\geqq 1$ with respect to this order. Then obviously all elements of $(x_1^{\varepsilon_1^0}, \dots, x_n^{\varepsilon_n^0})$ are $\geqq 1$, and all elements of $(x_1^{-\varepsilon_1^0}, \dots, x_n^{-\varepsilon_n^0})$ are $\leqq 1$. Therefore the intersection of these two sets is already equal to 1.

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1) Cf. K. Iwasawa, On linearly ordered groups. Journ. of Math. Soc. of Japan, 1 (1948).

Also, P. Lorenzen, Ueber halbgeordnete Gruppen. Math. Zeits. 52 (1949).

2) M. Ohnishi, On linearization of ordered groups. Osaka Math. Journ. 2 (1950).

(II)→(III). Let us consider the family of all subsets $\mathfrak{g}$ of G which have the properties:

(1) $\mathfrak{g}$ is an invariant sub-semigroup with 1,

(2) For any finite subset $\{x_1, \dots, x_n\}$ of G the intersection of all possible 2^n $\mathfrak{g}(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$, where $\varepsilon_i = \pm 1$, does not contain a.

This family being not empty by our assumption (II), and these above properties being clearly of finite character,[3] Zorn's lemma ascertains that there exists a maximal set $\mathfrak{g}'$ with respect to them. Then $\mathfrak{g}'$ contains either x or x^{-1} for any x of G. In fact, if neither $x(\neq 1)$ nor x^{-1} belongs to $\mathfrak{g}'$, then the invariant sub-semigroups $\mathfrak{g}'(x)$ and $\mathfrak{g}'(x^{-1})$ contain the maximal set $\mathfrak{g}'$ properly, hence they cannot satisfy the above property (2); in other words there exist some finite subsets $\{y_1, \dots, y_k\}$ and $\{z_1, \dots, z_s\}$, and the intersection of all $\mathfrak{g}'(x, y_1^{\pm 1}, \dots, y_k^{\pm 1})$ and that of all $\mathfrak{g}'((x^{-1}, z_1^{\pm 1}, \dots, z_s^{\pm 1})$ contain the element a.

Consequently the intersection of all $\mathfrak{g}'(x^{\pm 1}, y_1^{\pm 1}, \dots, y_k^{\pm 1}, z_1^{\pm 1}, \dots, z_s^{\pm 1})$ contains a, which contradicts the above property (2) of $\mathfrak{g}'$. Thus $\mathfrak{g}'$ containing either x or x^{-1} for any x, its complement $\mathfrak{g}_a = G - \mathfrak{g}'$ proves, as is easily seen, to be the desired $(*)$-ordering set containing the element a.

Finally (III)→(I). Again considering the family of all $(*)$-ordering sets, there also exists a maximal $(*)$-ordering set $\mathfrak{g}_0$ by applying of Zorn's lemma. We must show that $\mathfrak{g}_0$ is really a *linear*-ordering set. Let us now assume that $\mathfrak{g}_0$ contains neither a nor a^{-1} for some $a(\neq 1)$ of G. (III) assures us that we can find a $(*)$-ordering set $\mathfrak{g}_a$ containing a.

The set $\mathfrak{g}_0 + \mathfrak{g}_a - \mathfrak{g}_0^{-1}$ (here + and − are the usual set-operations, and $\mathfrak{g}_0^{-1}$ is the set composed of the inverses of $\mathfrak{g}_0$), or what is the same, $\mathfrak{g}_0 + \{x \in \mathfrak{g}_a ;\ x \notin \mathfrak{g}_0,\ x^{-1} \notin \mathfrak{g}_0\}$, obviously contains a, and by rather easy computations we know that this set is also a $(*)$-ordering set, which contradicts the maximal property of $\mathfrak{g}_0$, hence $\mathfrak{g}_0$ is a linear-ordering set, and G admits a linear order.

These three parts complete the proof of our theorem.

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3) Cf. J. Tukey, Convergence and Uniformity in Topology, Princeton Univ. Press. (1940), pp. 7-8.