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# GLOBAL EXISTENCE OF SOLUTIONS TO AN $n$ -DIMENSIONAL PARABOLIC-PARABOLIC SYSTEM FOR CHEMOTAXIS WITH LOGISTIC-TYPE GROWTH AND SUPERLINEAR PRODUCTION

Dedicated to Professor Masayasu Mimura on the occasion of his 75th birthday.

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## Abstract

We study the global existence of solutions to an  $n$ -dimensional parabolic-parabolic system for chemotaxis with logistic-type growth. We introduce superlinear production of a chemoattractant. We then show the global existence of solutions in  $L_p$  space ( $p > n$ ) under certain relations between the degradation and production orders.

## 1. Introduction

In the present paper we study a chemotaxis system with logistic growth:

$$(E) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & \text{in } \Omega \times (0, \infty), \\ \tau \frac{\partial v}{\partial t} = \Delta v - v + g(u) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases}$$

Here,  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary  $\partial \Omega$ , and the space dimension  $n \in \mathbb{N}$  is an arbitrary positive integer. The unknown functions  $u(x, t)$  and  $v(x, t)$  are the population density of bacteria and the concentration of a chemical substance at the position  $x$  and time  $t$ , respectively. The term  $-\chi \nabla \cdot (u \nabla v)$  expresses the advection of bacteria due to chemotaxis. The coefficient  $\chi$  is a positive constant, which shows chemotactic intensity. The function  $f(u)$  is the proliferation and the reduction in numbers due to death of bacteria (we refer to the combined effects of proliferation and reduction in numbers simply as growth). Typical  $f(u)$ 's are quadratic  $u(1 - u)$  and cubic  $u(1 - u)(u - \gamma)$ ,  $0 < \gamma < 1/2$ , logistic growth functions [12]. The coefficient  $\tau$  is a positive constant, which shows the time scale of reaction and diffusion of  $v$ . The function  $g(u)$  is the secretion of chemical substance  $v$  by bacteria. A typical  $g(u)$  is a linear function; and some nonlinear forms of  $g(u)$  have been proposed, such as the saturating function  $u/(1 + \gamma u)$ , as used in the nonlinear signal kinetics

model. For these topics, see the book by Murray [15], and the review articles by Hillen and Painter [4] and by Tindall, Maini, Porter and Armitage [23].

We consider the global existence of solutions to (E). In the context of global existence, the degradation of the growth  $f(u)$  can be considered as an inhibitory effect on the increase of  $u$ . Indeed, if there is no growth ( $f(u) \equiv 0$ ) and the production  $g(u)$  is linear, then the system (E) reduces to the classical parabolic-parabolic Keller-Segel system [10]. In the Keller-Segel system, it is known that when  $n = 2$ , a finite-time blow-up with a  $\delta$ -function singularity of  $u$  occurs if  $\chi \|u_0\|_{L^1}$  is sufficiently large [3, 7]. In contrast, when  $n \geq 3$ , no restriction on  $\chi$  and  $\|u_0\|_{L^1}$  is necessary for the occurrence of blow-up [26]. For other topics on the Keller-Segel system, see Horstmann's review papers [5, 6] and the references therein. On the contrary, if  $f(u)$  is quadratic and  $g(u)$  is linear, then blow-up does not occur and global existence of solutions is assured even if  $\|u_0\|_{L^1}$  and  $\chi$  are large. This has been shown for  $n = 2$  by one of the authors et al. [19] and for  $n \geq 1$  with convex  $\Omega$  and large  $\mu$  by Winkler [27]. See also the recent related works [1, 11, 14].

We henceforth assume that the function  $f(u)$  is a real, smooth function of  $u \in [0, \infty)$  such that  $f(0) \geq 0$  and

$$f(u) = u - \mu u^\alpha \quad \text{for sufficiently large } u \geq 0;$$

and the function  $g(u)$  is given by

$$g(u) = u(1 + u)^{\beta-1} \quad \text{for } u \geq 0,$$

where the exponents  $\alpha$  and  $\beta$  satisfy the relations

$$(1) \quad \alpha > 1 \quad \text{and} \quad 0 < \beta \leq 2,$$

and  $\mu$  is a positive constant. From the results quoted above, we find that in the  $n$ -dimensional domain ( $n \geq 2$ ), a blow-up can occur when  $\alpha = 1$  and  $\beta = 1$  with a special choice of  $\mu = 1$ , and the blow-up of solutions is prevented and the global existence of solutions is assured when  $\alpha = 2$  and  $\beta = 1$ . We can then conjecture that the critical degradation order  $\alpha_{\text{cr}}$  is in the interval  $1 \leq \alpha_{\text{cr}} \leq 2$  under linear production  $\beta = 1$ ; however, it has not been determined for the parabolic-parabolic chemotaxis-growth system (E). Recently, Xiang [29] showed global existence of solutions under  $\beta = 1$  when  $\alpha > 19/9$  if  $n = 3$  and when  $\alpha > n - 1$  if  $n > 3$ .

In the two- and three-dimensional cases, the authors [16, 17] introduced sublinear production order  $\beta < 1$ , and showed a sufficient condition  $2(n + 4)/(n + 6) < \alpha \leq 2$  and  $0 < \beta < (n + 6)(\alpha - 1)/[2(n + 2)]$  for the existence of global and bounded solutions to (E) in a Hilbert space  $H_2^{(n/2)-1}(\Omega) \times H_2^{(n/2)+\varepsilon}(\Omega) \subset L_n(\Omega) \times C(\bar{\Omega})$  (their results would include Xiang's results [29] when  $n = 3$  if the existence of local solutions were assured for  $\alpha > 2$ ). The authors have also shown in the previous paper [18] the global existence of solutions in  $L_p$ -space of arbitrary space dimension  $n$  with  $p > n$ , where  $(\alpha, \beta)$  is merely allowed for  $0 < \beta < (\alpha - 1)/2$ .

In this paper, we revise the results obtained in [18] considerably by combining the semi-group method and the energy estimates and by applying the technique of trace operator [9, 13] (see Step 2 of Proof of Lemma 9). The main theorem of this paper is as follows:

**Theorem 1.** *Assume that the exponents  $\alpha$  and  $\beta$  satisfy the relations (1) and*

$$(2) \quad \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha-1).$$

*Let  $p$  be an arbitrarily fixed exponent with*

$$(3) \quad \max\{2, n, (\alpha-2)n\} < p < \infty.$$

*Then, for each pair of nonnegative initial functions  $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$ , the system (E) admits a unique global solution  $(u, v)$  in the function space*

$$(4) \quad \begin{cases} 0 \leq u \in C([0, \infty); L_p(\Omega)) \cap C((0, \infty); H_{p,N}^2(\Omega)) \cap C^1((0, \infty); L_p(\Omega)), \\ 0 \leq v \in C([0, \infty); H_p^1(\Omega)) \cap C((0, \infty); H_{p,N}^3(\Omega)) \cap C^1((0, \infty); H_p^1(\Omega)). \end{cases}$$

*Moreover the solution satisfies the estimate*

$$(5) \quad \|u(t)\|_{L_p} + \|v(t)\|_{H_p^1} \leq \psi(\|u_0\|_{L_p} + \|v_0\|_{H_p^1}), \quad t \geq 0$$

*with some increasing function  $\psi(\cdot)$ .*

The definition and notation of function spaces will be given below and in Section 2. Theorem 1 above does not yet cover the case  $(\alpha, \beta) = (2, 1)$  for  $n \geq 2$  shown by Winkler [27], but the theorem requires no assumption on the largeness of  $\mu$  nor the convexity of  $\Omega$  considered in [27]. Our new results also contain the uniform boundedness of solutions with respect to the size of initial data.

We conclude this introduction by referring the results on the parabolic-elliptic chemotaxis systems. The parabolic-elliptic simplifications correspond to the situation where the chemical substance diffuses very quickly, which implies that the time scale  $\tau$  tends to 0 in (E). For the  $n$ -dimensional parabolic-elliptic system with  $\alpha$ -th order growth and linear secretion, that is, in the case of  $\tau = 0$  and  $\beta = 1$  in (E), the problem on the global existence and blow-up of solutions has largely been solved by Winkler [25, 28]: global existence and boundedness are assured when  $\alpha > \max\{n/2, 2 - (1/n)\}$  [25]; also, there exists a blow-up solution when  $1 < \alpha < 3/2 + 1/(2n-2)$  with  $n \geq 5$  [28].

This paper is organized as follows. We provide preliminary results that we utilize in subsequent sections. In Section 3 we show the local existence of solutions by using a semigroup method (Theorem 5). In the final section we construct several a priori energy estimates by combining semigroup and energy methods. After obtaining the a priori estimates, we give the proof of the main theorem.

**NOTATIONS.** Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ . For  $1 \leq p \leq \infty$ , the space of complex-valued  $L_p$  functions in  $\Omega$  is denoted by  $L_p(\Omega)$  with the usual norm  $\|\cdot\|_{L_p}$ . The complex Sobolev space in  $\Omega$  of order  $k$ ,  $k = 0, 1, 2, \dots$ , and exponent  $p$ ,  $1 \leq p \leq \infty$ , is denoted by  $H_p^k(\Omega)$  with norm  $\|\cdot\|_{H_p^k}$ . More generally, the Sobolev space of fractional order  $s > 0$  and exponent  $1 \leq p \leq \infty$  is denoted by  $H_p^s(\Omega)$  with norm  $\|\cdot\|_{H_p^s}$ . The space of complex-valued continuous functions on  $\overline{\Omega}$  is denoted by  $C(\overline{\Omega})$  with norm  $\|\cdot\|_C$ . Let  $X$  be a Banach space and  $I$  an interval of  $\mathbb{R}$ .  $C(I; X)$  and  $C^1(I; X)$  denote the space of  $X$ -valued continuous functions and of  $X$ -valued continuously differentiable functions, respectively.  $\mathcal{B}(I; X)$  denotes the space of  $X$ -valued bounded functions. For simplicity, we will use a universal notation  $C$  to denote various constants that are determined for each occurrence

by  $\Omega$  in a specific way. In a situation where  $C$  also depends on some parameter, say  $\eta$ , it will be denoted by  $C_\eta$ . In addition, by a universal notation  $\psi(\cdot)$  we will denote continuous increasing functions, which may change depending on the context.

## 2. Preliminaries

In this section we shall list some well-known results in the theories of function spaces and linear operators [19, 22, 24, 30].

**Interpolation of Sobolev spaces.** For  $0 \leq s_0 < s < s_1 < \infty$  and  $1 < p < \infty$ ,  $H_p^s(\Omega)$  is the interpolation space  $[H_p^{s_0}(\Omega), H_p^{s_1}(\Omega)]_\theta$  between  $H_p^{s_0}(\Omega)$  and  $H_p^{s_1}(\Omega)$ , where  $s = (1-\theta)s_0 + \theta s_1$ , with the estimate

$$(6) \quad \|w\|_{H_p^s} \leq C \|w\|_{H_p^{s_0}}^{1-\theta} \|w\|_{H_p^{s_1}}^\theta \quad \text{for } w \in H_p^{s_1}(\Omega).$$

See [30, Theorem 1.35].

**Embedding theorem of Sobolev spaces.** Let  $1 < p < \infty$ .

If  $0 \leq s < n/p$ , then  $H_p^s(\Omega) \subset L_r(\Omega)$  for any  $p \leq r \leq pn/(n-ps) = [(1/p) - (s/n)]^{-1}$  with continuous embedding

$$(7) \quad \|w\|_{L_r} \leq C_{s,p} \|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega).$$

If  $s = n/p$ , then  $H_p^s(\Omega) \subset L_r(\Omega)$  for any finite  $p \leq r < \infty$  with continuous embedding

$$(8) \quad \|w\|_{L_r} \leq C_{s,p} \|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega).$$

If  $n/p < s < \infty$ , then  $H_p^s(\Omega) \subset C(\overline{\Omega})$  with continuous embedding

$$(9) \quad \|w\|_C \leq C_{s,p} \|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega).$$

See [30, Theorem 1.36].

If  $1 \leq r \leq p < \infty$ , then  $L_r(\Omega)$  is embedded in  $(H_p^s(\Omega))'$ , the dual space of  $H_p^s(\Omega)$  with respect to  $L_2$ -inner product, for  $(n/r) - (n/p) \leq s < \infty$  and  $p' = p/(p-1)$  with continuous embedding

$$(10) \quad \|w\|_{(H_p^s)'} \leq C_r \|w\|_{L_r} \quad \text{for } w \in L_r(\Omega).$$

**Gagliardo-Nirenberg's inequality.** Let  $1 \leq q \leq p \leq \infty$ . Then the embedding  $H_p^1(\Omega) \cap L_q(\Omega) \subset L_r(\Omega)$  holds for

$$(11) \quad \begin{cases} q \leq r \leq pn/(n-p) & \text{if } 1 \leq p < n; \\ q \leq r < \infty & \text{if } p = n; \\ q \leq r \leq \infty & \text{if } n < p \leq \infty, \end{cases}$$

with the estimate

$$(12) \quad \|w\|_{L_r} \leq C_{p,q,r} \|w\|_{H_p^1}^a \|w\|_{L_q}^{1-a} \quad \text{for } w \in H_p^1(\Omega),$$

where  $a$  is given by

$$(13) \quad \frac{1}{r} = a \left( \frac{1}{p} - \frac{1}{n} \right) + \frac{1-a}{q}.$$

See [30, Theorem 1.37].

**Norms of a product of two functions.** For  $1 < p < \infty$  and  $s > n/p$ , from (9),

$$(14) \quad \|uv\|_{L_p} \leq C_p \|u\|_{L_p} \|v\|_{L_\infty} \leq C_{p,s} \|u\|_{L_p} \|v\|_{H_p^s} \quad \text{for } u \in L_p(\Omega), v \in H_p^s(\Omega).$$

As a corollary,

$$(15) \quad \begin{aligned} \|\nabla \cdot (u\nabla v)\|_{L_p} &\leq \|\nabla u \cdot \nabla v\|_{L_p} + \|u\Delta v\|_{L_p} \leq \|\nabla u\|_{L_p} \|\nabla v\|_{L_\infty} + \|u\|_{L_\infty} \|\Delta v\|_{L_p} \\ &\leq C_{p,s} (\|u\|_{H_p^1} \|v\|_{H_p^{1+s}} + \|u\|_{H_p^s} \|v\|_{H_p^2}) \\ &\quad \text{for } u \in H_p^1(\Omega) \cap H_p^s(\Omega), v \in H_p^2(\Omega) \cap H_p^{1+s}(\Omega). \end{aligned}$$

When  $n < p < \infty$ , since  $H_p^1(\Omega) \subset L_\infty(\Omega)$  by (9), it holds that

$$(16) \quad \|\nabla \cdot (u\nabla v)\|_{L_p} \leq C_p \|u\|_{H_p^1} \|v\|_{H_p^2} \quad \text{for } u \in H_p^1(\Omega), v \in H_p^2(\Omega).$$

**Domains of fractional powers of Laplace operators in  $L_p$ -spaces.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ , and  $A_0 = -\Delta + 1$ ,  $\Delta$  being the Laplace operator with Neumann boundary condition. Then, for each  $1 < p < \infty$ ,  $A_0$  is considered as a closed operator in  $L_p(\Omega)$ , the domain of which is  $H_{p,N}^2(\Omega)$  (see [2, Theorem 2.4.1.3], [24, Theorem 5.3.4] or [30, Theorem 2.15]). Let us denote  $A_p = A_0|_{L_p}$ ; then  $\mathcal{D}(A_p) = H_{p,N}^2(\Omega)$ . Moreover, by the shift property (see [2, Theorem 2.5.1.1] or [24, Theorems 5.3.4 and 5.4.1]) it holds that  $\mathcal{D}(A_p|_{H_p^1}) = H_{p,N}^3(\Omega)$  with norm equivalence.

The domains of fractional powers of  $A_p$  are characterized by

$$(17) \quad \mathcal{D}(A_p^\theta) = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \leq \theta < \frac{1}{2} + \frac{1}{2p} \\ H_{p,N}^{2\theta}(\Omega) & \text{for } \frac{1}{2} + \frac{1}{2p} < \theta \leq \frac{3}{2} \end{cases}$$

with norm equivalence. Here,  $H_{p,N}^s(\Omega)$  for  $s > 1 + (1/p)$  denotes a closed subspace of  $H_p^s(\Omega)$  such that

$$H_{p,N}^s(\Omega) = \left\{ w \in H_p^s(\Omega); \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{for } s > 1 + \frac{1}{p}.$$

Indeed, we can see that  $A_p$  has a bounded  $H_\infty$  functional calculus (see Yagi [30, Sec.16.1.2]) in  $L_p(\Omega)$  and  $H_p^1(\Omega)$ , and by Yagi [30, Theorem 16.5], that the interpolation  $\mathcal{D}(A_p^\theta) = [L_p(\Omega), H_{p,N}^2(\Omega)]_\theta$  and  $\mathcal{D}((A_p|_{H_p^1})^\theta) = [H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta$  hold for  $0 < \theta < 1$  with norm equivalence. Then, carefully following the proof of [30, Theorem 16.11], we can verify the rest part of (17). For the detail see Appendix.

**Analytic semigroups generated by Laplace operators in  $L_p$ -spaces.** For each  $1 < p < \infty$ ,  $A_0$  defined above generates in  $L_p$ -space an analytic semigroup  $e^{-tA_0}$  (it is independent of  $p$  in the sense that  $e^{-tA_p} w = e^{-tA_0} w$  for  $w \in L_p(\Omega) \cap L_2(\Omega)$ ). For  $\gamma \geq 0$  it satisfies the estimate

$$(18) \quad \|A_0^\gamma e^{-tA_0} w\|_{L_p} \leq C t^{-\gamma} e^{-\delta_0 t} \|w\|_{L_p}, \quad t > 0, w \in L_p(\Omega),$$

with some fixed constant  $\delta_0 > 0$ . See [8, Sec. 2] (see also [26, Lemma 1.3], [30, Theorems 2.19 and 2.27] and [22, Sec. 13.7]).

**A differential geometric property of functions with Neumann boundary condition.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with smooth boundary  $\partial\Omega$ . If the function  $w \in C^2(\overline{\Omega})$  satisfies  $\partial w / \partial \nu = 0$  on  $\partial\Omega$ , then it holds that

$$(19) \quad \frac{\partial |\nabla w|^2}{\partial \nu} \leq 2\kappa_\Omega |\nabla w|^2 \quad \text{on } \partial\Omega,$$

where  $\kappa_\Omega$  is an upper bound for the curvatures of  $\partial\Omega$ ;  $\kappa_\Omega = 0$  when  $\Omega$  is convex. See [13, Lemma 4.2]. See also [9].

**Boundedness of trace operators.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with Lipschitz boundary  $\partial\Omega$ . Let  $1 < p < \infty$  and  $s > 1/p$ . Then, the trace  $T : f \mapsto f|_{\partial\Omega}$  is a bounded linear operator from  $H_p^s(\Omega)$  to  $L_p(\partial\Omega)$ . Hence, we have

$$(20) \quad \|w\|_{L_p(\partial\Omega)} \leq C_{s,p} \|w\|_{H_p^s(\Omega)}, \quad w \in H_p^s(\Omega).$$

See [30, Theorem 1.39] or [24, Theorem 4.7.1].

### 3. Local solutions

By similar argument to that in [17, 18, 19] or [30, Chap. 12], we can show the existence of local solutions to (E). We first review the existence theorem by Yagi [30, Chap. 4] (see also [20]) for local solutions to an abstract equation in a Banach space. Let  $X$  be a Banach space with norm  $\|\cdot\|_X$ . We consider the following Cauchy problem for a semilinear abstract evolution equation in  $X$ :

$$(21) \quad \begin{cases} \frac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0. \end{cases}$$

Here  $A$  is a sectorial operator of  $X$  satisfying that its spectral set is contained in a sectorial domain  $\Sigma = \{\lambda \in \mathbb{C}; |\arg \lambda| \leq \phi\}$  with some  $0 \leq \phi < \pi/2$ , and  $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)$ ,  $\lambda \notin \Sigma$  with constant  $M$ . The nonlinear operator  $F$  is a mapping from  $D(A^\eta)$  to  $X$ , where  $0 < \eta < 1$ , and it also satisfies a Lipschitz condition:

$$(22) \quad \|F(U) - F(\tilde{U})\|_X \leq \varphi \left( \|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X \right) \\ \times \left[ \|A^\eta(U - \tilde{U})\|_X + \left( \|A^\eta U\|_X + \|A^\eta \tilde{U}\|_X \right) \|A^\gamma(U - \tilde{U})\|_X \right], \quad U, \tilde{U} \in D(A^\eta),$$

where  $\gamma$  is an exponent such that  $0 < \gamma \leq \eta < 1$ , and  $\varphi(\cdot)$  is some increasing continuous function. The initial value  $U_0$  is taken in  $D(A^\gamma)$ . Then, from [30, Theorem 4.1] (or [20, Theorem 3.1]) we have the existence theorem of the local solutions to (21):

**Theorem 2** ([30, Theorem 4.1]). *Under the above assumptions, for any  $U_0 \in D(A^\gamma)$ , (21) possesses a unique local solution  $U$  in the function space:*

$$\begin{cases} U \in C((0, T_{U_0}]; D(A)) \cap C([0, T_{U_0}]; D(A^\gamma)) \cap C^1((0, T_{U_0}]; X), \\ t^{1-\gamma} U \in B((0, T_{U_0}]; D(A)) \end{cases}$$

with the estimate

$$t^{1-\gamma} \|AU(t)\|_X + \|A^\gamma U(t)\|_X \leq C_{U_0}, \quad 0 < t \leq T_{U_0},$$

where  $T_{U_0}$  and  $C_{U_0}$  are positive constants depending only on the norm  $\|A^\gamma U_0\|_X$ .

By applying Theorem 2, we can show the existence of the local solutions to (E). The following proposition has been proved in [18].

**Proposition 3** ([18, Proposition 3]). *Let  $n \in \mathbb{N}$ , assume the relation (1) for  $\alpha$  and  $\beta$ , and let  $p$  be an exponent satisfying*

$$(23) \quad \max\{n, (\alpha - 2)n\} < p < \infty.$$

*Then, for each pair of initial functions  $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$ , the problem (E) admits a unique local solution  $(u, v)$  in the function space*

$$(24) \quad \begin{cases} u \in C((0, T]; H_p^1(\Omega)) \cap C([0, T]; L_p(\Omega)) \cap C^1((0, T]; (H_{p'}^1(\Omega))'), \\ v \in C((0, T]; H_{p,N}^2(\Omega)) \cap C([0, T]; H_p^1(\Omega)) \cap C^1((0, T]; L_p(\Omega)) \end{cases}$$

with the estimate

$$t^{\frac{1}{2}} \left\{ \|u(t)\|_{H_p^1} + \|v(t)\|_{H_p^2} \right\} + \left\{ \|u(t)\|_{L_p} + \|v(t)\|_{H_p^1} \right\} \leq C, \quad 0 < t \leq T,$$

where  $p' = p/(p - 1)$ , and  $T$  and  $C$  are positive constants depending only on the norm  $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$ .

By a solution  $(u, v)$  to (E) in the function space (24) we mean that the pair of functions  $(u, v)$  contained in (24) satisfies

$$\begin{cases} \frac{d}{dt} \langle u, w \rangle_{L_2} = -\langle \nabla u, \nabla w \rangle_{L_2} + \chi \langle u \nabla v, \nabla w \rangle_{L_2} + \langle f(u), w \rangle_{L_2} \\ \text{for any } w \in H_{p'}^1(\Omega) \text{ and } 0 < t < \infty, \\ \tau \frac{\partial v}{\partial t} = \Delta v - v + g(u) \quad \text{in } \Omega \times (0, \infty). \end{cases}$$

Next, we will show the local existence of solutions in the second function space:

**Proposition 4.** *Let  $n \in \mathbb{N}$ , assume the relation (1) for  $\alpha$  and  $\beta$ , and let  $p$  be an exponent satisfying  $n < p < \infty$ . Then, for each pair of initial functions  $(u_0, v_0) \in H_p^1(\Omega) \times H_{p,N}^2(\Omega)$ , the problem (E) admits a unique local solution  $(u, v)$  in the function space*

$$\begin{cases} u \in C((0, T]; H_{p,N}^2(\Omega)) \cap C([0, T]; H_p^1(\Omega)) \cap C^1((0, T]; L_p(\Omega)), \\ v \in C((0, T]; H_{p,N}^3(\Omega)) \cap C([0, T]; H_{p,N}^2(\Omega)) \cap C^1((0, T]; H_p^1(\Omega)) \end{cases}$$

with the estimate

$$t^{\frac{1}{2}} \left\{ \|u(t)\|_{H_p^2} + \|v(t)\|_{H_p^3} \right\} + \left\{ \|u(t)\|_{H_p^1} + \|v(t)\|_{H_p^2} \right\} \leq C, \quad 0 < t \leq T,$$

where  $T$  and  $C$  are positive constants depending only on the norm  $\|u_0\|_{H_p^1} + \|v_0\|_{H_p^2}$ .

**Proof.** The system (E) can be expressed as a semilinear parabolic equation



$$(25) \quad \begin{cases} \frac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \end{cases}$$

in a product Banach space  $X = L_p(\Omega) \times H_p^1(\Omega)$ . Here, we define the linear operator  $A$  by

$$A = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \tau^{-1}(-\Delta + 1) \end{bmatrix}, \quad \mathcal{D}(A) = H_{p,N}^2(\Omega) \times H_{p,N}^3(\Omega).$$

The nonlinear operator  $F$  is defined by

$$F(U) = \begin{bmatrix} -\chi \nabla \cdot (u \nabla v) + \bar{f}(u) + u \\ \bar{g}(u) \end{bmatrix}, \quad U = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(A^\eta) = H_p^1(\Omega) \times H_{p,N}^2(\Omega)$$

with  $\eta = 1/2$ . Here,  $\bar{f}(u)$  and  $\bar{g}(u)$  denote some smooth extensions of  $f(u)$  and  $g(u)$  for the variable  $u \in \mathbb{C}$  satisfying  $f(u) \geq 0$  for  $u < 0$  and  $g(u) = 0$  for  $u < -1$ , respectively. The initial value  $U_0$  is taken in the function space  $\mathcal{D}(A^\gamma) = \mathcal{D}(A^\eta)$ , that is  $\gamma = \eta$ . Under this setting, we need to verify only the Lipschitz condition (22). For  $U = \begin{bmatrix} u \\ v \end{bmatrix}, \tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{D}(A^\eta)$ ,

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X &\leq \chi \|\nabla \cdot (u \nabla v - \tilde{u} \nabla \tilde{v})\|_{L_p} \\ &\quad + \|u - \tilde{u}\|_{L_p} + \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} + \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H_p^1}. \end{aligned}$$

For the first term, applying (16), we see

$$\begin{aligned} \|\nabla \cdot (u \nabla v) - \nabla \cdot (\tilde{u} \nabla \tilde{v})\|_{L_p} &\leq \|\nabla \cdot ((u - \tilde{u}) \nabla v)\|_{L_p} + \|\nabla \cdot (\tilde{u} \nabla (v - \tilde{v}))\|_{L_p} \\ &\leq C_{p,s} (\|u - \tilde{u}\|_{H_p^1} \|v\|_{H_p^2} + \|\tilde{u}\|_{H_p^1} \|v - \tilde{v}\|_{H_p^2}) \\ &\leq C (\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X) \|A^\gamma (U - \tilde{U})\|_X. \end{aligned}$$

For the third and fourth terms, using (1) and  $H_p^s(\Omega) \subset L_\infty(\Omega)$  by (9), we can easily see that

$$\begin{aligned} \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})^{\alpha-1} \|u - \tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H_p^1} + \|\tilde{u}\|_{H_p^1})^{\alpha-1} \|u - \tilde{u}\|_{L_p}, \end{aligned}$$

$$\begin{aligned} \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H_p^1} &\leq \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{L_p} + \|\bar{g}'(u) \nabla (u - \tilde{u})\|_{L_p} + \|\{\bar{g}'(u) - \bar{g}'(\tilde{u})\} \nabla \tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty}) (\|u - \tilde{u}\|_{L_p} + \|\nabla (u - \tilde{u})\|_{L_p}) + C \|u - \tilde{u}\|_{L_\infty} \|\nabla \tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H_p^1} + \|\tilde{u}\|_{H_p^1}) \|u - \tilde{u}\|_{H_p^1}. \end{aligned}$$

Thus  $F(U)$  satisfies the Lipschitz condition (22). We complete the proof.  $\square$

Now we can state our main theorem of this section:

**Theorem 5.** *Let  $n \in \mathbb{N}$ , assume the relation (1) for  $\alpha$  and  $\beta$ , and let  $p$  be an arbitrarily fixed exponent satisfying (23). Then, for each pair of nonnegative initial functions  $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times C(\bar{\Omega})$ , the problem (E) admits a unique local solution  $(u, v)$  in the function space*

$$(26) \quad \begin{cases} 0 \leq u \in C([0, T]; L_p(\Omega)) \cap C((0, T]; H_{p,N}^2(\Omega)) \cap C^1((0, T]; L_p(\Omega)), \\ 0 \leq v \in C([0, T]; H_p^1(\Omega)) \cap C((0, T]; H_{p,N}^3(\Omega)) \cap C^1((0, T]; H_p^1(\Omega)) \end{cases}$$

with the estimate

$$(27) \quad \|u(t)\|_{L_p} + \|v(t)\|_{H_p^1} \leq C, \quad 0 < t \leq T,$$

where  $T$  and  $C$  are positive constants depending only on the norm  $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$ .

Proof. It is clear that the local solutions belong to the function space (26) from Propositions 3 and 4. The nonnegativity of solutions has been proved in [18, Theorem 4] with the aid of the truncation method [30, Section 12.1.3]. Hence we conclude the proof.  $\square$

#### 4. A priori estimates and global solutions

In this section we will construct several a priori estimates. The a priori estimates hold with each of the inequalities of  $\alpha$  and  $\beta$  in the lemmas. Throughout this section, except for in the global existence theorem, we assume that  $0 \leq u_0 \in H_{p,N}^2(\Omega) \subset H_\infty^1(\Omega)$  and  $0 \leq v_0 \in H_{p,N}^3(\Omega) \subset H_\infty^2(\Omega)$  with  $n < p < \infty$ . In this case, applying [30, Theorem 4.2], we can verify that  $0 \leq u \in C([0, T]; H_{p,N}^2(\Omega))$  and  $0 \leq v \in C([0, T]; H_{p,N}^3(\Omega))$  with the estimate  $\|u(t)\|_{H_p^2} + \|v(t)\|_{H_p^3} \leq C_{U_0}$  for  $0 \leq t \leq T$ , where  $C_{U_0}$  is some positive constant. For a local solution  $(u, v)$  to (E) and exponents  $z > 0$  and  $\omega > 0$ , we define

$$I_\omega^z(t) = \int_0^t \omega e^{-\omega(t-s)} \int_\Omega u^z dx ds.$$

The following lemma will be used frequently in this section.

**Lemma 6** (Gronwall's inequality). *Assume that a smooth real function  $h(t)$  satisfies the differential inequality*

$$h'(t) + ah(t) \leq K(t), \quad t_0 \leq t \leq T,$$

with a positive constant  $a$  and an integrable real function  $K(t)$ . Then,  $h(t)$  is estimated by

$$h(t) \leq h(t_0)e^{-a(t-t_0)} + \int_{t_0}^t e^{-a(t-s)} K(s) ds, \quad t_0 \leq t \leq T.$$

**Lemma 7.** *Let  $(u, v)$  be a local solution to (E), and assume that*

$$\alpha > 1.$$

Then, it holds that

$$(28) \quad \|u\|_{L_1} = \int_\Omega u dx \leq e^{-t} \|u_0\|_{L_1} + a_1 |\Omega|$$

with a constant  $a_1 = \max\{f(u) + u; u \geq 0\}$ . In addition, for an arbitrary constant  $\omega > 0$ ,

$$(29) \quad I_\omega^\alpha(t) \leq \frac{2}{\mu} \{(a + a_1\omega) |\Omega| + \omega \|u_0\|_{L_1}\} \equiv \bar{I}_\omega^\alpha$$

holds with a constant  $a = \max\{f(u) + \mu u^\alpha/2; u \geq 0\}$ .

Proof. (Just the same as [17, Lemmas 4.1 and 4.2] or the first half part of [18, Lemma 5].) Integrating the first equation of (E) over  $\Omega$ , we have

$$\frac{d}{dt} \int_{\Omega} u \, dx = \int_{\Omega} f(u) \, dx \leq \int_{\Omega} (a_1 - u) \, dx.$$

Then, by Lemma 6, we obtain (28). From these inequalities, we see that

$$\begin{aligned} \frac{\mu}{2} I_{\omega}^{\alpha}(t) &\leq \int_0^t \omega e^{-\omega(t-s)} \int_{\Omega} \{a - f(u)\} \, dx \, ds = \int_0^t \omega e^{-\omega(t-s)} \left\{ a|\Omega| - \frac{d}{ds} \|u\|_{L_1} \right\} \, ds \\ &\leq a|\Omega|(1 - e^{-\omega t}) + \omega e^{-\omega t} \|u_0\|_{L_1} + \int_0^t \omega^2 e^{-\omega(t-s)} \|u\|_{L_1} \, ds \\ &\leq (a + a_1\omega)|\Omega| + \omega \|u_0\|_{L_1}, \end{aligned}$$

which yields (29).  $\square$

**Lemma 8.** *Let  $(u, v)$  be a local solution to (E), and assume that*

$$\alpha > 1 \quad \text{and} \quad 0 < \beta \leq \frac{\alpha}{2}.$$

*Then, for any exponent  $2 \leq q \leq \alpha/\beta$ ,*

$$(30) \quad \|v\|_{H_q^1}^q \leq C_q e^{-\delta_q t} \|v_0\|_{H_q^1}^q + C_q (|\Omega| + \|u_0\|_{L_1})$$

*holds with some positive constants  $C_q$  and  $\delta_q$ .*

Proof. When  $q = 2$  (see [17, Proposition 4.4]), multiplying the second equation of (E) by  $-\Delta v + v$  and integrating it over  $\Omega$ , we see that

$$\frac{\tau}{2} \frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + v^2) \, dx \leq -\frac{1}{2} \int_{\Omega} (\Delta v)^2 \, dx - 2 \int_{\Omega} |\nabla v|^2 \, dx - \frac{1}{2} \int_{\Omega} v^2 \, dx + \int_{\Omega} (1 + u)^{2\beta} \, dx,$$

that is,

$$(31) \quad \begin{aligned} \tau \frac{d}{dt} \int_{\Omega} (|\nabla v|^2 + v^2) \, dx + \int_{\Omega} (|\nabla v|^2 + v^2) \, dx \\ + \int_{\Omega} (\Delta v)^2 \, dx + 3 \int_{\Omega} |\nabla v|^2 \, dx \leq 2 \int_{\Omega} (1 + u)^{2\beta} \, dx. \end{aligned}$$

Thus, by Lemma 6 again, we verify

$$\|v\|_{H_2^1}^2 \leq e^{-t/\tau} \|v_0\|_{H_2^1}^2 + 2 \int_0^t e^{-(t-s)/\tau} \int_{\Omega} (1 + u)^{2\beta} \, dx \, ds.$$

When  $q > 2$  (just in the same way as the second half part of [18, Lemma 5]), we utilize the semigroup  $e^{-tA_0/\tau}$  of  $A_0 = -\Delta + 1$ ,  $\Delta$  be the Laplace operator with Neumann boundary condition. Then the second equation of (E) gives

$$(32) \quad v(t) = e^{-tA_0/\tau} v_0 + \frac{1}{\tau} \int_0^t e^{-(t-s)A_0/\tau} g(u(s)) \, ds.$$

Operating  $A_0^{1/2}$  to this equality and applying (17) and (18), we have

$$\begin{aligned}
 \|v\|_{H_q^1} &\leq C_q \|A_0^{1/2} v\|_{L_q} \\
 &\leq C_q \|A_0^{1/2} e^{-tA_0/\tau} v_0\|_{L_q} + \frac{1}{\tau} \int_0^t C_q \|A_0^{1/2} e^{-(t-s)A_0/\tau} g(u)\|_{L_q} ds \\
 &\leq C_q e^{-\delta_0 t/\tau} \|A_0^{1/2} v_0\|_{L_q} + \int_0^t C_q (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|g(u)\|_{L_q} ds \\
 &\leq C_q e^{-\delta_0 t/\tau} \|v_0\|_{H_q^1} + \int_0^t C_q (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|(1+u)^\beta\|_{L_q} ds.
 \end{aligned}$$

The last term can be estimated as

$$\begin{aligned}
 &\int_0^t (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|(1+u)^\beta\|_{L_q} ds \\
 &= \int_0^t (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|1+u\|_{L_{q\beta}^\beta}^\beta ds \\
 &\leq \left( \int_0^t (t-s)^{-q'/2} e^{-\delta_0(t-s)/\tau} ds \right)^{1/q'} \left( \int_0^t e^{-\delta_0(t-s)/\tau} \|1+u\|_{L_{q\beta}^{q\beta}}^{q\beta} ds \right)^{1/q},
 \end{aligned}$$

where  $q' = q/(q-1)$ . Here we notice that  $q'/2 < 1$  and the singular integral converges. Hence we have

$$\|v\|_{H_q^1} \leq C_q e^{-\delta_0 t/\tau} \|v_0\|_{H_q^1} + C_q \left( \int_0^t e^{-\delta_0(t-s)/\tau} \int_\Omega (1+u)^{q\beta} dx ds \right)^{1/q}.$$

Combining both cases when  $q = 2$  and when  $q > 2$ , we have

$$(33) \quad \|v\|_{H_q^1}^q \leq C_q e^{-\delta_q t/\tau} \|v_0\|_{H_q^1}^q + C_q \int_0^t e^{-\delta_q(t-s)/\tau} \int_\Omega (1+u)^{q\beta} dx ds$$

for  $q \geq 2$  with some positive constants  $C_q$  and  $\delta_q$ . Applying (29), we prove (30) for  $2 \leq q \leq \alpha/\beta$ .  $\square$

**Lemma 9.** *Let  $(u, v)$  be a local solution to (E), and assume that*

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha-1).$$

*Then, for any  $2 \leq q \leq \alpha/\beta$  satisfying  $q > 2n\alpha/[(n+2)(\alpha-1)]$ , and for any exponent  $1 < \theta \leq \{q(n+2)/(2n) - 1\}(\alpha-1)$ ,*

$$(34) \quad \|1+u\|_{L_\theta}^\theta \leq e^{-q\theta/(2\tau)} \|1+u_0\|_{L_\theta}^\theta + \psi_{\theta,q} \left( \|1+u_0\|_{L_1} + \|v_0\|_{H_q^1} \right)$$

*holds with some increasing function  $\psi_{\theta,q}(\cdot)$ . In addition, for an arbitrary constant  $\omega > 0$ ,*

$$\begin{aligned}
 (35) \quad I_\omega^{\alpha+\theta-1}(t) &\leq \frac{4}{\mu} \left\{ \left( 1 + \frac{\omega}{\theta} \right) \psi_{\theta,q} \left( \|u_0\|_{L_1} + \|v_0\|_{H_q^1} \right) \right. \\
 &\quad \left. + \omega \left( \frac{1}{\theta} \|1+u_0\|_{L_\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right) \right\} \equiv \bar{I}_\omega^{\alpha+\theta-1}
 \end{aligned}$$

*holds with some constant  $\zeta > 0$ .*

Proof. We describe the proof in several steps.

*Step 1.* Multiplying the first equation of (E) by  $(1+u)^{\theta-1}$  and integrating it over  $\Omega$ , we see that

$$\begin{aligned} \frac{1}{\theta} \frac{d}{dt} \int_{\Omega} (1+u)^{\theta} dx &= -(\theta-1) \int_{\Omega} (1+u)^{\theta-2} |\nabla u|^2 dx \\ &\quad + \chi(\theta-1) \int_{\Omega} u(1+u)^{\theta-2} \nabla u \cdot \nabla v dx + \int_{\Omega} (1+u)^{\theta-1} f(u) dx \\ &\leq -\frac{\theta-1}{2} \int_{\Omega} (1+u)^{\theta-2} |\nabla u|^2 dx + \frac{\chi^2(\theta-1)}{2} \int_{\Omega} (1+u)^{\theta} |\nabla v|^2 dx \\ &\quad + \int_{\Omega} (1+u)^{\theta-1} f(u) dx. \end{aligned}$$

For the second term on the right-hand side, using (12), we note that

$$\begin{aligned} \frac{\chi^2(\theta-1)}{2} \int_{\Omega} (1+u)^{\theta} |\nabla v|^2 dx &\leq \frac{\chi^2(\theta-1)}{2} \|(1+u)^{\theta}\|_{L_{\kappa/(\kappa-1)}} \|\nabla v\|_{L_{\kappa}}^2 \\ &= \frac{\chi^2(\theta-1)}{2} \|(1+u)^{\theta}\|_{L_{\kappa/(\kappa-1)}} \|\nabla v\|_{L_{4\kappa/q}}^{4/q} \\ &\leq \frac{\chi^2(\theta-1)}{2} \|(1+u)^{\theta}\|_{L_{\kappa/(\kappa-1)}} \cdot C_q \|\nabla v\|_{H_2^1}^{2/\kappa} \|\nabla v\|_{L_2}^{(4/q)-(2/\kappa)} \\ &\leq C_q \eta^{-\kappa+1} \chi^{2\kappa} (\theta-1)^{\kappa} \|\nabla v\|_{H_2^1}^{2\kappa} + \eta \|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)} \int_{\Omega} (1+u)^{\theta\kappa/(\kappa-1)} dx \end{aligned}$$

with  $\kappa = q(n+2)/(2n)$  and an arbitrary  $\eta > 0$ . Hence we have

$$\begin{aligned} (36) \quad \frac{1}{\theta} \frac{d}{dt} \int_{\Omega} (1+u)^{\theta} dx &\leq -\frac{\theta-1}{2} \int_{\Omega} (1+u)^{\theta-2} |\nabla u|^2 dx + C_q \eta^{-\kappa+1} \chi^{2\kappa} (\theta-1)^{\kappa} \|\nabla v\|_{H_2^1}^{2\kappa} \\ &\quad + \int_{\Omega} \left[ \eta \|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)} (1+u)^{\theta\kappa/(\kappa-1)} + (1+u)^{\theta-1} f(u) \right] dx. \end{aligned}$$

*Step 2.* We present the differential inequality on  $\|v\|_{H_q^1}^q$  for  $q \geq 2$ . For the present assume  $q > 2$ . Firstly, multiplying the second equation of (E) by  $v^{q-1}$  and integrating it over  $\Omega$ , we see that

$$\begin{aligned} (37) \quad \frac{\tau}{q} \frac{d}{dt} \int_{\Omega} v^q dx &= -(q-1) \int_{\Omega} v^{q-2} |\nabla v|^2 dx - \int_{\Omega} v^q dx + \int_{\Omega} v^{q-1} g(u) \\ &\leq -(q-1) \int_{\Omega} v^{q-2} |\nabla v|^2 dx - \frac{1}{2} \int_{\Omega} v^q dx + \frac{C'_q}{2} \int_{\Omega} (1+u)^{q\beta} dx. \end{aligned}$$

Next, differentiating the second equation of (E), we have

$$\tau \frac{\partial}{\partial t} |\nabla v|^2 = 2\tau \nabla v \cdot \nabla v_t = 2\nabla v \cdot \nabla \Delta v - 2|\nabla v|^2 + 2\nabla v \cdot \nabla g(u).$$

Noting that  $\Delta |\nabla v|^2 = 2|D^2 v|^2 + 2\nabla v \cdot \nabla \Delta v$  and  $(\Delta v)^2 \leq n|D^2 v|^2$ , where  $|D^2 v|^2 = \sum_{i,j} |D_i D_j v|^2$ , we see

$$\tau \frac{\partial}{\partial t} |\nabla v|^2 \leq \Delta |\nabla v|^2 - \frac{2}{n} (\Delta v)^2 - 2 |\nabla v|^2 + 2 \nabla v \cdot \nabla g(u).$$

Multiplying this inequality by  $|\nabla v|^{q-2}$ , integrating it over  $\Omega$  and applying (19), we obtain

$$\begin{aligned} \frac{2\tau}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^q dx &\leq \int_{\Omega} |\nabla v|^{q-2} \left\{ \Delta |\nabla v|^2 - \frac{2}{n} (\Delta v)^2 - 2 |\nabla v|^2 + 2 \nabla v \cdot \nabla g(u) \right\} dx \\ &= \int_{\partial\Omega} |\nabla v|^{q-2} \frac{\partial |\nabla v|^2}{\partial \nu} dx - \int_{\Omega} \nabla |\nabla v|^{q-2} \cdot \nabla |\nabla v|^2 dx \\ &\quad - \int_{\Omega} |\nabla v|^{q-2} \left\{ \frac{2}{n} (\Delta v)^2 + 2 |\nabla v|^2 \right\} dx + \int_{\Omega} 2 |\nabla v|^{q-2} \nabla v \cdot \nabla g(u) dx \\ &\leq 2\kappa_{\Omega} \int_{\partial\Omega} |\nabla v|^q dx - \int_{\Omega} \frac{q-2}{2} |\nabla v|^{q-4} |\nabla |\nabla v|^2|^2 dx - \int_{\Omega} \frac{2}{n} |\nabla v|^{q-2} (\Delta v)^2 dx \\ &\quad - \int_{\Omega} 2 |\nabla v|^q dx + \int_{\Omega} 2 |\nabla v|^{q-2} \nabla v \cdot \nabla g(u) dx. \end{aligned}$$

For the first term on the right-hand side, applying (20) and (6) with any  $1/2 < s < 1$  and  $\varepsilon > 0$ , we see that

$$\begin{aligned} 2\kappa_{\Omega} \int_{\partial\Omega} |\nabla v|^q dx &= 2\kappa_{\Omega} \left\| |\nabla v|^{q/2} \right\|_{L_2(\partial\Omega)}^2 \leq C \left\| |\nabla v|^{q/2} \right\|_{H_2^s(\Omega)}^2 \\ &\leq C \left\| |\nabla v|^{q/2} \right\|_{H_2^1}^{2s} \left\| |\nabla v|^{q/2} \right\|_{L_2}^{2(1-s)} \leq \varepsilon \left\| \nabla (|\nabla v|^{q/2}) \right\|_{L_2}^2 + C_{\varepsilon} \left\| |\nabla v|^{q/2} \right\|_{L_2}^2. \end{aligned}$$

For the last term on the right-hand side, we see

$$\begin{aligned} \int_{\Omega} 2 |\nabla v|^{q-2} \nabla v \cdot \nabla g(u) dx &= - \int_{\Omega} \left\{ (q-2) |\nabla v|^{q-4} \nabla |\nabla v|^2 \cdot \nabla v + 2 |\nabla v|^{q-2} \Delta v \right\} g(u) dx \\ &\leq \int_{\Omega} (q-2) |\nabla v|^{q-3} |\nabla |\nabla v|^2| (1+u)^{\beta} dx + \int_{\Omega} 2 |\nabla v|^{q-2} |\Delta v| (1+u)^{\beta} dx \\ &\leq \frac{q-2}{4} \int_{\Omega} |\nabla v|^{q-4} |\nabla |\nabla v|^2|^2 dx + \frac{1}{n} \int_{\Omega} |\nabla v|^{q-2} (\Delta v)^2 dx + \int_{\Omega} |\nabla v|^q dx \\ &\quad + C_q'' (n+q-2)^{q/2} \int_{\Omega} (1+u)^{q\beta} dx. \end{aligned}$$

Hence, noting that  $\left| \nabla (|\nabla v|^{q/2}) \right|^2 = (q^2/16) |\nabla v|^{q-4} |\nabla |\nabla v|^2|^2$ , we have

$$\begin{aligned} (38) \quad \frac{2\tau}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^q dx &+ \frac{4(q-2)}{q^2} \int_{\Omega} \left| \nabla (|\nabla v|^{q/2}) \right|^2 dx \\ &+ \frac{1}{n} \int_{\Omega} |\nabla v|^{q-2} (\Delta v)^2 dx + \int_{\Omega} |\nabla v|^q dx \\ &\leq \varepsilon \int_{\Omega} \left| \nabla (|\nabla v|^{q/2}) \right|^2 dx + C_{\varepsilon} \int_{\Omega} |\nabla v|^q dx + C_q'' (n+q-2)^{q/2} \int_{\Omega} (1+u)^{q\beta} dx. \end{aligned}$$

Adding (38) to (37) and taking  $\varepsilon = 2(q-2)/q^2$ , we see

$$\begin{aligned}
(39) \quad & \frac{2\tau}{q} \frac{d}{dt} \int_{\Omega} (|\nabla v|^q + v^q) dx + \int_{\Omega} (|\nabla v|^q + v^q) dx \\
& + \frac{2(q-2)}{q^2} \int_{\Omega} \left| \nabla (|\nabla v|^{q/2}) \right|^2 dx + \frac{1}{n} \int_{\Omega} |\nabla v|^{q-2} (\Delta v)^2 dx + \frac{8(q-1)}{q^2} \int_{\Omega} \left| \nabla (v^{q/2}) \right|^2 dx \\
& \leq C \int_{\Omega} |\nabla v|^q dx + C_q \int_{\Omega} (1+u)^{q\beta} dx.
\end{aligned}$$

This inequality holds also for  $q = 2$  (see (31)). The right-hand side is bounded in terms of  $\|u_0\|_{L^1}$  and  $\|v_0\|_{H_q^1}$  in view of Lemmas 7 and 8 since  $q\beta \leq \alpha$ .

*Step 3.* Adding (39) multiplied by some weight  $\zeta > 0$  to (36), we see

$$\begin{aligned}
(40) \quad & \frac{1}{\theta} \frac{d}{dt} \int_{\Omega} (1+u)^\theta dx + \zeta \left( \frac{2\tau}{q} \frac{d}{dt} \|v\|_{H_q^1}^q + \|v\|_{H_q^1}^q + \frac{2(q-2)}{q^2} \|\nabla v\|_{H_2^1}^2 \right) \\
& \leq C_q \eta^{-\kappa+1} \chi^{2\kappa} (\theta-1)^\kappa \|\nabla v\|_{H_2^1}^2 + C \|\nabla v\|_{L_q}^q \\
& + \int_{\Omega} \left[ \eta \|\nabla v\|_{L_q}^{(2\kappa-q)/(\kappa-1)} (1+u)^{\theta\kappa/(\kappa-1)} + (1+u)^{\theta-1} f(u) + \zeta C_q (1+u)^{q\beta} \right] dx.
\end{aligned}$$

Since  $q\beta \leq \alpha < \alpha + \theta - 1$  and  $\theta\kappa/(\kappa-1) \leq \alpha + \theta - 1$  from the assumptions, suitable choice of  $\eta$  and  $\zeta$  yields

$$\begin{aligned}
(41) \quad & \frac{d}{dt} \left( \frac{1}{\theta} \|1+u\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) + \frac{q}{2\tau} \left( \frac{1}{\theta} \|1+u\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) \\
& \leq \psi_{\theta,q} (\|v\|_{H_q^1}) - \frac{\mu}{4} \int_{\Omega} u^{\alpha+\theta-1} dx,
\end{aligned}$$

and hence, by Lemma 6,

$$\begin{aligned}
\frac{1}{\theta} \|1+u\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q & \leq e^{-qt/(2\tau)} \left( \frac{1}{\theta} \|1+u_0\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right) \\
& + \int_0^t e^{-q(t-s)/(2\tau)} \psi_{\theta,q} (\|v\|_{H_q^1}) ds.
\end{aligned}$$

Application of (30) to the right-hand side of this inequality leads to (34).

*Step 4.* The proof of (35) is very similar to that of (29), as follows: using (41),

$$\begin{aligned}
\frac{\mu}{4} I_\omega^{\alpha+\theta-1}(t) & \leq \int_0^t \omega e^{-\omega(t-s)} \left\{ \psi_{\theta,q} (\|v\|_{H_q^1}) - \frac{d}{dt} \left( \frac{1}{\theta} \|1+u\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) \right\} ds \\
& \leq \psi_{\theta,q} \left( \sup_{t \geq 0} \|v\|_{H_q^1} \right) + \omega e^{-\omega t} \left( \frac{1}{\theta} \|1+u_0\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right) \\
& + \int_0^t \omega^2 e^{-\omega(t-s)} \left( \frac{1}{\theta} \|1+u\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v\|_{H_q^1}^q \right) ds \\
& \leq \left( 1 + \frac{\omega}{\theta} \right) \psi_{\theta,q} \left( \sup_{t \geq 0} \|v\|_{H_q^1} \right) + \omega \left( \frac{1}{\theta} \|1+u_0\|_{L^\theta}^\theta + \zeta \frac{2\tau}{q} \|v_0\|_{H_q^1}^q \right).
\end{aligned}$$

Thus we complete the proof of the lemma.  $\square$

**Lemma 10.** *Let  $(u, v)$  be a local solution to (E), and assume that*

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

*Suppose that for some exponent  $\sigma > 1$  and  $r > 2n\alpha/[(n+2)(\alpha-1)]$  the integral  $I_\omega^{\alpha+\sigma-1}(t)$  is bounded by*

$$(42) \quad I_\omega^{\alpha+\sigma-1}(t) \leq (1 + \omega) \psi_{\sigma,r} \left( \|1 + u_0\|_{L_\sigma} + \|v_0\|_{H^1} \right) \equiv \bar{I}_\omega^{\alpha+\sigma-1}$$

*for an arbitrary constant  $\omega > 0$ . Then, for any exponent  $q \geq 2$  satisfying  $r \leq q \leq (\alpha + \sigma - 1)/\beta$ ,*

$$(43) \quad \|v\|_{H^1_q}^q \leq C_q e^{-\delta_q t} \|v_0\|_{H^1_q}^q + C_q \psi_{\sigma,r} \left( \|1 + u_0\|_{L_\sigma} + \|v_0\|_{H^1} \right)$$

*holds with some positive constants  $C_q$  and  $\delta_q$ . Moreover, for any exponent  $\sigma \leq \theta \leq \{q(n+2)/(2n) - 1\}(\alpha - 1)$ ,*

$$(44) \quad \|1 + u\|_{L_\theta}^\theta \leq e^{-qt/(2\tau)} \|1 + u_0\|_{L_\theta}^\theta + \psi_{\theta,q} \left( \|1 + u_0\|_{L_\sigma} + \|v_0\|_{H^1_q} \right)$$

*holds with some increasing function  $\psi_{\theta,q}(\cdot)$ . In addition, for an arbitrary constant  $\omega > 0$ , it holds that*

$$(45) \quad I_\omega^{\alpha+\theta-1}(t) \leq (1 + \omega) \psi_{\theta,q} \left( \|1 + u_0\|_{L_\theta} + \|v_0\|_{H^1_q} \right) \equiv \bar{I}_\omega^{\alpha+\theta-1}.$$

*Proof.* We can prove the lemma in the similar argument as in Lemmas 8 and 9.

Firstly, the inequality (33) holds also in this case. Since  $q\beta \leq \alpha + \sigma - 1$ , by Lemma 6 again, we verify

$$\|v\|_{H^1_q}^q \leq C_q e^{-\delta_q t} \|v_0\|_{H^1_q}^q + C_q b_{q,\alpha+\sigma-1} \left( |\Omega| + I_{q/2\tau}^{\alpha+\sigma-1}(t) \right).$$

By (42), we obtain (43).

The estimate (44) is verified from the inequality (41) together with (43), since  $q\beta \leq \alpha + \sigma - 1 \leq \alpha + \theta - 1$  and  $\theta\kappa/(\kappa - 1) \leq \alpha + \theta - 1$  with  $\kappa = q(n+2)/(2n)$ . The proof of (45) is just the same as that of (35).

Thus we complete the proof of the lemma.  $\square$

For obtaining the final a priori estimate, we apply Lemma 10 iteratively. We then show the following a priori estimate.

**Proposition 11.** *Let  $(u, v)$  be a local solution to (E), and assume that*

$$\alpha > 1, \quad 0 < \beta \leq \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

*Then, for any exponent  $p > 2$ , it holds that*

$$(46) \quad \|1 + u\|_{L_p}^p + \|v\|_{H^1_p}^p \leq C e^{-pt/(2\tau)} \left( \|1 + u_0\|_{L_p}^p + \|v_0\|_{H^1_p}^p \right) + \psi_p \left( \|1 + u_0\|_{L_\sigma} + \|v_0\|_{H^1} \right)$$

*with some exponents  $1 < \sigma < p$ ,  $\alpha/\beta < r < p$  and some increasing function  $\psi_p(\cdot)$ .*

*Proof.* The proof is given by induction. Firstly we have estimates (28) on  $\|u\|_{L_1}$ . Let

$$\theta_0 = 1.$$



Secondly we have (30) on  $\|v\|_{H_q^1}$  for  $2 \leq q \leq \alpha/\beta$  by Lemma 8 and (34) on  $\|1 + u\|_{L_\theta}$  for  $1 < \theta \leq \{q(n+2)/(2n) - 1\}(\alpha - 1)$  by Lemma 9. Let

$$q_1 = \frac{\theta_0 + \alpha - 1}{\beta} = \frac{\alpha}{\beta}, \quad \theta_1 = \left( \frac{n+2}{2n} q_1 - 1 \right) (\alpha - 1).$$

For each integer  $k$  and given  $\theta_k$ , we can obtain by Lemma 10 the estimates (43) on  $\|v\|_{H_q^1}$  for  $2 < q \leq (\theta_k + \alpha - 1)/\beta$  and (44) on  $\|1 + u\|_{L_\theta}$  for  $1 < \theta \leq \{q(n+2)/(2n) - 1\}(\alpha - 1)$ . Define

$$q_{k+1} = \frac{\theta_k + \alpha - 1}{\beta} = \frac{(n+2)(\alpha - 1)}{2n\beta} q_k, \quad \theta_{k+1} = \left( \frac{n+2}{2n} q_{k+1} - 1 \right) (\alpha - 1).$$

Since  $(n+2)(\alpha - 1)/(2n\beta) > 1$  by assumption, we can easily see that

$$q_k \rightarrow \infty \quad \text{and} \quad \theta_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty.$$

Hence, for any  $p > 1$ , there exists a finite integer  $k_0$  such that  $q_{k_0} > p$  and  $\theta_{k_0} > p$ , and the desired estimates are obtained.  $\square$

By using the a priori estimates shown above, we prove the main theorem for the global existence of the solutions.

**Proof of Theorem 1.** From Theorem 5 for each pair of nonnegative initial functions  $(u_0, v_0)$  there exists a unique nonnegative local solution  $(u, v)$  on the interval  $[0, T]$  with the estimate (27), and the existence time  $T$  depends only on the norm  $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$ . In addition, from Proposition 11, the norm  $\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1}$ ,  $0 \leq t \leq T$ , is estimated from above by a uniform constant  $C_{U_0}$  also depending only on the norm  $\|u_0\|_{L_p} + \|v_0\|_{H_p^1}$ . Hence, the interval can be extended to  $[0, T + \tilde{T}]$ , where the extended time  $\tilde{T}$  and the norm  $\|u(t)\|_{L_p} + \|v(t)\|_{H_p^1}$ ,  $0 \leq t \leq T + \tilde{T}$ , are estimated by the same constant  $C_{U_0}$ . The existence interval can be again extended, to  $[0, T + 2\tilde{T}]$ . Repeating this procedure proves the global existence theorem with the estimate (5).  $\square$

### Appendix . On the domains of fractional powers of Laplace operators in $L_p$ -spaces

Here we discuss the characterization of the domains of definition of fractional powers of Laplace operator  $A_0 = -\Delta + 1$  with Neumann boundary condition on a bounded domain  $\Omega \subset \mathbb{R}^n$  with smooth boundary, as a closed operator in  $L_p(\Omega)$  for each  $1 < p < \infty$ .

We have already known the following facts.

**Theorem A.1** ([2, Theorem 2.4.1.3], [24, Theorem 5.3.4], [30, Theorem 2.15]). *For each  $1 < p < \infty$ ,  $A_0$  is considered as a closed operator in  $L_p(\Omega)$ , the domain of which is  $H_{p,N}^2(\Omega)$ . If we denote  $A_p = A_0|_{L_p}$ , then it holds that  $\mathcal{D}(A_p) = H_{p,N}^2(\Omega)$  with norm equivalence.*

**Theorem A.2** ([2, Theorem 2.5.1.1], [24, Theorems 5.3.4 and 5.4.1]). *Let  $k$  be a positive integer and  $1 < p < \infty$ . Then  $u \in H_p^{k+2}(\Omega) \cap H_{p,N}^2(\Omega)$  yields  $A_0 u \in H_p^k(\Omega)$ . Moreover, if  $u \in H_{p,N}^2(\Omega)$  satisfies  $A_0 u \in H_p^k(\Omega)$ , then  $u \in H_p^{k+2}(\Omega)$ . That means, as the first example, if we denote  $\mathfrak{A}_p = A_0|_{H_p^1}$ , that the identity  $\mathcal{D}(\mathfrak{A}_p) = H_{p,N}^3(\Omega)$  holds with norm equivalence.*

To interpolate these results between  $k = 0$  and  $k = 2$ , we apply the theory of bounded  $H_\infty$  functional calculus in  $L_p(\Omega)$  given by Yagi [30, Sec.16.1.2].

**Theorem A.3.** *For the operator  $A_p = A_0|_{L_p}$ , the identity*

$$(A.1) \quad \mathcal{D}(A_p^\theta) = [L_p(\Omega), H_{p,N}^2(\Omega)]_\theta = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \leq \theta < \frac{1}{2} + \frac{1}{2p} \\ H_{p,N}^{2\theta}(\Omega) & \text{for } \frac{1}{2} + \frac{1}{2p} < \theta \leq 1 \end{cases}$$

holds with norm equivalence.

Proof. Firstly, it is obvious that  $L_p(\Omega)$  is a reflexive Banach space and  $A_p$  is a sectorial operator in  $L_p(\Omega)$  with angle  $\omega_A = 0$ . Hence, we can directly verify the following condition given in [30, Theorem 16.5] with  $A = A_p$ ,  $X = L_p(\Omega)$ ,  $X^* = L_{p'}(\Omega)$  and  $\langle \cdot, \cdot \rangle$  their duality product:

(H) For every angle  $\omega_A < \omega < \pi$  and every exponent  $0 < \theta < 1$ , the integrable condition along the V-shaped contour  $\Gamma_\omega : \lambda = \rho e^{\pm i\omega}$  ( $0 \leq \rho < \infty$ )

$$(A.2) \quad I_{\omega,\theta} = \int_{\Gamma_\omega} |\lambda|^{2\theta-1} |\langle A^{2(1-\theta)}(\lambda - A)^{-2}F, G \rangle| |d\lambda| \leq C_{\omega,\theta} \|F\| \|G\|_*, \quad F \in X, G \in X^*,$$

holds with some constant  $C_{\omega,\theta} > 0$ .

We omit the detail here. Then, by [30, Theorem 16.5] it is verified that  $A_p$  has a bounded  $H_\infty$  functional calculus in  $L_p(\Omega)$ . Again by [30, Theorem 16.5], we have the first identity of (A.1). The rest part of the theorem has been already shown in [30, Theorem 16.11].  $\square$

The next theorem shows the interpolation result between  $k = 1$  and  $k = 3$ .

**Theorem A.4.** *For the operator  $\mathfrak{A}_p = A_0|_{H_p^1}$ , the identity*

$$(A.3) \quad \mathcal{D}(\mathfrak{A}_p^\theta) = [H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta = \begin{cases} H_p^{1+2\theta}(\Omega) & \text{for } 0 \leq \theta < \frac{1}{2p} \\ H_{p,N}^{1+2\theta}(\Omega) & \text{for } \frac{1}{2p} < \theta \leq 1 \end{cases}$$

holds with norm equivalence.

Proof. It is also obvious that  $H_p^1(\Omega)$  is a reflexive Banach space with duality product  $\langle \langle \cdot, \cdot \rangle \rangle$  of  $H_p^1(\Omega) \times H_{p'}^1(\Omega)$ , and that  $\mathfrak{A}_p$  is a sectorial operator in  $H_p^1(\Omega)$  with angle  $\omega_A = 0$ . Hence, again we can prove the condition (H) above by direct calculation to see that the Laplace operator  $\mathfrak{A}_p = A_0|_{H_p^1}$  has a bounded  $H_\infty$  functional calculus in  $H_p^1(\Omega)$ . Then, we have the first identity of (A.3) again by [30, Theorem 16.5]. For the proof of the rest part of the theorem, we must follow carefully the proof of [30, Theorem 16.11].

*Step 1:*  $\mathcal{D}(\mathfrak{A}_p^\theta) \subset H_{p,(N)}^{1+2\theta}(\Omega)$ . We can easily see that  $[H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta \subset [H_p^1(\Omega), H_p^3(\Omega)]_\theta = H_p^{1+2\theta}(\Omega)$ . To see the boundary condition for  $1/(2p) < \theta < 1$ , take a sequence  $\{u_k\} \subset \mathcal{D}(\mathfrak{A}_p) = H_{p,N}^3(\Omega)$  converging to  $u$  in  $\mathcal{D}(\mathfrak{A}_p^\theta)$ . This implies that every  $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$  satisfies the Neumann boundary condition on  $\partial\Omega$ .

*Step 2:*  $\mathcal{D}(\mathfrak{A}_p^\theta) \supset H_{p,(N)}^{1+2\theta}(\Omega)$ . We divide this step into three parts.

(i)  $\frac{1}{2} \leq \theta \leq 1$ . Let  $u \in H_{p,N}^{1+2\theta}(\Omega)$ . Then, for any  $v \in H_{p',N}^3(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$ , similarly in the proof of [30, Theorem 16.11], we see

$$\begin{aligned}
|\langle \langle u, (\mathfrak{A}_p^*)^\theta v \rangle \rangle| &= \left| \frac{1}{2\pi i} \int_{\Gamma} \lambda^\theta \langle \langle u, (\lambda - \mathfrak{A}_p^*)^{-1} v \rangle \rangle d\lambda \right| \\
&\leq \|A_0 u\|_{H_p^{2\theta-1}} \left\| \frac{1}{2\pi i} \int_{\Gamma} \lambda^\theta (\lambda - A_0)^{-1} v d\lambda \right\|_{(H_p^{2\theta-1})'} \\
&\leq C \|A_0 u\|_{H_p^{2\theta-1}} \left\| A_0^{\theta-\frac{1}{2}} v \right\|_{H_{p'}^{2-2\theta}} \leq C \|u\|_{H_p^{1+2\theta}} \|v\|_{H_{p'}^1}.
\end{aligned}$$

Here we utilize [30, Theorem 1.43] for the boundedness of  $A_0 = 1 - \sum_k D_k^2 : H_p^{2\theta+1}(\Omega) \rightarrow H_p^{2\theta-1}(\Omega)$  and Theorem 3 for the boundedness of the fractional powers of  $A_0|_{L_p} = A_p$ . This inequality yields that, for each fixed  $u \in H_{p,N}^{1+2\theta}(\Omega)$ , the linear form  $\langle \langle u, (\mathfrak{A}_p^*)^\theta v \rangle \rangle$  is a bounded linear functional of  $v \in H_{p'}^1(\Omega)$ , that is, there exists  $w \in H_p^1(\Omega)$  such that  $\langle \langle u, (\mathfrak{A}_p^*)^\theta v \rangle \rangle = \langle \langle w, v \rangle \rangle$  for any  $v \in H_{p'}^1(\Omega)$ . Hence,  $w = \mathfrak{A}_p^\theta u$  and  $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$ .

(ii)  $\frac{1}{2p} < \theta < \frac{1}{2}$ . Let  $u \in H_{p,N}^{1+2\theta}(\Omega)$ . Then, for any  $v \in H_{p',N}^3(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$ , by an argument quite similar to (i), we see

$$|\langle \langle u, (\mathfrak{A}_p^*)^\theta v \rangle \rangle| \leq C \|u\|_{H_p^{2\theta+1}} \left\| \frac{1}{2\pi i} \int_{\Gamma} \lambda^\theta (\lambda - A_0)^{-1} v d\lambda \right\|_{H_{p'}^{1-2\theta}} \leq C \|u\|_{H_p^{2\theta+1}} \|v\|_{H_{p'}^1}.$$

Here we utilize again [30, Theorem 1.43] for the boundedness of  $D_k : H_{p'}^{1-2\theta}(\Omega) \rightarrow (H_p^{2\theta}(\Omega))'$ . Thus, for each fixed  $u \in H_{p,N}^{1+2\theta}(\Omega)$ , there exists  $w \in H_p^1(\Omega)$  such that  $\langle \langle u, (\mathfrak{A}_p^*)^\theta v \rangle \rangle = \langle \langle w, v \rangle \rangle$  for any  $v \in H_{p'}^1(\Omega)$ . Hence,  $w = \mathfrak{A}_p^\theta u$  and  $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$ .

(iii)  $0 < \theta < \frac{1}{2p}$ . We can verify that  $u \in H_p^{1+2\theta}(\Omega)$  is contained in  $\mathcal{D}(\mathfrak{A}_p^\theta)$  by the same argument as (ii) except for the boundary conditions.

Hence we complete the proof.  $\square$

As the consequence, we conclude (17) in Section 2.

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