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CONSTRUCTION OF VERSAL GALOIS COVERINGS USING TORIC VARIETIES

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Abstract

In this article we give an explicit construction of versal Galois coverings for any given finite subgroup of $GL(n, \mathbb{Z})$. By this construction we give a positive answer to Question 1.4 in [5].

Introduction

Let X and Y be normal projective varieties. Let $\pi: X \rightarrow Y$ be a finite surjective morphism. We denote the rational function fields of X and Y by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, respectively. Under these circumstances, one can regard $\mathbb{C}(Y)$ as a subfield of $\mathbb{C}(X)$ by $\pi^*: \mathbb{C}(Y) \rightarrow \mathbb{C}(X)$.

DEFINITION 0.1. π is said to be a Galois covering if $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension. We call π a G -covering when the Galois group of the field extension is isomorphic to a finite group G .

REMARK 0.2. Note that there exists a natural G -action on X such that $Y = X/G$.

In [2], Namba gave a method for constructing new G -coverings from a given G -covering as follows: Let $\pi: X \rightarrow Y$ be a G -covering. Let W be a normal projective variety.

NOTATION 0.3. We denote the stabilizer of $x \in X$ by G_x . Also we define $\text{Fix}(X, G)$ by

$$\text{Fix}(X, G) = \{x \in X \mid G_x \neq \{1\}\}.$$

DEFINITION 0.4. A rational map $\nu: W \dashrightarrow Y$ is called a G -indecomposable rational map to Y if $\nu(W) \not\subset \pi(\text{Fix}(X, G))$ and ν does not factor through $\pi_H: X/H \rightarrow Y$ for any H , where X/H is the quotient variety of X by a subgroup $H \subset G$ and π_H is the quotient morphism.

Fix a G -indecomposable rational map $\nu: W \dashrightarrow Y$. Let W_0 be the graph of ν . Then we can obtain a G -covering Z over W by taking the $\mathbb{C}(W_0 \times_Y X)$ normalization of W . We also obtain a G -equivariant rational map from $\mu: Z \dashrightarrow X$ such that $\mu(Z) \not\subset \text{Fix}(X, G)$. We can construct many new G -coverings in this manner. However, we may not be able to construct every G -covering by this method, as the construction depends on the existence of a G -indecomposable rational map. This leads us to the notion of a versal G -covering introduced in [5] and [6].

DEFINITION 0.5. $\varpi: X \rightarrow Y$ is called a versal G -covering if, for any G -covering $\pi': Z \rightarrow W$, there exists a G -equivariant rational map $\mu: Z \dashrightarrow X$ such that $\mu(Z) \not\subset \text{Fix}(X, G)$.

REMARK 0.6. μ induces a G -indecomposable rational map ν from W to Y , and Z coincides with the G -covering constructed by the method above by using ν . Note that the versal G -covering here is not unique.

By the definition any G -covering can be obtained as a “rational pullback” from a versal G -covering. As for the existence of versal G -coverings, Namba proved the following.

Theorem 0.7 (Namba [2]). *For any finite group G , there exists a versal G -covering.*

Namba explicitly constructed a versal G -covering for each finite group G . However his method of construction gave versal coverings with dimensions equal to the order of the given group G , and it does not seem to be practical to use it in order to construct new Galois coverings. In [6], Tsuchihashi constructed versal G -coverings over the projective space \mathbb{P}^n for the symmetric groups and for a generalization of the symmetric groups using toric varieties. In this paper we generalize Tsuchihashi’s result partially and construct versal coverings of dimension n for any subgroup G of $GL(n, \mathbb{Z})$. Our result is the following.

Theorem 0.8. *Let N be a free \mathbb{Z} -module, Δ a projective fan in $N_{\mathbb{R}}$. Let $X(\Delta)$ be the toric variety associated to the fan Δ . Let G be a subgroup of $\text{Aut}_{\mathbb{Z}}(N)$ which keeps Δ invariant. Then G acts naturally on $X(\Delta)$ and*

$$\varpi: X(\Delta) \rightarrow X(\Delta)/G$$

is a versal G -covering.

1. Construction and proof of versality

In this section we will prove Theorem 0.8. We will first construct projective toric varieties with G -action and construct G -coverings by taking the quotient variety and the quotient morphism. Then we prove that the G -coverings that we have constructed are versal.

We will mostly follow Fulton [1] for notations concerning toric varieties. Let N be a free \mathbb{Z} -module of rank n . Let M be the dual module of N . We denote the dual pairing by $\langle u, v \rangle$ for $u \in M$ and $v \in N$. We denote a fan by Δ , and denote the toric variety associated to the fan Δ by $X(\Delta)$. We will say a fan to be a projective fan when $X(\Delta)$ is a projective variety. For basic properties of toric varieties, we refer the reader to Fulton [1] and Oda [3, 4].

A toric variety $X(\Delta)$ with G -action for a given finite subgroup G of $GL(n, \mathbb{Z})$ can be constructed as follows.

Suppose that Δ is a complete G -invariant fan (i.e. for any $g \in G$ and any $\sigma \in \Delta$ there exists $\sigma' \in \Delta$ such that $g(\sigma) = \sigma'$). Then $g: N \rightarrow N$, for any $g \in G$, induces an automorphism of varieties $g_{\sharp}: X(\Delta) \rightarrow X(\Delta)$. Thus we can define a G -action on $X(\Delta)$. We will abuse notation and denote g_{\sharp} by g . By the following lemma there exists a complete projective invariant fan for any finite subgroup G of $GL(n, \mathbb{Z})$.

Lemma 1.1. *For any finite subgroup G of $GL(n, \mathbb{Z})$, there exists a complete projective G -invariant fan.*

Proof. Take a fan Δ' of $N_{\mathbb{R}}$ corresponding to $(\mathbb{P}^1)^n$. It is a fan obtained by decomposing $N_{\mathbb{R}}$ with hyperplanes. By taking the images of these hyperplanes by G and by decomposing $N_{\mathbb{R}}$ with this new set of hyperplanes, we obtain a G -invariant fan Δ of $N_{\mathbb{R}}$. By the proof of Proposition 2.17 in [3], a complete fan obtained as a hyperplane decomposition is projective, hence Δ is projective. \square

By taking the quotient variety X/G of X by G , and taking the quotient morphism $\varpi: X \rightarrow X/G$ we obtain a G -covering. We will now prove some lemmas in order to show that the G -coverings constructed in the fashion above are versal.

Lemma 1.2. *Let $X(\Delta)$ be a complete projective toric variety with G -action. Then there exists a G -invariant T_N -invariant very ample divisor on $X(\Delta)$.*

Proof. Since $X(\Delta)$ is projective, there exists a T_N -invariant very ample divisor D on $X(\Delta)$. Let D' be

$$D' = \frac{1}{|G_{D'}|} \sum_{g \in G} g(D)$$

where $G_D = \{g \in G \mid g(D) = D\}$. Then D' is a G -invariant T_N -invariant divisor. It remains to show that D' is ample.

For any T_N -invariant ample divisors D_1 and D_2 the sum $D_1 + D_2$ is also ample. This is true since if D_1 and D_2 are ample, the piecewise linear functions ψ_{D_1} and ψ_{D_2} corresponding to D_1 and D_2 respectively are strictly convex. Then $\psi_{D_1+D_2}$ is also strictly convex which implies the ampleness of $D_1 + D_2$.

Each $g(D)$ is ample so D' is an ample divisor and for some m , mD' is a very ample G -invariant T_N -invariant divisor. \square

Let $\Delta(1)$ be the set of one dimensional cones of Δ . Let D_{τ_i} be the T_N -invariant divisor corresponding to $\tau_i \in \Delta(1)$. Let v_i be a primitive generator of τ_i . Let $D = \sum_{\tau_i \in \Delta(1)} a_i D_{\tau_i}$ be a G -invariant T_N -invariant cartier divisor (which implies $a_i = a_j$ if there exists $g \in G$ such that $g(\tau_i) = \tau_j$). Then $P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -a_i, \forall v_i \in \Delta(1)\} \subset M_{\mathbb{R}}$ is also G -invariant. From [1] p.66, the global sections of the sheaf $\mathcal{O}(D)$ is generated by ω^u , $u \in P_D \cap M$.

$$H^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u.$$

Hence we can define a (right) G -action on the global sections of the sheaf $\mathcal{O}(D)$ by $(\omega^u) \cdot g^* \mapsto \omega^{(u)g^*}$.

Define $u(\sigma) \in M$ by $\langle u(\sigma), v \rangle = \psi_D(u)|_{\sigma}$. Then from [1] p.62, $\Gamma(U_{\sigma}, \mathcal{O}(D)) = \chi^{u(\sigma)} \cdot A_{\sigma}$. Thus we have local trivialization isomorphisms $\eta_{\sigma}: \Gamma(U_{\sigma}, \mathcal{O}(D)) \cong A_{\sigma}$ given by $\omega^u \mapsto \chi^{u-u(\sigma)}$. Let σ and σ' be maximal cones of Δ and suppose there exists $g \in G$ such that $g(\sigma) = \sigma'$. Since D is G -invariant we have $(u(\sigma'))g^* = u(\sigma)$. Then

$$\eta_{\sigma}(\omega^u \cdot g^*) = \chi^{(u)g^* - u(\sigma)} = \chi^{(u - u(\sigma'))g^* - u(\sigma)} = \eta_{\sigma'}(\omega^u) \cdot g^*.$$

Hence this action on the global sections of $\mathcal{O}(D)$ coincides with the geometric action of G on $X(\Delta)$.

Lemma 1.3. *For a finite set of vectors $\{u_1, \dots, u_s \in M\}$, there exists $v \in N$ such that $\{\langle u_i, v \rangle\}_{i=1, \dots, s}$ are mutually distinct.*

Proof. We prove this by induction on the rank on M . For $\text{rank}(M) = 1$ take any $u \neq 0$.

Let $\text{rank}(M) = k$. Fix a basis for M and let $u_i = (a_{i1}, \dots, a_{ik})$. Define a projection p onto a lattice of rank $k-1$ by $(a_{i1}, \dots, a_{ik}) \mapsto (a_{i2}, \dots, a_{ik})$. Then by the hypothesis of induction there exists $v' = (b_2, \dots, b_k)$ such that $\langle p(u_i), v' \rangle$ are distinct for distinct $p(u_i)$. Let $b_1 = 2 \max\{|\langle p(u_i), v' \rangle|\}_{i=1, \dots, s} + 1$. Then $v = (b_1, \dots, b_s)$ satisfies the desired condition. This can be checked directly.

Let $u_i = (a_{i_1}, \dots, a_{i_k})$, $u_j = (a_{j_1}, \dots, a_{j_k})$, $i \neq j$. If $a_{i_1} > a_{j_1}$ then

$$\begin{aligned} \langle u_i, v \rangle - \langle u_j, v \rangle &= (a_{i_1} - a_{j_1})b_1 + \left(\sum_{t=2}^k a_{i_t} b_t \right) - \left(\sum_{t=2}^k a_{j_t} b_t \right) \\ &> b_1 + \left(\sum_{t=2}^k a_{i_t} b_t \right) - \left(\sum_{t=2}^k a_{j_t} b_t \right) \\ &\geq 1 \quad (\text{by the choice of } b_1). \end{aligned}$$

If $a_{i_1} = a_{j_1}$ then $p(u_i) \neq p(u_j)$ and

$$\begin{aligned} \langle u_i, v \rangle - \langle u_j, v \rangle &= 0 + \left(\sum_{t=2}^k a_{i_t} b_t \right) - \left(\sum_{t=2}^k a_{j_t} b_t \right) \\ &\neq 0 \quad (\text{by the choice of } v'). \end{aligned}$$

Hence $\{\langle u_i, v \rangle\}_{i=1, \dots, s}$ are distinct. □

Lemma 1.4. *Let $\pi': Z \rightarrow W$ be a G -covering. Let $G = \{g_1, \dots, g_{|G|}\}$.*

- (1) *There exists $z \in Z$ such that $z_i = g_i(z)$ ($i = 1, \dots, |G|$) are mutually distinct.*
- (2) *For any $\alpha_1, \dots, \alpha_{|G|} \in \mathbb{C}$ there exists a rational function f on Z such that $f(z_i) = \alpha_i$.*
- (3) *If $\alpha_i \neq 0$ for all i , then there exists a G -invariant affine open set U such that there exists a point z in U satisfying (1) and a function f satisfying (2) and in addition f and f^{-1} are regular on U .*

Proof. Let $U' = \text{Spec}(R)$ be an G -invariant affine open set of Z where G acts freely. Then clearly any point z of U' satisfies (1).

For any finite number of distinct points $z_i \in U'$, $i = 1, \dots, s$ and for any $\alpha_i \in \mathbb{C}$, $i = 1, \dots, s$, there exists a regular function f on U satisfying $f(z_i) = \alpha_i$. This is proved by induction on the number of points. The case where $s = 1$ is trivial. Let $s = k$ and let $\mathfrak{m}_i \subset R$ be the maximal ideal corresponding to the point z_i . Then $\mathfrak{m}_i \setminus \bigcup_{j \neq i} \mathfrak{m}_j \neq \emptyset$. For each i take a regular function $f_i \in \mathfrak{m}_i \setminus \bigcup_{j \neq i} \mathfrak{m}_j$. Then

$$f_1 \cdots f_{k-1}(z_i) \begin{cases} = 0, & i = 1, \dots, k-1 \\ \neq 0, & i = k \end{cases}.$$

By the hypothesis of induction, there exists regular functions h, h' satisfying $h(z_i) = \xi_i$ for $i = 1, \dots, k-1$, and $h'(z_k) = (\alpha_k - h(z_k))/(f_1 \cdots f_{k-1}(z_k))$. Then $f = h + f_1 \cdots f_k \cdot h'$ satisfies $f(z_i) = \alpha_i$ for $i = 1, \dots, k$. Hence we have a regular function satisfying (2).

Let $V = \text{Spec}(R_f)$. Then $U = \bigcap_{g \in G} g(V)$ satisfies (3). □

Let $\pi': Z \rightarrow W$ be any G -covering. Let f be a rational function on Z . For f , $u \in M$, $v \in N$, define $f^{u,v}$ by

$$f^{u,v} = \prod_{g \in G} f^{\langle u, g(v) \rangle} \cdot (g^{-1}).$$

Then $f^{u,v}$ satisfies the following properties (1) and (2) for any u_1 and $u_2 \in M$ and any $g' \in G$.

$$(1) \quad \begin{aligned} f^{u_1,v} \cdot f^{u_2,v} &= \prod_{g \in G} f^{\langle u_1+u_2, g(v) \rangle} \cdot (g^{-1}) \\ &= f^{u_1+u_2,v} \end{aligned}$$

$$(2) \quad \begin{aligned} f^{u,v} \cdot (g') &= \prod_{g \in G} f^{\langle u, g(v) \rangle} \cdot (g^{-1}(g')) \\ &= \prod_{g'' \in G} f^{\langle (u)g', g''(v) \rangle} (g''^{-1}) \\ &= f^{(u)g',v}. \end{aligned}$$

Let $V = \text{Spec}(R)$ be a G -invariant affine open set of Z where f and $1/f$ are regular. Define a ring homomorphism $\mu_f^v: R \rightarrow \mathbb{C}[M]$ by $\mu_f^v(\chi^u) = f^{u,v}$. Then from equations (1) and (2) above, μ_f^v is a G -equivariant ring homomorphism. Thus we obtain a G -equivariant morphism of varieties $\mu_f^{v,\sharp}: V \rightarrow T_N = \text{Spec}(\mathbb{C}[M])$.

We will show that we can choose a rational function f of Z and $v \in N$ so that $\mu_f^v(Z) \not\subset \text{Fix}(X(\Delta), G)$.

Let D be a G - T_N invariant very ample divisor of $X(\Delta)$. Then

$$H^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u$$

as before. Put $h = \dim(H^0(X(\Delta), \mathcal{O}(D)))$, and $\{u_1 = 0, u_2, \dots, u_h\} = P_D \cap M$. Put $g(i) = j$ when $(u_i)g = u_j$. Note again that P_D is G -invariant.

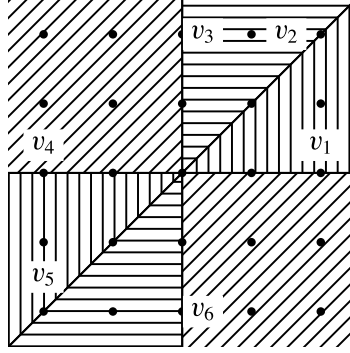
Let $\Phi_{|D|}$ be the morphism associated to the divisor D and embed $X(\Delta)$ into \mathbb{P}^{h-1} . For $x \in X(\Delta)$, $\Phi_{|D|}(x)$ is given by

$$\Phi_{|D|}(x) = [\omega^0(x) : \omega^{u_2}(x) : \dots : \omega^{u_h}(x)].$$

Restricting to T_N ,

$$\Phi_{|D|}|_{T_N}(x) = [1 : \chi^{u_2}(x) : \dots : \chi^{u_h}(x)]$$

since $\omega^0 \neq 0$ on T_N .



Take $z \in Z$ so that $\{g_i z \mid g_i \in G\}$ are distinct. Take $f \in \mathbb{C}(Z)$ so that $|f(z)| \neq 1, 0$ and $f(gz) = 1$ for $g \neq 1_G$. Let $V = \text{Spec}(R)$ be an affine G -invariant open set where f, f^{-1} are regular. Take $v \in N$ so that $\{\langle u_i, v \rangle = c_i\}_{i=1, \dots, h}$ are distinct. Then

$$\begin{aligned} \Phi_{|D|} \circ \mu_f^v(z) &= [1 : f(z)^{c_2} : \dots : f(z)^{c_h}] \\ \Phi_{|D|} \circ \mu_f^v(gz) &= [1 : f(gz)^{c_2} : \dots : f(gz)^{c_h}] \\ &= [1 : f(z)^{c_{g(2)}} : \dots : f(z)^{c_{g(h)}}] \end{aligned}$$

and we can see that $\{\mu_f^v(gz)\}_{g \in G}$ are distinct so $\mu_f^v(z) \notin \text{Fix}(X(\Delta), G)$.

Thus we have proved Theorem 0.8.

2. Examples

Here we give some examples of versal G -coverings. Generally it is difficult to compute the quotient, but in some cases it is possible.

EXAMPLE 2.1 (Namba). We will restate Namba's construction of versal G -coverings from our point of view. Let $G = \{g_1, \dots, g_n\}$ be any finite group of order n . Let N be a lattice of rank n and let $\{e_{g_1}, \dots, e_{g_n}\}$ be a basis of N . Then G can be identified to a subgroup of $\text{Aut}(N)$. The action of G on N is defined by $g(e_{g_i}) = e_{gg_i}$. Let Δ be the complete fan of N consisting of cones generated by $\{\pm e_{g_1}, \dots, \pm e_{g_n}\}$. Then Δ is a complete projective G -invariant fan and $X(\Delta) \cong (\mathbb{P}^1)^n$. Then $\varpi: X(\Delta) \rightarrow X(\Delta)/G$ is a versal galois covering from Theorem 0.8. Thus a versal G -covering exists for any finite group.

EXAMPLE 2.2. Let N be a lattice of rank 2 and $\{e_1, e_2\}$ be a basis of N . Let Δ be the complete fan of N generated by $v_1 = v_1, v_2 = e_2, v_3 = -e_1 + e_2, v_4 = -e_1, v_5 = -e_2, v_6 = e_1 - e_2$, as in the figure above. Thus $X(\Delta)$ is isomorphic to \mathbb{P}^2 blown-up along three points.

Let G be the subgroup of $\text{Aut}(N)$ generated by

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $G = \langle \alpha, \beta \mid \alpha^6 = \beta^2 = (\alpha\beta)^2 = 1 \rangle \cong D_{12}$ where D_{12} is the dihedral group of order 12. Δ is an invariant fan of G and by Theorem 0.8, $\varpi: X(\Delta) \rightarrow X(\Delta)/G$ is a versal Galois covering. One can compute the quotient as the weighted projective space $\mathbb{P}(1, 1, 2)$. This is done by taking the very ample divisor $D = \sum_{i=1}^6 D_{v_i}$ and compute the D_{12} -invariant ring of

$$\bigoplus_{i=1}^{\infty} H^0(X, \mathcal{O}(iD)).$$

It is generated by algebraically independent elements of weight 1, 1, and 2.

Proposition 2.3. *Example 2.2 gives a positive answer to Question 1.4 in [5].*

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