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## CONSTRUCTION OF VERSAL GALOIS COVERINGS USING TORIC VARIETIES

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### Abstract

In this article we give an explicit construction of versal Galois coverings for any given finite subgroup of  $GL(n, \mathbb{Z})$ . By this construction we give a positive answer to Question 1.4 in [5].

### Introduction

Let  $X$  and  $Y$  be normal projective varieties. Let  $\pi: X \rightarrow Y$  be a finite surjective morphism. We denote the rational function fields of  $X$  and  $Y$  by  $\mathbb{C}(X)$  and  $\mathbb{C}(Y)$ , respectively. Under these circumstances, one can regard  $\mathbb{C}(Y)$  as a subfield of  $\mathbb{C}(X)$  by  $\pi^*: \mathbb{C}(Y) \rightarrow \mathbb{C}(X)$ .

DEFINITION 0.1.  $\pi$  is said to be a Galois covering if  $\mathbb{C}(X)/\mathbb{C}(Y)$  is a Galois extension. We call  $\pi$  a  $G$ -covering when the Galois group of the field extension is isomorphic to a finite group  $G$ .

REMARK 0.2. Note that there exists a natural  $G$ -action on  $X$  such that  $Y = X/G$ .

In [2], Namba gave a method for constructing new  $G$ -coverings from a given  $G$ -covering as follows: Let  $\pi: X \rightarrow Y$  be a  $G$ -covering. Let  $W$  be a normal projective variety.

NOTATION 0.3. We denote the stabilizer of  $x \in X$  by  $G_x$ . Also we define  $\text{Fix}(X, G)$  by

$$\text{Fix}(X, G) = \{x \in X \mid G_x \neq \{1\}\}.$$

DEFINITION 0.4. A rational map  $\nu: W \dashrightarrow Y$  is called a  $G$ -indecomposable rational map to  $Y$  if  $\nu(W) \not\subset \pi(\text{Fix}(X, G))$  and  $\nu$  does not factor through  $\pi_H: X/H \rightarrow Y$  for any  $H$ , where  $X/H$  is the quotient variety of  $X$  by a subgroup  $H \subset G$  and  $\pi_H$  is the quotient morphism.

Fix a  $G$ -indecomposable rational map  $\nu: W \dashrightarrow Y$ . Let  $W_0$  be the graph of  $\nu$ . Then we can obtain a  $G$ -covering  $Z$  over  $W$  by taking the  $\mathbb{C}(W_0 \times_Y X)$  normalization of  $W$ . We also obtain a  $G$ -equivariant rational map from  $\mu: Z \dashrightarrow X$  such that  $\mu(Z) \not\subset \text{Fix}(X, G)$ . We can construct many new  $G$ -coverings in this manner. However, we may not be able to construct every  $G$ -covering by this method, as the construction depends on the existence of a  $G$ -indecomposable rational map. This leads us to the notion of a versal  $G$ -covering introduced in [5] and [6].

**DEFINITION 0.5.**  $\varpi: X \rightarrow Y$  is called a versal  $G$ -covering if, for any  $G$ -covering  $\pi': Z \rightarrow W$ , there exists a  $G$ -equivariant rational map  $\mu: Z \dashrightarrow X$  such that  $\mu(Z) \not\subset \text{Fix}(X, G)$ .

**REMARK 0.6.**  $\mu$  induces a  $G$ -indecomposable rational map  $\nu$  from  $W$  to  $Y$ , and  $Z$  coincides with the  $G$ -covering constructed by the method above by using  $\nu$ . Note that the versal  $G$ -covering here is not unique.

By the definition any  $G$ -covering can be obtained as a ‘‘rational pullback’’ from a versal  $G$ -covering. As for the existence of versal  $G$ -coverings, Namba proved the following.

**Theorem 0.7** (Namba [2]). *For any finite group  $G$ , there exists a versal  $G$ -covering.*

Namba explicitly constructed a versal  $G$ -covering for each finite group  $G$ . However his method of construction gave versal coverings with dimensions equal to the order of the given group  $G$ , and it does not seem to be practical to use it in order to construct new Galois coverings. In [6], Tsuchihashi constructed versal  $G$ -coverings over the projective space  $\mathbb{P}^n$  for the symmetric groups and for a generalization of the symmetric groups using toric varieties. In this paper we generalize Tsuchihashi’s result partially and construct versal coverings of dimension  $n$  for any subgroup  $G$  of  $GL(n, \mathbb{Z})$ . Our result is the following.

**Theorem 0.8.** *Let  $N$  be a free  $\mathbb{Z}$ -module,  $\Delta$  a projective fan in  $N_{\mathbb{R}}$ . Let  $X(\Delta)$  be the toric variety associated to the fan  $\Delta$ . Let  $G$  be a subgroup of  $\text{Aut}_{\mathbb{Z}}(N)$  which keeps  $\Delta$  invariant. Then  $G$  acts naturally on  $X(\Delta)$  and*

$$\varpi: X(\Delta) \rightarrow X(\Delta)/G$$

*is a versal  $G$ -covering.*

**1. Construction and proof of versality**

In this section we will prove Theorem 0.8. We will first construct projective toric varieties with  $G$ -action and construct  $G$ -coverings by taking the quotient variety and the quotient morphism. Then we prove that the  $G$ -coverings that we have constructed are versal.

We will mostly follow Fulton [1] for notations concerning toric varieties. Let  $N$  be a free  $\mathbb{Z}$ -module of rank  $n$ . Let  $M$  be the dual module of  $N$ . We denote the dual pairing by  $\langle u, v \rangle$  for  $u \in M$  and  $v \in N$ . We denote a fan by  $\Delta$ , and denote the toric variety associated to the fan  $\Delta$  by  $X(\Delta)$ . We will say a fan to be a projective fan when  $X(\Delta)$  is a projective variety. For basic properties of toric varieties, we refer the reader to Fulton [1] and Oda [3, 4].

A toric variety  $X(\Delta)$  with  $G$ -action for a given finite subgroup  $G$  of  $GL(n, \mathbb{Z})$  can be constructed as follows.

Suppose that  $\Delta$  is a complete  $G$ -invariant fan (i.e. for any  $g \in G$  and any  $\sigma \in \Delta$  there exists  $\sigma' \in \Delta$  such that  $g(\sigma) = \sigma'$ ). Then  $g: N \rightarrow N$ , for any  $g \in G$ , induces an automorphism of varieties  $g_{\sharp}: X(\Delta) \rightarrow X(\Delta)$ . Thus we can define a  $G$ -action on  $X(\Delta)$ . We will abuse notation and denote  $g_{\sharp}$  by  $g$ . By the following lemma there exists a complete projective invariant fan for any finite subgroup  $G$  of  $GL(n, \mathbb{Z})$ .

**Lemma 1.1.** *For any finite subgroup  $G$  of  $GL(n, \mathbb{Z})$ , there exists a complete projective  $G$ -invariant fan.*

*Proof.* Take a fan  $\Delta'$  of  $N_{\mathbb{R}}$  corresponding to  $(\mathbb{P}^1)^n$ . It is a fan obtained by decomposing  $N_{\mathbb{R}}$  with hyperplanes. By taking the images of these hyperplanes by  $G$  and by decomposing  $N_{\mathbb{R}}$  with this new set of hyperplanes, we obtain a  $G$ -invariant fan  $\Delta$  of  $N_{\mathbb{R}}$ . By the proof of Proposition 2.17 in [3], a complete fan obtained as a hyperplane decomposition is projective, hence  $\Delta$  is projective.  $\square$

By taking the quotient variety  $X/G$  of  $X$  by  $G$ , and taking the quotient morphism  $\varpi: X \rightarrow X/G$  we obtain a  $G$ -covering. We will now prove some lemmas in order to show that the  $G$ -coverings constructed in the fashion above are versal.

**Lemma 1.2.** *Let  $X(\Delta)$  be a complete projective toric variety with  $G$ -action. Then there exists a  $G$ -invariant  $T_N$ -invariant very ample divisor on  $X(\Delta)$ .*

*Proof.* Since  $X(\Delta)$  is projective, there exists a  $T_N$ -invariant very ample divisor  $D$  on  $X(\Delta)$ . Let  $D'$  be

$$D' = \frac{1}{|G_{D'}|} \sum_{g \in G} g(D)$$

where  $G_D = \{g \in G \mid g(D) = D\}$ . Then  $D'$  is a  $G$ -invariant  $T_N$ -invariant divisor. It remains to show that  $D'$  is ample.

For any  $T_N$ -invariant ample divisors  $D_1$  and  $D_2$  the sum  $D_1 + D_2$  is also ample. This is true since if  $D_1$  and  $D_2$  are ample, the piecewise linear functions  $\psi_{D_1}$  and  $\psi_{D_2}$  corresponding to  $D_1$  and  $D_2$  respectively are strictly convex. Then  $\psi_{D_1+D_2}$  is also strictly convex which implies the ampleness of  $D_1 + D_2$ .

Each  $g(D)$  is ample so  $D'$  is an ample divisor and for some  $m$ ,  $mD'$  is a very ample  $G$ -invariant  $T_N$ -invariant divisor.  $\square$

Let  $\Delta(1)$  be the set of one dimensional cones of  $\Delta$ . Let  $D_{\tau_i}$  be the  $T_N$ -invariant divisor corresponding to  $\tau_i \in \Delta(1)$ . Let  $v_i$  be a primitive generator of  $\tau_i$ . Let  $D = \sum_{\tau_i \in \Delta(1)} a_i D_{\tau_i}$  be a  $G$ -invariant  $T_N$ -invariant cartier divisor (which implies  $a_i = a_j$  if there exists  $g \in G$  such that  $g(\tau_i) = \tau_j$ ). Then  $P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -a_i, \forall v_i \in \Delta(1)\} \subset M_{\mathbb{R}}$  is also  $G$ -invariant. From [1] p.66, the global sections of the sheaf  $\mathcal{O}(D)$  is generated by  $\omega^u$ ,  $u \in P_D \cap M$ .

$$H^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u.$$

Hence we can define a (right)  $G$ -action on the global sections of the sheaf  $\mathcal{O}(D)$  by  $(\omega^u) \cdot g^* \mapsto \omega^{(u)g^*}$ .

Define  $u(\sigma) \in M$  by  $\langle u(\sigma), v \rangle = \psi_D(u)|_{\sigma}$ . Then from [1] p.62,  $\Gamma(U_{\sigma}, \mathcal{O}(D)) = \chi^{u(\sigma)} \cdot A_{\sigma}$ . Thus we have local trivialization isomorphisms  $\eta_{\sigma} : \Gamma(U_{\sigma}, \mathcal{O}(D)) \cong A_{\sigma}$  given by  $\omega^u \mapsto \chi^{u-u(\sigma)}$ . Let  $\sigma$  and  $\sigma'$  be maximal cones of  $\Delta$  and suppose there exists  $g \in G$  such that  $g(\sigma) = \sigma'$ . Since  $D$  is  $G$ -invariant we have  $(u(\sigma'))g^* = u(\sigma)$ . Then

$$\eta_{\sigma}(\omega^u \cdot g^*) = \chi^{(u)g^* - u(\sigma)} = \chi^{(u-u(\sigma'))g^*} = \eta_{\sigma'}(\omega^u) \cdot g^*.$$

Hence this action on the global sections of  $\mathcal{O}(D)$  coincides with the geometric action of  $G$  on  $X(\Delta)$ .

**Lemma 1.3.** *For a finite set of vectors  $\{u_1, \dots, u_s \in M\}$ , there exists  $v \in N$  such that  $\{\langle u_i, v \rangle\}_{i=1, \dots, s}$  are mutually distinct.*

*Proof.* We prove this by induction on the rank on  $M$ . For  $\text{rank}(M) = 1$  take any  $u \neq 0$ .

Let  $\text{rank}(M) = k$ . Fix a basis for  $M$  and let  $u_i = (a_{i_1}, \dots, a_{i_k})$ . Define a projection  $p$  onto a lattice of rank  $k-1$  by  $(a_{i_1}, \dots, a_{i_k}) \mapsto (a_{i_2}, \dots, a_{i_k})$ . Then by the hypothesis of induction there exists  $v' = (b_2, \dots, b_k)$  such that  $\langle p(u_i), v' \rangle$  are distinct for distinct  $p(u_i)$ . Let  $b_1 = 2 \max\{|\langle p(u_i), v' \rangle|\}_{i=1, \dots, s} + 1$ . Then  $v = (b_1, \dots, b_s)$  satisfies the desired condition. This can be checked directly.

Let  $u_i = (a_{i_1}, \dots, a_{i_k}), u_j = (a_{j_1}, \dots, a_{j_k}), i \neq j$ . If  $a_{i_1} > a_{j_1}$  then

$$\begin{aligned} \langle u_i, v \rangle - \langle u_j, v \rangle &= (a_{i_1} - a_{j_1})b_1 + \left( \sum_{t=2}^k a_{i_t} b_t \right) - \left( \sum_{t=2}^k a_{j_t} b_t \right) \\ &> b_1 + \left( \sum_{t=2}^k a_{i_t} b_t \right) - \left( \sum_{t=2}^k a_{j_t} b_t \right) \\ &\geq 1 \quad (\text{by the choice of } b_1). \end{aligned}$$

If  $a_{i_1} = a_{j_1}$  then  $p(u_i) \neq p(u_j)$  and

$$\begin{aligned} \langle u_i, v \rangle - \langle u_j, v \rangle &= 0 + \left( \sum_{t=2}^k a_{i_t} b_t \right) - \left( \sum_{t=2}^k a_{j_t} b_t \right) \\ &\neq 0 \quad (\text{by the choice of } v'). \end{aligned}$$

Hence  $\{\langle u_i, v \rangle\}_{i=1, \dots, s}$  are distinct. □

**Lemma 1.4.** *Let  $\pi': Z \rightarrow W$  be a  $G$ -covering. Let  $G = \{g_1, \dots, g_{|G|}\}$ .*

- (1) *There exists  $z \in Z$  such that  $z_i = g_i(z)$  ( $i = 1, \dots, |G|$ ) are mutually distinct.*
- (2) *For any  $\alpha_1, \dots, \alpha_{|G|} \in \mathbb{C}$  there exists a rational function  $f$  on  $Z$  such that  $f(z_i) = \alpha_i$ .*
- (3) *If  $\alpha_i \neq 0$  for all  $i$ , then there exists a  $G$ -invariant affine open set  $U$  such that there exists a point  $z$  in  $U$  satisfying (1) and a function  $f$  satisfying (2) and in addition  $f$  and  $f^{-1}$  are regular on  $U$ .*

*Proof.* Let  $U' = \text{Spec}(R)$  be an  $G$ -invariant affine open set of  $Z$  where  $G$  acts freely. Then clearly any point  $z$  of  $U'$  satisfies (1).

For any finite number of distinct points  $z_i \in U', i = 1, \dots, s$  and for any  $\alpha_i \in \mathbb{C}, i = 1, \dots, s$ , there exists a regular function  $f$  on  $U$  satisfying  $f(z_i) = \alpha_i$ . This is proved by induction on the number of points. The case where  $s = 1$  is trivial. Let  $s = k$  and let  $\mathfrak{m}_i \subset R$  be the maximal ideal corresponding to the point  $z_i$ . Then  $\mathfrak{m}_i \setminus \bigcup_{j \neq i} \mathfrak{m}_j \neq \emptyset$ . For each  $i$  take a regular function  $f_i \in \mathfrak{m}_i \setminus \bigcup_{j \neq i} \mathfrak{m}_j$ . Then

$$f_1 \cdots f_{k-1}(z_i) \begin{cases} = 0, & i = 1, \dots, k-1 \\ \neq 0, & i = k \end{cases}.$$

By the hypothesis of induction, there exists regular functions  $h, h'$  satisfying  $h(z_i) = \xi_i$  for  $i = 1, \dots, k-1$ , and  $h'(z_k) = (\alpha_k - h(z_k))/(f_1 \cdots f_{k-1}(z_k))$ . Then  $f = h + f_1 \cdots f_k \cdot h'$  satisfies  $f(z_i) = \alpha_i$  for  $i = 1, \dots, k$ . Hence we have a regular function satisfying (2).

Let  $V = \text{Spec}(R_f)$ . Then  $U = \bigcap_{g \in G} g(V)$  satisfies (3). □

Let  $\pi': Z \rightarrow W$  be any  $G$ -covering. Let  $f$  be a rational function on  $Z$ . For  $f$ ,  $u \in M$ ,  $v \in N$ , define  $f^{u,v}$  by

$$f^{u,v} = \prod_{g \in G} f^{\langle u, g(v) \rangle} \cdot (g^{-1}).$$

Then  $f^{u,v}$  satisfies the following properties (1) and (2) for any  $u_1$  and  $u_2 \in M$  and any  $g' \in G$ .

$$(1) \quad \begin{aligned} f^{u_1,v} \cdot f^{u_2,v} &= \prod_{g \in G} f^{\langle u_1+u_2, g(v) \rangle} \cdot (g^{-1}) \\ &= f^{u_1+u_2,v} \end{aligned}$$

$$(2) \quad \begin{aligned} f^{u,v} \cdot (g') &= \prod_{g \in G} f^{\langle u, g(v) \rangle} \cdot (g^{-1}(g')) \\ &= \prod_{g'' \in G} f^{\langle u, g'(g''(v)) \rangle} (g''^{-1}) \\ &= f^{(u)g',v}. \end{aligned}$$

Let  $V = \text{Spec}(R)$  be a  $G$ -invariant affine open set of  $Z$  where  $f$  and  $1/f$  are regular. Define a ring homomorphism  $\mu_f^v: R \rightarrow \mathbb{C}[M]$  by  $\mu_f^v(\chi^u) = f^{u,v}$ . Then from equations (1) and (2) above,  $\mu_f^v$  is a  $G$ -equivariant ring homomorphism. Thus we obtain a  $G$ -equivariant morphism of varieties  $\mu_f^{v,\sharp}: V \rightarrow T_N = \text{Spec}(\mathbb{C}[M])$ .

We will show that we can choose a rational function  $f$  of  $Z$  and  $v \in N$  so that  $\mu_f^v(Z) \not\subset \text{Fix}(X(\Delta), G)$ .

Let  $D$  be a  $G$ - $T_N$  invariant very ample divisor of  $X(\Delta)$ . Then

$$H^0(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \omega^u$$

as before. Put  $h = \dim(H^0(X(\Delta), \mathcal{O}(D)))$ , and  $\{u_1 = 0, u_2, \dots, u_h\} = P_D \cap M$ . Put  $g(i) = j$  when  $(u_i)g = u_j$ . Note again that  $P_D$  is  $G$ -invariant.

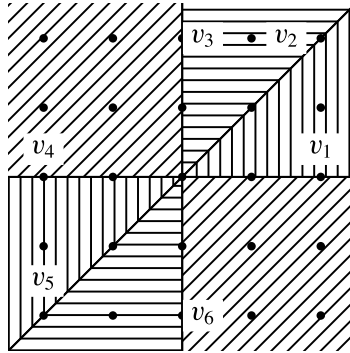
Let  $\Phi_{|D|}$  be the morphism associated to the divisor  $D$  and embed  $X(\Delta)$  into  $\mathbb{P}^{h-1}$ . For  $x \in X(\Delta)$ ,  $\Phi_{|D|}(x)$  is given by

$$\Phi_{|D|}(x) = [\omega^0(x) : \omega^{u_2}(x) : \dots : \omega^{u_h}(x)].$$

Restricting to  $T_N$ ,

$$\Phi_{|D|}|_{T_N}(x) = [1 : \chi^{u_2}(x) : \dots : \chi^{u_h}(x)]$$

since  $\omega^0 \neq 0$  on  $T_N$ .



Take  $z \in Z$  so that  $\{g_i z \mid g_i \in G\}$  are distinct. Take  $f \in \mathbb{C}(Z)$  so that  $|f(z)| \neq 1, 0$  and  $f(gz) = 1$  for  $g \neq 1_G$ . Let  $V = \text{Spec}(R)$  be an affine  $G$ -invariant open set where  $f, f^{-1}$  are regular. Take  $v \in N$  so that  $\{u_i, v\} = c_i\}_{i=1, \dots, h}$  are distinct. Then

$$\begin{aligned} \Phi_{|D|} \circ \mu_f^v(z) &= [1 : f(z)^{c_2} : \dots : f(z)^{c_h}] \\ \Phi_{|D|} \circ \mu_f^v(gz) &= [1 : f(gz)^{c_2} : \dots : f(gz)^{c_h}] \\ &= [1 : f(z)^{c_{g(2)}} : \dots : f(z)^{c_{g(h)}}] \end{aligned}$$

and we can see that  $\{\mu_f^v(gz)\}_{g \in G}$  are distinct so  $\mu_f^v(z) \notin \text{Fix}(X(\Delta), G)$ .

Thus we have proved Theorem 0.8.

## 2. Examples

Here we give some examples of versal  $G$ -coverings. Generally it is difficult to compute the quotient, but in some cases it is possible.

EXAMPLE 2.1 (Namba). We will restate Namba's construction of versal  $G$ -coverings from our point of view. Let  $G = \{g_1, \dots, g_n\}$  be any finite group of order  $n$ . Let  $N$  be a lattice of rank  $n$  and let  $\{e_{g_1}, \dots, e_{g_n}\}$  be a basis of  $N$ . Then  $G$  can be identified to a subgroup of  $\text{Aut}(N)$ . The action of  $G$  on  $N$  is defined by  $g(e_{g_i}) = e_{gg_i}$ . Let  $\Delta$  be the complete fan of  $N$  consisting of cones generated by  $\{\pm e_{g_1}, \dots, \pm e_{g_n}\}$ . Then  $\Delta$  is a complete projective  $G$ -invariant fan and  $X(\Delta) \cong (\mathbb{P}^1)^n$ . Then  $\varpi: X(\Delta) \rightarrow X(\Delta)/G$  is a versal galois covering from Theorem 0.8. Thus a versal  $G$ -covering exists for any finite group.

EXAMPLE 2.2. Let  $N$  be a lattice of rank 2 and  $\{e_1, e_2\}$  be a basis of  $N$ . Let  $\Delta$  be the complete fan of  $N$  generated by  $v_1 = v_1, v_2 = e_2, v_3 = -e_1 + e_2, v_4 = -e_1, v_5 = -e_2, v_6 = e_1 - e_2$ , as in the figure above. Thus  $X(\Delta)$  is isomorphic to  $\mathbb{P}^2$  blown-up along three points.



Let  $G$  be the subgroup of  $\text{Aut}(N)$  generated by

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then  $G = \langle \alpha, \beta \mid \alpha^6 = \beta^2 = (\alpha\beta)^2 = 1 \rangle \cong D_{12}$  where  $D_{12}$  is the dihedral group of order 12.  $\Delta$  is an invariant fan of  $G$  and by Theorem 0.8,  $\varpi: X(\Delta) \rightarrow X(\Delta)/G$  is a versal Galois covering. One can compute the quotient as the weighted projective space  $\mathbb{P}(1, 1, 2)$ . This is done by taking the very ample divisor  $D = \sum_{i=1}^6 D_{v_i}$  and compute the  $D_{12}$ -invariant ring of

$$\bigoplus_{i=1}^{\infty} H^0(X, \mathcal{O}(iD)).$$

It is generated by algebraically independent elements of weight 1, 1, and 2.

**Proposition 2.3.** Example 2.2 gives a positive answer to Question 1.4 in [5].

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