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δ-HOMOGENEITY IN FINSLER GEOMETRY AND THE POSITIVE CURVATURE PROBLEM

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Abstract

In this paper, we explore the similarity between normal homogeneity and δ-homogeneity in Finsler geometry. They are both non-negatively curved Finsler spaces. We show that any connected δ-homogeneous Finsler space is G-δ-homogeneous, for some suitably chosen connected quasi-compact G. So δ-homogeneous Finsler metrics can be defined by a bi-invariant singular metric on G and submersion, just as normal homogeneous metrics, using a bi-invariant Finsler metric on G instead. More careful analysis shows, in the space of all Finsler metrics on G/H, the subset of all G-δ-homogeneous ones is in fact the closure for the subset of all G-normal ones, in the local C^0-topology (Theorem 1.1). Using this approximation technique, the classification work for positively curved normal homogeneous Finsler spaces can be applied to classify positively curved δ-homogeneous Finsler spaces, which provides the same classification list. As a by-product, this argument tells more about δ-homogeneous Finsler metrics satisfying the (FP) condition (a weaker version of positively curved condition).

1. Introduction

The concept of δ-homogeneity is introduced by V. N. Berestovskii and C. P. Plaut [4], and extensively studied in Riemannian geometry [5] [6] [7] [8]. Recall that a connected metric space (M, d) is called G-δ-homogeneous, where G is a closed connected subgroup of the isometry group $I(M, d)$, if for any point $x, y \in M$, there exists an isometry $g \in G$ satisfying $g(x) = y$ and the displacement function $f(\cdot) = d(\cdot, g(\cdot))$ on M reaches its maximum at x (such an isometry $g \in G$ is called a $\delta(x)$-translation).

In [20], L. Zhang and S. Deng defined and studied δ-homogeneity in Finsler geometry. They provided several equivalent descriptions for the G-δ-homogeneity. Instead of the isometry $g \in G$ which achieves the transitivity and maximal displacement, we can also use $\delta(\cdot)$-Killing vector fields or δ-vectors from $\mathfrak{g} = \mathrm{Lie}(G)$ (see [20] or Section 2). Their alternative description for a G-δ-homogeneous Finsler metric F on G/H, is the foundation for defining the Chebyshev metric $\tilde{F}$, i.e. a bi-invariant singular metric on G induced by F on $M = G/H$. Then F is canonically determined by the Chebyshev metric and the Finsler submersion $\pi : (G, \tilde{F}) \to (G/H, F)$. This alternative description for δ-homogeneity in Finsler geometry is very similar to that for normal homogeneity, which uses smooth bi-invariant Finsler metrics on G instead. Though defining the Chebyshev metric, or any other bi-invariant smooth or singular metric on G, will require G to be quasi-compact, i.e. $\mathfrak{g}$ is

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compact, we prove this is not an essential obstacle, i.e. for any connected δ-homogeneous Finsler space (M, F), we can choose a suitable connected quasi-compact Lie group G, such that (M, F) is G-δ-homogeneous (see Theorem 3.6 below).

So both normal homogeneity and δ-homogeneity can be defined by submersion. This explains the phenomenon that they share many geometric properties, for example, both have 0 S-curvature and non-negative flag curvature. This similarity in their defining patterns can be formulated as the following approximation theorem, where we apply the fundamental technique of approximating a singular metric (or a singular norm) by smooth ones in the local C^0-topology.

Theorem 1.1. *Assume G is a connected quasi-compact Lie group which acts effectively on the smooth coset space G/H. In the space of all smooth Finsler metric on G/H, the subset of all G-δ-homogeneous ones coincides with the closure for all G-normal homogeneous ones in the local C^0-topology.*

Here the local C^0-topology means, on the fixed manifold M, the sequence of singular or smooth Finsler metrics $\{F_n : TM \to \mathbb{R}\}$ converge to $F : TM \to \mathbb{R}$, iff they uniformly converge to F when restricted each compact subset of TM.

Theorem 1.1 brings a natural question: what geometric properties of G-normal homogeneous Finsler metrics can be passed to their local C^0-limit metrics, i.e. those G-δ-homogeneous ones? Studying this question in the general context seems to be very intriguing. But local C^0-convergence is weak for studying most curvature properties, and there is no convenient comparison theorems in Finsler geometry as in Riemannian geometry.

In this paper, we will use Theorem 1.1, and the method in [17] to study the classification of positively curved δ-homogeneous Finsler spaces. The key observation in [17] is that a flat splitting subalgebra (see [17] or Section 4) provides totally geodesic flat subspace with a dimension bigger than 1 in a normal homogeneous Finsler space. So it is an obstacle for positive flag curvature. It still works for δ-homogeneous Finsler spaces (see Lemma 4.1), so we have the following theorem.

Theorem 1.2. *Let G be a compact connected Lie group which acts effectively on the smooth coset space G/H. Then G/H admits positively curved G-δ-homogeneous Finsler metrics iff it admits positively curved G-normal homogeneous Riemannian metrics.*

Notice a positively curved homogeneous Finsler space (M, F) is compact by Bonnet-Myers Theorem, so does its isometry group $I(M, F)$, through which G acts on G/H. The classification for positively curved homogeneous spaces is only up to local isometries. We can "cancel" some local product factor of G contained in H without changing the metric, and at the same time with the effectiveness of G satisfied. So Theorem 1.2 indicates the classification list for positively curved δ-homogeneous Finsler spaces coincides with that in [17], or that of M. Berger in [2] (plus B. Wilking's space [15]). To be precise, it consists of

(1) Rank one compact symmetric spaces

$$S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1), \mathbb{C}\mathrm{P}^{n-1} = \mathrm{SU}(n)/S(\mathrm{U}(n-1)\mathrm{U}(1)),$$
$$\mathbb{H}\mathrm{P}^{n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)\mathrm{Sp}(1), \text{ and } \mathbb{F}_4/\mathrm{Spin}(9).$$

(2) Other homogeneous spheres and complex projective spaces

$$
\begin{aligned}
S^{2n-1} &= \mathrm{SU}(n)/\mathrm{SU}(n-1) = \mathrm{U}(n)/\mathrm{U}(n-1),\\
S^{4n-1} &= \mathrm{Sp}(n)/\mathrm{Sp}(n-1) = \mathrm{Sp}(n)\mathrm{U}(1)/\mathrm{Sp}(n-1)\mathrm{U}(1)\\
&= \mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n-1)\mathrm{Sp}(1),\\
S^6 &= \mathrm{G}_2/\mathrm{SU}(3), S^7 = \mathrm{Spin}(7)/\mathrm{G}_2,\\
S^{15} &= \mathrm{Spin}(9)/\mathrm{Spin}(7), \text{ and } \mathbb{C}\mathrm{P}^{2n-1} = \mathrm{Sp}(n)/\mathrm{Sp}(n-1)\mathrm{U}(1).
\end{aligned}
$$

(3) Berger's spaces $\mathrm{SU}(5)/\mathrm{Sp}(2)\mathrm{U}(1)$ and $\mathrm{Sp}(2)/\mathrm{SU}(2)$.
(4) Wilking's space $S_{1,1} = \mathrm{SU}(3) \times \mathrm{SO}(3)/\mathrm{U}(2)$.

The flat splitting subalgebra and the totally geodesic technique can tell more when we study δ-homogeneous Finsler spaces satisfying the (FP) condition (a weaker version of positive flag curvature condition, which is also called the flag-wise positively curved condition; see [18] or Section 4). We prove a flag-wise positively curved δ-homogeneous Finsler space G/H must be compact, and satisfy the rank inequality as positively curved homogeneous spaces [19]. When $\dim G/H$ is even, it admits flag-wise positively curved δ-homogeneous (or normal homogeneous) Finsler metrics iff it admits positively curved δ-homogeneous (or normal homogeneous) Finsler metrics. But when $\dim G/H$ is odd, there are many more examples of flag-wise positively curved δ-homogeneous Finsler spaces than those positively curved ones.

2. Preliminaries

In this section, we collect some fundamental knowledge on Finsler geometry from [1] and [13], and Finsler submersion from [12].

2.1. Minkowski norm and Finsler metric. A *Minkowski norm* on a real vector space $\mathbf{V}$, $\dim \mathbf{V} = n$, is a continuous function $F : \mathbf{V} \to [0, +\infty)$ satisfying the following conditions:

(1) Positiveness and smoothness: F is a positive smooth function on $\mathbf{V}\backslash\{0\}$.
(2) Positively homogeneity of degree one: $F(\lambda y) = \lambda F(y)$ when $\lambda \geq 0$.
(3) Strong convexity: given any basis $\{e_1, \ldots, e_n\}$ of $\mathbf{V}$ and correspondingly the linear coordinates $y = y^i e_i$, the Hessian matrix

$$(g_{ij}(y)) = \left(\frac{1}{2}[F^2(y)]_{y^i y^j}\right)$$

is positive definite whenever $y \neq 0$.

We will call $(\mathbf{V}, F)$ a *Minkowski space*.

The Hessian matrix defines an inner product $\langle \cdot, \cdot \rangle_y^F$ on $\mathbf{V}$ by

$$\langle u, v \rangle_y^F = g_{ij}(y) u^i v^j = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}s} F^2(y + tu + sv)|_{s=t=0},$$

from which we see it is independent of the choice for the linear basis.

A *Finsler metric* on a smooth manifold M is a continuous function F on TM such that its restriction to the slit tangent bundle $TM\backslash 0$ is smooth, and its restriction to each tangent space is a Minkowski norm. We will also call (M, F) a *Finsler space*. Given any smooth tangent field Y on M which is non-vanishing everywhere in an open set $\mathcal{U} \subset M$, the Hessian matrices $(g_{ij}(x, Y(x)))$ or equivalently the inner products $\langle \cdot, \cdot \rangle_{Y(x)}^F$ at each $x \in \mathcal{U}$ define a

smooth Riemannian metric on $\mathcal{U}$. We will simply call this metric the *localization of F at Y*, and denote it as g_Y^F.

Important examples of Finsler metrics include Riemannian metrics, Randers metrics, (α,β)-metrics, etc. Riemannian metrics are a special class of Finsler metrics whose Hessian matrices at each point only depends on $x \in M$ rather than $y \in T_xM$. A Riemannian metric can also be defined as a global smooth section $g_{ij}dx^i dx^j$ of $\mathrm{Sym}^2(T^*M)$. Randers metrics are the most simple and important class of non-Riemannian metrics in Finsler geometry. They are defined by $F = \alpha + \beta$, in which α is a Riemannian metric and β is a 1-form. They can be naturally generalized as (α,β)-metrics which have the form $F = \alpha\phi(\beta/\alpha)$ with a positive smooth function ϕ and similar α and β as Randers metrics. In recent years, there have been a lot of research works for (α,β)-metrics as well as for Randers metrics.

2.2. Geodesic, Riemannian curvature, and totally geodesic subspace. On a Finsler space (M,F), we usually choose the *standard local coordinates* (x^i,y^j), where $x=(x^i)\in M$ and $y=y^j\partial_{x^j}\in T_xM$, to present the connections, curvatures, and other geometric quantities.

The geodesics are important geometric subjects in Finsler geometry. They are smooth curves which satisfies the locally minimizing principle with respect to the distance function $d_F(\cdot,\cdot)$ induced by the Finsler metric F. Notice $d_F(\cdot,\cdot)$ may not be reversible in general. We will always parametrize a geodesic $c(t)$ on (M,F) to have a nonzero constant speed, i.e. the F-length of the tangent field $\dot{c}(t)=\frac{d}{dt}c(t)$ is a positive constant for all t. Then the geodesics can be equivalently defined as following.

First, we have a globally defined smooth vector field $\mathbf{G}$ on $TM\backslash 0$, called the *the geodesic spray*. For any standard local coordinates, it can be presented as

$$\mathbf{G} = y^i\partial_{x^i} - 2\mathbf{G}^i\partial_{y^i}, \tag{2.1}$$

where

$$\mathbf{G}^i = \frac{1}{4}g^{il}([F^2]_{x^k y^l}y^k - [F^2]_{x^l}). \tag{2.2}$$

Then a curve $c(t)$ on M is a geodesic of nonzero constant speed if and only if $(c(t),\dot{c}(t))$ is an integration curve of $\mathbf{G}$. For standard local coordinates, a geodesic $c(t)=(c^i(t))$ satisfies the equations

$$\ddot{c}^i(t) + 2\mathbf{G}^i(c(t),\dot{c}(t)) = 0. \tag{2.3}$$

The coefficients $\mathbf{G}^i$ of the geodesic spray $\mathbf{G}$ are important for us to present curvatures in Finsler geometry.

For example, we have a similar curvature as in the Riemannian case, which is called the *Riemann curvature*. It can be defined either by the Jacobi field equation for the variation of a constant speed geodesic, or by the structure equation for the curvature of the Chern connection.

Using any standard local coordinates, the Riemannian curvature can be presented as $R_y^F = R_k^i(y)\partial_{x^i}\otimes dx^k : T_xM \to T_xM$, where

$$R_k^i(y) = 2\partial_{x^k}G^i - y^j\partial^2_{x^j y^k}G^i + 2G^j\partial^2_{y^j y^k}G^i - \partial_{y^j}G^i\partial_{y^k}G^j. \tag{2.4}$$

Using the Riemannian curvature, the sectional curvature can be generalized to Finsler geometry, which is called the *flag curvature*. We call $(x,y,\mathbf{P})$ a *flag triple*, if x is a point in

M, $\mathbf{P}$ is a tangent plane in some T_xM, and y is a nonzero vector in $\mathbf{P}$. Assume $\mathbf{P}$ is linearly spanned by y and v, then the flag curvature for $(x, y, \mathbf{P})$ is

$$K^F(x, y, \mathbf{P}) = \frac{\langle R^F_y v, v\rangle^F_y}{\langle y, y\rangle^F_y \langle v, v\rangle^F_y - (\langle y, v\rangle^F_y)^2}. \tag{2.5}$$

When F is a Riemannian metric, it is just the sectional curvature for $(x, \mathbf{P})$, which is independent of the choice of y.

Using Riemann curvature or flag curvature, the Ricci curvature can also be generalized to Finsler geometry, i.e. for any $x \in M$ and nonzero tangent vector $y \in T_xM$,

$$\mathrm{Ric}^F(x, y) = \mathrm{tr}(R^F_y) = \sum_{i=1}^{n} F(y)^2 K^F(x, y, y \wedge v_i),$$

where $\{v_1, \ldots, v_n\}$ is an orthogonal basis for the g^F_y-orthogonal complement of y in T_xM.

Z. Shen has an important observation that all these curvatures described above can be closely related to Riemannian geometry (see Proposition 6.2.2 in [13]), so we may call them *Riemannian curvatures*. In this work, we will need the following refinement of his observation.

Proposition 2.1. *Let Y be a smooth vector field on a Finsler space (M, F), such that $y = Y(x) \neq 0$, and Y generates a geodesic of constant speed through x, then $R^F_y = R^{g^F_Y}_y$, $\mathrm{Ric}^F(x, y) = \mathrm{Ric}^{g^F_Y}(x, y)$, and for any tangent plane $\mathbf{P}$ in T_xM containing y, $K^F(x, y, \mathbf{P}) = K^{g^F_Y}(x, \mathbf{P})$.*

The statements for R^F_y and $K^F(x, y, \mathbf{P})$ are contained in Theorem 4.2 in [19]. The statement for $\mathrm{Ric}^F(x, y)$ then follows easily.

A submanifold N of a smooth Finsler space (M, F) can be naturally endowed with a smooth submanifold metric, denoted as $F|_N$. At each $x \in N$, the Minkowski norm $F|_N(x, \cdot)$ is just the restriction of the Minkowski norm $F(x, \cdot)$ from T_xM to T_xN. We say that $(N, F|_N)$ is a *Finsler submanifold* or a *Finsler subspace*.

For the study of Riemann curvature and flag curvature, the most important Finsler subspaces are totally geodesic subspaces. A Finsler subspace $(N, F|_N)$ is totally geodesic iff, for any standard local coordinate system (x^i, y^j) such that N is locally defined by $x^{k+1} = \cdots = x^n = 0$ where $n = \dim M$, we have

$$G^i(x, y) = 0, \quad k < i \leq n, y \in T_xN.$$

A direct calculation shows that in this case, the Riemann curvature $R^{F|_N}_y : T_xN \to T_xN$ of $(N, F|_N)$ is just the restriction of the Riemann curvature R^F_y of (M, F), when y is a nonzero tangent vector in T_xN at $x \in N$. Therefore, we have [17]

Proposition 2.2. *Let $(N, F|_N)$ be a totally geodesic submanifold of (M, F). Then for any flag $(x, y, \mathbf{P})$ in N (i.e. $x \in N$, y is nonzero tangent vector in T_xN, and $\mathbf{P}$ is a tangent plane containing y), we have*

$$K^{F|_N}(x, y, \mathbf{P}) = K^F(x, y, \mathbf{P}).$$

2.3. Finsler Submersion. A linear map $\pi : (\mathbf{V}_1, F_1) \to (\mathbf{V}_2, F_2)$ between two Minkowski spaces is called a *Finsler submersion*, if we have

$$\pi(\{w \in \mathbf{V} \text{ with } F_1(w) \leq 1\}) = \{u \in \mathbf{V} \text{ with } F_2(u) \leq 1\}. \tag{2.6}$$

It is obvious that a submersion map π must be surjective, and that the Minkowski norm F_2 on $\mathbf{V}_2$ is uniquely determined by the following equality,

$$F_2(u) = \inf\{F_1(w) | \pi(w) = u\}.$$

Given the Minkowski space $(\mathbf{V}_1, F_1)$ and a surjective linear map $\pi : \mathbf{V}_1 \to \mathbf{V}_2$, there exists a unique Minkowski norm F_2 on $\mathbf{V}_2$ such that π is a submersion. We usually say that F_2 is induced by F_1 and submersion.

Given the submersion $\pi : (\mathbf{V}_1, F_1) \to (\mathbf{V}_2, F_2)$, the horizonal lift of a nonzero vector $u \in \mathbf{V}_2$ is the unique $w \in \mathbf{V}_1$ satisfying the conditions

$$\pi(w) = u \text{ and } F_1(w) = F_2(u). \tag{2.7}$$

Then $\pi : (\mathbf{V}_1, \langle\cdot,\cdot\rangle_w^{F_1}) \to (\mathbf{V}_2, \langle\cdot,\cdot\rangle_u^{F_2})$ is also a submersion between Euclidean spaces.

A smooth map $\pi : (M_1, F_1) \to (M_2, F_2)$ between two Finsler spaces is called a submersion, if for any $x \in M_1$ the induced tangent map $\pi_* : (T_xM_1, F_1(x,\cdot)) \to (T_{\rho(x)}M_2, F_2(\rho(x),\cdot))$ is a Finsler submersion between Minkowski spaces. Given the surjective smooth map $\pi : M_1 \to M_2$, and a Finsler metric F_1 on M_1, if there exists a metric F_2 on M_2 which makes π a Finsler submersion, we will call F_2 the *induced metric by F_1 and the submersion π*. The induced metric must be unique, even though it usually does not exist.

For a Finsler submersion $\pi : (M_1, F_1) \to (M_2, F_2)$, we can define the *horizonal lift for flag triples*. We call the flag triple $(x_1, y_1, \mathbf{P}_1)$ on M_1 the horizonal lift of the flag triple $(x_2, y_2, \mathbf{P}_2)$ on M_2, iff $\pi(x_1) = x_2$, y_2 is the horizonal lift of y_1, and $\mathbf{P}_1$ is the horizonal lift of $\mathbf{P}_2$ with respect to $\pi_* : (T_{x_1}M_1, \langle\cdot,\cdot\rangle_{y_1}^{F_1}) \to (T_{x_2}M_2, \langle\cdot,\cdot\rangle_{y_2}^{F_2})$.

The importance of Finsler submersion for the study of flag curvature is implied by the following theorem in [12].

Theorem 2.3. *Let $\pi : (M_1, F_1) \to (M_2, F_2)$ be a Finsler submersion, and the flag $(x_1, y_1, \mathbf{P}_1)$ on M_1 is the horizonal lift for the flag $(x_2, y_2, \mathbf{P}_2)$ on M_2. Then we have*

$$K^{F_1}(x_1, y_1, y_1 \wedge v_1) \leq K^{F_2}(x_2, y_2, y_2 \wedge v_2). \tag{2.8}$$

3. δ-homogeneous Finsler metrics

3.1. Singular Minkowski norms and singular metrics. In this work, we also need to consider singular norms and singular metrics. If not specified, Minkowski norms and Finsler metrics are referred to the smooth ones defined in Subsection 2.1. Notice in some literatures, for the notions of norms and metrics, people use "continuous" instead of "singular", "regular" instead of "smooth".

A non-negative continuous function F on a real vector space $\mathbf{V}$ is called a singular norm, if it satisfies the following conditions:

(1) Positiveness: F is a positive function on $\mathbf{V}\backslash\{0\}$.
(2) Positive homogeneity of degree one: $F(\lambda y) = \lambda F(y)$ when $\lambda \geq 0$.
(3) Convexity: $F(\lambda y_1 + (1-\lambda)y_2) \leq \lambda F(y_1) + (1-\lambda)F(y_2)$ whenever $\lambda \in [0, 1]$.

We will also call $(\mathbf{V}, F)$ a *singular normed space*.

A geometric way to describe the convexity condition for a singular norm F on $\mathbf{V}$ is pro-

vided by its indicatrix $\mathcal{I}_{F,\mathbf{V}} = \{y \in \mathbf{V} | F(y) = 1\}$. The indicatrix $\mathcal{I}_{F,\mathbf{V}}$ for a singular norm F on $\mathbf{V}$ is a sphere surrounding the origin $o \in \mathbf{V}$, which bounds a convex region in $\mathbf{V}$. The indicatrix $\mathcal{I}_{F,\mathbf{V}}$ is a smooth sub-manifold iff F is smooth on $\mathbf{V}\backslash\{0\}$. Notice even in this situation, F has not been guaranteed to be a Minkowski norm, because we only have the semi positive definiteness for the Hessian $(g_{ij}(y)) = (\frac{1}{2}[F^2(y)]_{y^i y^j})$, rather than the positive definiteness for the strong convexity condition.

Similarly, a *singular metric* F on a smooth manifold M is a continuous function F on TM, such that it is positive on the slit tangent bundle $TM\backslash 0$, and its restriction to each tangent space is a singular norm. We will call (M, F) a *singular metric space*. In practice, for the homogeneous singular metrics we will consider in this work, the smoothness is broken in the tangent direction (i.e. within T_xM for each $x \in M$), but still kept in some sense along the manifold directions.

Geodesics of singular metric spaces can still be defined using locally minimizing principle. But it is not hard to find examples that there exists more than one minimizing geodesic from x to y within a neighborhood $\mathcal{U}$ of x, no matter how small $\mathcal{U}$ is. On the other hand, most curvature concepts in Finsler geometry can not be easily generalized to the singular situation.

Submersions for singular normed spaces and singular metric spaces can be similarly defined as in Subsection 2.3. We can still use (2.7) to define the horizonal lift of a tangent vector. But the horizonal lift may not be unique. Further more, we do not have a flag curvature inequality for Finsler submersion as in Theorem 2.3.

3.2. Definition and Properties. In [20], connected δ-homogeneous Finsler spaces (we only consider connected Lie groups and connected Finsler spaces in this work) are defined as following.

Let G be a connected Lie group and $(G/H, F)$ a G-homogeneous Finsler space. We call $(G/H, F)$ a *G-δ-homogeneous Finsler space*, if for any $x, y \in G/H$, there exists an element $g \in G$, such that $g(x) = y$ and the displacement function $f(\cdot) = d_F(\cdot, g(\cdot))$ of g reaches its maximum at x. More generally, a connected Finsler space (M, F) is called *δ-homogeneous* if it is G-δ-homogeneous for $G = I_0(M, F)$.

We may always assume G is a closed connected subgroups of the connected isometry group $I_0(G/H, F)$, i.e. G acts effectively on G/H. We apply the following fundamental algebraic setup. Choose an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{m}$ can be identified with the tangent space $T_o(G/H)$ at $o = eH$. We will denote $\mathrm{pr}_{\mathfrak{h}}$ and $\mathrm{pr}_{\mathfrak{m}}$ the corresponding projections to $\mathfrak{h}$ and $\mathfrak{m}$ respectively. In this setup, a G-homogeneous Finsler metric F on G/H is one-to-one determined by an $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$, which for simplicity, will still be denoted as F [9]. This setup is also correct when F is singular.

S. Deng and L. Zhang gave an equivalent description for δ-homogeneous Finsler spaces in [20], i.e. the following theorem,

Theorem 3.1. *Let G be a connected Lie group and $(G/H, F)$ a G-homogeneous Finsler space. Then $(G/H, F)$ is G-δ-homogeneous iff one of two equivalent conditions is satisfied:*

(1) *For any $u \in \mathfrak{m}$, we can find a δ-vector $\tilde{u} \in \mathfrak{g}$ for u, i.e. $\mathrm{pr}_{\mathfrak{m}}(\tilde{u}) = u$, and the function $f(\cdot) = F(\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(\cdot)\tilde{u}))$ achieves its maximum at e.*

(2) *For any $x \in G/H$ and any tangent vector $u \in T_x(G/H)$, we can find a $\delta(x)$-Killing*

vector field X from $\mathfrak{g}$ for the tangent vector u, i.e. $X(x) = u$, and the function $f(\cdot) = F(X(\cdot))$ achieves its maximum at x.

The δ-vectors in (1) provides the $\delta(o)$-Killing vectors in (2), and their $\mathrm{Ad}(G)$-orbits provides the $\delta(\cdot)$-Killing vectors for all points and all tangent directions.

Here are some fundamental properties of δ-homogeneous Finsler spaces.

Theorem 3.2. *A δ-homogeneous Finsler space (M, F) has non-negative flag curvature.*

Proof. By Theorem 3.1, for any $x \in M$ and any nonzero tangent vector $u \in T_xM$, there is a $\delta(x)$-Killing vector field X of (M, F) for the tangent vector u. Denote g^F_X the localization of F at X defined by the Hessian matrices evaluated with the base vector $X(\cdot)$ at each point. This Riemannian metric is well defined in a neighborhood $\mathcal{U}$ of x where X is non-vanishing at each point. Because X is a Killing vector field of (M, F), it is also a Killing vector field of $(\mathcal{U}, g^F_X)$. By Lemma 3.1 in [10], the integration curve of X passing x is a geodesic of (M, F). Applying Proposition 2.1, we have $K^F(x, u, u \wedge v) = K^{g^F_X}(x, u \wedge v)$, where $v \in T_xM$ is linearly independent of u. Because the length function of X for the metric F coincides with that for the metric g^F_X inside $\mathcal{U}$, i.e.

$$F(X(\cdot))^2 \equiv \langle X(\cdot), X(\cdot)\rangle^F_{X(\cdot)} \text{ in } \mathcal{U}.$$

So the length function of the Killing vector field X for $(\mathcal{U}, g^F_X)$ also achieves its maximum at x, by [3] or Lemma 2.2 in [14], we get $K^{g^F_X}(x, u \wedge v) \geq 0$ for any $v \in T_xM$ which is linear independent with u. So $K^F(x, u, u \wedge v) \geq 0$, i.e. (M, F) is non-negatively curved. □

Proposition 3.3. *Assume $(G/H, F)$ is a G-δ-homogeneous Finsler metric, in which G is a connected Lie group and H is the compact isotropy subgroup at $o = eH \in G/H$. Then we have the following:*

(1) *Denote H_0 the identity component of H. Then the metric F naturally induces a G-δ-homogeneous metric on G/H_0, still denoted as F, such that the canonical projection $\pi : G/H_0 \to G/H$ is locally isometric.*

(2) *For any closed subgroup K of G containing H_0, we have a G-δ-homogeneous Finsler metric F' on G/K induced by F on G/H_0 such that the canonical projection map $\pi : G/H_0 \to G/K$ is a Finsler submersion.*

Proof. The proof for (1) is very easy. We only need to prove (2) with $H_0 = H$. We have an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}'$ such that $\mathfrak{m} = \mathfrak{m}' + \mathfrak{p}$, with projections $\mathrm{pr}_{\mathfrak{p}}$ and $\mathrm{pr}_{\mathfrak{m}}$ accordingly.

Denote $\mathcal{C}$ the union of all the $\mathrm{Ad}(G)$-orbits of δ-vectors, for all vectors u in $\mathfrak{m}$ with $F(u) \leq 1$. Then the G-δ-homogeneous Finsler metric F can be uniquely determined by

$$\mathrm{pr}_{\mathfrak{m}}(\mathcal{C}) = \{u \in \mathfrak{m} \text{ with } F(u) \leq 1\}. \tag{3.9}$$

Let F' be Minkowski norm on $\mathfrak{p}$ induced by the Minkowski norm F on $\mathfrak{m}$ and the Finsler submersion $\mathrm{pr}_{\mathfrak{p}}|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{p}$. Then we also have

$$\mathrm{pr}_{\mathfrak{p}}(\mathcal{C}) = \{u \in \mathfrak{p} \text{ with } F'(u) \leq 1\}. \tag{3.10}$$

By similar arguments as Lemma 3.1 in [17], and all the essential conditions: the $\mathrm{Ad}(G)$-invariance of $\mathcal{C}$, and the similarity among (3.9), (3.10) and (2.6), we see $\mathcal{C}$ defines a G-

homogeneous Finsler metric F' on G/K, and using the translations of G, the canonical projection map $\pi : (G/H, F) \to (G/K, F')$ is a Finsler submersion. Finally, we can see that for each nonzero $v \in \mathfrak{p}$, it has a horizonal lift $\tilde{v} \in \mathfrak{m}$. Then the δ-vector for $\tilde{v}$ with respect to G/H, is also a δ-vector for v with respect to G/K. Thus $(G/K, F')$ is G-δ-homogeneous. □

Notice the closed subgroups H and K in Proposition 3.3 is not required to be compact, so the G-action on G/K may not be effective. But it will not affect the δ-homogeneity here or later discussions.

3.3. Quasi-compactness and the submersion construction. When the connected Lie group G is a quasi-compact Lie group, i.e. $\mathfrak{g}$ is a compact Lie algebra, or equivalently, the universal cover of G is a product of compact semisimple Lie group and an Abelian group $\mathbb{R}^k$, then a G-homogeneous Finsler space $(G/H, F)$ is called G-normal homogeneous, when F is induced by a bi-invariant Finsler metric $\bar{F}$ on G such that the projection $\pi : G \to G/H$ is a Finsler submersion [17]. It is not hard to see all G-normal homogeneous Finsler metrics are G-δ-homogeneous as well.

On the other hand, consider a G-δ-homogeneous Finsler space $(G/H, F)$ with a connected quasi-compact G. We may choose an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, with respect to which the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is orthogonal. In this context, we have the following lemma.

Lemma 3.4. *Assume $(G/H, F)$ is a G-δ-homogeneous Finsler space with the connected quasi-compact Lie group G acting effectively on G/H. Then*

$$\tilde{F}(w) = \max_{g \in G} F(\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(g)w)) \tag{3.11}$$

defines an $\mathrm{Ad}(G)$-invariant singular norm on $\mathfrak{g}$ and correspondingly a bi-invariant singular metric on G, such that the canonical projection $\pi : (G, \tilde{F}) \to (G/H, F)$ is a submersion.

Proof. Firstly we observe $\tilde{F}$ is well defined when G is quasi-compact, because only the compact semisimple factor is relevant for defining $\tilde{F}$. Secondly, we need to prove $\tilde{F}$ satisfies all the conditions for singular norms. The argument is standard and easy. Thirdly, it is obvious to see the singular norm $\tilde{F}$ on $\mathfrak{g}$ is $\mathrm{Ad}(G)$-invariant, thus it defines a bi-invariant singular metric on G. Finally, we prove the canonical projection $\pi : (G, \tilde{F}) \to (G/H, F)$ is a Finsler submersion. The tangent map $\pi_* : (\mathfrak{g}, \tilde{F}) \to (\mathfrak{m}, F)$ coincides with $\mathrm{pr}_{\mathfrak{m}}$. It is a Finsler submersion by Theorem 3.1. By the bi-invariance of $\tilde{F}$, the G-homogeneity of F, and the argument for proving Lemma 3.1 in [17], the tangent map for the projection $\pi : (G, \tilde{F}) \to (G/H, F)$ is a submersion everywhere. □

According to [6], we call the singular norm $\tilde{F}$ defined by (3.11) the *Chebyshev norm*, and the corresponding bi-invariant singular metric the *Chebyshev metric*. To summarize, when G is quasi-compact, a G-δ-homogeneous Finsler metrics can be induced by a bi-invariant singular metric and the submersion. It is easy to check any Finsler metric produced in this process on G/H, whenever it is smooth, must be a G-δ-homogeneous Finsler metric. The horizonal lifts may not be unique, but they provides the $\delta(x)$-Killing vector fields for all $x \in G/H$ and all tangent vectors in $T_x(G/H)$. V. N. Berestovskii and Yu. G. Nikonorov

have already made these observations when G is compact or G/H is compact [6]. Now the problem is

Question 3.5. Do we still have this construction when the connected δ-homogeneous Finsler space (M, F) is not compact?

The answer is positive. We just need to choose a suitable closed connected quasi-compact subgroup $G \subset I_0(M, F)$ which acts transitively and effectively on M.

Theorem 3.6. *Assume (M, F) is a connected δ-homogeneous Finsler space. Let G be the smallest closed connected subgroup of $I_0(M, F)$ which Lie algebra $\mathfrak{g}$ contains all the $\delta(o)$-Killing vector fields in $\mathrm{Lie}(I_0(M, F))$ for a fixed $o \in M$ and all tangent vectors $u \in T_oM$, then G is quasi-compact and (M, F) is G-δ-homogeneous.*

The group G in Theorem 3.6 can be constructed as following. First, we fix $o \in M$ and use all the $\delta(o)$-Killing vectors (or all the δ-vectors) in $\mathrm{Lie}(I_0(M, F))$ to generate a subalgebra $\mathfrak{g}'$, which corresponds to a connected subgroup $G' \subset I_0(M, F)$. Then G is the closure of G' in $I_0(M, F)$. To prove Theorem 3.6, we will need the following lemma.

Lemma 3.7. *Let $G' \subset I_0(M, F)$ be the connected Lie group generated by all δ-vectors for a δ-homogeneous Finsler space (M, F), and G be the closure of G' in $I_0(M, F)$. Denote their Lie algebras as $\mathfrak{g}'$ and $\mathfrak{g}$ respectively. Then we have the following:*

(1) *If $\mathfrak{g}'' \subset \mathfrak{g}$ is a real linear subspace satisfying $[\mathfrak{g}'', \tilde{u}] \subset \mathfrak{g}''$ for all δ-vectors $\tilde{u} \in \mathfrak{g}'$, then $\mathfrak{g}''$ is an ideal of $\mathfrak{g}$.*

(2) *If $\mathfrak{g}'' \subset \mathfrak{g}$ is a real linear subspace satisfying $[\mathfrak{g}'', \tilde{u}] = 0$ for all δ-vectors $\tilde{u} \in \mathfrak{g}'$, then $\mathfrak{g}''$ is contained in the center of $\mathfrak{g}$.*

Proof. Because $[\mathfrak{g}'', \tilde{u}] \subset \mathfrak{g}''$ for all δ-vectors $\tilde{u}$ which generate $\mathfrak{g}'$, we have $\mathrm{Ad}(G')\mathfrak{g}'' \subset \mathfrak{g}''$. It is still valid when we replace G' with its closure G in $I_0(M, F)$, i.e. $\mathrm{Ad}(G)\mathfrak{g}'' \subset \mathfrak{g}''$. So $\mathfrak{g}''$ is an ideal of $\mathfrak{g}$. The proof for (1) is done. The proof for (2) is similar. □

Proof of Theorem 3.6. Because $\mathfrak{g} = \mathrm{Lie}(G)$ contains all $\delta(o)$-Killing vector fields, G acts transitively around a neighborhood of o, and thus transitively everywhere. By Theorem 3.1, (M, F) is G-δ-homogeneous. Denote $M = G/H$ with an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

We only need to prove $\mathfrak{g}$ is compact. We take Levi decomposition $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ where $\mathfrak{g}_1$ is a semisimple subalgebra, and $\mathfrak{g}_2$ is a solvable ideal. If we can prove $\mathfrak{g}_1$ is compact, and $\mathfrak{g}_2$ is the center of $\mathfrak{g}$, then the compactness of $\mathfrak{g}$ is done.

First we prove $\mathfrak{g}_2$ is the center of $\mathfrak{g}$. Consider any δ-vector $\tilde{u}$ in $\mathfrak{g}$ and $v \in [\mathfrak{g}_2, \mathfrak{g}_2]$. Because $[\mathfrak{g}_2, \mathfrak{g}_2]$ is a nilpotent ideal of $\mathfrak{g}$, the right side of

$$\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(\exp tv)\tilde{u}) = \mathrm{pr}_{\mathfrak{m}}(\tilde{u}) + t\mathrm{pr}_{\mathfrak{m}}([v, \tilde{u}]) + \frac{1}{2}t^2\mathrm{pr}_{\mathfrak{m}}([v, [v, \tilde{u}]]) + \cdots, \tag{3.12}$$

is a finite sum. Because $\tilde{u}$ is a δ-vector, the function $f(t) = F(\mathrm{pr}_{\mathfrak{m}}(\mathrm{Ad}(\exp tv)\tilde{u}))$ is bounded for $t \in \mathbb{R}$. So the vector coefficient in (3.12) for each positive power of t must vanish. In particular, $\mathrm{pr}_{\mathfrak{m}}([v, \tilde{u}]) = 0$, i.e. $[v, \tilde{u}] \in \mathfrak{h}$ for any $v \in [\mathfrak{g}_2, \mathfrak{g}_2]$ and δ-vector $\tilde{u}$. In fact, we have

$$[[\mathfrak{g}_2, \mathfrak{g}_2], \tilde{u}] \subset \mathfrak{h} \cap [\mathfrak{g}_2, \mathfrak{g}_2]$$

because $[\mathfrak{g}_2, \mathfrak{g}_2]$ is an ideal of $\mathfrak{g}$. By (1) of Lemma 3.7, $\mathfrak{h} \cap [\mathfrak{g}_2, \mathfrak{g}_2]$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. It must be 0 because $G \subset I_0(M, F)$ acts effectively on M. By (2) of Lemma 3.7, $[\mathfrak{g}_2, \mathfrak{g}_2]$ is contained in the center of $\mathfrak{g}$.

Now we consider (3.12) for any $v \in \mathfrak{g}_2$ and δ-vector $\tilde{u}$. Because $[\mathfrak{g}_2, \mathfrak{g}_2]$ is contained in the center of $\mathfrak{g}$, the right side of (3.12) is a finite sum. Similar arguments as above proves $\mathfrak{g}_2$ is contained in the center of $\mathfrak{g}$. Because $\mathfrak{g}_1$ is semi-simple, this proves $\mathfrak{g}_2$ is the center of $\mathfrak{g}$.

Then we prove $\mathfrak{g}_1$ must be compact. Assume conversely it is not. Let G_1 be the semisimple Lie group corresponding to $\mathfrak{g}_1$. We can find a maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}_1$ such that $\mathfrak{h} \subset \mathfrak{h}' = \mathfrak{k} + \mathfrak{g}_2$. Denote H' the closed subgroup of G corresponding to $\mathfrak{h}'$. By Proposition 3.3 and Theorem 3.2, G/H' admits a G-δ-homogeneous Finsler metric F'', which is a non-negatively curved. By the Iwasawa decomposition $G_1 = NAK$, where A is Abelian and N is nilpotent, the metric F' on G/H' is a left invariant Finsler metric on the solvable Lie group $G'' = NA$. Because we have assumed that $\mathfrak{g}_1$ is not compact, both N and A has positive dimensions, i.e. $\dim G'' \geq 2$. Denote $\mathfrak{g}'' = \mathfrak{a} + \mathfrak{n}$, then $[\mathfrak{g}'', \mathfrak{g}''] = \mathfrak{n}$. We can find a nonzero vector $u \in \mathfrak{g}''\backslash\mathfrak{n}$, such that $\langle u, \mathfrak{n}\rangle_u^{F''} = 0$. By Lemma 4.3 in [11], $\mathrm{Ric}^{F''}(u) \leq 0$ with equality only happens when $\mathrm{ad}(u)$ is skew symmetric with respect to $g_u^{F''}$. Because $(G/K, F'')$ has non-negative flag curvature, so $\mathrm{Ric}^{F''}(u) = 0$, and $\mathrm{ad}(u)$ is skew symmetric with respect to $g_u^{F''}$. It implies the nonzero eigenvalues of $\mathrm{ad}u : \mathfrak{g}'' \to \mathfrak{g}''$ are pure imaginary numbers. On the other hand $\mathrm{ad}(u)$ only has real eigenvalues, so u must be a nilpotent element in $\mathfrak{n}$. This is a contradiction to our choice of u from $\mathfrak{g}''\backslash\mathfrak{n}$. □

All above discussions can be summarized as the following theorem.

Theorem 3.8. *Let (M, F) be a connected δ-homogeneous Finsler metric, either compact or noncompact. Then we can find a closed connected quasi-compact subgroup $G \in I_0(M, F)$ such that (M, F) is G-δ-homogeneous, and F can be induced by the Chebyshev metric on G (i.e. the singular bi-invariant Finsler metric $\tilde{F}$ on G determined by F and* (3.11)*) and submersion.*

3.4. Normal homogeneity and δ-homogeneity. Theorem 3.8 implies all connected δ-homogeneous Finsler metrics, no matter compact or noncompact, can be constructed by the same process as normal homogeneous Finsler metrics. The only difference is that for the suitably chosen connected quasi-compact group G, we use (smooth) bi-invariant Finsler metrics for G-normal homogeneity, but singular bi-invariant Finsler metrics (i.e. the Chebyshev metrics) for G-δ-homogeneity.

Notice for any connected G-δ-homogeneous Finsler space $(G/H, F)$ where G is connected and quasi-compact, we may have many different bi-invariant singular metrics on G which defines the same F, among which the Chebyshev metric $\tilde{F}$ is the smallest one. In particular, when $(G/H, F)$ is G-normal homogeneous, there exists a smooth one, i.e. a bi-invariant Finsler metric $\bar{F}$, which meets our purpose. But generally speaking, these smooth $\bar{F}$ may not exist, i.e. G-normal-homogeneity and G-δ-homogeneity are essentially different.

Here is an example. Let M be coset space $S^3 = \mathrm{U}(2)/\mathrm{U}(1)$ where $\mathrm{U}(1)$ corresponds the right down corner. We denote each matrix

$$\begin{pmatrix} \sqrt{-1}(a+d) & b+\sqrt{-1}c \\ -b+\sqrt{-1}c & \sqrt{-1}(-a+d) \end{pmatrix} \in \mathfrak{u}(2)$$

as (a, b, c, d). Then $\mathfrak{h} = \mathbb{R}(1, 0, 0, -1)$ and $\mathfrak{m} = \{(a, b, c, a) \text{ for all } a, b, c \in \mathbb{R}\}$. We have a family of U(2)-homogeneous Riemannian metrics F_ϵ, defined by

$$F_\epsilon(a, b, c, a) = (4 - \epsilon)a^2 + b^2 + c^2$$

on $\mathfrak{m}$. When ϵ is a positive number, sufficiently close to 0, F_ϵ is an U(2)-normal homogeneous Riemannian metric. Their limit when ϵ approaches 0, i.e. F_0, is U(2)-δ homogeneous (see Lemma 3.10 below).

Now we show F_0 can not be induced by a bi-invariant Finsler metric $\bar{F}$ on U(2) and submersion. Assume conversely such a smooth metric exists. Then the indicatrix $\mathcal{I}_{\bar{F},\mathfrak{g}}$ in $\mathfrak{g}$ must be contained in the region

$$\{(a, b, c, d) | (a + d)^2 + b^2 + c^2 \leq 1\} \subset \mathfrak{g}.$$

Because it is invariant under right SU(2)-translations, $\bar{F}$ must be an (α, β)-metric corresponding to the Lie algebra decomposition $\mathfrak{u} = \mathfrak{su}(2) \oplus \mathbb{R}$. So $\mathcal{I}_{\bar{F},\mathfrak{g}}$ is contained in the region

$$\{(a, b, c, d) | \sqrt{a^2 + b^2 + c^2} \leq 1 - |d|\}.$$

On the other hand, $\mathcal{I}_{\bar{F},\mathfrak{g}}$ must contain the round sphere

$$\{(a, b, c, 0) | a^2 + b^2 + c^2 = 1\} \subset \mathfrak{su}(2),$$

where $\mathcal{I}_{\bar{F},\mathfrak{g}}$ lost its smoothness. This provides the contradiction.

3.5. Proof of Theorem 1.1. It is a fundamental observation that singular norms and singular metrics can be approximated by smooth ones. For proving Theorem 1.1, we will only use the following lemma for singular norms.

Lemma 3.9. *For any singular norm F on an n-dimensional real vector space $\mathbf{V}$, we can find a sequence of smooth Minkowski norm F_n on $\mathbf{V}$ which converge to F in the local C^0-topology.*

Proof. Denote $|\cdot|$ and $d\text{vol}_x$ the standard Euclidean norm and volume form on $\mathbf{V}$ respectively. We can find a family of smooth non-negative functions $\psi_\epsilon(x)$ on $\mathbb{R}^n$, with the parameter $\epsilon > 0$, such that each $\psi_\epsilon(x)$ is supported in $\mathcal{B}_\epsilon = \{x \in \mathbf{V} \text{ with } |x| \leq \epsilon\}$, and $\int_{x \in \mathbf{V}} \psi_\epsilon(x) dx = 1$. the convolution between F and each ψ_ϵ defines a family of smooth functions

$$F_{1;\epsilon}(x) = \int_{y \in \mathbf{V}} F(x - y)\psi_\epsilon(y) d\text{vol}_y$$

on $\mathbb{R}^n$. When the positive parameter ϵ approaches 0, $F_{1;\epsilon}$ is locally C^0-convergent to F. Because F is convex, i.e. for any $x_1, x_2 \in \mathbf{V}$ and $\lambda \in [0, 1]$, $F(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda F(x_1) + (1 - \lambda)F(x_2)$, we also have

$$\begin{aligned} F_{1;\epsilon}(\lambda x_1 + (1 - \lambda)x_2) &= \int_{y \in \mathbf{V}} F(\lambda x_1 + (1 - \lambda)x_2 - y)\psi_\epsilon(y) d\text{vol}_y \\ &\leq \int_{y \in \mathbf{V}} (\lambda F(x_1 - y) + (1 - \lambda)F(x_2 - y))\psi_\epsilon(y) d\text{vol}_y \\ &= \lambda F_{1;\epsilon}(x_1) + (1 - \lambda)F_{1;\epsilon}(x_2). \end{aligned}$$

So for each $\epsilon > 0$, $F_{1;\epsilon}$ is convex as well.

Now the problem is $F_{1;\epsilon}$ is not positively homogeneous in general. Suppose the positive ϵ is sufficiently closed to 0, then the derivative of $F_{1;\epsilon}$ in the radius direction is non-vanishing everywhere in the region $\{x \in \mathbb{R}^n \text{ with } 1/2 \leq |x| \leq 2\}$. The pre-image $\mathcal{S}_\epsilon = F_{1;\epsilon}^{-1}(1)$ is a smooth $n-1$-dimensional sphere surrounding the origin. Because $F_{1;\epsilon}$ is convex, $\mathcal{S}_\epsilon$ bounds a convex region in $\mathbf{V}$. We define the second family of perturbation functions $F_{2;\epsilon}$ on $\mathbf{V}$, such that each $F_{2;\epsilon}$ is positively homogeneous of degree one, and $F_{2;\epsilon}(x) = 1$ iff $x \in \mathcal{S}_\epsilon$. Then $F_{2;\epsilon}$ is a positive smooth function on $\mathbb{R}^n\backslash\{0\}$. The convexity of $\mathcal{S}_\epsilon$ implies the Hessian $(g_{ij}(y)) = (\frac{1}{2}[F^2_{2;\epsilon}(y)]_{y^i y^j})$ is semi positive definite at least.

At last we define $F_\epsilon = \sqrt{F^2_{2;\epsilon} + \epsilon|x|^2}$, then F_ϵ satisfies all the conditions for smooth Minkowski norms, when the positive parameter ϵ is sufficiently close to 0. When ϵ approaches 0, F_ϵ uniformly converges to F in each compact subset in $\mathbf{V}$, i.e. in the local C^0-topology. □

We will also need the following simple technical facts.

Lemma 3.10. *Assume G is a connected quasi-compact Lie group which acts transitively on the smooth coset space G/H. Let F_n be a sequence of G-δ-homogeneous Finsler metrics on G/H induced by submersion and the bi-invariant singular metrics $\bar{F}$ respectively. Denote $\tilde{F}_n$ the Chebyshev metric defined F_n respectively. Then we have the following.*

(1) *If $\bar{F}_n$ converges to a singular metric $\bar{F}$ on G in the local C^0-topology, then F_n converges to a singular metric F on G/H in the local C^0-topology. Further more, $\bar{F}$ is bi-invariant, F is G-homogeneous, and the canonical projection $\pi : (G, \bar{F}) \to (G/H, F)$ is a submersion.*

(2) *If F_n converges to a singular metric F on G/H in the local C^0-topology, then F can be induced by its Chebyshev metric on G and submersion. If F is Finsler metric, then it must be a G-δ-homogeneous Finsler metric.*

Proof. (1) Assume $\bar{F}_n$ converges to the singular metric $\bar{F}$ on G in the local C^0-topology. Then $\bar{F}$ is bi-invariant because each $\tilde{F}_n$ is. Similar arguments as for Lemma 3.1 in [17], $\bar{F}$ defines a G-homogeneous singular metric F on G/H. To show F_n locally C^0-converges to F, we only need to prove it in $\mathfrak{m}$, viewing all $\tilde{F}_n$, F_n, $\tilde{F}$, and F as singular norms. For each sufficiently small $\epsilon > 0$, we can find $N > 0$, whenever $n > N$, $|\bar{F}_n(\tilde{w}) - \bar{F}(\tilde{w})| < \epsilon$ when $\bar{F}(\tilde{w}) \leq 1$. It implies the following inclusions among subsets of $\mathfrak{g}$ for each $n > N$,

$$\begin{aligned}\{\tilde{w} \in \mathfrak{g} \text{ with } \bar{F}_n(\tilde{w}) \leq \frac{1}{1+\epsilon}\} &\subset \{\tilde{w} \in \mathfrak{g} \text{ with } \bar{F}(\tilde{w}) \leq 1\} \\ &\subset \{\tilde{w} \in \mathfrak{g} \text{ with } \bar{F}_n(\tilde{w}) \leq \frac{1}{1-\epsilon}\}.\end{aligned}$$

Because of submersion, $\mathrm{pr}_{\mathfrak{m}}$ maps these subsets onto the following subsets in $\mathfrak{m}$ with corresponding inclusions,

$$\begin{aligned}\{w \in \mathfrak{m} \text{ with } F_n(w) \leq \frac{1}{1+\epsilon}\} &\subset \{w \in \mathfrak{m} \text{ with } F(w) \leq 1\} \\ &\subset \{w \in \mathfrak{m} \text{ with } F_n(w) \leq \frac{1}{1-\epsilon}\}.\end{aligned}$$

So we also have $|F_n(w) - F(w)| \leq \epsilon$ when $n > N$ and $F(w) = 1$. So as singular norms on $\mathfrak{m}$, F_n converges to F in the local C^0-topology. Using the G-translations, this convergence

is still valid when all F_n and F are viewed as singular metrics.

(2) Viewed as singular norms on $\mathfrak{m}$, F_n converges to F in the local C^0-topology. Then the Chebyshev norms $\tilde{F}_n$ also converge to the Chebyshev norm $\tilde{F}$ defined by F. Because $\mathrm{pr}_{\mathfrak{m}} : (\mathfrak{g}, \tilde{F}_n) \to (\mathfrak{m}, F_n)$ is a submersion for each n, i.e.

$$\mathrm{pr}_{\mathfrak{m}}(\{\tilde{u} \in \mathfrak{g} \text{ with } \tilde{F}_n(\tilde{u}) \leq 1\}) = \{u \in \mathfrak{m} \text{ with } F_n(u) \leq 1\},$$

passing to their limits, we also have

$$\mathrm{pr}_{\mathfrak{m}}(\{\tilde{u} \in \mathfrak{g} \text{ with } \tilde{F}(\tilde{u}) \leq 1\}) = \{u \in \mathfrak{m} \text{ with } F(u) \leq 1\}.$$

That proves F can be induced by its Chebyshev metric on G and submersion. The last assertion is an obvious observation we have made. □

Now we prove Theorem 1.1.

We only need to prove two assertions for this theorem. The first one is that, inside the space of all smooth Finsler metrics on G/H, the subset of all G-δ-homogeneous Finsler metrics is closed in the local C^0-topology. This is already done by (2) of Lemma 3.10.

The second one is that for any G-δ-homogeneous Finsler metric F, we can find a sequence of G-normal homogeneous metrics F_n, such that F_n converges to F in the C^0-topology.

Now we prove the second claim. Consider a G-δ-homogeneous Finsler metric F on G/H, which is defined by submersion from the Chebyshev metric $\tilde{F}$ on G. By Lemma 3.9, we can find a sequence of Minkowski norms $\{\tilde{F}'_n\}$ on G which converge to $\tilde{F}$ in the C^0-topology. Average each $\tilde{F}'_n$ with all $\mathrm{Ad}(G)$-actions as following,

$$\tilde{F}_n(w) = \sqrt{\frac{\int_{g\in G} \tilde{F}'_n(\mathrm{Ad}(g)w)^2 d\mathrm{vol}_g}{\int_{g\in G} d\mathrm{vol}_g}},$$

where vol_g is a bi-invariant volume form of G, we get a sequence of $\mathrm{Ad}(G)$-invariant Minkowski norms $\{\tilde{F}_n\}$ on $\mathfrak{g}$, such that the bi-invariant Finsler metrics $\tilde{F}_n$ converge to the bi-invariant singular metric $\tilde{F}$. Each bi-invariant smooth Finsler metric $\tilde{F}_n$ on G defines a G-normal homogeneous metric F_n. By (1) of Lemma 3.10, F_n converges to F in the local C^0-topology.

This ends the proof of Theorem 1.1.

4. The positive curvature problem for δ-homogeneous Finsler spaces

Let $(G/H, F)$ be a G-δ-homogeneous Finsler space, where G is a connected quasi-compact Lie group which acts effectively on G/H. We choose an $\mathrm{Ad}(G)$-invariant inner product, and denote $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the corresponding orthogonal decomposition. Then according to [17], a subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is called a *flat splitting subalgebra* (FSS in short), when it satisfies the following conditions:

(1) $\mathfrak{s}$ is the intersection of a family of Cartan subalgebras of $\mathfrak{g}$.

(2) $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{h} + \mathfrak{s} \cap \mathfrak{m}$.

(3) $\dim \mathfrak{s} \cap \mathfrak{m} \geq 2$.

Because of Condition (1) for a FSS, $\mathfrak{s}$ generates a closed connected abelian subgroup $\exp \mathfrak{s}$ of G. The orbit $\exp \mathfrak{s} \cdot o = \exp(\mathfrak{s} \cap \mathfrak{m}) \cdot o$ can be viewed as the connected abelian Lie

group $S = \exp(\mathfrak{s} \cap \mathfrak{m})$. The subspace metric $F|_{S\cdot o}$ is a left invariant Finsler metric on S, which is obviously flat. The crucial observation here is that $(S \cdot o, F|_{S\cdot o})$ is totally geodesic.

Lemma 4.1. *Keep all above notations and assumptions. Then* $(S \cdot o, F|_{S\cdot o})$ *is a totally geodesic subspace of* $(G/H, F)$.

Proof. By Theorem 1.1, we assume F is the limit of a sequence F_n of G-normal homogeneous metrics on G/H in the local C^0-topology. Denote $\tilde{F}$ and $\tilde{F}_n$ their Chebyshev norms on $\mathfrak{g}$ respectively. For each n, we have another bi-invariant Minkowski norm $\bar{F}_n$ defining the G-normal homogeneity of F_n. Consider any nonzero vector $u \in \mathfrak{s} \cap \mathfrak{m}$. By Lemma 3.4 in [17], there exists a unique vector $\tilde{u}_n \in \mathfrak{s}$ such that $\mathrm{pr}_{\mathfrak{m}}(\tilde{u}_n) = u$ and $\bar{F}_n(\tilde{u}_n) = \tilde{F}_n(\tilde{u}_n) = 1$. Because $\tilde{F}_n$ converges to $\tilde{F}$ in the local C^0-topology, $\tilde{F}(\tilde{u}_n)$ is bounded. We can find a convergent subsequence converging to some $\tilde{u} \in \mathfrak{s}$ with $\mathrm{pr}_{\mathfrak{m}}(\tilde{u}) = u$ and $\tilde{F}(\tilde{u}) = 1$. So $\tilde{u}$ defines a $\delta(o)$-Killing vector field X of $(G/H, F)$ for the tangent vector $u \in \mathfrak{m}$. The integration curve of X passing o is then a geodesic of $(G/H, F)$. On the other hand, it coincides with $\exp(t\tilde{u})\cdot o = \exp(tu)\cdot o$ which is a geodesic of $(S \cdot o, F|_{S\cdot o})$. By left S-translations, this proves $(S \cdot o, F|_{S\cdot o})$ is totally geodesic. □

The observation that a FSS is a totally geodesic flat subspace for a normal homogeneous Finsler space is the key technique that we reduce the classification for positively curved normal homogeneous Finsler spaces to a totally algebraic problem.

To be precise, in [17], Theorem 3.3, the preparation lemmas, and the case by case discussion from Section 4 to Section 6, proves the following.

If G/H does not admit positively curved G-normal homogeneous Riemannian metrics, i.e. there exists a linearly independent commuting pair of vectors in $\mathfrak{m}$, then we can find a closed subgroup K in G with $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, with the corresponding orthogonal decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, such that there exists a FSS for G/K. Then we see G/H does not admit positively curved G-normal homogeneous Finsler metrics either. In particular, when G/H is even dimensional, we can choose $K = H$. In most cases, the FSS can be found among Cartan subalgebras, which will be called a *flat splitting Cartan subalgebra* or FSCS in short.

Because of Theorem 1.1 and Lemma 4.1, this theory can be applied to prove Theorem 1.2, which implies the classification for positively curved δ-homogeneous Finsler spaces coincides with that for positively curved normal homogeneous Riemannian spaces.

Proof of Theorem 1.2. Let $(G/H, F)$ be a positively curved G-δ-homogeneous Finsler spaces. Assume conversely G/H does not admit a positively curved G-normal homogeneous Riemannian metric, then we can find a closed subgroup K of G with $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, such that there exists a FSS for G/K. By Theorem 2.3 and Proposition 3.3, G/K also admits a positively curved G-δ-homogeneous Finsler metric. But by Lemma 4.1 implies the FSS provides a totally geodesic flat subspace of dimension bigger than 1. This is the contradiction. □

Lemma 4.1 can tell us more when we turn to the (FP) condition [18].

A Finsler space (M, F) is called *flag-wise positively curved* or satisfying the *(FP) condition*, if for any $x \in M$, any tangent plane $\mathbf{P} \subset T_xM$, we can find a $y \in \mathbf{P}$ such that the flag curvature $K^F(x, y, \mathbf{P}) > 0$.

The (FP) condition itself is very weak [16]. But its combination with the non-negatively curved condition seems much stronger, very like the positively curved condition. We guess

a flag-wise positively curved and non-negatively curved homogeneous Finsler space G/H must be compact. Further more, when G is compact, we guess $\mathrm{rank}G \leq \mathrm{rank}H + 1$.

By Theorem 3.2, a flag-wise positively curved δ-homogeneous Finsler space belongs to this special category. We can prove

Theorem 4.2. *Let (M, F) be a connected flag-wise positively curved δ-homogeneous Finsler space which dimension is bigger than 1, then M is compact. If $M = G/H$ where G is a compact connected Lie group acting effectively on M, then* $\mathrm{rank}G \leq \mathrm{rank}H + 1$.

Proof. By Theorem 3.6, we can assume the flag-wise positively curved δ-homogeneous Finsler space (M, F) is G-δ-homogeneous, where G is a connected and quasi-compact. We can present $G = G_1 \times \mathbb{R}^k$ where G_1 is a maximal compact subgroup of G, $M = G/H$ where H is a closed subgroup of G_1. First we prove then $\mathrm{rank}G \leq \mathrm{rank}H + 1$. If this is not true, any Cartan subalgebra of $\mathfrak{h}$ can be enlarged to a FSCS in $\mathfrak{g}$. By Lemma 4.1, (M, F) can not satisfy the (FP) condition because of the totally geodesic flat subspace. This argument proves the second statement in the theorem. Now we prove the first statement, i.e. the compactness of M. Assume conversely M is not compact, then $k = 1$ and $\mathrm{rank}G_1 = \mathrm{rank}H$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{h}$, with respect to which $\mathfrak{g}_1 = \mathrm{Lie}(G_1)$ and $\mathfrak{h}$ can be decomposed as the sum of $\mathfrak{t}$ and root planes $\mathfrak{g}_{\pm\alpha}$. Because $\mathrm{rank}G_1 = \mathrm{rank}H$, Each root plane $\mathfrak{g}_{\pm\alpha}$ is contained in either $\mathfrak{h}$ or $\mathfrak{m}$. Because $\dim G/H > 1$, we can find a root plane $\mathfrak{g}_{\pm\alpha}$ in $\mathfrak{m}$. Let u be any nonzero vector in $\mathfrak{g}_{\pm\alpha}$, v any nonzero vector in the $\mathbb{R}$-factor, and $\ker\alpha$ the codimension one subspace of $\mathfrak{t}$ where α takes zero values. Then u, v and $\ker\alpha \subset \mathfrak{t}$ span a FSCS, which will be a contradiction to the (FP) condition. □

As the end, we remark that, FSS can be viewed as an obstacle for normal or δ-homogeneous Finsler metric either positively curved or satisfying the (FP) condition. For the positively curved condition, this obstacle can be passed downward by submersion, i.e. when G/H admits a positively curved G-normal or G-δ-homogeneous Finsler metric, then for any closed subgroup $K \subset G$ containing H, the naturally induced G-normal or G-δ-homogeneous Finsler metric is also positively curved. But this observation is not true for the (FP) condition, i.e. this obstacle can not be passed downward for the (FP) condition. When $\dim G/H$ is even, H itself can be taken as K, so the classification work in [17] proves

Corollary 4.3. *The following conditions are equivalent for an even dimensional smooth coset space G/H:*

(1) *It admits positively curved G-normal homogeneous Finsler metrics.*
(2) *It admits positively curved G-δ-homogeneous Finsler metrics.*
(3) *It admits flag-wise positively curved G-normal homogeneous Finsler metrics.*
(4) *It admits flag-wise positively curved G-δ-homogeneous Finsler metrics.*

Notice when $\dim G/H$ is odd, Corollary 4.3 is not true. Theorem 1.2 in [18] provides many examples of smooth coset spaces admitting flag-wise positively curved and non-negatively curved homogeneous Finsler metrics. By Corollary 1.4 in [18], most of them do not admit positively curved homogeneous Finsler metrics. In each example $(G/H, F)$ in Theorem 1.2 of [18], F is defined by the navigation with respect to a normal homogeneous Riemannian metric F' and a nonzero Killing vector field V with $F'(V(\cdot)) \equiv \mathrm{const} < 1$. Then for any $x \in G/H$ and any F'-unit tangent vector $u' \in T_x(G/H)$, we have a $\delta(x)$-Killing vector

field X of $(G/H, F')$ from $\mathfrak{g}$, for the tangent vector u'. Then $X + V$ is a $\delta(x)$-Killing vector field of $(G/H, F)$ from $\mathfrak{g} \oplus \mathbb{R}$, for the F-unit tangent vector $u = u' + V(x)$, which can exhaust all the tangent directions. So F is $(G \times S^1)$-δ-homogeneous.

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