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LATTICES AND AUTOMORPHISMS OF LIE GROUPS

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Introduction

Let G be a connected Lie group. A discrete subgroup L of G is said to be a lattice if G/L has a G-invariant measure with finite total mass.

We denote by $\mathrm{Aut}(G)$ the totality of the bi-continuous automorphisms of G. $\mathrm{Aut}(G)$ is endowed with a Lie group structure in the natural manner. For a lattice L of G, we define a subgroup $F(L)$ of $\mathrm{Aut}(G)$ by

$$F(L) = \{\alpha \in \mathrm{Aut}(G);\ \alpha(x) = x \quad \text{for every} \quad x \in L\}\ .$$

Assume for a moment that G is simply connected. Let L be a lattice of G. It is known that if G is semi-simple without compact factors or nilpotent $F(L)$ is trivial. Triviality of $F(L)$ implies that each automorphism is determined by its values on L. However if G is a simply connected compact semi-simple group and L is the trivial subgroup, L is a lattice of G and $F(L)=\mathrm{Aut}(G)$ is not trivial. Except for such a trivial case, even if G has no normal connected compact subgroup, or if G has no compact subgroup, $F(L)$ is not always trivial [See Appendix (B)]. In the case where G is a simply connected Lie group without normal compact semi-simple subgroups, $F(L)$ is a closed vector subgroup of $\mathrm{Aut}(G)$ consisting of inner automorphisms by elements of the center of the largest connected normal nilpotent subgroup of G [See Coro. 2.6]. In general, for a simply connected Lie group G and a lattice L of G, $F(L)$ has only finitely many components and the identity component $(F(L))_0$ of $F(L)$ is the direct product of a connected compact subgroup and a vector subgroup consisting of inner automorphisms [See Theorem 2.10].

If G is not simply connected, the structure of $F(L)$ is more complicated. However, it is shown that the structure of $(F(L))_0$ is the same as in the simply connected case [See Theorem 3.1].

1. Notations

Throughout this paper we will use the following notations;

G : a connected Lie group,

R : the largest connected normal solvable subgroup of G,
N : the largest connected normal nilpotent subgroup of G,
S : a Levi subgroup of G, that is, a maximal connected semi-simple subgroup,
Q : the product of all the noncompact factors of S,
C : the product of all the compact factors of S,
C_1 : the product of the factors of C which are not normal in G,
C_2 : the product of the factors of C which are normal in G,
G_1 : the product of C_1, Q and R.

For a Lie group G, we denote by G_0 its connected component containing the unit element. For a subgroup H of G, $Z(H)$ and $C(H)$ denote the center of H and the centralizer of H, respectively. For $x \in G$, the inner automorphism by x is denoted by i_x, that is,

$$i_x\colon G \ni y \mapsto xyx^{-1} \in G .$$

For a subset A of a topological space, $Cl(A)$ denotes the topological closure of A. By $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Z}$ we denote the real number field, the complex number field and the integer ring, respectively.

A subgroup H of G is said to be characteristic if H is invariant under the action by $\mathrm{Aut}(G)$.

Lemma 1.1. *G_1, C_2, CR and C_1N are all characteristic.*

Proof. Let α be in $\mathrm{Aut}(G)$. Since $\alpha(S)$ is a Levi subgroup, there exists $n \in N$ such that $\alpha(S)=nSn^{-1}$. Both $\alpha(C_2)$ and $nC_2n^{-1}=C_2$ coincide with the product of those compact factors of $\alpha(S)=nSn^{-1}$, which are normal in G. Thus we have that $\alpha(C_2)=C_2$. Similarly $\alpha(C_1)=nC_1n^{-1}$. This implies that $\alpha(C_1N)=nC_1n^{-1}N=C_1N$. By the same argument as above we can prove that $G_1=C_1QR$ and CR are characteristic. q.e.d.

REMARK. C_2 is characterized as the largest normal connected compact semi-simple subgroup of G.

2. Structure of $F(L)$ in the case that G is simply connected

In this section we assume that G is simply connected and let L be a lattice of G.

Since G is simply connected, G is the semi-direct product of Q and CR: $G=Q \ltimes CR$. Let $p\colon G \to Q$ be the projection map with respect to the semi-direct product decomposition. Set $M=p(L)$. M plays an important role in the following arguments. The next two lemmas are essentially due to H. C. Wang. For the proof the reader can refer [4, Propositions 8 and 9].

Lemma 2.1. *$CR/(L\cap CR)$ is compact.*

Lemma 2.2. *M is a lattice of Q.*

In order to prove Lemma 2.2, we must note only that Q/M is homeomorphic to $G/(LCR)$ as G-spaces.

Lemma 2.3. *Let r be a continuous representation of G on a finite dimensional $\boldsymbol{R}$-vector space V and f a continuous map of G to V such that for $x, y\in G$,*

$$f(xy)=f(x)+r(x)f(y)\,. \tag{2.1}$$

Assume that $f(L)=0$. Then we have that;

(1) *$f(G)$ is compact and,*

(2) *there exists $u\in V$ such that for every $x\in G$, $f(x)=u-r(x)u$.*

Proof. By Lemma 2.1, there exists a compact set $K\subset CR$ such that $CR=K(L\cap CR)$. Since $f(L)=0$, by Lemma 1.1 and (2.1)

$$f(LCR)=f(CRL)=f(CR)=f(K(L\cap CR))=f(K)\,.$$

Since $M=p(L)\subset LCR$, we have that

$$f(MCR)=f(K)\,. \tag{2.2}$$

We define a new representation r' of G on $V\oplus\boldsymbol{R}$ by

$$r'(x)(v\oplus t)=(r(x)v+tf(x))\oplus t$$

where $x\in G$, $v\in V$, and $t\in\boldsymbol{R}$. That r' is a representation follows by the condition (2.1). Note that for any subset $A\subset G$

$$r'(A)(f(e)\oplus 1)=f(A)\oplus 1 \tag{2.3}$$

where e denotes the unit element of G, since $r(G)f(e)=0$ by (2.1). $f(K)\oplus 1$ is $r'(MCR)$-invariant because we have that, by (2.2) and (2.3),

$$\begin{aligned} r'(MCR)(f(K)\oplus 1) &= r'(MCR)\,r'(K)(f(e)\oplus 1)\\ &= r'(MCR)(f(e)\oplus 1)\\ &= f(MCR)\oplus 1=f(K)\oplus 1\,. \end{aligned}$$

Let W be the vector subspace of $V\oplus\boldsymbol{R}$ spanned by all the elements of $f(K)\oplus 1$. W is $r'(MCR)$-invariant and especially $r'(M)$-invariant. Thus, by Lemma 2.2 and [1, Coro. 4.5], W is $r'(Q)$-invariant and $r'(G)=r'(QCR)$-invariant.

Let $W_{\boldsymbol{C}}$ be the complexification of W. We consider $GL(W)$ as a subset of $GL(W_{\boldsymbol{C}})$ in the natural manner. For a subset $B\subset GL(W)$, we denote by $B^{\sharp}$ the intersection of $GL(W)$ and the Zariski-closure of B in $GL(W_{\boldsymbol{C}})$. By Lemma 2.2 and [5, 5.16], $(r'(Q)|W)^{\#}=(r'(M)|W)^{\#}$, where $r'(Q)|W$ and $r'(M)|W$ denote the restrictions of $r'(Q)$ and $r'(M)$ to W, respectively.

Since $f(K)\oplus 1$ is compact and invariant under $r'(MCR)$, $r'(MCR)|W$ has the compact closure in $GL(W)$. Thus, by Chevalley's theory on compact Lie groups [2, Chap. 6],

$$Cl(r'(MCR)|W)=(r'(MCR)|W)^{\#}$$

Consequently, we have that

$$\begin{aligned} r'(G)|W &= (r'(Q)|W)(r'(MCR)|W) \\ &\subset (r'(Q)|W)^{\#}(r'(MCR)|W)^{\#} \\ &= (r'(M)|W)^{\#}(r'(MCR)|W)^{\#} \\ &= (r'(MCR)|W)^{\#} = Cl(r'(MCR)|W) \end{aligned}$$

and, using (2.2) and (2.3),

$$\begin{aligned} f(G)\oplus 1 &= (r'(G)|W)(f(e)\oplus 1) \\ &\subset Cl(r'(MCR)|W)(f(e)\oplus 1) \\ &\subset Cl(f(MCR)\oplus 1) \\ &= f(K)\oplus 1 \end{aligned}$$

Thus, $f(G)=f(K)$. The proof of (1) is completed.

Since $f(L)=0$, f induces a continuous map $\dot{f}$ of G/L to V. Let μ be a G-invariant measure on G/L with total mass $\mu(G/L)=1$. Since $\dot{f}(G/L)=f(G)$ is compact, $\dot{f}$ is a μ-integral map with values in V. Set $u=\int_{G/L}\dot{f}(\dot{x})\,d\mu(\dot{x})$, where $\dot{x}$ denotes an element of G/L. For every $y\in G$, by (2.1),

$$\begin{aligned} u-r(y)\,u &= \int_{G/L}(\dot{f}(\dot{x})-r(y)\dot{f}(\dot{x}))\,d\mu(\dot{x}) \\ &= \int_{G/L}(\dot{f}(\dot{x})-(\dot{f}(y\dot{x})-f(y)))\,d\mu(\dot{x}) \\ &= f(y)\,. \end{aligned}$$

The proof of (2) is completed.

REMARK. In Lemma 2.3, the assumption of simply connectedness for G is not essential. We can easily reduce the proof for non simply connected groups to the one for simply connected groups.

Since G is simply connected, by Lemma 1.1, G has Aut(G)-stable direct product decomposition $G=G_1\times C_2$. We denote by id_1 and id_2 the identity maps of G_1 and C_2, respectively. Set

$$\mathrm{Aut}(G_1) = \{\alpha \in \mathrm{Aut}(G);\ \alpha | C_2 = id_2\}$$

and

$$\mathrm{Aut}(C_2) = \{\alpha \in \mathrm{Aut}(G);\ \alpha | G_1 = id_1\}$$

where $\alpha | G_1$ and $\alpha | C_2$ denote the restriction of α to G_1 and C_2, respectively. Lemma 1.1 implies that $\mathrm{Aut}(G)$ is the direct product of $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(C_2)$.

Lemma 2.4. *$F(L)$ is the direct product of $F(L) \cap \mathrm{Aut}(G_1)$ and $F(L) \cap \mathrm{Aut}(C_2)$.*

Proof. In order to prove the lemma, it is sufficient to show that for any $\alpha \in F(L)$ there exist $\alpha_1 \in F(L) \cap \mathrm{Aut}(G_1)$ and $\alpha_2 \in F(L) \cap \mathrm{Aut}(G_2)$ such that $\alpha = \alpha_1 \cdot \alpha_2$. Set

$$\alpha_1 = \begin{cases} \alpha & \text{on} \quad G_1 \\ id_2 & \text{on} \quad C_2 . \end{cases}$$

Note that $\alpha_1 \in \mathrm{Aut}(G_1)$. For any $x \in L$, there exist uniquely $x_1 \in G_1$ and $x_2 \in C_2$ such that $x = x_1 \cdot x_2$. Since G_1 and C_2 are characteristic and the decomposition $x = x_1 \cdot x_2$ is unique, $x_1 \cdot x_2 = x = \alpha(x) = \alpha(x_1) \cdot \alpha(x_2)$ implies that $x_1 = \alpha(x_1)$ and $x_2 = \alpha(x_2)$. Thus $\alpha_1(x) = \alpha_1(x_1) \cdot \alpha_1(x_2) = \alpha(x_1) \cdot x_2 = x_1 \cdot x_2 = x$. This shows that $\alpha_1 \in F(L) \cap \mathrm{Aut}(G_1)$. Set $\alpha_2 = (\alpha_1)^{-1} \cdot \alpha$. By the definition $\alpha_2 \in F(L) \cap \mathrm{Aut}(C_2)$.

q.e.d.

The following proposition makes clear the structure of $F(L) \cap \mathrm{Aut}(G_1)$.

Proposition 2.5. *$F(L) \cap \mathrm{Aut}(G_1)$ is a closed vector subgroup and consists of inner automorphisms by elements of $Z(N)$.*

Corollary 2.6. *Let G be a simply connected Lie group without connected normal compact semi-simple subgroups and L a lattice of G. Then $F(L)$ is a closed vector subgroup consisting of inner automorphisms by elements of $Z(N)$.*

In fact even if G is a simply connected Lie group without normal compact subgroups, or even if G has no compact subgroup, $F(L)$ is not always trivial, see Appendix (B). In order to prove Proposition 2.5, we need the following three lemmas.

Lemma 2.7. *Let α be in $F(L)$. Then α acts on N trivially and*

$$\{\alpha(x)\, x^{-1};\ x \in G\} \subset C(N) .$$

Proof. By Lemma 1.1 and Lemma 2.1, $\{\alpha(x)\, x^{-1};\ x \in CR\}$ is compact. In particular $\{\alpha(x)\, x^{-1};\ x \in N\}$ has the compact closure. Thus by [7, lemme 2] α acts on N trivially. For $x \in G$ and $y \in N$, $x^{-1} y x = \alpha(x^{-1} y x) = \alpha(x)^{-1} y \alpha(x)$.

Hence $\alpha(x)\,x^{-1}$ centralizes N. q.e.d.

Lemma 2.8. $Z(N)=(C(N)\cap R)_0$.

Lemma 2.9. $Z(N)=(C(N)\cap C_1N)_0$.

Though it is not difficult to prove Lemmas 2.8 and 2.9, for completeness we shall prove them in Appendix (A).

Proof of Proposition 2.5. Let $\alpha\in F(L)\cap \mathrm{Aut}(G_1)$. The assumption on α implies that α acts trivially on LC_2. $\{\alpha(x)\,x^{-1};\, x\in C_1\}$ is connected and, by Lemma 1.1, contained in C_1N. Thus Lemmas 2.7 and 2.9 imply that

$$\{\alpha(x)\,x^{-1};\, x\in C_1\}\subset Z(N)\,. \tag{2.4}$$

Similarly by Lemmas 2.7 and 2.8 we obtain

$$\{\alpha(x)\,x^{-1};\, x\in R\}\subset Z(N)\,. \tag{2.5}$$

Let x be in $LCR=RC_1C_2L$, cf. Lemma 1.1. There exist $r\in R$, $y\in C_1$ and $z\in C_2L$ such that $x=ryz$. Then $\alpha(x)\,x^{-1}=\alpha(r)\,\alpha(y)\,\alpha(z)\,z^{-1}y^{-1}r^{-1}=(\alpha(r)\,r^{-1})\,r(\alpha(y)\,y^{-1})\,r^{-1}$, because α acts trivially on C_2L. Since $Z(N)$ is characteristic, by (2.4) and (2.5), both $\alpha(r)\,r^{-1}$ and $r(\alpha(y)\,y^{-1})\,r^{-1}$ are contained in $Z(N)$. Consequently we have that

$$\{\alpha(x)\,x^{-1};\, x\in LCR\}\subset Z(N)\,. \tag{2.6}$$

Set $G'=G/Z(N)$. By Ad we denote the adjoint representation of G' on its Lie algebra $\hat{G}$. Let $\pi\colon G\to G'$ be the canonical projection and α' the automorphism of G' induced by α. Note that by (2.6) α' acts trivially on $\pi(LCR)$. Since $M=p(L)\subset LCR$, for $x\in\pi(M)$ $\alpha'(x)=x$. It follows that for $x\in\pi(M)$,

$$d\alpha'\circ Ad(x) = Ad(x)\circ d\alpha'$$

where $d\alpha$ denotes the automorphism of $\hat{G}'$ corresponding with α'. Since π maps Q onto $\pi(Q)$ isomorphically, by Lemma 2.2 and [1, Coro. 4.5], for $x\in\pi(Q)$

$$d\alpha'\circ Ad(x) = Ad(x)\circ d\alpha'$$

It follows that

$$\{\alpha'(x)\,x^{-1};\, x\in\pi(Q)\}\subset Z(G')\,. \tag{2.7}$$

Define the map q of $\pi(Q)$ to $Z(G')$ by

$$q\colon \pi(Q)\ni x\mapsto \alpha'(x)\,x^{-1}\in Z(G')\,.$$

Since by (2.7) q is a homomorphism of a semi-simple group $\pi(Q)$ to an abelian group $Z(G')$, q maps $\pi(Q)$ to the unit element of G'. Thus α' acts tiivially on G'. Consequently we have that

$$\{\alpha(x)\,x^{-1};\ x\in G\}\subset Z(N)\,.$$

Since $Z(N)$ is a vector group, we denote the multiplication in $Z(N)$ by $x+y$. Let r be a representation of G on the vector group $Z(N)$ defined by

$$r(x)\,v = xvx^{-1}$$

where $x\in G$ and $v\in Z(N)$. Define a map f of G to $Z(N)$ by

$$f\colon G\ni x\mapsto \alpha(x)\,x^{-1}\in Z(N)\,.$$

Since they satisfy the condition (2.1) in Lemma 2.3 and $f(L)=0$, by Lemma 2.3 there exists $u\in Z(N)$ such that for $x\in G$

$$f(x) = u - r(x)\,u\,.$$

It follows that $\alpha(x)\,x^{-1}=u(xux^{-1})^{-1}=uxu^{-1}\,x^{-1}$. Consequently, we have that $F(L)\cap \mathrm{Aut}(G_1)\subset \{i_u;\ u\in Z(N)\}$.

Define a continuous homomorphism $\Phi\colon Z(N)\to \mathrm{Aut}(G)$ by

$$\Phi\colon Z(N)\ni u\mapsto i_u\in \mathrm{Aut}(G)\,.$$

Set $T=\Phi^{-1}(F(L)\cap \mathrm{Aut}(G_1))$. We have already shown that $\Phi(T)=F(L)\cap \mathrm{Aut}(G_1)$. Note that $u\in Z(N)$ is contained in T if and only if $xux^{-1}=u$ for every $x\in L$. Let $u\in T$ and let X be an element of the Lie algebra of $Z(N)$ such that $\exp X=u$. Since the exponential map of $Z(N)$ is bijective and $Z(N)$ is normal, for every $x\in L$, $Ad(x)\,X=X$, where Ad denotes the adjoint representation of G. For an arbitrary $s\in \boldsymbol{R}$ and $x\in L$, $x(\exp s\,X)\,x^{-1}=\exp s\,Ad(x)\,X=\exp s\,X$. Thus $\exp s\,X\in T$. It follows that T is connected. Similar arguments show that $\ker \Phi$ is connected and contained in T.

Since $F(L)\cap \mathrm{Aut}(G_1)$ is closed in $\mathrm{Aut}(G)$, $F(L)\cap \mathrm{Aut}(G_1)$ is locally compact. $Z(N)$ is locally compact and σ-compact. Thus $F(L)\cap \mathrm{Aut}(G_1)$ is isomorphic to $T/\ker \Phi$ and is a connected and simply connected abelian Lie group, i.e, a vector group. Proof of Proposition 2.5 is completed.

On the other hand, $F(L)\cap \mathrm{Aut}(C_2)$ is a closed subgroup of the automorphism group of the compact semi-simple group C_2. Thus $F(L)\cap \mathrm{Aut}(C_2)$ is compact and $(F(L)\cap \mathrm{Aut}(C_2))_0$ consists of inner automorphisms by elements of C_2.

From the above results we obtain the following theorem.

Theorem 2.10. *Let G be a connected and simply connected Lie group and L a lattice of G. Then*

(1) $F(L)=(F(L)\cap \mathrm{Aut}(G_1))\cdot(F(L)\cap \mathrm{Aut}(C_2))$,

(2) $F(L)\cap \mathrm{Aut}(G_1)$ *is a closed vector subgroup consisting of inner automor-*

phisms by elements of $Z(N)$,

(3) $F(L)\cap \mathrm{Aut}(C_2)$ *is a compact subgroup and* $(F(L)\cap \mathrm{Aut}(C_2))_0$ *consists of inner automorphisms by elements of* C_2,

(4) $(F(L))_0$ *consists of inner automorphisms and* $F(L)/(F(L))_0$ *is a finite group.*

Proof. (1), (2) and (3) have been already proved. Since the decomposition in (1) is topological direct product, $(F(L))_0=[(F(L)\cap \mathrm{Aut}(G_1))\cdot(F(L)\cap \mathrm{Aut}(C_2))]_0=(F(L)\cap \mathrm{Aut}(G_1))\cdot(F(L)\cap \mathrm{Aut}(C_2))_0$. Thus $F(L)/(F(L))_0$ is isomorphic to $(F(L)\cap \mathrm{Aut}(C_2))/(F(L)\cap \mathrm{Aut}(C_2))_0$. Since $F(L)\cap \mathrm{Aut}(C_2)$ is a compact Lie group, $(F(L)\cap \mathrm{Aut}(C_2))/(F(L)\cap \mathrm{Aut}(C_2))_0$ is a finte group. Proof of (4) is completed.

Cororally 2.11. $F(L)$ *and* $(F(L))_0$ *are real algebraic groups (as subgroups of the automorphism group of the Lie algebra of* G).

Proof. By (2) of Theorem 2.10, $F(L)\cap \mathrm{Aut}(G_1)$ consists of unipotent endomorphisms on the Lie algebra of G. Thus $F(L)\cap \mathrm{Aut}(G_1)$ is an algebraic group. $F(L)\cap \mathrm{Aut}(C_2)$ is also an algebraic group, because it is compact. Thus $F(L)=(F(L)\cap \mathrm{Aut}(G_1))\cdot(F(L)\cap \mathrm{Aut}(C_2))$ is an algebraic group. Similarly, $(F(L))_0=(F(L)\cap \mathrm{Aut}(G_1))\cdot(F(L)\cap \mathrm{Aut}(C_2))_0$ is also an algebraic group. q.e.d.

3. Structure of $F(L)$ in general cases

Let G be a connected Lie group and L a lattice of G. Let (G', π) be the universal covering group of G and D the kernel of π. We can identify G with G'/D. Under this identification $\mathrm{Aut}(G)$ may be considered as the subgroup of $\mathrm{Aut}(G')$ determined by

$$\mathrm{Aut}(G)=\{\alpha\in \mathrm{Aut}(G');\ \alpha(D)=D\}\ .$$

Set $L'=\pi^{-1}(L)$. Note that L' is a lattice of G' and that $D\subset L'$. If $\alpha\in F(L')$, for $d\in D$ $\alpha(d)=d$. Thus $F(L')\subset \mathrm{Aut}(G)$ and $F(L')\subset F(L)$. We have that $(F(L'))_0\subset(F(L))_0$. Conversely, if $\alpha\in(F(L))_0$, $\alpha(L')=L'$, because α commutes with π. Since L' is discrete and $(F(L))_0$ is connected, α fixes L' pointwise. Thus $(F(L))_0\subset F(L')$. Consequently, we have that $(F(L))_0=(F(L'))_0$. From Theorem 2.10, we obtain the following theorem.

Theorem 3.1. *Let* G *be a connected Lie group and* L *a lattice of* G. *Then,* $(F(L))_0$ *is the product of a vector subgroup and a connected compact subgroup and consists of inner automorphisms.*

Corollary 3.2. $(F(L))_0$ *is a real algebraic group.*

REMARK. In general $F(L)/(F(L))_0$ is not a finite group. Let G be a torus $\boldsymbol{R}^n/\boldsymbol{Z}^n$ and L the trivial subgroup consisting of only the unit element. L is a lattice of G. In this case $F(L)=\mathrm{Aut}(G)$ is isomorphic to the discrete infinite group $SL(n, \boldsymbol{Z})$. Thus $F(L)/(F(L))_0=SL(n, \boldsymbol{Z})$ is not a finite group.

Appendix (A)

Let $\hat{C}$, $\hat{Z}$, $\hat{N}$ and $\hat{R}$ be the Lie algebras of $C(N)$, $Z(N)$, N and R, respectively.

Proof of Lemma 2.8. Since $Z(N)$ is connected, $Z(N)$ is contained in $(C(N) \cap R)_0$. If $Y \in \hat{C} \cap \hat{R}$, each eigen value of the adjoint representation of Y on $\hat{R}$ equals 0. Thus $Y \in \hat{N}$. It follows that $\hat{C} \cap \hat{R} \subset \hat{N} \subset \hat{C}=\hat{Z}$. Consequently, $(C(N) \cap R)_0 \subset Z(N)$. q.e.d.

In order to prove Lemma 2.9, we use a well known result;

Sublemma. *Let $\hat{R}$ be a solvable Lie algebra and $\hat{N}$ its largest nilpotent ideal. Assume that α is contained in* $(\mathrm{Aut}(\hat{R}))_0$. *Then for $X \in \hat{R}$, $\alpha(X) \equiv X$* (mod. $\hat{N}$).

Proof. The Lie algebra of $\mathrm{Aut}(\hat{R})$ consists of all the derivations on $\hat{R}$. The derivation of $\hat{R}$ sends every element of $\hat{R}$ into $\hat{N}$ [3, p. 51]. Thus we have Sublemma.

Proof of Lemma 2.9. Since $Z(N)$ is connected and contained in $C(N) \cap C_1N$, $Z(N) \subset (C(N) \cap C_1N)_0$.

Assume that $x \in C_1$, $y \in N$ and $xy \in C(N)$. By Ad we denote the adjoint representation of G. The restriction $Ad(xy) \mid \hat{N}$ of $Ad(xy)$ to $\hat{N}$ is the identity map of $\hat{N}$, because $xy \in C(N)$. Thus $Ad(x) \mid \hat{N}=Ad(y^{-1}) \mid \hat{N}$,. Since $Ad(x) \mid \hat{N}$ is an element of the compact linear group $Ad(C_1) \mid \hat{N}$ and $Ad(y^{-1}) \mid \hat{N}$ is a unipotent element, both $Ad(x) \mid \hat{N}$ and $Ad(y^{-1}) \mid \hat{N}$ coincide with the identity map of $\hat{N}$. Thus $y \in Z(N)$ and $Ad(x)$ acts trivially on $\hat{N}$. We shall show that $Ad(x)$ acts trivially on $\hat{R}$. Since $Ad(C_1) \mid \hat{R}$ is compact, $\hat{R}$ has an $Ad(C_1)$-invariant inner product. Let $\hat{N}^{\perp}$ be the orthogonal complement of $\hat{N}$ with respect to the invariant inner product. Since $Ad(C_1)$ is connected, by Sublemma, we have that for $X \in \hat{R}$ $Ad(x)\, X \equiv X$ (mod. $\hat{N}$). Thus $Ad(x)$ acts trivially on $\hat{N}^{\perp}$, because $\hat{N}^{\perp}$ is $Ad(C_1)$-invariant. Consequently, $Ad(x)$ acts trivially on $\hat{R}=\hat{N}+\hat{N}^{\perp}$. Consider a homomorphism $h\colon C_1 \ni z \mapsto Ad(z) \mid \hat{R} \in GL(\hat{R})$. The definition of C_1 implies that the kernel D of h is discrete. Hence we have that $C(N) \cap C_1N$ is contained in $Z(N) \cdot D$ and $(C(N) \cap C_1N)_0$ is contained in $Z(N)$. q.e.d.

Appendix (B)

For a connected and simply connected Lie group G and a lattice L of G, $F(L)$ is not alwyas trivial if G has no normal compact subgroup or even if G does not have any compact subgroup. Typical examples are the followings.

(1) Let n be an integer$\geqq 3$. Let π: $Spin(n)\rightarrow SO(n)$ be the universal covering group of $SO(n)$. Since $SO(n)$ acts naturally on $\boldsymbol{R}^n$, $Spin(n)$ acts on $\boldsymbol{R}^n$ via π. By this action we construct the semi-direct product $G=Spin(n)\ltimes\boldsymbol{R}^n$. G is a connected simply connected Lie group with no normal connected compact subgroup. Let L be $\boldsymbol{Z}^n\subset\boldsymbol{R}^n$. L is a lattice of G. A calculation shows that

$$F(L)=\{i_x;\ x\in\boldsymbol{R}^n\}\ ,$$

which is isomorphic to $\boldsymbol{R}^n$.

(2) Define an action of $\boldsymbol{R}$ on $\boldsymbol{R}^2$ by

$$\begin{aligned}&\boldsymbol{R}\times\boldsymbol{R}^2\ni(t,(x,y))\\&\qquad\rightarrow(x\cdot\cos t-y\cdot\sin t,\ x\cdot\sin t+y\cdot\cos t)\in\boldsymbol{R}^2\,.\end{aligned}$$

By this action we construct the semi-direct product $G=\boldsymbol{R}\ltimes\boldsymbol{R}^2$. G is a simply connected solvable Lie group with no compact subgroup. Set

$$L=\{(\pi l,(m,n))\in\boldsymbol{R}\times\boldsymbol{R}^2;\ l\in\boldsymbol{Z},\ (m,n)\in\boldsymbol{Z}^2\}\ .$$

L is a lattice of G. In this case

$$F(L)=\{i_x;\ x\in\boldsymbol{R}^2\}\ ,$$

which is isomorphic to $\boldsymbol{R}^2$.

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