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CHARACTERIZATIONS OF p-NILPOTENT GROUPS

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Introduction

Let G be a finite group and p a prime. For a p-block B of G, let $\mathrm{Irr}^0(B)$ be the set of irreducible characters of height 0 in B. Most results in this paper are related with the characters of height 0 in the principal p-block $B_0(G)$. In section 1 we shall show that G is p-nilpotent if and only if every $\chi \in \mathrm{Irr}^0(B_0(G))$ is modularly irreducible (Theorem 1.3). This result is in a sense analogous to a theorem of Okuyama and Tsushima [8]. We shall give also a characterization of p-nilpotent groups via weights [1]. In section 2 several normal subgroups associated to $\mathrm{Ker}\,\chi$, $\chi \in \mathrm{Irr}^0(B)$, are shown to be p-nilpotent. Also p-nilpotent groups are characterized via their character values (Corollary 2.10). In section 3 a question arising from a paper of Ono [9] will be discussed. Throughout this paper (K, R, k) denotes a p-modular system. We assume that K contains the $|G|$-th roots of unity. The maximal ideal of R is denoted by (π).

1. Characterizations of p-nilpotent groups

Let

$$\Lambda(G) = \{\chi;\ \chi \in \mathrm{Irr}^0(B_0(G)),\ o(\det \chi) \not\equiv 0 \pmod{p}\}\ ,$$

where $o(\det \chi)$ denotes the determinantal order of χ. For an irreducible Brauer character ϕ of G and a subset Λ of $\Lambda(G)$, let $\delta(\Lambda, \phi) = \Sigma\, d(\chi, \phi)\, \chi(1)$, where $d(\chi, \phi)$ is the decomposition number and the sum is taken over all $\chi \in \Lambda$. For brevity, put $\delta(G, \phi) = \delta(\Lambda(G), \phi)$.

The following lemma will be used frequently in the sequel.

Lemma 1.1. *If $\delta(G, \phi) \not\equiv 0 \pmod{p}$ for some irreducible Brauer character ϕ in $B_0(G)$ with $\phi(1) \not\equiv 0 \pmod{p}$, then G is p-nilpotent.*

Proof. Put $N = O^p(G)$. Since $\phi(1)$ is prime to p, $\psi := \phi_N$ is irreducible. The same is true for $\chi \in \Lambda(G)$, and the restriction gives a bijection from $\Lambda(G)$ onto the subset Ξ of G-invariant members of $\Lambda(N)$, cf. Corollary 6.28 in Isaacs [5]. From this it follows that $\delta(G, \phi) = \delta(\Xi, \psi)$. Now let Ψ be the character of the projective cover of the module affording ψ. Since Ψ and ψ are G-invariant, it follows that $\Psi(1)\equiv\delta(\Xi, \psi)$ (mod p) (consider the natural action of G on Irr (N)). Hence we get $\Psi(1)\not\equiv 0$ (mod p), which shows that N is a p'-group. This completes the proof.

Remark 1.2. Although the above lemma was inspired by the proof of Theorem 12.1 in Isaacs [5], it turned out that a similar idea had appeared, cf. the proof of Theorem 2 in Pahlings [10].

Theorem 1.3. *The following conditions are equivalent.*

(i) *G is p-nilpotent.*

(ii) $l(B_0(G))=1$.

(iii) *Every irreducible character of height* 0 *in $B_0(G)$ is linear.*

(iv) *Every irreducible character of height* 0 *in $B_0(G)$ is modularly irreducible.*

(v) $\Lambda(G)=\{1_G\}$.

Proof. (i)$\Rightarrow$(ii): This is obvious

(ii)$\Rightarrow$(iii): Let $\chi\in\mathrm{Irr}(B_0(G))$ and assume that $\chi(1)>1$, then $\chi(g)=0$ for some $g\in G$ (Burnside). If s is the p'-part of g, then $\chi(g)\equiv\chi(s)$ (mod π), and $\chi(s)=\chi(1)$ by (ii). Hence $\chi(1)\equiv 0$ (mod p), completing the proof.

(iii)$\Rightarrow$(iv): Obvious.

(iv)$\Rightarrow$(i): Let $\chi\in\Lambda(G)$. If $d(\chi, 1_G)\neq 0$, (iv) implies that when considered as a Brauer character, χ is the trivial Brauer character. In particular $\chi(1)=1$ and then $o(\det\chi)\not\equiv 0$ (mod p) implies $\chi=1_G$. So we get $\delta(G, 1_G)=1$ and G is p-nilpotent by Lemma 1.1.

(i)$\Rightarrow$(v): This is obvious.

(v)$\Rightarrow$(i): (v) implies $\delta(G, 1_G)=1$, so G is p-nilpotent as above.

Remark. 1.4. The condition (ii) is due to R. Brauer and the condition (iii), which strengthens Thompson's condition [12], is due to Isaacs and Smith [6]. See also Pahlings [10]. In [6] the implication (iii)$\Rightarrow$(i) is proved through a characterization of groups of p-length 1. For a generalization of their characterization, cf. [7]. The condition (iv) may be considered as a special case of nonabelian version of a theorem of Okuyama and Tsushima [8].

We give still another characterization of p-nilpotent groups, which is related to the notion of weight introduced by Alperin [1].

Theorem 1.5. *The following conditions are equivalent.*

(i) *G is p-nilpotent.*

(ii) *$N_G(P)$ is p-nilpotent for a p-Sylow subgroup P of G, and there is no weight (Q, S) for G with $Q<P$ and $S\in B_0(N_G(Q))$.*

Proof. (i)$\Rightarrow$(ii): For any p-subgroup Q, $N_G(Q)$ is p-nilpotent. If there exists a simple $kN_G(Q)$-module S with vertex Q lying in $B_0(N_G(Q))$, then S must be the trivial module. So Q is a p-Sylow subgroup of $N_G(Q)$ and hence of G.

(ii)$\Rightarrow$(i): Assume false and let G be a counterexample of minimal order. Take a p-subgroup $Q\neq 1$ which is maximal under the condition that $N_G(Q)$ is not p-nilpotent. (Recall that G is p-nilpotent if $N_G(Q)$ is p-nilpotent for all p-subgroups $Q\neq 1$ of G.) Put $H=N_G(Q)$. By the choice of Q, $Q=O_p(H)$. We claim that H satisfies the same assumption as G. To see this let R be a p-Sylow subgroup of H. Then $R>Q$, so $N_H(R)\leqq N_G(R)$ is p-nilpotent. Next let (R, S) be a weight for H with S in $B_0(N_H(R))$. If $R\leqq Q$, $N_Q(R)\leqq O_p(N_H(R))=R$, cf. [1]. Hence $R=Q$, which contradicts the assumption (ii). Thus $RQ\neq Q$, so $N_H(R)\leqq N_G(QR)$ is p-nilpotent and R is a p-Sylow subgroup of H, cf. the proof of (i)$\Rightarrow$(ii). Thus the claim is proved. By the choice of G, we get that $G=H$. It is not difficult to see that $G/O_{p'}(G)$ and G/Q satisfy the same assumption as G. Hence $O_{p'}(G)=1$ and G/Q is p-nilpotent. In particular, G is p-solvable and $C_G(Q)\leqq Q$. Let $R/Q\neq 1$ be any p-subgroup of G/Q and K the normal p-complement of $N_G(R)$, then $[K, Q]=1$ and $K\leqq C_G(Q)\leqq Q$, so $K=1$. Hence $N_{G/Q}(R/Q)$ is a p-group. This shows that G/Q is a Frobenius group whose Frobenius complement is a p-Sylow subgroup. So G/Q has a simple kG/Q-module of p-defect 0. Since G has a unique block, this contradicts the assumption (ii). This completes the proof.

2. Block kernels

Throughout this section P is a p-Sylow subgroup of the group G.

Lemma 2.1. *Let N be a normal subgroup of G and B a block of G covering $B_0(N)$.*

(i) *Assume the following:*

(*) *there exists $\zeta\in \mathrm{Irr}^0(B)$ with $N\leqq \mathrm{Ker}\,\zeta$.*

Let ξ be a P-invariant member of $\Lambda(N)$. Then for some $\chi\in \mathrm{Irr}^0(B)$, we have $(\chi, \xi)_N\neq 0$.

(ii) *Assume that for any P-invariant member $\xi\neq 1_N$ of $\Lambda(N)$, $d(\xi, 1_N)=0$. Then N is p-nilpotent.*

Proof. (i) There exists an extension $\hat{\xi}$ of ξ to PN, as before. With ζ as in (*), let θ be the class function on G defined by

$$\theta(g)=\begin{cases} p^d\,\zeta(g) & \text{if } g \text{ is } p\text{-regular},\\ 0 & \text{otherwise,}\end{cases}$$

where d is the defect of B. We have $(\hat{\xi}^G, \theta)_G=(\hat{\xi}, \theta)_{PN}=p^d\,\zeta(1)\;|PN|^{-1}\,a$, where a denotes $\Sigma\,\xi(y)$ (y runs through $N_{p'}$). As is well-known ([2]) $a\not\equiv 0 \pmod{\pi}$, so $(\hat{\xi}^G, \theta)_G\not\equiv 0 \pmod{\pi}$. Hence $(\hat{\xi}^G, \chi)_G\neq 0$ for some $\chi\in \mathrm{Irr}^0(B)$ ([2]).

By Frobenius reciprocity, ξ appears as an irreducible constituent of χ_N. This completes the proof.

(ii) As in the proof of Lemma 1.1, we have that $\delta(N, 1_N) \equiv \delta(\Xi, 1_N)$ (mod p), where Ξ is the set of P-invariant members of $\Lambda(N)$. By assumption, $\delta(\Xi, 1_N)=1$, so the result follows from Lemma 1.1.

REMARK 2.2. The condition (*) is always satisfied and the assertion (i) itself could be extended, cf. Corollary 4.6 and Theorem 4.4 in [7].

For an arbitrary block B of G, we let $\mathrm{Ker}^0(B)=\bigcap \mathrm{Ker}\,\chi$, where χ runs through $\mathrm{Irr}^0(B)$.

Theorem 2.3. $\mathrm{Ker}^0(B)$ *is p-nilpotent.*

Proof. Put $N=\mathrm{Ker}^0(B)$. Let P be as above and ξ a P-invariant member of $\Lambda(N)$ and choose $\chi \in \mathrm{Irr}^0(B)$ with $(\chi, \xi)_N \neq 0$ (Lemma 2.1 (i)). Since $N \leqq \mathrm{Ker}\,\chi$, $\xi=1_N$. So N is p-nilpotent by Lemma 2.1 (ii).

Let $\mathfrak{N}(G)$ be the set of normal subgroups N of G such that for any $\chi \in \mathrm{Irr}^0(B_0(G))$, χ_N is a sum of linear characters of N. The following theorem gives a characterization of $O_{p',p}(G)$ via (ordinary) irreducible characters. We remark that $O_{p',p}(G)$ has been characterized by R. Brauer via irreducible modular representations.

Theorem 2.4. $O_{p',p}(G)$ *is the unique maximal member of* $\mathfrak{N}(G)$.

Proof. Let $N \in \mathfrak{N}(G)$. Let ξ be a P-invariant member of $\Lambda(N)$. Choose $\chi \in \mathrm{Irr}^0(B_0(G))$ with $(\chi, \xi)_N \neq 0$. (The condition (*) in Lemma 2,1 (i) is satisfied with $\zeta=1_G$.) By definition of $\mathfrak{N}(G)$, ξ must be linear, and then $o(\det \xi) \not\equiv 0$ (mod p) implies that $d(\xi, 1_N)=0$ unless $\xi=1_N$. So N is p-nilpotent by Lemma 2.1 (ii), and $N \subseteq O_{p',p}(G)$. Conversely, let ξ be an irreducible constituent of χ_N, where $N=O_{p',p}(G)$ and $\chi \in \mathrm{Irr}^0(B_0(G))$. Then ξ lies in $B_0(N)$ and $\xi(1)$ is prime to p, so $\xi(1)=1$, since N is p-nilpotent. This completes the proof.

REMARK 2.5. The implication (iii)$\Rightarrow$(i) in Theorem 1.3 follows also from the above theorem.

We can restate Theorem 2.4 as follows:

Corollary 2.6. $O_{p',p}(G)/\mathrm{Ker}^0(B_0(G))$ *is the unique maximal normal abelian subgroup of* $G/\mathrm{Ker}^0(B_0(G))$.

For the principal block, let

$$Z^0(G) = \{g \in G;\ |\chi(g)| = \chi(1) \quad \text{for any} \quad \chi \in \mathrm{Irr}^0(B_0(G))\}\ ,$$

where $|\cdot|$ denotes the absolute value. Then $Z^0(G) \in \mathfrak{N}(G)$, so we get:

Corollary 2.7. *$Z^0(G)$ is p-nilpotent.*

REMARK 2.8. This corollary could be used in the proof of Z^*-Theorem, cf. Step VI of the proof of Theorem 1 in Glauberman [3].

Theorem 2.9. $O_{p'}(G/Z^0(G))=1$.

Proof. Put $Z=Z^0(G)$. Let N be the inverse image in G of $O_{p'}(G/Z)$. We claim that N is p-nilpotent. Assume this, then N/Z is a p-group, since $O_{p'}(N)=O_{p'}(G)\leqq Z$. Hence $N/Z=1$, as required. To prove the claim, let ξ be any P-invariant member of $\Lambda(N)$ and choose $\chi\in\mathrm{Irr}^0(B_0(G))$ such that $(\chi, \xi)_N\neq 0$ as above. By definition of Z, χ_Z is a multiple of a linear character. So $\xi_Z=e\,\eta$, where $e=\xi(1)$ and η is a linear character of Z. Since χ is trivial on $O_{p'}(G)=O_{p'}(N)$, so is ξ. Hence $(\det\xi)_Z$ (which equals η^e) and η are inflated from $Z/O_{p'}(N)$. Since this group is a p-group by Corollary 2.7 and $o(\det\xi)$ is prime to p, it follows that $(\det\xi)_Z=1_Z$. Then $\eta=1_Z$, since e is prime to p. So ξ is inflated from N/Z. Hence $d(\xi, 1_N)=0$ unless $\xi=1_N$, since N/Z is a p'-group. This implies that N is p-nilpotent as before.

Now we give a characterization of p-nilpotent groups via their character values, from which the implication (iii)$\Rightarrow$(i) in Thoerem 1.3 follows again.

Corollary 2.10. *The following conditions are equivalent.*

(i) *G is p-nilpotent.*

(ii) $|\chi(u)|=\chi(1)$ *for all p-elements u of G and all* $\chi\in\mathrm{Irr}^0(B_0(G))$.

Proof. (i)$\Rightarrow$(ii): This is obvious.
(ii)$\Rightarrow$(i): (ii) implies that $G/Z^0(G)$ is a p'-group, so $G=Z^0(G)$ by Theorem 2.9. Then G is p-nilpotent by Corollary 2.7.

3. Conjugacy classes of Ono type

For any irreducible character χ of the group G, we let, as usual, ω_χ be the central character associated to χ.

DEFINITION 3.1. Let α be an element of the center of $\boldsymbol{Z}G$, where $\boldsymbol{Z}$ is the ring of integers. α is said to be of *Ono type* if for any $\chi\in\mathrm{Irr}(G)$ there holds either $\omega_\chi(\alpha)=0$ or $|\omega_\chi(\alpha)|=\varepsilon(\alpha)$, where ε: $\boldsymbol{Z}G\to\boldsymbol{Z}$ is the augmentation map. A conjugacy class K is said to be of Ono type if the class sum $\hat{K}$ is of Ono type. (If $g\in K$, the condition is the same as saying that either $\chi(g)=0$ or $|\chi(g)|=\chi(1)$ for all $\chi\in\mathrm{Irr}(G)$.) A group G is said to be of Ono type if every conjugacy class of G is of Ono type.

Groups of Ono type has appeared in Ono [9]. First we prove:

Psoposition 3.2. *Groups of Ono type are nilpotent.*

Proof. Let G be a group of Ono type and p any prime. For any p-element g of G, either $\chi(g)=0$ or $|\chi(g)|=\chi(1)$ holds, for any $\chi\in\mathrm{Irr}^0(B_0(G))$. Since $\chi(g)\equiv\chi(1)\not\equiv 0 \pmod{\pi}$, the latter holds. So G is p-nilpotent by Corollary 2.10. Since p is arbitrary, G is nilpotent.

REMARK 3.3. The above proposition could be proved by induction on the group order (without using block theory).

One may well conjecture that the subgroup generated by a conjugacy class of Ono type is solvable, as will be explained below.

For a subset H of G, let $\hat{H}=\Sigma_{h\in H}\, h$. For an element $\alpha=\Sigma_g\, \alpha_g\, g$ of $\boldsymbol{Z}G$, put $\mathrm{Supp}\,\alpha=\{g\in G;\ \alpha_g\neq 0\}$.

Lemma 3.4. *For an element $\alpha(\neq 0)$ of the center of $\boldsymbol{Z}G$ with $\alpha_g>0$ for all $g\in\mathrm{Supp}\,\alpha$, the following conditions are equivalent.*

(i) *α is of Ono type.*

(ii) *$\alpha=mg\hat{H}$, for a positive integer m and a subgroup H of G.*

(iii) *$\alpha=mg\hat{H}$, for a positive integer m and a normal subgroup H of G.*

Proof. (i)$\Rightarrow$(ii): This is proved by induction on $|G|$. First assume that there is $\chi\in\mathrm{Irr}(G)$ such that $|\omega_\chi(\alpha)|=\varepsilon(\alpha)$ and that $\chi(1)>1$. Then $\mathrm{Supp}\,\alpha\subseteq Z(\chi)\neq G$, where $Z(\chi)=\{g\in G;\ |\chi(g)|=\chi(1)\}$. For any $\zeta\in\mathrm{Irr}(Z(\chi))$, take $\chi\in\mathrm{Irr}(G)$ such that $(\chi,\zeta)_N\neq 0$, then $\omega_\chi(\alpha)=\omega_\zeta(\alpha)$. So we get the conclusion by the induction hypothesis applied to $Z(\chi)$. So we may assume that $\omega_\chi(\alpha)=0$ for any $\chi\in\mathrm{Irr}(G)$ with $\chi(1)>1$. Then $\alpha=\Sigma\,\omega_\lambda(\alpha)\,e_\lambda$, where the sum is taken over the linear characters λ of G and e_λ is the central idempotent associated to λ. Replacing α by $g^{-1}\alpha, g\in\mathrm{Supp}\,\alpha$, if necessary, we may further assume $1\in\mathrm{Supp}\,\alpha$. Assume that for some λ, $|\omega_\lambda(\alpha)|=\varepsilon(\alpha)$. Then for any $g\in\mathrm{Supp}\,\alpha$, $\lambda(g)=\lambda(1)$, so $g\in\mathrm{Ker}\,\lambda$, and if $\mathrm{Ker}\,\lambda\neq G$, we get the conclusion by induction as above. So we may assume that $\omega_\lambda(\alpha)=0$ for $\lambda\neq 1_G$. This implies α is a multiple of e_{1_G}, so (ii) holds.

(ii)$\Rightarrow$(iii): Let $\alpha=mg\hat{H}$ as above. For any $x\in G$, $m\,g^x\,\hat{H}^x=\alpha^x=\alpha=mg\hat{H}$. So $g^x\in gH$ and $g^x\,\hat{H}=g\hat{H}=g^x\,\hat{H}^x$. Hence $\hat{H}^x=\hat{H}$, so H is normal.

(iii)$\Rightarrow$(i): Let $\chi\in\mathrm{Irr}(G)$. If $\mathrm{Ker}\,\chi\geqq H$, $|\omega_\chi(\alpha)|=\varepsilon(\alpha)$, because gH is central in G/H. Otherwise, $\omega_\chi(\alpha)=0$, as is well-known. This completes the proof.

From this lemma we get:

Corollary 3.5. *A conjugacy class K of G is of Ono type if and only if $K=gH$ for some $g\in G$ and a (normal) subgroup H of G.*

Lemma 3.6. *Let K be a conjugacy class of Ono type consisting of p-elements for some prime p. Then the subgroup generated by K is p-nilpotent.*

Proof. Let $g \in K$. As in the proof of Proposition 3.2, we get that $g \in Z^0(G)$, and the conclusion follows from Corollary 2.7.

Now we have:

Theorem 3.7. *The following assertions are equivalent.*

(i) *Any conjugacy class of Ono type consisting of elements of prime power order generates a solvable subgroup.*

(ii) *Let G be a semi-direct product of groups A and N with N normal. If $C_N(A)=1$ and A is cyclic of prime power order, then G is solvable.*

Proof. (i)$\Rightarrow$(ii): Let g be a generator of A and K the conjugacy class containing g of G. Obviously $K \subseteq gN$ and we have $|K|=|N|$, since $C_N(g)=1$. So $K=gN$, and K is of Ono type by Corollary 3.5. The subgroup generated by K is G, so G is solvable. (ii)$\Rightarrow$(i): Let K be a conjugacy class of a group G consisting of p-elements for a prime p. The proof is done by induction on $|G|$. Since K is of Ono type, $K=gH$ for a normal subgroup H of G and $g \in K$. We see that $\langle K\rangle=\langle g\rangle H$ and that $H=\{g^{-1}g^x; x \in G\}$. Let N be the normal p-complement of H, cf. Lemma 3.6. We may assume that $N \neq 1$. Let C be the inverse image in G of $C_{G/N}(gN)$. We claim that the conjugacy class K' containing g of C is of Ono type. Since $N \leqq H$, it follows that $N=\{g^{-1}g^x; x \in C\}$. Then $K'=gN$ and the claim follows from Corollary 3.5. If $C \neq G$, $\langle K'\rangle$ and hence N is solvable by induction. Since the image of K in G/N is of Ono type, the image of $\langle K\rangle$ in G/N is solvable by induction. So $\langle K\rangle$ is solvable. Now assume $C=G$. Then $N=H$ by the above. On the other hand, we must have $G=C_G(g)N$, since N is a p'-group. This implies $C_G(g) \cap N=1$, since $|N|=|K|$. Taking $A=\langle g\rangle$ in (ii), we get that G is solvable.

The assertion (ii) is a longstanding conjecture (see for example [4], p 487)

From the above (proof) and a theorem of Thompson [11], we get:

Corollary 3.8. *Let K be a conjugacy class of Ono type consisting of elements of prime order. Then K generates a solvable subgroup.*

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