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On the Pseudo-Harmonic Functions

By Yukinari TÔKI and Kôichi TARUMOTO

Introduction. Let F be an orientable surface. Let $u(p)$ be a real-valued function in a neighborhood N_{p_0} of p_0 on F where N_{p_0} corresponds to the unit circular disc in the complex plane by the topological mapping $z = T_{p_0}(p)$, $z = x + iy$.

Set
$$u(p) = u(T_{p_0}(p)) = U(z).$$

Then $u(p)$ is termed *pseudo-harmonic* at p_0 , if $U(z)$ is harmonic and not identically constant in $|z| < 1$. A real-valued function on F is termed *pseudo-harmonic* if it is pseudo-harmonic on each point of F . In this paper we will prove that there exist the local parameters such that F is a Riemann surface with respect to them and $u(p)$ is harmonic on F .

1. Terminologies and notations.

Let $u(p)$ be a pseudo-harmonic function on F . By the *level-curve* of $u(p)$ with the *height* c , we mean the locus of the equation $u(p) = c$. It is well known that with each point $p_0 \in F$, there exists a suitably chosen neighborhood N_{p_0} of p_0 and a topological mapping $z = T_{p_0}(p)$ of N_{p_0} onto $|z| < 1$ under which p_0 goes into $z = 0$ and the level-curves of $u(p)$ in N_{p_0} go into the level-curves of $Re z^n$ in $|z| < 1$ ¹⁾. we shall term this N_{p_0} a *canonical neighborhood* of p_0 . When $n = 1$, we shall call p_0 a *regular point* and N_{p_0} a *simple canonical neighborhood*. When $n \geq 2$, we shall call p_0 a *saddle-point of order* n . A real-valued function $v(p)$ on F is called "*pseudo-conjugate to a pseudo-harmonic function $u(p)$* ", if it satisfies the following condition.

There exists a topological mapping $z = T_{p_0}(p)$ by which N_{p_0} corresponds to $|z| < 1$, and $U(z) = u(T_{p_0}(p))$ is conjugate-harmonic to $V(z) = v(T_{p_0}(p))$ in $|z| < 1$.

1) Y. Tôki, A topological characterization of pseudo-harmonic functions, Osaka Mathematical J. 3 (1951), 101-122. See also J. Jenkins and M. Morse, Topological methods on Riemann surface, pseudoharmonic function. Contributions to the theory of Riemann surfaces 1953 p. 114.

2. The triangulation of a surface.

Let F be an orientable surface and $u(p)$ be a pseudo-harmonic function on it. In the first place, we can easily triangulate the surface F such that each saddle-point of $u(p)$ is a vertex of a triangle and each triangle of F is contained in a canonical neighborhood, especially any triangle without the saddle-points is contained in a simple canonical neighborhood. We shall prove the following lemmas on this triangulation.

Lemma 1. *We can triangulate the surface F such that each side of any triangle of F intersects every one of the level-curves of $u(p)$ at most at the finite number of points.*

Proof. Let Δ be any triangle on F and a, b, c , be the three vertices of it. Let L_i ($i=1, 2 \dots n$) and M_j ($j=1, 2 \dots m$) be the sides of the triangles with the common vertex a and b respectively: especially L_1 denotes the arc \widehat{ab} , M_1 denotes the arc \widehat{ba} . There exists a canonical neighborhood N_Δ ($N_\Delta \supset \Delta$) and a topological mapping $z = T_\Delta(p)$ under which Δ is mapped onto a curvilinear triangle Δ' in $|z| < 1$. Let the points a', b', c' , be the three vertices of Δ' and L'_i ($i=1, 2 \dots n$) and M'_j ($j=1, 2 \dots m$) the mapped images of the arc L_i ($i=1, 2 \dots n$) and M_j ($j=1, \dots m$) in N_Δ . Let $C_{a'}$ and $C_{b'}$ be the sufficiently small circles with the center a', b' and contained in $|z| < 1$ respectively. Let a'_i ($i=1, 2, \dots n$) be the points at which the arc L'_i cut the circle $C_{a'}$ for the last time. We can choose the points b'_j ($j=1, 2 \dots m$) on $C_{b'}$ similarly. We can connect a'_1 and b'_1 by a polygon without intersecting L'_i ($i=1, 2 \dots n$) and M'_j ($j=1, 2 \dots m$) out side of the circles

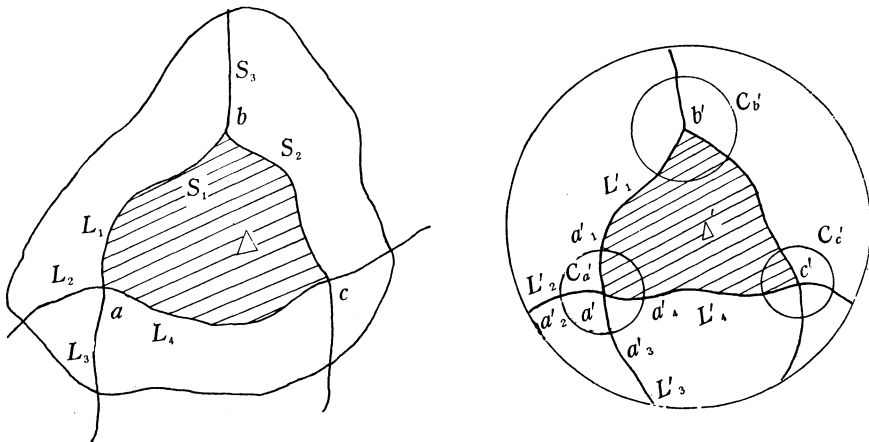


Fig. 2.

$C_{a'}$ and $C_{b'}$. We also connect a'_j and a' by the radius in the circle $C_{a'}$. We connect b'_j and b' similarly. We repeat this deformation with respect to every side of the triangles on F . In this repetition, each side of the triangles are varied in finite times: for instance, side \widehat{ab} varies in $(m+n-1)$ -times. When some part of a side of a triangle lies on a level-curve, then we can deform slightly it such that each one of sides of the deformed triangle cut the level-curves at most once.

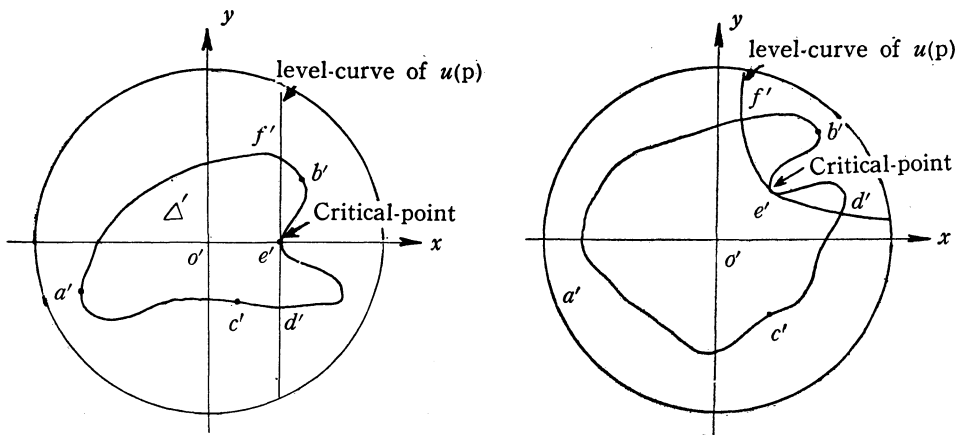
Therefore we have after a finite number of time the desired triangulation. A point p on the sides of a triangles is termed a critical point when the side through the point p is on one side of the level-curve $u(p)$ except to the point p in the neighborhood of p from now.

Lemma 2. *We can triangulate the surface F such that each side of any triangle of F intersects every one of the level-curves of $u(p)$ at most at one point.*

Proof. Let Δ be any triangle of F such that each side of it intersects every one of the level-curves of $u(p)$ at most at a finite number of points. When Δ have critical points or saddle-points on its boundary. Let us subdivide Δ into triangles and polygonal domains by the level-curves through the critical points and the saddle-point.

Let one of these polygonal domains be Σ . The polygonal domain Σ can be mapped onto a rectangle Σ^* by the topological mapping $z = S_\Sigma(p)$ under which the level-lines in Σ go into the lines parallel to the y -axis and the vertices of Σ go into points on the boundary of Σ^* .

The polygonal domain Σ^* can be subdivided into triangles by lines connecting the center of Σ^* to the vertices. Let us subdivide Σ into triangles which are the inverse images of the triangles of Σ^* .



simple-canonical neighborhood.

Fig. 1.

Saddle-point of order 2.

Subdivide each polygonal domain of F into triangles similarly. We can easily deform the above triangulation slightly such that each side of the triangles intersect the level-lines at most once.

Theorem. *Let $u(p)$ be pseudo-harmonic on F . We can associate the local parameters of F such that F is a Riemann surface with respect to them and $u(p)$ is harmonic on it.*

Proof. By the lemma 2, we can subdivide the surface F such that each side of any triangle of F intersects every one of the level-curves of $u(p)$ at most at one point. Therefore each triangle of $\{\Delta\}$ can be mapped onto the rectilinear one in the z -plane and at the same time the level-curves of $u(p)$ can be mapped onto the lines parallel to the y -axis.

Let these transformations be $z = \tau_{\Delta}(p)$. It is clear that the function $u(\tau_{\Delta}^{-1}(z))$ is harmonic. Let p_0 be any point on F and Δ_{p_0} be a triangle such that $\Delta_{p_0} \ni p_0$. The following three cases will arise:

- (i) p_0 is contained in Δ_{p_0} .
- (ii) p_0 lies on one of the sides of Δ_{p_0} .
- (iii) p_0 is a vertex of Δ_{p_0} .

We can associate the local parameters as follows, corresponding to the above three cases.

(i) We associate the function $z = \tau_{\Delta_{p_0}}(p)$ as a local parameter to p_0 .

(ii) There exists the two neighboring triangles Δ_j and Δ_k such that the point p_0 is contained in the common side of Δ_j and Δ_k . We can transform Δ_j and Δ_k onto the rectilinear ones S_j and S_k by the transformation $z = \tau_{\Delta_j}(p)$ and $z = \tau_{\Delta_k}(p)$ respectively. We can also map S_j and S_k onto the triangles R_j and R_k lying on the upper and the lower half-plane with common side of the interval $0 \leq x \leq 1$ by two linear transformations respectively. Any point on the common side of Δ_j and Δ_k is mapped on the different points on the side of S_j and S_k respectively. Since these two points lie on the same level-curve parallel to the x -axis, it is clear that these are mapped on the same point on the interval $0 \leq x \leq 1$ by the two linear transformations. Thus we can map the curvilinear quadrilateral $\Delta_j \cup \Delta_k$ onto the rectilinear quadrilateral $R_j \cup R_k$ topologically and the common side of Δ_j and Δ_k can be mapped onto the interval $0 \leq x \leq 1$. Let this transformation be $z = \tau_{\Delta_j, \Delta_k}(p)$. We associate this function to p_0 as a local parameter of p_0 .

(iii) Let $\Delta_{i_1}, \Delta_{i_2}, \dots, \Delta_{i_n}$ be the triangles with the common vertex p_0 . Each Δ_{i_k} ($k=1, 2, \dots, n$) is mapped onto a rectilinear one S_{i_k} ($k=1, 2, \dots, n$) and p_0 goes into z_{i_k} by the transformation $z = T_{\Delta_{i_k}}(p)$. Let the vertical angle of z_{i_k} of S_{i_k} be α_{i_k} . The triangle S_{i_k} is mapped onto S'_{i_k}

and z_{i_k} goes into w_{i_k} by the transformation $w = z^{2\pi/(\alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_n})}$. Let the vertical angle of w_{i_k} of S'_{i_k} be β_{i_k} . Then $\sum_{k=1}^n \beta_{i_k} = 2\pi$. Accordingly, we can map S'_{i_k} and $S'_{i_{k+1}}$ onto S''_{i_k} and $S''_{i_{k+1}}$ by linear transformations respectively such that w_{i_k} and $w_{i_{k+1}}$ go into $\zeta = 0$ and the common side of the two neighboring triangles Δ_{i_k} and $\Delta_{i_{k+1}}$ goes into the common side of S''_{i_k} and $S''_{i_{k+1}}$. Thus the polygonal domain composed of Δ_{i_k} ($k=1, 2 \dots n$) is mapped onto the polygonal domain consisting of S''_{i_k} ($k=1, 2 \dots n$) in the ζ -plane. Let this mapping be $\zeta = \tau_{\Delta_{i_1}, \Delta_{i_2}, \dots, \Delta_{i_n}}(p)$.

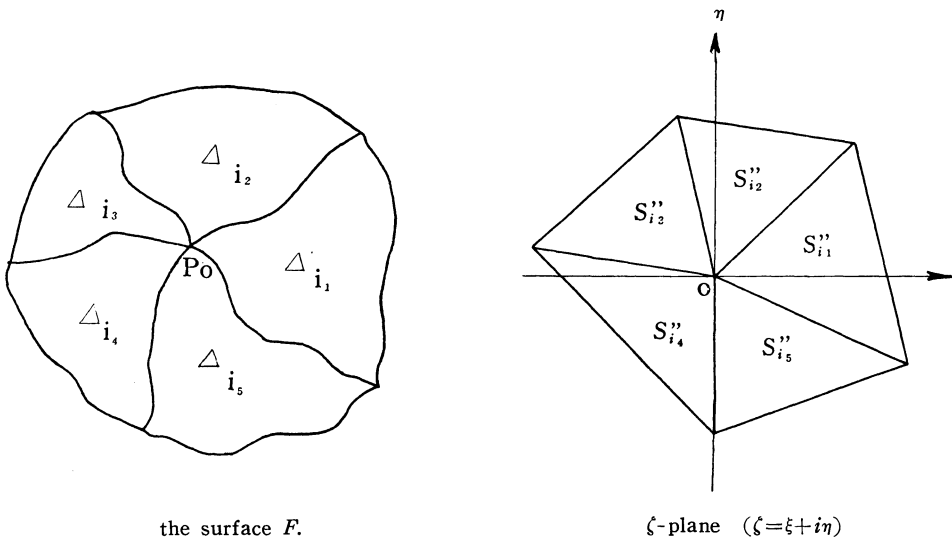


Fig. 3.

We associate the function $\zeta = \tau_{\Delta_{i_1}, \dots, \Delta_{i_n}}(p)$ to p_0 as a local parameter. These local parameters $\tau_{\Delta}(p)$, $\tau_{\Delta_{i_1}, \Delta_{i_2}}(p)$ and $\tau_{\Delta_{i_1}, \dots, \Delta_{i_n}}(p)$ satisfy the conformal neighboring relation and $u(p)$ is harmonic on F with respect to them.

Corollary. *Let $u(p)$ be a pseudo-harmonic function on F . Then there exists always a conjugate pseudo-harmonic function to $u(p)$ on F .*

Proof. We can assume that the function $u(p)$ is harmonic on F with respect to the suitably chosen local parameters by the theorem. Then there exists always a conjugate harmonic function to $u(p)$ on F . The corollary follows at once. This conjugate pseudo-harmonic function $v(p)$ is multiple-valued on F in general.

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