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<th><strong>Title</strong></th>
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<tr>
<td><strong>Author(s)</strong></td>
<td>Washio, Takumi; Kasue, Atsushi</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 27(4) P.899-P.928</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1990</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/6799">https://doi.org/10.18910/6799</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/6799</td>
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Osaka University
GROWTH OF EQUIVARIANT HARMONIC MAPS AND HARMONIC MORPHISMS

Dedicated to Prof. Tadashi Nagano on his 60th birthday

ATSUSHI KASUE and TAKUMI WASHIO

(Received December 4, 1989)

0. Introduction

The purpose of the present paper is to study the growth of certain harmonic maps in relation with the geometry of the domains and ranges.

Let \( \phi: M \to N \) be a harmonic map between complete noncompact Riemannian manifolds \( M \) and \( N \). We fix a point \( o \) of \( M \) (resp. a point \( o' \) of \( N \)) and denote by \( r_M \) (resp. \( r_N \)) the distance to \( o \) in \( N \) (resp. \( o' \) in \( N \)). Set \( \mu(\phi; t) := \max \{ r_N(\phi(x)) : x \in M, r_M(x) = t \} \). We want to know the growth of \( \phi \), or the asymptotic behavior of \( \mu(\phi; t) \) as \( t \) goes to infinity. We first recall the following result by Cheng [8] (cf. also [3] [31: Chap. 6]): Suppose that \( M \) has nonnegative Ricci curvature and \( N \) is a Hadamard manifold, namely, \( N \) is a simply connected and nonpositively curved manifold manifold. Then the energy density \( e(\phi) \) of the map \( \phi \) satisfies:

\[
e(\phi)(o) \leq c_M \mu(\phi; t)^{\frac{2}{m}},
\]

where \( c_M \) is a constant depending only on the dimension \( m \) of \( M \). It follows that \( \phi \) is a constant map if \( \phi \) has sublinear growth, that is, \( \lim \inf \frac{\mu(\phi; t)}{t} = 0 \). We are interested in a (nonconstant) harmonic map \( \phi: M \to N \) which has linear growth, namely, which has the property that \( \lim \sup \frac{\mu(\phi; t)}{t} < +\infty \). For instance, it turns out that a harmonic map \( \phi: M \to N \) of linear growth must be totally geodesic if \( M \) has volume growth of at most quadratic order (cf. [9]). It has been also proved in [24] that a \( d \)-closed harmonic 1-form of bounded length on \( M \) must be parallel if the sectional curvature of \( M \) is nonnegative and decays quadratically. Moreover Li and Tam [26] have shown that the dimension on the space of linear growth harmonic functions on \( M \) is less than or equal to \( k + 1 \) if the volume of the metric ball of radius \( t \) around \( o \) is bounded by \( ct^k \) for some constant \( c \).

On the other hand, we can construct a noncompact complete manifold \( M \) of positive Ricci curvature and a harmonic map \( \phi_F: M \to F \) of bounded energy density from \( M \) onto a complete manifold \( F \) of nonnegative Ricci curvature (cf. Example in Section 2). It turns out from the construction that \( \phi \) is a harmonic morphism from \( M \) onto \( F \) with totally geodesic fibers, namely, it is a
map carrying the germs of harmonic functions to the germs of harmonic functions such that the inverse image of $\phi$ for each $p$ of $F$ is a totally geodesic submanifold. For example, in case $F = R$, $\phi_R: M \to R$ is a harmonic function of linear growth which is not totally geodesic, and in case $F = S^1$, $\phi_{S^1}: M \to S^1$ defines a harmonic 1-form of bounded length which is not parallel. Moreover we observe that if $F$ admits a harmonic function $h$ of linear growth, then the composition $h \circ \phi_F: M \to R$ is also a harmonic function of linear growth on $M$. It is not clear whether there exist (nonconstant) harmonic maps with linear growth between manifolds of nonnegative Ricci curvature and Hadamard manifolds of negative curvature.

We shall explain briefly the contents of this paper. In Section 1, we construct equivariant harmonic maps by solving certain ordinary differential equations and discuss their growth in some cases (cf. Theorem 1.1). Section 2 is devoted to the study of harmonic morphisms and their growth. For example, we shall give a lower bound for the growth of harmonic morphisms under certain conditions (cf. Theorem 2.10). In Section 3, we consider harmonic maps with linear growth between manifolds of nonnegative Ricci curvature and Hadamard manifolds and get sufficient conditions for such maps to be totally geodesic (cf. Theorem 3.2).

1. Examples of equivariant harmonic maps and their growth

In this section, we shall first show some examples of equivariant harmonic maps and then discuss the asymptotic behavior of them at infinity. See e.g., [4: Chap. 6] for a general theory on equivariant harmonic maps.

1.1. Let us first consider simple equivariant harmonic maps between rotationally symmetric spaces. To begin with, take a smooth function $\eta$ on $[0, \infty)$ such that

$$\eta(0) = 0, \quad \eta'(0) = 1 \quad \text{and} \quad \eta > 0 \quad \text{on} \quad (0, \infty),$$

and also a smooth function $\xi$ on $[0, \infty)$ with the same property (1.1) as $\eta$. We denote by $g_\eta$ a Riemannian metric on $\mathbb{R}^n$ which can be expressed as $g_\eta = dt^2 + \eta(t)^2d\theta^2$ in the polar coordinates $(t, \theta)$. Let us denote by $\mathbb{R}^n_\eta$, $\mathbb{R}^n_\eta(T)$ (resp., the metric ball of $\mathbb{R}^n_\eta$ with radius $T$ around the origin) for simplicity. Let $\phi: S^{n-1} \to S^{n-1}$ be a harmonic map from the unit sphere $S^{n-1}$ of $\mathbb{R}^n$ to the unit sphere $S^{n-1}$ of $\mathbb{R}^n$ with constant energy $e$. For a positive smooth function $\alpha_1(t)$ on $(0, T)$ ($0 < T \leq +\infty$), define a map $F_1: \mathbb{R}^n_\eta(T) \setminus \{0\} \to \mathbb{R}^n_\eta$ by

$$F_1(t, \theta) = (\alpha_1(t), \phi(\theta)).$$

Then it turns out from direct computations that $F_1$ is harmonic if and only if $\alpha_1$ satisfies an ordinary differential equation:
Let us next consider equivariant harmonic maps from the Riemannian 4-manifold $R^4_{\Omega_1\Omega_2\Omega_3}$ described below to $R^3_\theta$ or $R^3_{\theta_1\theta_2\theta_3}$. Let $Z_1, Z_2, Z_3$ be a left invariant orthonormal frame field on the unit 3-sphere $S^3$ such that $[Z_1, Z_2] = 2Z_3$, $[Z_2, Z_3] = 2Z_1$, $[Z_3, Z_1] = 2Z_2$. We denote by $\Omega_i$ ($i = 1, 2, 3$) the dual forms of $Z_i$ and consider a Riemannian metric $g_{\Omega_1\Omega_2\Omega_3}$ on $R^4$ of the form:

$$g_{\Omega_1\Omega_2\Omega_3} = dt^2 + \eta_1(t)^2\Omega_1^2 + \eta_2(t)^2\Omega_2^2 + \eta_3(t)^2\Omega_3^2,$$

where $\eta_i$ ($i = 1, 2, 3$) satisfy (1.1). As before, $R^4_{\Omega_1\Omega_2\Omega_3}$ (resp., $R^4_{\Omega_1\Omega_2\Omega_3}(T)$) stands for the Riemannian manifold $(R^4, g_{\Omega_1\Omega_2\Omega_3})$ (resp., the metric ball around the origin with radius $T$).

Let $\psi: S^3 \to S^2$ be the Hopf fibering and assume that $\text{Ker} \psi_* = Z_3$. We consider the case $\eta_1 = \eta_2$, and set $\eta_1 = \eta_2$ and $\lambda = \eta_3$. Given a smooth function $\alpha_2: (0, T) \to (0, \infty)$ ($0 < T \leq +\infty$), define a map $F_2: R^4_{\eta_1\eta_2\eta_3}(T) \setminus \{0\} \to R^3_\theta$ by

$$F_2(t, \theta) = (\alpha_2(t), \psi(\theta)).$$

Then direct computations show that $F_2$ is harmonic if and only if $\alpha_2$ satisfies the following ordinary differential equation:

$$(E_1) \quad \alpha_2''(t) + \frac{1}{t} P_2(t) \alpha_2'(t) - \frac{1}{t^2} Q_2(t, \alpha_2(t)) = 0$$
on $(0, T)$, where $P_2(t) = (m - 1)\eta_1'(t)\eta_1(t)^{-1}$ and $Q_2(t, s) = e t^2 \eta(t)^{-2}\xi(s)\xi'(s)$.

Let $F_3: R^4_{\eta_1\eta_2\eta_3}(T) \setminus \{0\} \to R^3_{\theta_1\theta_2\theta_3}$ be a map defined by

$$F_3(t, \theta) = (\alpha_3(t), \theta),$$

where $\alpha_3(t)$ is a positive smooth function on $(0, T)$. Then the equation for the harmonicity of the map $F_3$ is given by

$$(E_3) \quad \alpha_3''(t) + \frac{1}{t} P_3(t) \alpha_3'(t) - \frac{1}{t^2} Q_3(t, \alpha_3(t)) = 0$$
on $(0, T)$, where $P_3(t) = t \sum_i \eta_i'(t)\eta_i(t)^{-1}$ and $Q_3(t, s) = t^2 \sum_i \eta_i(t)^{-2}\xi_i(s)\xi_i'(s)$.

Finally let us consider equivariant harmonic maps between $R^2$-bundles over 2-sphere $S^2$ with certain metrics. Take first a smooth function $g$ on $[0, \infty)$ such that

$$(1.2) \quad g(0) = 1, \quad g'(0) = 0, \quad \text{and} \quad g > 0 \quad \text{on} \quad [0, \infty)$$

and also a smooth function $h$ on $[0, \infty)$ with the same property (1.2) as $g$. Let
\( \eta \) be a function satisfying (1.1). Then define a Riemannian metric \( \hat{G}_{g, k, \eta} \) on \( S^2 \times \mathbb{R}^2 \) by

\[
\hat{G}_{g, k, \eta} = g(t)^2 (\Omega_1^2 + \Omega_2^2) + h(t)^2 \Omega_3^2 + d^2 + \eta(t)^2 d\theta^2,
\]

where \( \{ \Omega_1, \Omega_2, \Omega_3 \} \) are, as before, left invariant 1-forms over the unit 3-sphere \( S^3 \) in \( \mathbb{R}^4 = \mathbb{C}^2 \). Given an integer \( k \), we consider the quotient space \( L_k \), or the \( \mathbb{R}^2 \)-bundle over \( S^2 \), by the action of \( S^1 \) on \( S^3 \times \mathbb{R}^2 \): \((p, t, \theta) e^{i\omega} = (pe^{i\omega}, t, \theta - k\omega)\). Then the metric \( \hat{G}_{g, k, \eta} \) descends to \( L_k \) and define a Riemannian metric \( G_{g, k, \eta} \) on \( L_k \), since the action preserves the metric. We denote by \( L'_k \) and \( \pi_k \): \((S^3 \times \mathbb{R}^2, \hat{G}_{g, k, \eta}) \to L'_k \), respectively, the resulting Riemannian manifold \( (L^*, G_{g, k, \eta}) \) and the Riemannian submersion. Moreover the radial function \( t \) on \( S^3 \times \mathbb{R}^2 \), which is the distance function to \( S^3 \times \{0\} \) with respect to \( \hat{G}_{g, k, \eta} \), descends also to the quotient manifold \( L_k \) and defines the distance function to the zero section \( S^3 \) of \( L_k \) with respect to \( G_{g, k, \eta} \). We put \( L_k(T) = \{ x \in L_k : t(x) < T \} \). For a pair \((k, l)\) of integers such that \( k \) divides \( l \), i.e., \( l = nk \) for some integer \( n \), we take two Riemannian manifolds \( L_k \) and \( L'_k \) described just above, and consider a map \( F_\alpha : L_k \to S^2 \) defined by

\[
F_\alpha(\pi_k(p, t, \theta)) = \pi_\alpha(f(p), \alpha_\eta(t), \phi_n(\theta)),
\]

where \( f : S^3 \to S^3 \) is a symmetry of the Hopf fibration (i.e., a unitary transformation or the composition of a unitary transformation and the conjugation), \( \alpha_\eta \) is a positive smooth function on \((0, T)\), and \( \phi_n : S^1 \to S^1 \) is the rotation given by \( \phi_n(e^{i\omega}) = e^{i\omega} \). Then direct computations show that \( F_\alpha : L_k \to S^2 \) is harmonic if and only if \( \alpha_\eta \) satisfies

\[
(\text{E}_4) \quad \alpha_\eta''(t) + \frac{1}{t} P_\eta(t) \alpha_\eta'(t) - \frac{1}{t^2} Q_\eta(t, \alpha_\eta(t)) = 0
\]
on \((0, T)\), where

\[
P_\eta(t) = t \left( \frac{2 g'(t)^2 + h^2 \eta(t)^2 h'(t) + h(t)^2 \eta'(t)}{g(t)^2} \right),
\]

\[
Q_\eta(t, s) = t^2 \left( \frac{g(s)^2 g'(s)^2 + n^2(h^2 \eta(t)^2 + h(t)^2)(\eta(s)^2 \eta'(s)^2 + \eta(s)^2 \eta'(s)^2)}{h(t)^2 \eta(t)^2 (l^2 \eta(s)^2 + \eta(s)^2)^2} \right).
\]

We note here that \( P_\eta(t) \) and \( Q_\eta(t, s) \) \((i=1, 2, 3, 4)\) have the following properties:

\[
(1) \quad \lim_{t \to 0} P_\eta(t) = p_1 + 1,
\]

where \( p_1 = m - 2, p_2 = p_3 = 2 \) and \( p_4 = 0 \);

\[
(2) \quad \lim_{t, s \to 0} \frac{1}{s} Q_\eta(t, s) = q_i,
\]
where \( q_1 = 8, q_2 = 3, q_3 = 3 \) and \( q_4 = n^2 \);

(3) there exist positive numbers \( t_0 \) and \( s_0 \) such that

\[
Q_i(t, 0) = 0, \quad Q_i(0, s) > 0 \quad (0 < s \leq s_0),
\]

\[
0 < Q_i(t, s_1) \leq Q_i(t, s_2) \quad (0 < t \leq t_0, \ 0 < s_1 < s_2 < s_0).
\]

In the last two inequalities for the case: \( i = 4 \), it is assumed that

\[
(1.4) \quad \log s \geq 0 \quad \text{and} \quad \log s' \geq 0 \quad \text{on} \ [0, s_0].
\]

Let us now state some results on the existence of certain solutions \( \alpha_i(t) \) of equation \((E_i) \ (i = 1, 2, 3, 4)\) and their asymptotic behavior as \( t \) goes to zero and also tends to infinity, under some conditions. The proofs will be found in the next subsection.

We assume first that \( n \neq 0 \) for the case of \( i = 4 \). Fix an \( i \in \{1, 2, 3, 4\} \). Then given two positive numbers \( t_0 \) and \( s_0 \) for which \((1.3) \) and \((1.4) \) hold, there exists a monotonically increasing, positive solution \( \alpha_i: (0, t_0) \rightarrow (0, \infty) \) with \( \alpha_i(t_0) = s_0 \) such that

\[
\lim_{t \to 0} \frac{\log \alpha_i(t)}{\log t} = \frac{1}{2} \left( -p + \sqrt{p^2 + 4q} \right) \quad \text{cf. Lemmas 1.3 in 1.2}.
\]

Hence the harmonic map \( F_i \) defined by \( \alpha_i \) as before turns out to be a continuous weakly harmonic maps defined around the origin \( o \) for the cases: \( i = 1, 2, 3 \) and the zero-section \( S^2_0 \) for the case: \( i = 4 \). Thus the fundamental regularity theory (cf. e.g., [11], [17]) shows that \( F_i \) is actually smooth over the origin for the cases: \( i = 1, 2, 3 \) and the zero-section \( S^2_0 \) for the case: \( i = 4 \).

In what follows, we assume that the solution \( \alpha_i \) is defined maximally on \((0, T_i)\) for some \( T_i \in (t_0, \infty) \). Then we have the following

**Proposition 1.1.** Let \( P_i, Q_i \) and \( \alpha_i \) be as above. Suppose that \( Q_i(t, s) \) is non-negative on \((0, \infty) \times (0, \infty) \). Then \( \alpha_i \) is positive on \((0, T_i) \). In particular, \( \alpha_i(t) \) tends to infinity as \( t \) goes to \( T_i \) if \( T_i \) is finite. Moreover suppose that for some constants \( A_i \in [-\infty, +\infty) \) and \( B_i \in [0, \infty] \), \( P_i(t) \) converges to \( A_i \) as \( t \) goes to infinity and \( Q_i(t, s) \) tends to \( B_i \) as \( t \) and \( s \) go to infinity. Then the following assertions hold:

1. If \(-\infty < A_i < +\infty \) and \( 0 < B_i < +\infty \), or if \( 0 \leq A_i < 1 \) and \( 0 \leq B_i < +\infty \), then \( T_i = +\infty \) and

\[
\lim_{t \to +\infty} \frac{\log \alpha_i(t)}{\log t} = \frac{1}{2} \left( -p + \sqrt{p^2 + 4q} \right) \quad (\in (0, \infty)).
\]
(2) If \( B_i = +\infty \), then

\[
\lim_{t \to x_i} \frac{\log \alpha_i(t)}{\log t} = +\infty.
\]

Moreover \( T_i = +\infty \) if \( Q_i(t, s) \leq B_i(t)s \) for some continuous function \( B_i(t) \) on \((0, \infty)\).

(3) If \( 1 \leq A_i < +\infty \) and \( B_i = 0 \), then \( T_i = +\infty \) and

\[
\lim_{t \to +\infty} \frac{\log \alpha_i(t)}{\log t} = 0.
\]

(cf. Lemmas 1.4 and 1.7 in 1.2).

Before concluding this subsection, let us consider the case: \( i = 4 \) and determine the order \( \text{ord}(\alpha_4) = \mu(A_4-1, B_4) \) of \( \alpha_4 \) in some cases. For simplicity, we assume that

\[
g(t) = at + a', \quad h(t) = bt + b', \quad \eta(t) = ct + c',
\]

for large \( t \) and

\[
g(s) = as + a', \quad h(s) = bs + b', \quad \eta(s) = cs + c'
\]

for large \( s \), where \( a, \bar{a}, b, \bar{b}, c, \) and \( \bar{c} \) are nonnegative constants and the others are arbitrary constants.

Case 1: \( a > 0, \bar{a} \geq 0, b > 0, \bar{b} > 0, c < 0, \bar{c} > 0 \).

\[
\text{ord}(\alpha_4) = -1 \left[ 1 + 2 \left( \frac{a}{\bar{a}} \right)^2 + n \frac{b^2 \bar{c}^2 (b^2 + \bar{b}^2)}{\bar{b}^2 \bar{c}^2 (\bar{b}^2 + b^2)} \right]^{1/2}
\]

Case 2: \( a = 0, \bar{a} > 0, b > 0, \bar{b} > 0, c > 0, \bar{c} > 0 \).

\[
\text{ord}(\alpha_4) = +\infty.
\]

Case 3: \( a = \bar{a} = 0, b > 0, \bar{b} > 0, c > 0, \bar{c} > 0 \).

\[
\text{ord}(\alpha_4) = \frac{n}{bc} \left( \frac{b^2 \bar{c}^2 + \bar{b}^2}{\bar{b}^2 \bar{c}^2 + b^2} \right)^{1/2}.
\]

Case 4: \( a > 0, \bar{a} \geq 0, b = \bar{b} = c = \bar{c} = 0 \).

\[
\text{ord}(\alpha_4) = \frac{1}{2} \left[ -1 + \left( 1 + 8 \left( \frac{a}{\bar{a}} \right)^2 \right) \right]^{1/2}.
\]

Case 5: \( \bar{a} > 0, a = \bar{b} = \bar{c} = c = \bar{c} = 0 \).

\[
\text{ord}(\alpha_4) = +\infty.
\]

Case 6: \( b = c = 0, \bar{b} > 0, \bar{c} > 0 \).
Case 7: \( b > 0, c > 0, \beta = \tau = 0 \).

\[
\text{ord}(\alpha_i) = \begin{cases} 0 & \text{if } a = \bar{a} = 0, \\ -1 + \left[1 + 2\left(\frac{\bar{a}}{a}\right)^2\right]^{1/2}, & \text{otherwise}. \end{cases}
\]

1.2. In this subsection, we shall consider a nonlinear ordinary differential equation which contains equations \((E_i)\) \((i = 1, 2, 3, 4)\) as special cases.

Given a smooth function \(P(t)\) on \([0, \infty)\) and a smooth function \(Q(t, s)\) on \([0, \infty) \times [0, \infty)\), we consider an equation as follows:

\[
(E_0) \quad \alpha''(t) + \frac{1}{t} P(t) \alpha'(t) - \frac{1}{P^2} Q(t, s) = 0.
\]

In what follows, we assume that for some \(p^*>(), q^*>0, \tau^*>0, \text{ and } s_0>0,\)

\[
\lim_{t \to 0} P(t) = p^* + 1 \quad \text{if } p^*>0,
\]

\[
(1.3) \quad \frac{1}{t} |P(t) - 1| \text{ is bounded on } (0, \tau_0) \quad \text{if } p^* = 0,
\]

\[
Q(t, 0) = 0 \quad (0 < t < \tau_0), \quad Q(0, s) > 0 \quad (0 < s < s_0),
\]

\[
(1.4) \quad \lim_{t \to \tau_0, s \to 0} -\frac{1}{s} Q(t, s) = q^* \text{, and}
\]

\[
0 < Q(t, s_1) \leq Q(t, s_2) \text{ for } t \in (0, \tau_0), s_1 \text{ and } s_2 \in (0, s_0) \text{ with } s_1 \leq s_2.
\]

We want a positive solution \(\alpha(t)\) of equation \((E_0)\) which converges monotonically to zero as \(t\) goes to zero. To begin with, let us reparametrize equation \((E_0)\) with parameter \(u = \log t \in (-\infty, +\infty)\) as follows:

\[
(E'_0) \quad \beta''(u) + (P(e^u) - 1) \beta'(u) - Q(e^u, \beta(u)) = 0.
\]

Let us take two positive numbers \(\tau_0\) and \(s_0\) for which \((1.3)\) and \((1.4)\) hold, and set \(u_0 = \log \tau_0\). For any \(v \in \mathbb{R}\), we denote by \(\beta_v\) a unique solution of \((E'_0)\) subject to the conditions: \(\beta_v(u_0) = s_0\) and \(\beta'_v(u_0) = v\). Define a set \(\mathcal{A}\) of \(\mathbb{R}\) by \(\mathcal{A} = \{v: \beta_v(u)\text{ decreases monotonically to zero in finite time as } u\} \), and put \(v_0 = \inf \mathcal{A}\). Then \(v_0 > 0\) and \(\mathcal{A}\) is open (cf. [4: Chap.6]). Moreover we have

Lemma 1.1. For any \(u_0\) and \(s_0>0\) as above, the solution \(\beta = \beta_{v_0}\) of equation \((E'_0)\) has the following properties:

(1) \(\beta > 0, \beta' > 0\) on \((-\infty, u_0]\),

(2) \(\lim_{t \to -\infty} \beta(t) = \lim_{t \to -\infty} \beta'(t) = \lim_{t \to -\infty} \beta''(t) = 0\).

Proof. The first assertion (1) follows from the same arguments as in [4:
As for the second one, we first prove that \( \lim_{u \to \pm \infty} \beta'(u) = 0 \).

Fix a positive constant \( r_\ast \) and choose \( v_0 \in (-\infty, u_0] \) so that \( p_* + r_\ast > 0 \) and \( P(e^u) < p_* + 1 + r_\ast \) on \((-\infty, v_0]\). For any \( v_1 \in (-\infty, v_0 - 1] \), let \( \gamma(u) \) be a unique solution of equation:

\[
\gamma''(u) + (p_* + r_\ast)\gamma'(u) = 0
\]

subject to the conditions: \( \gamma(v_1) = \beta(v_1) \) and \( \gamma'(v_1) = \beta'(v_1) \). Then \( \gamma(u) = \beta(v_1) + (p_* + r_\ast)^{-1} \beta'(v_1)[1 - \exp\{(p_* + r_\ast)(v_1 - u)\}] \). Define a function \( R \) by

\[
R(u) = \beta(u)\gamma(u) - \beta(u)\gamma'(u).
\]

Then we have

\[
R'(u) = -(p_* + r_\ast)R(u) + (p_* + 1 + r_\ast - P(e^u))\beta'(u)\gamma(u) + Q(e^u, \beta(u))\gamma'(u).
\]

Hence \( R(v_1) = 0 \) and \( R'(v_1) > 0 \). Moreover \( R \geq 0 \) on \([v_1, v_0] \). In fact, if \( R > 0 \) on \((v_1, v_2)\) and \( R(v_1) = 0 \) for some \( v_1 \in [v_1, v_0) \), then \( R'(v_2) \leq 0 \), which contradicts (1.5). Since \( (\beta/\gamma)' = R/\gamma^2 > 0 \) on \((v_1, v_0)\), we have \( \beta \geq \gamma \) on \([v_1, v_0] \). Thus we get

\[
\beta(v_1 + 1) - \beta(v_1) \geq \gamma(v_1 + 1) - \gamma(v_1)
\]

\[
= \frac{\beta'(v_1)}{p_* + r_\ast} \{1 - e^{-p_* - r_\ast}\}
\]

for any \( v_1 \in (-\infty, v_0 - 1] \). Suppose that \( \limsup_{u \to \pm \infty} \beta'(u) > 0 \). Then there exists a positive constant \( \delta \) and a sequence \( \{v_j\}_{j=1,2,\ldots} \) such that \( \beta'(v_j) > \delta \) and \( v_{j+1} \leq v_j - 1 \). It follows from (1.6) that for each \( j \),

\[
\beta(v_{j+1}) - \beta(v_j) \geq \frac{\delta}{p_* + r_\ast} \{1 - e^{-p_* - r_\ast}\}.
\]

This is absurd, because \( \beta \) is positive. Thus we have shown that \( \lim_{u \to \pm \infty} \beta'(u) = 0 \).

Set \( \beta_\ast = \lim_{u \to \mp \infty} \beta(u) \). Then we have

\[
0 = \lim_{u \to \mp \infty} \beta''(u) = \lim_{u \to \mp \infty} \{-(P(e^u) - 1)\beta'(u) + Q(e^u, \beta(u))\}
\]

\[
= Q(0, \beta_\ast),
\]

and hence \( \beta_\ast = 0 \) by (1.4). This proves the second assertion (2) of the lemma. //

**Lemma 1.2.** Let \( u_0, s_0 \) and \( \beta \) be as in Lemma 1.1. Given \( \varepsilon > 0 \), suppose that \( |s^{-1}Q(e^s, s) - q_*| < \varepsilon \), and \( |P(e^s) - 1 - p_*| < \varepsilon \) on \((-\infty, u_0] \times (-\infty, s_0] \). Then \( \beta(u) \leq c \exp \delta_* u \) and \( \beta'(u) \leq c \exp \delta_* u \) on \((-\infty, u_0) \), for some positive constant \( c \), where \( \delta_* = \frac{1}{2} \{-(p_* + \varepsilon) + \sqrt{(p_* + \varepsilon)^2 + 4(q_* - \varepsilon)}\} \).
Proof. For any \( v \in (-\infty, u_0] \), let \( \gamma_{\alpha}(u) \) be a unique solution of equation: 
\[
\gamma_{\alpha}(u) = \delta \gamma_{\alpha}(u),
\]
subject to the condition: \( \gamma_{\alpha}(v) = \beta(v) \). That is, \( \gamma_{\alpha}(u) = \beta(v) \exp \delta(u-v) \). Then we claim that \( \beta'(v) \geq \gamma_{\alpha}(v) = \delta \beta(v) \). In fact, we have

\[
\gamma_{\alpha}(u) = \delta^2 \gamma_{\alpha}(u)
\]

(1.7)

Suppose that \( \gamma_{\alpha}(v) < \gamma_{\alpha}(v) \). Then by (1.7) we get
\[
\gamma_{\alpha}(v) < -(p_\alpha + \varepsilon) \gamma_{\alpha}(v) + O(\varepsilon, \beta(v))
\]

Hence we have

\[
(1.8) \quad \gamma_{\alpha} < \beta, \quad \gamma_{\alpha} = \beta', \quad \gamma_{\alpha}'' < \beta''
\]
on \([v', v] \) for some \( v' \in (-\infty, v] \). If (1.8) holds on \((-\infty, v)\), then \( \gamma_{\alpha} - \beta \) \( \gamma_{\alpha}'' < 0 \) on \((-\infty, v)\), so that \( \gamma_{\alpha} - \beta \) \( \gamma_{\alpha}''(v) < 0 \) for any \( u \in (-\infty, v) \). This yields a contradiction, because \( \lim_{v \to -\infty} \beta' = \lim_{v \to -\infty} \gamma_{\alpha}'(u) = 0 \). Therefore there exists \( v_1 \in (-\infty, v) \) such that (1.8) ceases to hold at \( v_1 \). Then it turns out that \( \gamma_{\alpha}(v_1) = \beta(v_1), \gamma_{\alpha}'(v_1) = \beta'(v_1), \) and \( \gamma_{\alpha}''(v) = \beta''(v_1) \). By (1.7) and (1.4), we obtain
\[
\gamma_{\alpha}(v_1) = -(p_\alpha + \varepsilon) \gamma_{\alpha}(v_1) + O(\varepsilon, \beta(v_1))
\]

This yields a contradiction. Thus we have shown that \( \beta'(v) \geq \gamma_{\alpha}'(v) = \delta \beta(v) \) on \((-\infty, u_0] \), which implies that \( \beta'(v) \leq c_1 \exp \delta u \) on \((-\infty, u_0] \) for some constant \( c_1 > 0 \). As for the estimate on \( \beta' \), consider first the case \( p_\alpha > 0 \) and then assume that \( p_\alpha > e \). Then, we have
\[
\beta''(u) \leq -(p_\alpha - \varepsilon) \beta'(u) + (q_\alpha + \varepsilon) \beta(u)
\]

on \((-\infty, u_0] \). Integrating the both sides, we obtain
\[
\beta'(u) \leq c_2((p_\alpha - \varepsilon) \delta + q_\alpha + \varepsilon) \delta^{-1} e^{\delta u}
\]
on \((-\infty, u_0] \). As for the case \( p_\alpha = 0 \), we have by (1.3)

\[
(1.9) \quad \beta''(u) \leq c_2(e^u \beta' + \beta)
\]
for some constant \( c_2 > 0 \). Integrating the both sides, we get

\[ \beta'(u) \leq c_2 e^{c_1 u} \]

on \((-\infty, u_0]\), for some constant \( c_2 > 0 \), where \( \delta_3(1) = \min \{1, \delta_x\} \). If \( \delta_x > \delta_3(1) \), then inserting (1.10) into (1.9) and integrating the resulting inequality, we obtain

\[ \beta'(u) \leq c_4 e^{c_3 u} \]

for some constant \( c_4 > 0 \), where \( \delta_4(2) = \min \{2, \delta_x\} \). Thus repeating the same argument, we have

\[ \beta'(u) \leq c_5 e^{c_6 u} \]

on \((-\infty, u_0]\) for some constant \( c_5 > 0 \). This completes the proof of Lemma 1.2. //

Let us now return to equation \((E_0)\). Let \( \beta \) and \( u_0 \) be as in Lemma 1.2. Define a solution \( \alpha(t) \) of equation \((E_0)\) by \( \alpha(t) = \beta(\log t) \). Then by Lemma 1.2, we have

Lemma 1.3. Let \( \alpha \) be as above. Then \( \alpha \) satisfies

1. \( \alpha(t_0) = s_0 \) \( (t_0 = e^{s_0}) \), \( \alpha > 0 \), \( \alpha' > 0 \) on \((0, t_0)\);

2. \( \lim_{t \to 0} \frac{\log \alpha(t)}{\log t} \leq \mu(p_*, q_*) \);

3. \( \lim_{t \to 0} \frac{\log \alpha'(t)}{\log t} \leq \mu(p_*, q_*) - 1 \);

where \( \mu(p_*, q_*) = \frac{1}{2} \left\{ -p_* + \sqrt{p_*^2 + 4q_*} \right\} \) \( (\geq 0) \).

In what follows, we shall study the asymptotic behavior at infinity of solutions of equation \((E_0)\) under certain conditions. Let \( t_0 \) and \( s_0 \) be positive numbers and \( \alpha(t) \) a solution of equation \((E_0)\) with \( \alpha(t_0) > s_0 \) and \( \alpha'(t_0) > 0 \). Here \( \alpha(t) \) is assumed to be defined at least on \([t_0, T_\alpha)\), where \( T_\alpha = \sup \{T: \alpha \text{ is positive and bounded on } [t_0, T]\} \) \( (\leq +\infty) \). In order to study the asymptotic behavior at infinity of \( \alpha \), we employ elementary comparison arguments just used in the proof for the preceding lemmas. Let \( A \) and \( B \geq 0 \) be two numbers chosen appropriately later and \( \gamma \) a unique solution of equation:

\[ \gamma''(t) + \frac{A}{t} \gamma'(t) - \frac{B}{t^2} \gamma(t) = 0, \]

subject to the condition: \( \gamma(t_0) = \alpha(t_0) \) and \( \gamma'(t_0) = \alpha'(t_0) \). Define a function \( R_{AB} : [t_0, T_\alpha] \to \mathbb{R} \) by

\[ R_{AB}(t) = \alpha(t) \gamma'(t) - \alpha'(t) \gamma(t). \]
Then obviously $R_{AB}(t_0)=0$ and

$$R_{AB}'(t) = -\frac{1}{t} P(t)R_{AB}(t) + \frac{1}{t} (P(t) - A) \alpha(t) \gamma(t) + \frac{1}{t^2} (B \alpha(t) - Q(t, \alpha(t))) \gamma(t).$$

**Lemma 1.4.** Let $\alpha$ be as above. Suppose that $Q(t, s) \geq 0$ on $[t_0, T_\alpha] \times [s_0, \infty)$. Then $\alpha'>0$ on $[t_0, T_\alpha)$. In particular, $\lim_{t \to +\infty} \alpha(t) = +\infty$ if $T_\alpha < +\infty$.

Proof. Suppose that $\alpha'(t)>0$ on $[t_0, t_1]$ and $\alpha'(t_1)=0$, for some $t_1 \in (t_0, T_\alpha)$. Take two numbers $A$ and $B$ as above so that $P(t)<A$ on $[t_0, t_1]$ and $B=0$. Then $R_{AB}(t_0)=0$ and $R_{AB}'(t)<0$. We claim that $R_{AB}(t)<0$ on $(t_0, t_1)$. In fact if we have $t_2 \in (t_0, t_1)$ such that $R_{AB}(t)<0$ on $(t_0, t_2)$ and $R_{AB}(t_2)=0$, then

$$R_{AB}'(t_2) = \frac{1}{t_2} (P(t_2) - A) \alpha(t_2) \gamma(t_2) - \frac{1}{t_2} Q(t_2, \alpha(t_2)) \gamma(t_2) < 0.$$ 

This yields a contradiction. Hence we have $R_{AB}(t)<0$ on $(t_0, t_1)$, that is,

$$\alpha'(t) \gamma(t) < \alpha(t) \gamma(t)$$
on $(t_0, t_1)$. Putting $t=t_1$, we get

$$0 < \alpha(t_1) \gamma(t_1) < \alpha'(t_1) \gamma(t_1) = 0.$$ 

This is absurd. Thus we have shown that $\alpha'>0$ on $[t_0, T_\alpha)$. //

**Lemma 1.5.** Let $\alpha$ be as above. Suppose that for some $A$ and $B \geq 0$, $P(t)<A$ on $[t_0, \infty)$ and $Q(t, s) \geq Bs$ on $[t_0, \infty) \times [s_0, \infty)$, or for some $A$ and $B \geq 0$, $P(t) \leq A$ on $[t_0, \infty)$ and $Q(t, s) > Bs$ on $[t_0, \infty) \times [s_0, \infty)$. Then:

$$\liminf_{t \to +\infty} \frac{\log \alpha(t)}{\log t} \geq \frac{1}{2} \left\{ -(A-1) + \sqrt{(A-1)^2 + 4B} \right\}.$$ 

where $\mu(A-1, B) = \frac{1}{2} \left\{ -(A-1) + \sqrt{(A-1)^2 + 4B} \right\}$.

Proof. Obviously $R_{AB}(t_0)=0$ and $R_{AB}'(t_0)<0$. The same arguments as in the proof of Lemma 1.4 show that $R_{AB}<0$ on $(t_0, T_\alpha)$, namely, $\alpha(t) \gamma(t) < \alpha'(t) \gamma(t)$ on $(t_0, T_\alpha)$. Then it turns out that $\gamma(t) < \alpha(t)$ on $(t_0, T_\alpha)$, which implies that

$$\liminf_{t \to +\infty} \frac{\log \alpha(t)}{\log t} \geq \liminf_{t \to +\infty} \frac{\log \gamma(t)}{\log t} = \mu(A-1, B).$$

This completes the proof of Lemma 1.5. //

**Lemma 1.6.** Let $\alpha$ be as above. Suppose that for some $A$ and $B \geq 0$, $P(t)>A$ on $[t_0, \infty)$ and $Bs \geq Q(t, s) \geq 0$ on $[t_0, \infty) \times [s_0, \infty)$, or for some $A$ and
$B > 0$, $P(t) \geq A$ on $[t_0, \infty)$ and $B_s > Q(t, s) \geq 0$ on $[t_0, \infty) \times [s_0, \infty)$. Then, $T_\alpha = +\infty$ and

$$\limsup_{t \to \infty} \frac{\log \alpha(t)}{\log t} \leq \mu(A-1, B).$$

Proof. Obviously $R_{AB}(t_0) = 0$ and $R_{AB}'(t_0) > 0$. The same arguments as in the proof of Lemma 1.4 again show that $R_{AB} > 0$ on $(t_0, T_\alpha)$, namely, $\alpha(t)\gamma'(t) > \alpha'(t)\gamma(t)$ on $(t_0, T_\alpha)$. Hence $\gamma(t) > \alpha(t) > \alpha(t_0)$ on $(t_0, T_\alpha)$. This shows that $T_\alpha = +\infty$ and

$$\limsup_{t \to \infty} \frac{\log \alpha(t)}{\log t} \leq \lim_{t \to \infty} \frac{\log \gamma(t)}{\log t} = \mu(A-1, B).$$

This completes the proof of Lemma 6. ///

By Lemmas 1.5 and 1.6, we have

**Lemma 1.7.** Let $\alpha$ and $T_\alpha$ be as before. Suppose that $Q(t, s) \geq 0$ for any $t > 0$ and $s > 0$, and suppose that for some $A \in [0, +\infty]$ and $B \in [0, +\infty]$, $P(t)$ converges to $A$ as $t$ goes to infinity and $Q(t, s)$ tends to $B$ as $t$ and $s$ go to infinity.

1. If $-\infty < A < +\infty$ and $0 < B \leq +\infty$, or if $0 \leq A < 1$ and $0 \leq B < +\infty$, then $T_\alpha = +\infty$ and

$$\lim_{t \to \infty} \frac{\log \alpha(t)}{\log t} = \mu(A-1, B) \quad (\in (0, \infty)).$$

2. If $B = +\infty$, then

$$\lim_{t \to \infty} \frac{\log \alpha(t)}{\log t} = +\infty.$$  

Moreover $T_\alpha = +\infty$ if $Q(t, s) \leq B(t)s$ for some continuous function $B(t)$.

3. If $1 \leq A < +\infty$ and $B = 0$, then $T_\alpha = +\infty$ and

$$\lim_{t \to \infty} \frac{\log \alpha(t)}{\log t} = 0.$$

2. Growth of harmonic morphisms

In this section, we shall first show a generalization of the O’Neill’s curvature formula for Riemannian submersions to horizontally conformal maps, and then discuss the growth of harmonic morphisms (cf. e.g., [4] [11] for the general theory on such maps).

2.1. Let $\phi: M \to N$ be a smooth map between Riemannian manifolds $(M, g)$ and $(N, h)$. For a point $x$ of $M$, we set $\mathcal{U}_x = \ker(d\phi_x)$. The space $\mathcal{U}_x$ is called
the vertical space at $x$. Let $\mathcal{H}_x$ denote the orthogonal complement of $\mathcal{C}V_x$ in the tangent space $T_xM$. For a tangent vector $E \in T_xM$, we denote by $\mathcal{C}V E$ and $\mathcal{H} E$, respectively, the vertical component and the horizontal component of $E$. Let $\mathcal{C}U$ and $\mathcal{H}$, respectively, denote the corresponding vertical and horizontal distributions in the tangent bundle $TM$. We say that $\phi$ is *horizontally conformal* if, for each point $x \in M$ at which $d\phi_x \neq 0$, the restriction $d\phi_{x,\mathcal{H}} : \mathcal{H}_x \to T_{\phi(x)}N$ is conformal and surjective. Thus for some nonnegative function $\lambda_\phi$ on $M$, $\phi^* h_{\mathcal{E},\mathcal{E}} = \lambda_\phi^2 g_{\mathcal{E},\mathcal{E}}$. The function $\lambda_\phi$ is called the dilation of $\phi$. Then $\lambda_\phi^2$ is smooth on $M$ and actually equal to $e(\phi)/n$, where $n = \dim N$. Fuglede [13] and Ishihara [20] showed that a smooth map $\phi : M \to N$ is a harmonic morphism if and only if $\phi$ is both harmonic and horizontally conformal. Here $\phi$ is called a harmonic morphism if for every function $f$ harmonic on an open subset $U$ of $N$, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(U)$.

A horizontal conformal map is obviously a Riemannian submersion if (and only if) the dilation is constantly equal to 1. For Riemannian submersions, we have the O’Neill’s formula on curvature [28]. We will first state the corresponding formula for horizontally conformal maps, after some definitions and auxilia results which follow from the same direct computations as in [28].

Let $\phi : M \to N$ be a horizontally conformal map between Riemannian manifolds $(M, g)$ and $(N, h)$. We call the points $x \in M$ where $d\phi_x = 0$ the critical points of $\phi$ and denote by $C_\phi$ (resp., $M_\phi$) the critical points (resp., the complement of $C_\phi$, namely $M_\phi = \{x \in M : \lambda_\phi^2(x) > 0\}$, where $\lambda_\phi$ is the dilation of $\phi$). After [28], we define two tensors $T$ and $A$ of type $(1,2)$ over $M_{\phi}$ by

$$T_E F = \mathcal{H}_{\mathcal{C}V E} \mathcal{C}V F + \mathcal{C}V_{\mathcal{C}V E} \mathcal{H} F$$
$$A_E F = \mathcal{C}V_{\mathcal{H} E} \mathcal{H} F + \mathcal{H}_{\mathcal{H} E} \mathcal{C}V F,$$

for vector fields $E$ and $F$ on $M_\phi$, where $\nabla$ denotes the Levi-Civita connection of $M$. Then we have

**Lemma 2.1** (cf. [28]).

1. Both $T$ and $A$ are skew-symmetric operators on the tangent space of $M$ reversing horizontal and vertical subspaces.
2. $T_E = T_{\mathcal{C}V E}$, and $A_E = A_{\mathcal{H} E}$.
3. For vertical vector fields $V$ and $W$, $T$ is symmetric, i.e., $T_V W = T_W V$.

For horizontal vector fields $X$ and $Y$,

$$A_X Y = \frac{1}{2} \mathcal{C}V [X, Y] + \frac{1}{2} g(X, Y) \mathcal{C}V \Lambda_\phi,$$

where $\Lambda_\phi$ denotes a vector field on $M_\phi$ defined by

$$\Lambda_\phi = \text{grad} \log \lambda_\phi^2.$$

A basic vector field is by definition a horizontal vector field $X$ on $M_\phi$ which is $\phi$-related to a vector field $X_\phi$ on $N$, namely $d\phi_x(X) = X_{\phi(\phi(x))}$ for all $x \in M_\phi$. 

Lemma 2.2 (cf. [28]). Suppose $X$ and $Y$ are, respectively, basic vector fields on $M_0$ which are $\phi$-related to $X_*$ and $Y_*$. Then:

1. $\lambda^2 g(X, Y) = h(X_*, Y_*) \circ \phi$.
2. $\mathcal{J}[X, Y]$ is basic and $\phi$-related to $[X_*, Y_*]$.
3. The basic vector field which is $\phi$-related to $\nabla^*_X Y_*$ is given by

$$\mathcal{J} \nabla_X Y + \frac{1}{2} \{g(\Lambda_\phi, X)Y + g(\Lambda_\phi, Y)X - g(X, Y)\mathcal{J}\Lambda_\phi\},$$

where $\nabla^*$ denotes the Levi-Civita connection of $N$.

For linearly independent vectors $E$ and $F$, we denote by $P_{E, F}$ the tangent plane spanned by these two vectors. Moreover $K_M$, $K_N$, and $K$, respectively, stand for the sectional curvature of $M$, $N$, and the fibres (in $M_0$). Then making use of Lemmas 2.1 and 2.2, we can derive the curvature formula stated below for horizontally conformal maps as in [28].

Theorem 2.3. Let $\phi: M \to N$ be a horizontally conformal map between Riemannian manifolds $M$ and $N$. Then for orthonormal horizontal vectors $X, Y$, and orthonormal vertical vectors $V, W$ on $M_0$ one has the following relations:

1. $K_M(P_{V, W}) = \hat{K}(P_{V, W}) - g(T_V V, T_W W) + g(T_V W, T_W W)$.
2. $K_M(P_{X, Y}) = g(A_X V, A_X V) - g(T_Y X, T_Y X) + g(\nabla_X T) V, X) - \frac{1}{2} g(\Lambda_\phi, V)^2 + \frac{1}{2} g(\nabla V \mathcal{J} \Lambda_\phi, V)$.
3. $K_M(P_{X, Y}) = \lambda^2 K_N(P_{X, Y}) - 3g(A_X Y, A_X Y)$

$$- \frac{1}{4} \{g(\mathcal{J} \Lambda_\phi, \mathcal{J} \Lambda_\phi) - g(\mathcal{J} \Lambda_\phi, \mathcal{J} \Lambda_\phi) + g(\Lambda_\phi, X)^2 + g(\Lambda_\phi, Y)^2\}$$

$$+ \frac{1}{2} \{g(\nabla_X \mathcal{J} \Lambda_\phi, X) + g(\nabla_Y \mathcal{J} \Lambda_\phi, Y)\},$$

where $X_* = d\phi(X)$ and $Y_* = d\phi(Y)$.

2.2. Before showing a few applications of Theorem 2.3, let us first make some observations. Let $\phi: M \to N$ be a horizontally conformal map and $\nabla d\phi$ the second fundamental form of $\phi$. Then the tension field $\tau(\phi)$ of $\phi$, i.e., the trace of $\nabla d\phi$, is given by

$$\tau(\phi) = d\phi \left( \left( 1 - \frac{n}{2} \right) \Lambda_\phi - (m-n) \eta \right)$$

on $M_0$, where $n = \dim N$, $m = \dim M$, and $\eta$ is the mean curvature normal of the fibres, namely, $\eta = (m-n)^{-1} \text{trace} T_{1_{C^1}}$. Hence if $n \geq 3$ then it follows that any two of the conditions below imply the other one:

(i) $\phi$ is harmonic on $M$.
(ii) $\phi$ has minimal fibres on $M_0$.
(iii) $\text{grad } \lambda^2$ is vertical.
(cf. [4: Chap. 7]). We mention here a theorem by Fuglede [14], which states that a nonconstant harmonic morphism $\phi: M \to N$ is an open map and furthermore if $\nabla^2 \phi$ is vertical, then the set of singular points of $\phi$ is empty, that is, $\phi: M \to N$ is a submersion.

Let $\phi: M \to N$ be a nonconstant harmonic morphism with $\nabla^2 \phi$ vertical. Let $\eta: [a, b] \to N$ be a regular curve lying in the image of $\phi$. For any point $x \in \phi^{-1}(\eta(a))$, take the horizontal lift $\eta_x: [a, b] \to M$ of $\eta$ with $\eta_x(a) = x$, and then define a map $\mathcal{D}_{ab}: \phi^{-1}(\eta(a)) \to \phi^{-1}(\eta(b))$ by $\mathcal{D}_{ab}(x) = \eta_x(b)$. Since $\nabla^2 \phi$ is vertical, $d\nabla^2 \phi(\eta_x(t))/dt = 0$, and hence $\lambda^2 \circ \mathcal{D}_{ab} = \lambda^2$. Moreover it is easy to see that if $M$ is complete, then so is $N$ and $\phi$ is surjective. Now suppose that the fibres are totally geodesic. Then $\mathcal{D}_{ab}$ induces an isometry between two fibers $\phi^{-1}(\eta(a))$ and $\phi^{-1}(\eta(b))$, and hence $|\nabla^2 \phi| \circ \mathcal{D}_{ab} = |\nabla^2 \phi|$. In fact, let $\sigma: (c, d) \to \phi^{-1}(\eta(a))$ be a regular curve in $\phi^{-1}(\eta(a))$, and define a map $F: [a, b] \times (c, d) \to M$ by $F(t, s) = \eta_{\sigma(t)}(s)$. Set $V = \partial F/\partial s$ and $X = \partial F/\partial t$. Then obviously $V$ is vertical and $X$ is horizontal. Since $\phi$ has totally geodesic fibres, we have

\[
\frac{d}{dt} g(V, V) = 2 g(\nabla_X V, V) = 2 g(\nabla_V X, V) = 2 g(T_V X, V) = 0.
\]

This proves that $\mathcal{D}_{ab}$ is an isometry from $\phi^{-1}(\eta(a))$ onto $\phi^{-1}(\eta(b))$. Thus we have shown

**Lemma 2.4.** Let $\phi: M \to N$ be a nonconstant harmonic morphism with $\nabla^2 \phi$ vertical.

1. If $M$ is complete, then so is $N$ and $\phi$ is surjective.
2. If the fibres of $\phi$ are totally geodesic, then the map $\mathcal{D}_{ab}: \phi^{-1}(\eta(a)) \to \phi^{-1}(\eta(b))$ defined as above induces an isometry and furthermore one has

$$
\lambda^2 \circ \mathcal{D}_{ab} = \lambda^2, \quad |\nabla^2 \phi| \circ \mathcal{D}_{ab} = |\nabla^2 \phi|.
$$

2.3. We are now in a position to show an example of a harmonic morphism of a complete noncompact manifold of positive Ricci curvature, and then give a few applications of Theorem 2.3 (3), in connection with the example.

**Example.** Let us denote by $R^*_n$ the Riemannian manifold $R^k$ equipped with a rotationally symmetric metric $g_\theta = dr^2 + \eta(r)^2 d\theta^2$. Assume that $k$ is greater than or equal to 3 and take a smooth function $\xi(t)$ on $[0, \infty)$ such that $\xi(t) = t$ on $[0, 1]$, $\xi(t) > 0$ on $[1, 2]$ and $\xi(t) = t^{-1-\alpha}$ on $[2, \infty)$, where $\alpha \in (0, 1)$ is a constant. Then define a smooth function $\xi_\alpha(t)$ on $[0, \infty)$ by $\xi_\alpha(t) = \int_0^t \xi(t) ds$, and set
\[ \eta(r) = \frac{1}{2} r + \frac{1}{2a} \int_0^r \xi(t) \, dt, \]
where \( a = \xi_2(0) = \int_0^\infty \xi(t) \, dt \). We choose two constants \( \alpha, \beta \in (0, 1) \) so that \( k - \beta > 2 + \alpha \), and define a smooth function \( f(r) \) on \( \mathbb{R}^k \) by
\[ f(r) = (b + r^2)^{(b + r^2 - k)/2} + c, \]
where \( b \) and \( c \) are positive constants sufficiently large. Given a complete manifold \( F \) of nonnegative Ricci curvature, we consider the warped product \( M = \mathbb{R}^k \times \phi F \) of \( \mathbb{R}^k \) and \( F \) with a warping function \( f(r) \). We denote by \( \phi^F \) the projection of \( M \) onto \( F \). Then \( M \) has positive Ricci curvature and \( \phi \) defines a harmonic morphism from \( M \) onto \( F \) with bounded energy density and totally geodesic fibers. We note that if \( F \) is flat and compact, then the sectional curvature of \( M \) decays quadratically in the absolute values.

The first application of Theorem 2.3 is an immediate consequence from the assertion (3):

**Proposition 2.5.** Let \( \phi: M \to N \) be a (nonconstant) harmonic morphism with \( \nabla \lambda_\phi \) vertical, where \( \lambda_\phi \) denotes the dilation of \( \phi \). Suppose that the sectional curvature \( K_M \) is nonnegative. Then the sectional curvature \( K_N \) is also nonnegative on the image \( \phi(M) \). Moreover if \( K_N(\pi) = 0 \) for a plane \( \pi \) tangent to \( N \) at a point of the image \( \phi(M) \), then \( \lambda_\phi \) is constant, i.e., \( \phi \) is a Riemannian submersion up to homothety.

We shall state further results.

**Proposition 2.6.** Let \( \phi: M \to N \) be as in Proposition 2.5. Suppose that the following conditions hold:
1. \( M \) is complete and \( N \) is noncompact.
2. The Ricci curvature \( \text{Ricci}_M \) of \( M \) is nonnegative.
3. The scalar curvature \( \text{Scal}_M \) of \( N \) is nonpositive.
4. The sectional curvature \( K_M \) of \( M \) satisfies
   \[ K_M \geq -\frac{c}{r_M^{2+k}}, \]
where \( c \) and \( k \) are positive constants, and \( r_M \) stands for the distance on \( M \) to a fixed point \( o \) of \( M \). Then \( \phi \) is totally geodesic and \( N \) is Ricci-flat.


**Proposition 2.7.** Let \( \phi: M \to N \) be as in Proposition 2.5. Suppose that \( M \) is complete and \( N \) is noncompact, and suppose that \( M \) has nonnegative sectional...
curvature and the fibres of $\phi$ are totally geodesic. Then the dilation $\lambda_{\phi}$ of $\phi$ is constant, i.e., $\phi$ is a Riemannian submersion up to homothety.

**Proposition 2.8.** Let $\phi : \mathbb{R}^n \to N$ be a harmonic morphism of Euclidean space $\mathbb{R}^n$ onto a Riemannian manifold $N$ of dimension $n \geq 3$. Suppose the fibers are totally geodesic, i.e., affine subspaces of codimension $n$. Then $N$ is an affine subspace and $\phi$ is the orthogonal projection.

Proof of Proposition 2.6. To begin with, we state the Weitzenböck formula for harmonic morphisms, which reads

\[(2.1) \quad \frac{1}{2n} \Delta_M \lambda_{\phi}^2 = |\nabla d\phi|^2 + \lambda_{\phi}^2 \text{trace Ricci}_M - \lambda_{\phi}^4 \text{Scal}_N \phi ,\]

where $\text{trace Ricci}_M$ denotes the trace of the Ricci tensor of $M$ on the horizontal distribution $\mathcal{H}$ and $\text{Scal}_N$ stands for the scalar curvature of $N$. On the other hand, in the case of $\text{grad} \lambda_{\phi}^2$ vertical, as we noted in 2.2, $d\phi$ has maximal rank $n = \dim N$ everywhere. Moreover by Theorem 2.3 (3), we have

\[(2.2) \quad \sum_{1 \leq i < j \leq n} K_M(P_{x_i,x_j}) = \frac{\lambda_{\phi}^2}{2} \text{Scal}_N \phi - 3 \sum_{1 \leq i < j \leq n} g(A_{x_i}, X_i, A_{x_j}, X_j) - \frac{n(n-1)}{8} g(\Lambda_{\phi}, \Lambda_{\phi}) ,\]

where $\{X_1, \ldots, X_n\}$ is an orthonormal basis of the horizontal subspace.

Now it turns out from the completeness of $M$ and the assumption (4) on the sectional curvature of $M$ that for large $R$, the outside of the metric ball $B_M(R)$ around $o$ with radius $R$ is homeomorphic to $[R, \infty) \times \partial B_M(R)$ and furthermore any pair of points belonging to the same connected component of $M \setminus B_M(R)$ can be joined by a smooth curve which lies in $M \setminus B_M(R)$ and the length of which is bounded by $c_M R$, where $c_M$ is a positive constant depending only on $M$ (cf. [23]). Based on this observation, we shall show that $\lambda_{\phi}$ must be constant. In fact, take two points $x, y$ in a connected component, say $E$, of $M \setminus B_M(R)$ and assume that $r_M(x) \geq r_M(y) (\geq R)$. Let $\gamma : [0, a] \to M$ be a distance minimizing geodesic joining $o$ to $x$ ($a = r_M(x)$). Since we have by the assumptions (3), (4) and (2.2)

\[(2.3) \quad |\Lambda_{\phi}| \leq \frac{c_1}{(1 + r_M)^{1+\delta}} ,\]

on $M$ for some positive constants $c_1$ and $\delta$, we get

\[\frac{d}{dt} \log \lambda_{\phi}^2(\gamma(t)) \leq \frac{c_1}{(1 + t)^{1+\delta}} .\]
and hence integrating the both sides, we obtain

\[ \frac{\lambda_\phi^2(\gamma(t))}{\lambda_\phi^2(\gamma(s))} \leq \exp \left( \frac{c_1}{\delta} \left( \frac{1}{(1+s)^\delta} - \frac{1}{(1+t)^\delta} \right) \right) \]

\((0 \leq s \leq t \leq a)\). This shows that

\[ \lambda_\phi^2(x) \leq c_2 \lambda_\phi^2(y) \]

for some positive constant \(c_2\). Let us take a smooth arclength-parametrized curve \(\sigma: [0, b] \to M\) which lies in \(M/B_M(d)\) \((d=r_M(y))\) and joins \(y\) to \(\gamma(d)\). Then it follows from (2.3) again that

\[ \frac{\lambda_\phi^2(\gamma(d))}{\lambda_\phi^2(y)} \leq \exp \left( \frac{c_1 b}{(1+d)^{1+\delta}} \right) \leq \exp \left( \frac{c_1 c_M d}{(1+d)^{1+\delta}} \right). \]

Hence by (2.4) and (2.5), we have

\[ \frac{\lambda_\phi^2(x)}{\lambda_\phi^2(y)} \leq \exp \left( \frac{c_1}{\delta} \left( \frac{1}{(1+d)^\delta} - \frac{1}{(1+a)^\delta} \right) + \frac{c_1 c_M d}{(1+d)^{1+\delta}} \right). \]

This implies that \(\lambda_\phi^2(x)\) goes to a constant \(e_E\) as \(x \in E\) tends to infinity. Moreover for any point \(x \in M\), there is a smooth curve \(\eta_x: [0, \infty) \to M\) which is the horizontal lift of a ray \(\gamma\) in \(N\) starting at \(\phi(x)\), since \(N\) is complete and noncompact. Then

\[ \frac{d}{dt} \lambda_\phi^2(\eta_x(t)) = g(\nabla \lambda_\phi^2, \dot{\eta}_x(t)) = 0, \]

so that we have

\[ \lambda_\phi^2(x) = \lim_{t \to \infty} \lambda_\phi^2(\eta_x(t)) = e_E, \]

if \(\eta_x(t)\) goes to infinity through \(E\). This proves that \(\lambda_\phi\) is constant. Now it turns out from the assumption (2) and the Weitzenböck formula (2.1) that \(\phi\) is totally geodesic. //

Proof of Proposition 2.7. By Theorem 2.3 (3) and Lemma 2.4 (1), \(N\) is a complete manifold of nonnegative curvature \(K_N\). Since \(N\) is assumed to be noncompact, there exists a sequence of points \(\{p_i\}\) of \(N\) and tangent planes \(P_i\) at \(p_i\) such that \(K_N(P_i)\) goes to zero as \(i\) tends to infinity. Let \(p\) be any fixed point of \(N\) and take smooth paths \(\eta_i: [0, b_i] \to N\) joining \(p\) with \(p_i\). Then it follows from Theorem 2.3 (3) and Lemma 2.4 (2) that for any \(x \in \phi^{-1}(\hat{p})\),

\[ |\text{grad} \lambda_\phi^2(x)|^2(x) = |\text{grad} \lambda_\phi^2(p_i)|^2 \leq 4 \lambda_\phi^2(x)K_N(P_i), \]

where \(\mathcal{O}_0: \phi^{-1}(p) \to \phi^{-1}(p_i)\) is the isometry as in Lemma 2.4. Thus \(\text{grad} \lambda_\phi^2\)
vanishes everywhere on $M$. This completes the proof of Proposition 2.7. //

Proof of Proposition 2.8. Since the fibres are affine subspaces, it is easy to see that $N$ is diffeomorphic to $\mathbb{R}^n$. Thus it turns out from Proposition 2.7 that the fibres are parallel, and hence $\phi$ is totally geodesic. This completes the proof of Proposition 2.8. //

REMARK. In Propositions 2.7 and 2.8, if we assume, instead of the fibres being totally geodesic, that the second fundamental form $\alpha$ of the fibres satisfies

$$\sup_{ \phi^{-1}(p)} |\alpha| \leq \frac{c}{r_N(p)^{1+\varepsilon}} \quad (p \in N)$$

for some constants $c > 0$ and $\varepsilon > 0$, then the same assertions hold. See [5] for related results to Proposition 2.8.

2.4. As an application of the Weitzenböck formula (2.1) and a generalized maximum principle, we can derive Schwarz lemma for harmonic morphisms. To be precise, let $\phi: M \to N$ be a harmonic morphism. Suppose $M$ is complete, the Ricci curvature of $M$ is bounded from below by a nonpositive constant $-k_1$ and the scalar curvature of $N$ is bounded from above by a negative constant $-k_2$. Then the dilation $\lambda_\phi$ of $\phi$ satisfies:

$$\sup \lambda_\phi \leq \frac{n-k_1}{k_2}$$

where $n = \dim N$ (cf. [32]).

We are also able to give an upper bound of the growth of a harmonic morphism, comparing the decay order of the Green functions of the domain and the target.

**Proposition 2.9.** Let $\phi: M \to N$ be a (nonconstant) harmonic morphism between complete noncompact Riemannian manifolds $M$ and $N$. Suppose that the conditions below hold:

1. The Ricci curvature $\text{Ricci}_M$ of $M$ satisfies:

$$\text{Ricci}_M \geq -\frac{c}{r_M^{2+\varepsilon}},$$

where $c$ and $\varepsilon$ are positive constants.

2. The dimension $n$ of $N$ is greater than or equal to 3, $N$ is simply connected, and the sectional curvature of $N$ is nonpositive.

Then one has

$$\limsup_{t \to \infty} \frac{\log \mu(\phi, t)}{\log t} \leq \frac{m-2}{n-2},$$

where $m$ and $n$ are the dimensions of $M$ and $N$, respectively.
where \( m = \text{dim}\ M, \ \mu(\phi, t) = \max \{r_N(\phi(x)) : x \in M, r_M(x) = t\}, \) and \( r_N \) stands for the distance in \( N \) to a fixed point \( o' \) of \( N \).

Proof. Let us consider the Dirichlet problem outside the metric ball \( B_N(a) \) of \( N \) around \( o' \) with radius \( a \): \( \Delta_N u = 0 \) on \( N \setminus B_N(a), u = 1 \) on \( \partial B_N(a) \). Then there exists a unique solution \( u \) of the problem which satisfies:

\[
(2.7) \quad u(y) \leq \frac{c_1}{r_N^{n-2}(y)}
\]

for some positive constant \( c_1 \). In fact, the Rauch comparison theorem says that the Laplacean \( \Delta_N r_N \) of the distance function \( r_N \) is bounded from below by \( (n-1)r_N^{-1} \), so that \( r_N^{2-n} \) is superharmonic (i.e., \( \Delta_N r_N^{2-n} \leq 0 \) (cf. e.g. [16]). Hence by the maximum principle, the solution \( u_s \) of the Dirichlet problem: \( \Delta_N u_s = 0 \) on \( B_N(s) \setminus B_N(a), u_s = 1 \) on \( \partial B_N(a) \) and \( u_s = 0 \) on \( \partial B_N(s) \), is bounded from above by \( a^{n-2}r_N^{2-n} \). Thus \( u = \lim_{s \to a} u_s \) satisfies inequality (2.7). By setting \( u(y) = 1 \) on \( B_N(a) \), we assume that \( u \) is defined on \( N \). Then the composition \( u \circ \phi \) is a positive superharmonic function on \( M \). Let us now consider the same Dirichlet problem outside the metric ball \( B_M(b) \) around \( o \in M \) with radius \( b \): \( \Delta_M v = 0 \) on \( M \setminus B_M(b), v = 1 \) on \( \partial B_M(b) \). Taking the radius \( a \) sufficiently large, we may assume that \( B_M(b) \subseteq \phi^{-1}(B_N(a)) \). Then there exists a unique solution \( v \) of the problem such that

\[
(2.8) \quad v \leq u \circ \phi \quad \text{on} \quad M \setminus B_M(b)
\]

On the other hand, by the assumption (2.6), we have

\[
(2.9) \quad \frac{c_2}{r_M^{n-2}} \leq v \quad \text{on} \quad M \setminus B_M(b)
\]

for some positive constant \( c_2 \) (cf. [22, 24]). Thus we have by (2.7) and (2.8)

\[
\log \mu(\phi, t) \leq \frac{m-2}{n-2} \log t + c_3
\]

for some positive constant \( c_3 \) and large \( t \). This proves Proposition 2.9. //

Remarks. (1) In case \( \text{Ricci}_N \geq 0 \), estimate (2.9) is due to Calabi [7]. Our condition (2.6) is rather technical, but from the view point of the problem discussed here, it seems to be natural (cf. [16, 22, 23, 24]). (2) Proposition 2.9 is a generalization of Proposition 8.1.1 in [4] where harmonic morphisms of homogenous polynomials between Euclidean spaces are discussed. In particular, the upper bound in Proposition 2.9 is sharp as noted in [4].

2.5. Finally we shall show a lower bound for the growth of a harmonic morphism under certain conditions.
Let $M$ be a complete noncompact Riemannian manifold. We assume that $M$ is connected at infinity, namely, for any compact set $K$, there is a compact set $\mathcal{K}$ such that $K \subseteq \mathcal{K}$ and $M \setminus \mathcal{K}$ is connected. Since $M$ is complete, $M$ is connected at infinity if and only if for any metric ball $B_M(t)$ centered at a point $o \in M$ of radius $t$, $M \setminus B_M(t)$ has only one noncompact component, say $\Sigma(t)$. We fix a point $o$ of $M$ as a base point and denote by $\text{diam}(\partial \Sigma(t), \Sigma(t))$ the diameter of $\partial \Sigma(t)$ measured with respect to the intrinsic distance of the open manifold $\Sigma(t)$. Set

$$\delta_\infty = \limsup_{t \to \infty} \frac{1}{t} \text{diam}(\partial \Sigma(t), \Sigma(t)) \in [0, \infty].$$

We shall now prove the following

**Theorem 2.10** Let $\phi: M \to N$ be a (nonconstant) harmonic morphism between complete noncompact Riemannian manifolds $M$ and $N$ both of which are connected at infinity. Suppose that the conditions below hold:

1. The Ricci curvature $\text{Ricci}_M$ of $M$ satisfies:

   $$\text{Ricci}_M \geq -\frac{c}{r_M^{1+\varepsilon}}$$

   for some positive constants $c$ and $\varepsilon$.

2. The dimension of $N$ is equal to 2 and $N$ has finite total curvature $\int K_N dA < +\infty$.

   Then if $\delta_\infty > 0$ or $2\pi \chi(N) - \int K_N dA > 0$, one has

   $$\liminf_{t \to \infty} \frac{\log \mu(\phi, t)}{\log t} \geq \left( \chi(N) - \frac{1}{2\pi} \int K_N dA \right) \log \left[ \frac{\exp(c_M \delta_\infty) + 1}{\exp(c_M \delta_\infty) - 1} \right],$$

where $\mu(\phi, t)$ is as in Proposition 2.9, $\chi(N)$ denotes the Euler characteristic of $N$ and $\delta_\infty$ is defined as above.

**Remarks.**

1. The classical Cohn-Vossen inequality says that $2\pi \chi(N) \geq \int K_N dA$. Finn [12] and Huber [18, 19] studied the difference: $\chi(N) - \int K_N dA/2\pi$ from the viewpoint of conformal geometry. On the other hand, Shiohama [29] discussed the same problem in a different way and showed that

   $$\chi(N) - \frac{1}{2\pi} \int K_N dA = \lim_{t \to \infty} \frac{\text{Length}(\partial B_N(t))^2}{4\pi \text{Area}(B_N(t))}$$

   $$= \lim_{t \to \infty} \frac{\text{Area}(B_N(t))}{\pi t^2}$$

   $$= \lim_{t \to \infty} \frac{\text{Length}(\partial B_N(t))}{2\pi t}.$$
(2) If we replace the Ricci curvature of $M$ with the sectional curvature in the condition (1) of Theorem 2.10, then the scaled metric spheres $\{\partial B_M(t), t^{-1} d_t\}$ (where $d_t$ is the intrinsic distance of $\partial B_M(t)$ induced from that of $M$) converge to a compact metric space $M(\infty)$ with respect to the Gromov-Hausdorff distance, and $\delta_\infty$ is equal to the diameter of $M(\infty)$ (cf. [23]).

(3) When $M$ has nonnegative Ricci curvature and $M \setminus B_M(t)$ is homeomorphic to $(0, \infty) \times \partial B_M(t)$ for large $t$, $\delta_\infty$ is finite (cf. [1]).

The proof of Theorem 2.10 is carried out by the same idea as in [24]. To begin with, we shall show

**Lemma 2.11.** Let $M$ be as in Theorem 2.10 and let $h$ be a harmonic function defined outside a compact set $K$ of $M$ such that $h$ is not bounded from below nor above. For large $t$, set $\bar{m}(h, t) = \max \{h(x): x \in \partial B_M(t)\}$, $\underline{m}(h, t) = \min \{h(x): x \in \partial B_M(t)\}$, and $\mu(h, t) = \bar{m}(h, t) - \underline{m}(h, t)$. Then,

$$\limsup_{t \to \infty} \frac{\log \mu(h, t)}{\log t} \geq \log \left[ \frac{\exp(c_m \delta_\infty) + 1}{\exp(c_m \delta_\infty) - 1} \right],$$

where $c_m$ is a positive constant depending only on the dimension $m$ of $M$.

**Proof.** It suffices to show the lemma in case $\delta_\infty < +\infty$. For large $t$, we take two points $p_i, q_i$ of $\partial B_M(t)$ such that $h(p_i) = \bar{m}(h, t)$ and $h(q_i) = \underline{m}(h, t)$. By the maximum principle, both $p_i$ and $q_i$ belong to $\partial \Sigma(t)$. Join $q_i$ to $p_i$ by an arclength-parametrized smooth curve $\tau_i: [0, a_i] \to \Sigma(t)$ whose length $a_i$ satisfies:

$$a_i \leq \operatorname{diam}(\partial \Sigma(t)), \Sigma(t) + \varepsilon_i(t),$$

where $\varepsilon_i(t)$ goes to zero as $t$ tends to infinity. Let us fix here a positive integer $k$ which is greater than $\delta_\infty$ and set $p_{i, t} = \tau_i(ia_i/3k)$ ($i=0, 1, \ldots, 3k$). Observe that

$$\limsup_{t \to \infty} \frac{a_i}{t} \leq \delta_\infty \tag{2.10}$$

$$\limsup_{t \to \infty} \frac{1}{t} \operatorname{dist}_M(p_{i, t}, p_{i, t+1}) \leq \frac{\delta_\infty}{3k} < \frac{1}{3}.$$

By the assumption on $h$, $\bar{m}(h, t)$ is monotone increasing and hence $\bar{m}(h, (\delta_\infty + 3/2)t) - h$ is a positive harmonic function on the metric ball $B_M(p_{i, t}, t/2)$ around $p_{i, t}$ with radius $t/2$. By applying the Harnack inequality due to Cheng and Yau [9] to $\bar{m}(h, (\delta_\infty + 3)t) - h$, and by the assumption on the Ricci curvature, we have

$$\bar{m}(h, (\delta_\infty + 3)t) - h(p_{i, t}) \leq \exp\left( c_m(1 + \varepsilon_2(t)) \frac{a_i}{3kt} \right) \{\bar{m}(h, (\delta_\infty + 3)t) - h(p_{i, t})\},$$

where $\varepsilon_2(t)$ goes to zero as $t$ tends to infinity. This implies that
(2.11) \[ \bar{m}(h, (\delta_n+3)t) - m(h, t) \leq \exp \left( c_m(1+\varepsilon_z(t)) \frac{a_t}{t} \right) \{m(h, (\delta_n+3)t) - m(h, t)\}. \]

Moreover by the assumption on \( h \), \( m(h, t) \) is monotone decreasing and \( h - m(h, (\delta_n+3)t) \) is a positive harmonic function on \( B_{\delta}(p, t/2) \). Hence the same reasoning as above shows that

(2.12) \[ \bar{m}(h, t) - m(h, (\delta_n+3)t) \leq \exp \left( c_m(1+\varepsilon_z(t)) \frac{a_t}{t} \right) \{m(h, t) - m(h, (\delta_n+3)t)\}. \]

Now it follows from (2.11) and (2.12) that

\[ \mu(h, (\delta_n+3)t) + \mu(t) \leq \exp \left( c_m(1+\varepsilon_z(t)) \frac{a_t}{t} \right) \{\mu(h, (\delta_n+3)t) - \mu(t)\}. \]

This implies that

\[ \mu(h, t) \leq \frac{\exp(c_m(1+\varepsilon_z(t))a_t/t) - 1}{\exp(c_m(1+\varepsilon_z(t))a_t/t) + 1} \mu(h, (\delta_n+3)t). \]

Thus it turns out from the above inequality, (2.10) and the standard iteration argument that

\[ \limsup_{t \to \infty} \frac{\log \mu(h, t)}{\log t} \geq \log \left[ \frac{\exp(c_m(\delta_n+1)}{\exp(c_m(\delta_n-1)} \right]. \]

This completes the proof of Lemma 2.11. \( /// \)

Proof of Theorem 2.10. Since \( N \) has finite total curvature \( \int K_N dA \), it follows from [18] that the end of \( N \) is conformally equivalent to that of \( C \), to be precise, there is a conformal diffeomorphism \( \Psi: N \backslash K \to C \backslash D_R \) from the complement of a compact set \( K \) in \( N \) onto that of a disk \( D_R = \{z \in C: |z| \leq R\} \). Then applying the argument in Theorems 11 and 13 of [12] and Théorém 1 of [19] to \( N \backslash K \), we have

(2.13) \[ \lim_{x \to \infty} \frac{\log r_N(x)}{\log |\Psi(x)|} = \chi(N) - \frac{1}{2\pi} \int K_N dA, \]

where \( r_N \) denotes the distance to a fixed point in \( N \). Moreover there exist harmonic functions \( f \) and \( g \) on \( N \) such that both \( |f - \text{Re}(\Psi)| \) and \( |g - \text{Im}(\Psi)| \) are bounded on \( N \backslash K \) (cf. [3: Chap.III]). Define a harmonic map \( \bar{\Psi}: N \to C \) by \( \bar{\Psi} = f + \sqrt{-1}g \). Then by Lemma 2.11, we get

(2.14) \[ \liminf_{t \to \infty} \frac{\log \mu(\bar{\Psi} \circ \phi, t)}{\log t} \geq \log \left[ \frac{\exp(c_m(\delta_n)+1)}{\exp(c_m(\delta_n)-1)} \right]. \]

Thus Theorem 2.10 follows from (2.13) and (2.14). This completes the proof of the theorem. \( /// \)
The same argument as in the proof of Theorem 2.10 will yield the following

**Proposition 2.12.** Let \( \phi: M \to N \) be a (nonconstant) harmonic morphism between complete noncompact Riemannian manifolds \( M \) and \( N \) both of which are connected at infinity. Suppose the following conditions hold:

1. \( \phi \) has at most linear growth.
2. \( M \) has nonnegative Ricci curvature.
3. The dimension of \( N \) is equal to 2, the Gaussian curvature \( K_N \) of \( N \) is non-positive and the total curvature \( \int K_N \, dA \) is finite.

Then \( N \) must be flat.

**Proof.** We observe first that \( N \) is conformally equivalent to \( \mathbb{C} \), because \( M \) admits no nonconstant harmonic functions. Hence there exists a conformal diffeomorphism \( \Psi: N \to \mathbb{C} \) such that

\[
\lim_{x \to \infty} \frac{\log |\Psi(x)|}{\log r_N(x)} = \left(1 - \frac{1}{2\pi} \int K_N \, dA \right)^{-1}.
\]

Therefore by the condition (1), we have

\[
\liminf_{t \to \infty} \frac{\log \mu(\Psi \circ \phi, t)}{\log t} = \liminf_{t \to \infty} \frac{\log \mu(\Psi \circ \phi, t)}{\log \mu(\phi, t)} \geq \left(1 - \frac{1}{2\pi} \int K_N \, dA \right)^{-1}.
\]

The right side of the above inequality is greater than or equal to 1, because of the Cheng's theorem [8] mentioned in the introduction. Thus we see that the total curvature of \( N \) vanishes, which implies that \( N \) must be flat. This completes the proof of Proposition 2.12.

**Corollary.** Let \( \phi: M \to N \) be as in Proposition 2.12. Suppose the conditions

1. and (3) of the proposition hold, and suppose that
2. the sectional curvature \( K_M \) of \( M \) is nonnegative and decays quadratically (i.e., \( 0 \leq K_M \leq c/r_M^2 \)).

Then \( \phi \) is totally geodesic.

**Proof.** This follows from Proposition 2.12 and Theorem B in [24] quoted in the introduction.

### 3. Harmonic maps of linear growth

Let \( M \) be a complete noncompact Riemannian manifold of nonnegative Ricci curvature, and let \( N \) be a Hadamard manifold. In Theorem 3.2 below,
we shall give sufficient conditions for a nonconstant harmonic map \( \phi: M \to N \) to be totally geodesic. The main ingredients in the proof of Theorem 3.2 are a scaling argument and harmonic coordinates with specific properties (cf. e.g., [2,6,21,24,25]). Especially we make use of the following recent result due to Anderson [2]:

**Fact 3.1** ([2: Main Lemma 2.1]). Let \( X=(X,G) \) be a Riemannian manifold (not necessarily complete) of dimension \( d \) such that

\[
|\text{Ricci}_X| \leq \lambda, \quad \text{inj}_x \geq \iota_0 > 0. 
\]

Then given any \( C > 1 \) and \( \sigma \in (0,1) \), there is an \( \varepsilon_0 = \varepsilon_0 (\lambda, \iota_0, d, \sigma) \) with the following property: given any \( x \in X \), there is a harmonic coordinate system \( U=(u_1, \ldots, u_d) \) defined on the metric ball \( B_x(\varepsilon(x)) \) of \( X \) around \( x \) with radius \( \varepsilon(x) \) such that \( U(x)=0 \) and if \( G_{ij}=G(\nabla u_i, \nabla u_j) \), then \( G_{ij}(x)=\delta_{ij} \) and

\[
C^{-1} \leq G(y) \leq CI \quad \text{(as bilinear forms)}
\]

\[
\varepsilon(x)^{1+\sigma} \|G_{ij}(y)\|_{c^1, \sigma} \leq C
\]

for all \( y \in B_x(\varepsilon(x)) \), where

\[
\frac{\varepsilon(x)}{\text{dis} (x, \partial X)} \geq \varepsilon_0 > 0.
\]

We shall now prove the following

**Theorem 3.2.** Let \( M \) be a complete noncompact manifold of nonnegative Ricci curvature and \( N \) a Hadamard manifold. Let \( \phi: M \to N \) be a nonconstant harmonic map between \( M \) and \( N \). Suppose the conditions below hold:

1. The Ricci curvature \( \text{Ricci}_M \) of \( M \) satisfies

\[
(0 \leq) \text{Ricci}_M \leq \frac{c_1}{r_M^2}
\]

for some constant \( c_1 > 0 \), and moreover \( M \) has maximal volume growth, i.e.,

\[
\text{Vol} (B_M(t)) \geq c_2 t^m
\]

for some constant \( c_2 > 0 \), where \( m = \dim M \).

2. The sectional curvature \( K_N \) of \( N \) satisfies

\[
-\frac{c_3}{r_N^2} \leq K_N \quad (\leq 0)
\]

for some constant \( c_3 > 0 \).

3. The map \( \phi \) satisfies

\[
c_4 (r_M - 1) \leq r_M \circ \phi \leq c_5 (r_M + 1)
\]
for some constants $c_4 > 0$ and $c_5 > 0$, and moreover $\phi$ has maximal rank $n (= \dim N)$ outside a compact set $K$, i.e.,

\[(3.5) \quad \text{the rank of } d\phi_x = n \quad (x \in M \setminus K).\]

Then both $M$ and $N$ are isometric to Euclidean $m$-space $\mathbb{R}^m$ and $\phi$ is an affine map.

We make here some remarks before the proof of Theorem 3.2.

REMARKS. (1) As mentioned in the introduction, the second inequality in
(3.4) is equivalent to the condition that the energy density $e(\phi)$ of $\phi$ is bounded on $M$. Moreover $e(\phi)$ is subharmonic on $M$ because of the Weitzenböck formula which reads:

\[(3.6) \quad \frac{1}{2} \Delta_M e(\phi) = |\nabla d\phi|^2 + \sum_{i=1}^n \langle \phi_* \text{Ricci}_M e_i, \phi_* e_j \rangle
- \sum_{i,j=1}^n \langle \text{R}^N(\phi_* e_i, \phi_* e_j) \phi_* e_i, \phi_* e_j \rangle\]

where $\{e_1, \ldots, e_m\}$ is an orthonormal basis at the point under consideration on $M$.

(2) By a theorem due to Croke [10] and by the nonnegativity of the Ricci curvature of $M$ and the condition (3.2) on the volume growth of $M$, the injectivity radius $\text{inj}_M(x)$ of $M$ at a point $x \in M$ satisfies

\[(3.7) \quad \text{inj}_M(x) \geq c_6 r_M(x)\]

for some constant $c_6 > 0$.

Proof of Theorem 3.2. We shall divide the proof into three steps.

Step 1. Given a positive number $t$, consider the scaled metric $g_t = t^{-2} g_M$ of the metric $g_M$ of $M$. Denote by $B^t_M(x, a)$ (resp., $A^t_M(x, a)$) the metric ball around a point $x$ of radius $a$ with respect to $g_t$ (resp., the annular domain $B^t_M(o, b) \setminus B^t_M(o, c)$ ($b < c$)). Given a number $k > 1$, by (3.7), we can apply Fact 3.1 to $A^t_M(k, k^{-1})$ and find constants $a$ and $b$ independent of $t$ such that for any $x \in A^t_M(k, k^{-1})$, there is a harmonic coordinate system $U = (u_1, \ldots, u_m)$ on $B^t_M(x, a)$ which has the property described in Fact 3.1 and the image of which contains the Euclidean $m$-ball $B^m(b)$ of radius $b$, i.e. $U(B^t_M(x, a)) \supset B^m(b) = \{w \in \mathbb{R}^m : |w| < b\}$. Moreover since $M$ has nonnegative Ricci curvature and maximal volume growth (3.2), we can employ the simple covering argument based on the Bishop comparison theorem (cf. [15]) and obtain a finite collection of balls $\{B^t_M(x_i, a)\}_{1 \leq i \leq I}$, the union of which covers $A^t_M(k, k^{-1})$ and the number $I$ of which is bounded by a constant independent of $t$. Therefore it turns out from a version of the Gromov's Lipschitz convergence theorem and the standard diagonal argument that arbitrary divergent sequences $\{t_i\}$ and $\{k_i\}$ respectively contain some subsequences, denoted by the same letters, for which $A^t_M(k_{ij}, k_{ij}^{-1})$ converges to an $m$-manifold $C^*(M(\infty))$ of a $C^1$-metric $g_\omega$ in the
C^1\tau\)-topology (0<\sigma<\tau<1) as \( j \) goes to infinity (cf. e.g., [6, 25]). As for \( N \), the same observations are valid. To be precise, if we take two (large) numbers \( t \) and \( \hat{k} \), and consider the scaled metric \( h_t=\gamma^2 h_N \) of the metric \( h_N \) of \( N \), then we can find positive constants \( \hat{a} \) and \( \hat{b} \) independent of \( t \) such that for any \( y \in A_t^\gamma(\hat{k}, \hat{k}^{-1}) \), there is a harmonic coordinate system \( V=(v_1, \ldots, v_n) \) on \( B_t(y, \hat{a}) \) which has the property described in Fact 3.1 and the image of which contains the Euclidean \( n \)-ball \( B^n(\hat{b}) \) of radius \( \hat{b} \). Moreover by (3.4) and (3.5), we will assume that

\[
A_t^\gamma(\hat{k}, (\gamma k)^{-1}) \subset \phi(A_t^\gamma(k, k^{-1})) \subset A_t^\gamma(\hat{k}, (\hat{k}k)^{-1})
\]

for some constants \( c \) and \( \hat{c} \) independent of \( t \).

Step 2. Set \( e_\infty=\sup e(\phi) \) and take a family of points \( \{x_t\} \) in \( M \) such that \( r_M(x_t)=t \) and \( e(\phi)(x_t) \) goes to \( e_\infty \) as \( t \) tends to infinity. In what follows, we will assume that \( a<\hat{a} \) and \( B_t(\phi(x), \hat{a}) \subset A_t^\gamma(\hat{k}, \hat{k}^{-1}) \) for any \( x \in A_t^\gamma(k, k^{-1}) \), where \( a, \hat{a}, k, \hat{k} \) are as in Step 1. Let \( U_t=(u_1, \ldots, u_m) \) be a harmonic coordinate system on \( B_t(x_t, a) \) with the property described in Fact 3.1 and let \( V_t=(v_1, \ldots, v_u) \) be such a system on \( B_t(\phi(x_t), \hat{a}) \). Set \( \phi_t=V_t \circ \phi \circ U_t^{-1} \) and assume that \( \phi_t(B^m(b)) \subset B^n(\hat{b}) \), where \( b \) and \( \hat{b} \) are as in Step 1. Then the components \( \{\phi_t^n\}_{n=1,\ldots,N} \) of \( \phi_t \) satisfy

\[
\sum_i \sum_{j=1}^m g_{ij} \frac{\partial^2 \phi_t^n}{\partial u_i \partial u_j} - \sum_i \sum_{j=1}^m \Gamma_{ij}^{\gamma} (g_t)^{ij} \frac{\partial \phi_t^n}{\partial u_i} + \sum_i \sum_{j=1}^m \Gamma_{ij}^{\gamma} (h_t)^{ij} \frac{\partial \phi_t^n}{\partial u_i} - \frac{\partial \phi_t^n}{\partial u_j} = 0,
\]

where \( \{\Gamma_{ij}^{\gamma}\} \) (resp., \( \{\Gamma_{ij}^{\gamma}\} \)) are the Christoffel symbols of \( g_t \) (resp. \( h_t \)) with respect to the harmonic coordinates \( U_t \) (resp. \( V_t \)). It follows from the standard elliptic regularity theory that the \( C^{2,\gamma} \)-norms of \( \phi_t^n \) are bounded uniformly in \( t \). Thus for any divergent sequence \( \{t_j\} \), there exist a subsequence, denoted by the same letters \( \{t_j\} \), a \( C^{1,\gamma} \) metric \( g_{\infty} \) on \( B^m(b) \), a \( C^{1,\gamma} \) metric \( h_{\infty} \) on \( B_n(\hat{b}) \) and a \( C^{2,\gamma} \) map \( \phi_{\infty}: B^m(b) \to B^n(\hat{b}) \) such that \( U_{t_j} \circ \phi_{t_j} \) (resp., \( V_{t_j} \circ h_{t_j} \)) converges to \( g_{\infty} \) (resp., \( h_{\infty} \)) in the \( C^{1,\tau} \) topology and so does \( \{\phi_{t_j}\} \) to \( \phi_{\infty} \) in the \( C^{2,\gamma} \) topology. Moreover if we use the apriori estimates in the Sobolev space, then we may assume that the limit metrics \( g_{\infty} \) and \( h_{\infty} \) have, respectively, the curvature tensors \( R^m_{\infty} \) and \( R^n_{\infty} \) in \( L^2 \)-sense, to which the curvature tensors of \( g_{t_j} \) and \( h_{t_j} \), respectively, converge weakly in \( L^2 \)-sense. Thus it follows that the limit map \( \phi_{\infty} \) defines a harmonic map with respect to the limit metrics and the energy density \( e(\phi_{\infty}) \) of \( \phi_{\infty} \) satisfies weakly the Weitzenböck formula (3.6) in \( L^2 \)-sense. Applying the maximum principle to \( e(\phi_{\infty}) \), we see that \( e(\phi_{\infty}) \) must be constant, because \( e(\phi_{\infty})(0)=\max e(\phi_{\infty})=e_\infty \). This implies that \( \phi_{\infty} \) is totally geodesic, i.e., the
second fundamental form of \( \phi_w \) with respect to the limit metrics vanishes identically. Thus in particular, it turns out that \( e(\phi)(x) \) converges to the constant \( c_w \) as \( x \in M \) tends to infinity and the second fundamental form \( \nabla d\phi \) of satisfies:
\[
\lim_{s \to \infty} r_M(x)|\nabla d\phi| (x) = 0.
\]

Step 3. Let \( \{t_j\}, \{k_j\} \) and \( \{k_j\} \) be divergent sequences as in Step 1 such that (3.8) is kept for any \( j \), and \( A^M_{kj}(k_j, k_j^{-1}) \) and \( A^N_{kj}(k_j, k_j^{-1}) \) converge respectively to the Riemannian manifolds \( C^*(M(\infty)) \) and \( C^*(N(\infty)) \) as \( j \) goes to infinity. As observed in Step 2, we may assume that as \( j \) goes to infinity, the restriction \( \phi_{t_j} \) of \( \phi \) to \( A^M_{kj}(k_j, k_j^{-1}) \) converges to a totally geodesic map, say again \( \phi_w \), from \( C^*(M(\infty)) \) to \( C^*(N(\infty)) \). Moreover it follows from (3.8) that \( \phi_w \) is a diffeomorphism. Hence by the Weitzenböck formula (3.6), we see that both \( C^*(M(\infty)) \) and \( C^*(N(\infty)) \) are isometric to \( R^m \setminus \{0\} \). This shows in particular that \( m = n \) and
\[
\lim_{t \to \infty} \frac{\text{Vol}(B^M_M(t))}{\omega_m t^m} = \lim_{t \to \infty} \frac{\text{Vol}(B^N_N(t))}{\omega_m t^m} = 1
\]
where \( \omega_m \) denotes the volume of the unit sphere in \( R^m \) (cf. [24, 25]). Thus it turns out from the equality discussion of Bishop and Rauch comparison theorems that both \( M \) and \( N \) are isometric to \( R^m \). Hence it is obvious that \( \phi \) is an affine map. This completes the proof of Theorem 3.2. //

References


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