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<th>Generalizations of Nakayama ring. I</th>
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<tr>
<td>Author(s)</td>
<td>Harada, Manabu</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 23(1) P.181–P.200</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1986</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6809">https://doi.org/10.18910/6809</a></td>
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<tr>
<td>DOI</td>
<td>10.18910/6809</td>
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T. Nakayama found a very important ring in ring theory, which we call a generalized uniserial ring [6]. He showed that a left and right artinian ring \( R \) is a generalized uniserial ring if and only if every (finitely generated) left (resp. right) \( R \)-module is a direct sum of uniserial modules. We shall generalize further such a ring from this point of view.

We shall define Conditions \((\ast, 3)\) and \((\ast\ast, 3)\) (see §1) for a direct sum \( D(3) \) of three hollow modules. If \( R \) is a generalized uniserial ring, \((\ast, 3)\) and \((\ast\ast, 3)\) are satisfied ([2] and [3]). In §2 we shall give a characterization of a right artinian ring \( R \) which satisfies \((\ast\ast, 3)\) for any \( D(3) \). In §3 we shall expose several examples related to the results in the previous section.

We shall study Condition \((\ast, 3)\) in a forthcoming paper.

1. Definitions. Let \( R \) be throughout a right artinian ring with identity. Modules in this note are unitary right \( R \)-modules with finite length. Let \( e \) be a primitive idempotent in \( R \).

\[
eR \supsetneq e^2R \supsetneq \cdots \supsetneq e^nR = 0
\]

is a unique chain of the submodules of \( eR \) for each \( e \), \( R \) is called a right (generalized uni-) serial ring (Nakayama ring), where \( J=J(R) \) is the Jacobson radical of \( R \).

As a generalization of a serial ring, we have considered the following two conditions [1]:

\((\ast, n)\) Every (non-zero) maximal submodule of a direct sum \( D(n) \) of \( n \) non-zero hollow modules is also a direct sum of hollow modules, and

\((\ast\ast, n)\) Every (non-zero) maximal submodule of the \( D(n) \) above contains a non-trivial direct summand of \( D(n) \).

By Nakayama [6], if \( R \) is a right and left serial ring, \((\ast, n)\) holds for any \( D(n) \) and any \( n \) as right (resp. left) \( R \)-modules. Further \( R \) is a right serial ring if and only if \((\ast, n)\), replaced hollow by uniserial, holds for any \( D(n) \) and any \( n \) as right \( R \)-modules [5].

In general, if \( J^2=0 \), \((\ast, 2)\) holds for any \( D(2) \) by [3], Proposition 3. Let \( \{N_i\}_{i=1}^\ell \) be a set of hollow modules, and put \( D(n)=\sum_{i=1}^\ell N_i \). Let \( M_i \) be a maximal submodule of \( D(n-1)=\sum_{i=1}^{n-1} N_i \). Then \( M=M_i\oplus N_* \) is a maximal submodule of \( D \). If \( D \) satisfies \((\ast, n)\), \( M_i \) is also a direct sum of hollow modules.
by Krull-Remak-Schmidt’s theorem. Hence $D(n-1)$ satisfies $(\ast, n-1)$. Contrarily, if $D(n-1)$ satisfies $(\ast\ast, n-1)$, $D(n)$ does $(\ast\ast, n)$ by [2], Lemma 1.

**Definition.** A ring $R$ is called a right US-$n$ ring if $(\ast\ast, n)$ is satisfied for any $D(n)$ (US is an abbreviation of uni-serial).

We have obtained the following theorem [2]:

**Theorem 1.** Let $R$ be a right artinian ring. Then $R$ is a right US-1 ring if and only if $R$ is a semi-simple ring. $R$ is a right US-2 ring if and only if $R$ is a right generalized uni-serial ring.

Hence the next problem concerning $(\ast\ast, n)$ is to study the structure of right US-3 rings.

2. **US-3 rings.** Let $N_1$ and $N_2$ be two hollow modules. Assume that $N_1 \cong eR/eA_i$ and $N_2 \cong eR/A_{i+1}$, where $e$ is a primitive idempotent and the $A_i$ are submodules of $eR$. If there exists an epimorphism of $N_1$ to $N_2$, then there exists a unit element $x$ in $eRe$ such that $xA_1 \subset A_2$. If there exists an epimorphism of one to another between $N_1$ and $N_2$, we indicate it by $N_1 \sim N_2$ or $A_1 \sim A_2$, namely there exists a unit element $y$ in $eRe$ such that $yA_1 \subset A_2$ or $yA_2 \subset A_1$. Since $y(eR/A_i) = eR/yA_i$, we may assume, in this case, that $A_1 \subset A_2$ or $A_2 \subset A_1$.

Now we put $\Delta = eRe/eJe$, a division ring, and $\Delta(A_i) = \{x \mid xA_i \subset A_1\}$, there exists $x'$ in $eRe$ such that $x'A_1 \subset A_1$, and $x - x' \in eJe$. If we put $S(A_i) = \{x \mid x \in eRe, xA_1 \subset A_1\}$, then $S(A_i)$ is a subring of $eRe$. Let $v$ be the natural epimorphism of $eRe$ onto $eRe/eJe = \Delta$. Then $\Delta(A_i) = v(S(A_i))$, and $\Delta(A_i)$ is a sub-division ring of $\Delta$. We may regard $\Delta$ as a right $\Delta(A_i)$-module, and hence we denote the dimension of $\Delta$ over $\Delta(A_i)$ by $[\Delta : \Delta(A_i)]$ (see [1]). $|A_i|$ means the length of $A_i$.

**Lemma 1.** Let $\{A_i\}_{i=1}^3$ be a set of three submodules in $eR$. If $A_i = eJ$ for some $i$, $D = \sum_{i=1}^3 eR/A_i$ satisfies $(\ast\ast, 3)$. Conversely, if $D$ satisfies $(\ast\ast, 3)$, $A_1 \sim A_j$ for some pair $(i, j)$.

Proof. If $A_i = eJ$, $eR/A_i$ is simple. Let $M$ be a maximal submodule of $D$. Then $M \supseteq eR/A_i$ or $D = M \oplus eR/A_j$. Hence $(\ast\ast, 3)$ holds. If $A_i = eJ$ for some $i$, then $A_i \sim A_j$ for certain $j$, since $eJ$ is a unique maximal submodule of $eR$. Assume that $A_i \neq eJ$ for all $i$. Then any $eR/A_i$ does not satisfy $(\ast\ast, 1)$. Hence $A_i \sim A_{i'}$ for some pair $(i, i')$ by [4], Corollary 2.

**Proposition 1.** Assume that $R$ is a US-3 ring. Then 1) $[\Delta : \Delta(A)] \leq 2$ for any submodule $A$ in $eR$. 2) If there exists a submodule $B$ in $eR$ such that $[\Delta : \Delta(B)] = 2$, then $B \sim C$ for any submodule $C$ in $eR$. 3) $t = |eJ^i| |eJ^{i+1}| \leq 2$ for all $i$. 4) Assume that $eJ^i$ contains a maximal submodule $A_i$ ($\supseteq eJ^{i+1}$) with $\Delta(A_i) = \Delta$. Then $i)$ $eJ^i$ contains at most two maximal submodules $A_1$ and $A_2$. $\Delta(A_i)$
GENERALIZATIONS OF NAKAYAMA RING

\[ \Delta(A_k) = \Delta. \]

iii) \( A_1 \) and \( A_2 \) are characteristic in \( eR \).

iv) Either \( A_1 \) or \( A_2 \) is hollow, provided \( t = 2 \).

Proof. 1) and 2). They are immediate consequences of [2], Theorem 2 and [4], Corollary 1, respectively.

3) Let \( eJ^i/eJ^{i+1} = \sum C_i \), where the \( C_i \) are simple modules and \( eJ^i \subseteq eJ^{i+1} \). It is clear that \( |C_i| = |C_{k}| = 1 + |eJ^{i+1}| \) for all \( k \). Assume that \( \Delta(C_i) = \Delta \) and \( C_i \sim C_2 \). Then there exists a unit \( x \) in \( eRe \) such that \( xC_i = C_2 \). Since \( \Delta(C_i) = \Delta \), put \( x = x_i + j \), where \( x_i \) and \( j \) are elements of \( eJ^{i+1} \). Then \( C_2 = xC_2 = (x_i + j)C_i \subseteq xC_i + jC_i \subseteq C_1 + eJ^{i+1} = C_1 \). Hence, if \( \Delta(C_i) = \Delta \) for all \( i \), \( t = 2 \) by Lemma 1. Next assume that \( \Delta(C_i) \neq \Delta \) for all \( i \) by 2) and the above proof. There exists, from 2), a unit \( x_i \) in \( eRe \) with \( C_2 = x_i C_2 \). Then \( x_i \in \Delta(C_i) \).

4) ii) and iii). They are clear from the first part of the proof of 3). 4), i). Assume that \( A_1 \) is a maximal submodule of \( eJ^i \) with \( \Delta(A_1) = \Delta \). Then \( A_1 \) is characteristic from iii). Let \( A_2 \) be another maximal submodule of \( eJ^i \). Then \( \Delta(A_2) \neq \Delta \) from 1) and 2). Hence \( A_1 \sim A_2 \), so \( eJ^i \) contains at most two maximal submodules \( A_1 \) and \( A_2 \). 4), iv). Assume that \( A_1 \neq A_2 \). Then \( A_1 \cap A_2 = eJ^{i+1} \). Let \( B \) be a maximal submodule in \( A_1 \). If \( B \sim eJ^{i+1} \), then \( eJ^{i+1} \) is characteristic. Hence, if \( |A_1/A_1 J| \geq 2 \), \( A_1 \) contains a maximal submodule \( B_1 \) such that \( B_1 \sim eJ^{i+1} \). Let \( B' \) be a maximal submodule of \( A_2 \). If \( B' \neq eJ^{i+1} \) and \( |A_1/A_1 J| \geq 2 \), then \( B' \sim B_1 \). Hence there exists a unit element \( x \) in \( eRe \) such that \( B' = xB_1 \subseteq xA_1 \), and so \( B' \subseteq A_1 \cap A_2 = eJ^{i+1} \), which is a contradiction. Hence \( A_2 \) is hollow.

**Corollary.** Let \( R \) be a right artinian ring. Then \( R \) is a right US-3 ring if and only if we have the following properties for each primitive idempotent \( e \):

1) For any three submodules \( A_i \) of \( eR \) such that \( \Delta(A_i) = \Delta \), \( A_1 \sim A_2 \) for some pair \( (i, j) \).
2) \( \Delta(A) \leq 2 \) for any submodule \( A \) of \( eR \).
3) If \( \Delta(A) = 2 \) for a submodule \( B \) of \( eR \), then \( B \sim A \).

Proof. "Only if" part is clear from Lemma 1 and Proposition 1. Assume 1), 2) and 3) and put \( D = eR/A_1 \oplus eR/A_2 \oplus eR/A_3 \). If the \( A_i \) satisfy 1), then \( D \) satisfies (**) and hence (**, 3) by [4], Corollary 3. Assume that \( \Delta(A) = 2 \). Then we may assume that \( A_1 \subset A_2 \) or \( A_2 \subset A_1 \) from 3). If \( \Delta(A_1) = \Delta(A_2) \), then \( \Delta(A_1, A_2) = \{ x \mid x \in eRe, xA_1 \subseteq A_2 \} = \Delta \) or \( \Delta(A_2, A_1) = \Delta \), respectively. Hence \( D \) satisfies (**) by [4], Theorem 2. Finally assume that \( \Delta(A) = 2 \) for all \( i \). Then \( D \) satisfies (**) by 3) and [4], Corollary 4.

In Corollary to Proposition 1, we have given the condition under which (**) holds for any \( D(3) \). Using this corollary, we shall give the complete form of the lattice of submodules of \( eR \), provided \( R \) is a right US-3 ring.
First we consider the following situation:

**X ⊃ Y are characteristic submodule of eJ, X/Y is simple and Y is hollow.**

Let \( \{D_i\} \) be the set of submodules of \( eR \) containing \( Y \) such that \( |D_i| = |X| \) and \( D_i \cap X = Y \). Then \( D_i \cap X = Y \) and \( D_i \prec X \), so \( D_i \sim D_j \) provided \( i \geq 2 \). Since \( D_1 \cap D_2 = Y \), \( D_1 \) and \( D_2 \) cannot contain a common maximal submodule except \( Y \). Let \( B_i = B_j \) be a maximal submodule of \( D_i \) for each \( i \) (note that \( D_1 \approx D_i \)). Then \( B_1 \neq B_2 \), and put \( E = B_1 + B_2 \). Then \( B_i \cap B_j = (B_1 + Y)/B_1 \approx Y,(B_1 \cap Y) \) is simple and \( Y \) is hollow by assumption, and so \( B_i \cap Y = J(Y) (= Z) \). Similarly, \( B_2 \cap Y = Z \). Since \( D_i/Y \approx B_i/Y \) and \( B_1 \cap B_2 = Z \), \( |E| = |D_1| = |D_2| \), and \( E \sim D_i \), for \( E \neq X \). From the fact: \( D_1 \supseteq Y \) and \( Y \) is characteristic, \( E \supseteq Y \), and so \( E = D_j \) for some \( j \). However, \( D_i \cap D_j \supseteq B_1 \) implies \( D_j = D_1 \), and similarly \( D_j = D_2 \), which is a contradiction. Hence, whenever \( i \geq 2 \), each \( D_i \) is hollow. Thus we obtain

\[
\begin{align*}
\text{i)} &- 1 \quad \alpha \geq 2 \\
& \text{(X is characteristic and \( D \) is hollow)} \\
& \text{(Y is hollow and characteristic)}
\end{align*}
\]

or

\[
\begin{align*}
\text{i)} &- 2 \\
& \text{(\( eJ^i \) is hollow)} \\
& \text{(\( eJ^{i+1} \) is hollow)} \\
& \text{(\( eJ^i \) is hollow)} \\
& \text{\( (C_1 \text{ and } C_2 \text{ are characteristic and either } C_1 \text{ or } C_2 \text{ is hollow and } \Delta(C_1) = \Delta (i=1, 2)) \)
}
\end{align*}
\]

Now following Proposition 1, we divide the situations into the following cases:

\[
\begin{align*}
I' \quad & eJ^i \\
& eJ^{i+1} \quad (\text{\( eJ^{i+1} \) is hollow}) \\
& eJ^{i+2} \\
II' \quad & eJ^i \\
& eJ^{i+1} \\
& eJ^{i+2} \quad (\text{\( eJ^i \) is hollow}) \\
& C_1 \prec C_2 \quad (\text{\( C_1 \text{ and } C_2 \text{ are characteristic and either } C_1 \text{ or } C_2 \text{ is hollow and } \Delta(C_1) = \Delta (i=1, 2))}
\end{align*}
\]
III' \[ \begin{array}{c} A_1 \\ \sim \\ eJ^i \\ \sim \\ eJ^{i+1} \\ \sim \\ A_2 \\ \downarrow eJ^{i+2} \end{array} \]

(A_1 and A_2 are characteristic, either A_1 or A_2 is hollow and $\Delta (A_i) = \Delta (i=1, 2)$) 

(e$J^{i+1}$ is hollow)

IV' \[ \begin{array}{c} C_1 \\ \sim \\ C_2 \\ \downarrow eJ^{i+2} \end{array} \]

(C_1 and C_2 are characteristic, either C_1 or C_2 is hollow and $\Delta (C_i) = \Delta (i=1, 2)$)

V' \[ \begin{array}{c} C_1 \\ \sim \\ C_2 \\ \sim \\ C_\alpha \alpha \geq 2 \\ \downarrow eJ^{i+2} \end{array} \]

($\Delta: \Delta (C_i) = 2$ for all $i$)

VI' \[ \begin{array}{c} A_1 \\ \sim \\ A_2 \\ \sim \\ A_\alpha \alpha \geq 2 \\ \downarrow eJ^{i+2} \\ \downarrow eJ^{i+1} \end{array} \]

($\Delta: \Delta (A_i) = 2$ for all $i$) 

(e$J^{i+1}$ is hollow)
\[ \alpha \geq 2 \]

\[ ([\Delta: \Delta(A_i)] = 2 \text{ for all } i) \]

\[ \beta \geq 2 \]

\[ ([\Delta: \Delta(C_i)] = 2 \text{ for all } i) \]

\( C_1 \) and \( C_2 \) are characteristic, either \( C_1 \) or \( C_2 \) is hollow and \( \Delta = \Delta(C_i) \) (\( i = 1, 2 \))

\( A_1 \) and \( A_2 \) are characteristic, either \( A_1 \) or \( A_2 \) is hollow and \( \Delta = \Delta(A_i) \) (\( i = 1, 2 \))

\[ ([\Delta: \Delta(C_i)] = 2 \text{ for all } i) \]
In the above and following observations, every chain of a diagram means a composition series, all modules located on the same horizontal line in the diagram have the same length, and all modules with same length appear in the diagrams below. It may happen that some modules in the diagrams do not appear. Further we always consider a case where \( eJ^i \) is a waist (every composition series contains \( eJ^i \)) or \( D_1 \) in the diagram exists.

I' Let \( D' (\neq eJ^i) \) be a submodule of \( eR \) with \( |D'| = |eJ^i| \). Since there exists \( D_1 \) containing \( eJ^{i+1} \) with \( |D_1| = |eJ^i| \) by assumption, \( D' \sim D_1 \) by Lemma 1. Hence \( D_1 \supset eJ^{i+1} \) implies \( D' \supset eJ^{i+1} \). Thus we obtain from i)

\[
\begin{array}{c} eJ^i \\ \downarrow \ \\
\begin{array}{c} eJ^{i+1} \\ \downarrow \ \\
eJ^{i+2}
\end{array}
\end{array}
\]

(\( eJ^i \) is hollow and a waist)

I)-1

\[
\begin{array}{c} eJ^i \\ \downarrow \ \\
eJ^{i+1} \\ \downarrow \ \\
eJ^{i+2}
\end{array}
\]

(\( eJ^{i+1} \) is hollow and a waist)

I)-2

\[
\begin{array}{c} eJ^i \\ \downarrow \ \\
eJ^{i+1} \\ \downarrow \ \\
eJ^{i+2}
\end{array}
\]

(\( \Delta = \Delta(D) \))

(\( B_i \) is hollow and \( \Delta = \Delta(B_i) \) for all \( i \))

(\( eJ^{i+2} \) is a waist)

(We shall know later that the \( B_i \) are hollow)

I)-3

\[
\begin{array}{c} eJ^i \\ \downarrow \ \\
eJ^{i+1} \\ \downarrow \ \\
eJ^{i+2}
\end{array}
\]

(\( D_i \) is hollow and \( \Delta = \Delta(D_i) \) for all \( i \))

(\( eJ^{i+1} \) is a waist)

(\( eJ^{i+2} \) is waist)

II' Let \( \{D_p (\neq eJ^i)\}_{p=1}^\infty \) be a set of submodules of \( eJ \) such that \( D_p \supset eJ^{i+1} \) and \( |D_p| = |eJ^i| \) (see the initial part of I'). Let \( B_k \) (\( k=1, 2, \ldots \)) be maximal submodules of \( D_1 \) different from \( eJ^{i+1} \). Since \( D_1/B_1 \) is simple and \( eJ^{i+1} \) contains at most two maximal submodules \( C_j \), we may assume that \( B_1 \cap eJ^{i+1} = C_1 \). Then since \( B_2 = xB_1 \) for a unit \( x \), \( B_2 \cap eJ^{i+1} = x(B_1 \cap eJ^{i+1}) = xC_1 = C_1 \). Let \( C \) be a
maximal submodule of $B_1$. Since $B_1 \cap e^{j+i+1} = C_1 \approx C_2$, $C \sim C_1$ or $C \sim C_2$. However, since $C_1$ and $C_2$ are characteristic and $|C| = |C_1| = |C_2|$, $C = C_1$ or $C = C_2$. Therefore $C_1$ is a unique maximal submodule of $B_1$. Next assume that $n \geq 2$, i.e. $D_1 \neq D_2$. If $D_1$ is not hollow, $D_2$ contains a maximal submodule $B_1' (\neq e^{j+i+1})$ which contains also a unique maximal submodule $C_2$ from the above argument ($k = 1$ or 2). Since $B_1' \sim B_1$ by Lemma 1 and $C_1 \approx C_2$, $k = 1$. Therefore $B_1 \cap B_1' = C_1$. If we replace $J(Y)$ by $C_1$ in the proof of i), we obtain the same situation (put $E = B_1 + B_1'$) and the $D_i$ are hollow. Thus we have

$$B_i \sim B_2 \quad e^{j+i} \sim \alpha \geq 2$$

(B$_i$ is hollow and $\Delta = \Delta(B_i)$)

$$D_1 \sim D_2 \quad e^{j+i} \sim \alpha \geq 2$$

(D$_i$ is hollow and $\Delta = \Delta(D_i)$

for all i))

($e^{j+i+1}$ is a waist)

III' The $A_i$ are characteristic. Replacing $C_i$ in II) by $A_i$, we obtain from i)

$$D_1 \sim D_2 \quad e^{j+i} \sim \alpha \geq 2$$

III)-1
or

\[
\begin{align*}
D_1 \sim & \sim D_2 \sim D_3 \sim D_4 \\
& \sim A_1 \sim A_2 \\
& \sim B \sim eJ^{i+1} \\
& \sim eJ^{i+2}
\end{align*}
\]

\(\Delta = \Delta(D_i)\) for all \(i\)

III)–2

\[
\begin{align*}
D_1 \sim & \sim D_2 \sim D_3 \\
& \sim A_1 \sim A_2 \\
& \sim B_1 \sim B_2 \sim B_3 \\
& \sim eJ^{i+1} \\
& \sim eJ^{i+2}
\end{align*}
\]

\(\beta \geq 2\)

\(B_i\) is hollow and \(\Delta = \Delta(B_i)\) for all \(i\)

or

\[
\begin{align*}
D_1 \sim & \sim D_2 \sim D_3 \sim D_4 \\
& \sim A_1 \sim A_2 \\
& \sim B \sim eJ^{i+1} \\
& \sim eJ^{i+2}
\end{align*}
\]

(We shall know later that the \(B_i\) are hollow.)

IV’ Similarly to II’ and the above, we obtain
In order to study the remaining cases, we consider the following:

(We omit other forms similar to the second form in III)

Since $\Delta(A_i) \neq \Delta$ by Proposition 1, 4), for any submodule $E$ of $eR$, $E \sim A_i$ by Proposition 1, 2). Hence $E \supset eJ^{i+1}$ provided $|E| = |A_i|$, since $A_i \supset eJ^{i+1}$ and $eJ^{i+1}$ is characteristic, and so maximal submodules of $D$ consist of a subset of $\{A_i\}$. Therefore $D$ is hollow. On the other hand, $\Delta = \Delta(D)$ and $\Delta = \Delta(A_i)$ by assumption and Proposition 1, 2). Therefore there exists a unit $x$ in $eRe$ such that $(x+j)A_i \neq A_i$ for any $j$ in $eJe$. Since $\Delta = \Delta(D)$, there exists $j'$ in $eJe$ with $(x+j')D = D$. Hence $D$ contains $A_i$ and $(x+j')A_i$, and so $D \supset A_i + (x+j')A_i = eJ^i$, which is a contradiction. Hence $D = eJ^i$. Therefore $eJ^i$ is a waist from the assumption of this observation. If further $A_i$ is not hollow, there exists a maximal submodule $B_i := eJ^{i+1}$ of $A_i$. Then $\Delta(B_i) = \Delta$ and we obtain the same situation as above. Hence $B_i \subset A_i \cap (x+j')A_i = eJ^{i+1}$, a contradiction. Therefore $eJ^{i+1}$ is also a waist. Thus from the argument above and the consideration of I) - IV) we obtain
ii) \( eJ^i \) (is a waist)

\( A_i \) is hollow and \([\Delta: \Delta(A_i)] = 2\) for all \( i \)

\( eJ^{i+1} \) is a waist

V) \( D_i \) is hollow and \( \Delta = \Delta(D_i) \) for all \( i \)

\( eJ^{i+1} \) is a waist

\( C_i \) is hollow and \([\Delta: \Delta(C_i)] = 2\) for all \( i \)

\( eJ^{i+2} \) is a waist

VI) \( A_i \) is hollow and \([\Delta: \Delta(A_i)] = 2\) for all \( i \)

\( eJ^{i+1} \) is a waist

\( eJ^{i+2} \) is a waist
VII) \( e^{f^{i+1}} \) is a waist

\( A_i \) is hollow and 
\[ [\Delta : \Delta(A_i)] = 2 \text{ for all } i \]

\( e^{f^{i+2}} \) is a waist

VIII) \( e^{f^{i+1}} \) is a waist

\( A_i \) is hollow and 
\[ [\Delta : \Delta(A_i)] = 2 \text{ for all } i \]

\( e^{f^{i+2}} \) is a waist
IX) 

\[ (D_i \text{ is hollow and } \Delta = \Delta(D_i) \text{ for all } i) \]

\[ (eJ_{i+1}^{i+1} \text{ is a waist}) \]

\[ (C_i \text{ is hollow and } [\Delta: \Delta(C_i)] = 2 \text{ for all } i) \]

\[ (eJ_{i+2}^{i+2} \text{ is a waist}) \]

X) 

\[ (D_i \text{ is hollow and } \Delta = \Delta(D_i) \text{ for all } i) \]
If the $B_i$ in I)–3 or III)–2 are not hollow, we should obtain a circle around $eJ^{i+2}$ (note that $\beta \geq 2$), which is directly connected to the circle around $eJ^{i+1}$ like a cylinder. However, there are no circles directly connected in the diagrams. Hence $B_i$ is hollow.

If we start from $eJ$ and use the diagrams and induction on the nilpotency of $J$, we know that either $eJ^k$ is a waist or $D_i$ in each diagram exists. Thus we obtain the lattice of all submodules in $eR$ by connecting the diagrams I)–IX). For example see X), XI) and

\begin{align*}
\text{XI)} & \\
\text{XII)} &
\end{align*}
If we consider the lattice of isomorphism classes given by the left-sided multiplication of unit elements in $eRe$, we obtain Diagrams XI) and XIII). Checking each diagram, we know that $|A/AJ| \leq 2$ for any submodule $A$ of $eR$ and that the $A$ satisfy the conditions in Corollary to Proposition 1.

**Theorem 2.** Let $R$ be a right artinian ring. Then the following conditions are equivalent:

1) $R$ is a right US-3 ring.

2) Let $e$ be a primitive idempotent in $R$. For any submodule $A$ of $eR$, $A/AJ = \overline{A}_1 \oplus \overline{A}_2$, where $A \supseteq A_1 \supseteq AJ$ and $\overline{A}_1$ is simple or zero, and one of the following situations occurs:

   i) $\overline{A}_2 = 0$, i.e., $A$ is hollow.

   ii) $\overline{A}_1 \not\cong \overline{A}_2$ and $A_1J = AJ$.

   iii) There exists a unique characteristic and maximal submodule $C$ in $A$, and for any maximal submodule $B_i \supset C$, there exists a unit element $x_i$ in $eRe$ such that $x_iB_i = B_i$, and further $B_iJ = AJ$ and $\Delta(B_i) = \Delta$.

   iv) There are no characteristic and maximal submodules in $A$, and for any maximal submodules $B_i$ in $A$, there exists a unit element $x_i$ in $eRe$ such that $x_iB_i = B_i$, and further $B_iJ = AJ$ and $[\Delta(\Delta(B_i)) = 2$.

   In case of iii) and iv) $\overline{A}_1 \cong \overline{A}_2$.

3) The lattice of the submodules of $eR$ is obtained by connecting Diagrams I)~IX).

**Proof.** We note that $A$ contains exactly two maximal submodules if and only if $A/AJ = \overline{A}_1 \oplus \overline{A}_2$ and $\overline{A}_1 \not\cong \overline{A}_2$. Hence 1$\iff$3) and 3$\implies$2) are clear from the argument before Theorem 2. 2$\implies$3). We can easily see by induction on $eJ^i$ that $D$ in Diagram ii') is hollow and $(D+eJ)/D$ is simple. Hence we obtain this implication from the same argument.

Finally we consider a case of $\Delta = \Delta(A)$ for any submodule $A$ of $eR$. If $eR$ is uniserial, every submodule of $eR$ is characteristic. From this point of view, we consider
Condition II’. Every hollow module is quasi-projective [3].

If \( R \) is a US-3 ring and Condition II’ fulfils, every submodule \( A \) of \( eR \) contains at most two maximal submodules \( A_1 \) and \( A_2 \) by Lemma 1. Put \( B = A_1 \cap A_2 \). If \( A_i/B \) is isomorphic to \( A_i/B \) via \( f, A_3 = \{ a_1 + a_2; f(a_1 + B) = a_2 + B \} \) is a submodule of \( A \) and \( A_i/B \approx A_i/B \). It is clear that \( A_1 \supset A_3 \) and \( A_2 \supset A_3 \) which contradicts the assumption. Hence \( A/A_1 \approx A/A_2 \) provided \( A_1 \neq A_2 \). Thus we obtain the following diagrams:

\[
\begin{array}{c}
\text{a)} \quad A \\
\text{b)} \quad A_1 \quad A_2 \\
\end{array}
\]

where the \( A/A_i \) are simple and \( A/A_1 \not\approx A/A_2 \), and \( \{ A_i \} \) is the set of maximal submodules of \( A \). Conversely, assume that \( A \) is characteristic in the diagrams a) and b). Then \( A_1 = J(A) \) for a). Hence \( A_1 \) is also characteristic. Consider the diagram b). Let \( x \) be any element in \( eRe \). Since \( \{ A_i \} \) is the set of maximal submodules of \( A \), \( xA_1 \subset A_1 \) or \( xA_1 \subset A_2 \). Assume \( xA_1 \subset A_1 \). Since \( (e+x)A_1 \subset A_1 \) and \( (e+x)A_1 \subset A_2 \), \( xA_1 \subset A_1 \) implies \( (e+x)A_1 \subset A_2 \). On the other hand, \( xA_2 \subset A_2 \) for \( xA_1 \subset A_1 \). Hence \( A_1 \subset A_2 \), which is a contradiction. Therefore the \( A_i \) are also characteristic.

**Theorem 3.** Let \( R \) be as in Theorem 2. Then the following conditions are equivalent:

1) \( R \) is a right US-3 ring and Condition II’ holds.

2) For each primitive idempotent \( e \), any submodule \( A \) of \( eR \) contains at most two maximal submodules \( A_1 \) and \( A_2 \), and either \( A_1 \) or \( A_2 \) is hollow, provided \( A_1 \neq 0 \) and \( A_2 \neq 0 \).

3) The lattice of the submodules of \( eR \) is of Diagram XIV) below, where each parallelogram is of Diagram b).
Proof. The first two conditions are equivalent from the argument before Theorem 3.

2) $\Rightarrow$ 3) is trivial from Diagram XIV).

3) $\Rightarrow$ 2). It is clear from the same argument that we obtain Diagram a) or b).

We can show by induction on $eJ^i$ that there exists one of the following situations:

Hence we obtain Diagram XIV).

3. Examples. Let $R$ be a ring with $J^2 = 0$. We shall give the complete list of the lattice of submodules of $eR$, when $R$ is US-3.
(It may happen that some modules do not appear)
GENERALIZATIONS OF NAKAYAMA RING I

We shall construct a ring for each case. Let \( L \supset K \) be fields with \([L: K] = 2\).

\( a_1 \)
\[
R = \begin{pmatrix}
K & K & K & K \\
0 & K & K & K \\
0 & 0 & K & 0 \\
0 & 0 & 0 & K
\end{pmatrix}
\]

\( a_2 \)
\[
R = \begin{pmatrix}
L & L & L \\
0 & L & L \\
0 & 0 & L \\
0 & 0 & 0 & L
\end{pmatrix}
\]

\( b_1 \)
\[
R = \begin{pmatrix}
L & L & L & L \\
0 & K & L & L \\
0 & 0 & L & 0 \\
0 & 0 & 0 & L
\end{pmatrix}
\]

\( b_2 \)
\[
R = \begin{pmatrix}
L & L & L \\
0 & K & L \\
0 & 0 & K \\
0 & 0 & 0 & K
\end{pmatrix}
\]

\( c_1 \) ([3], Example 2). Let \( R \) be a vector space over \( K \) with basis \( \{e_1, x_{11}, y_{12}, x_{12}, e_2, x_{22}, y_{21}, x_{21}\} \). Define \( e_i e_j = e_i \delta_{ij}, e_i x_j e_j = x_j \delta_{ij}, e_i y_j e_j = y_j \delta_{ij}, x_{11} x_{12} = x_{12}, x_{12}, x_{21} = y_{21} \). Putting other multiplications to be zero, we see that \( R \) is a ring with \( J^2 = 0 \). Put \( e = e_1, A_1 = x_{11}K + y_{12}K, A_2 = y_{12}K \) and \( B_1 = x_{12}K \). Then

\[
\begin{array}{c}
eR \\ (e+x_{11}k)B_1 \\
A_1 \\
\sim \\
A_1 \oplus B_1 \\
eJ= A_1 \oplus B_1 \\
eJ^2 = A_2 \\
B_1 \\
\end{array}
\]

where \( k \) are in \( K \).

\( c_2 \)
\[
R = \begin{pmatrix}
L & L & L \\
0 & L & 0 \\
0 & 0 & L \\
0 & 0 & 0 & K
\end{pmatrix}
\]

\( c_3 \)
\[
R_1 = \begin{pmatrix}
K & K & K & K \\
0 & K & 0 & K \\
0 & 0 & K & K \\
0 & 0 & 0 & K
\end{pmatrix}
\] (this is a case \( \alpha = 1 \))

However if
\[
R_2 = \begin{pmatrix}
L & L & L & L \\
0 & K & 0 & 0 \\
0 & 0 & L & L \\
0 & 0 & 0 & L \\
0 & 0 & 0 & 0 & L
\end{pmatrix},
\]
then the lattice of submodules of $e_1 R_2$ has the same form as $e_3$, but $[\Delta: \Delta(B_i)] = 2$. Hence $R_4$ is not US-3.

Let $R_3$ be a vector space over $K$ with basis \{e, f, a, b, x, y, u\}. Define the multiplication of elements in the basis as follows: $e^2 = e$, $f^2 = f$, $eae = a$, $ebf = b$, $eef = x$, $eye = y$, $fue = u$, $ab = x$, $bu = y$ and $aa = y$. Other multiplications are zero. Then $R_3$ is a ring and the lattice of the submodules of $eR_3$ is

\[
\begin{align*}
\text{eR}_3 & \\
\langle a, f = \langle a, b, x, y \rangle & \\
\langle a, x, y \rangle & \sim \langle -b, y \rangle \oplus \langle x \rangle & \\
\langle e + ka \rangle \langle b, y \rangle & \sim \langle x, y \rangle = ej^2 & \langle b, y \rangle \\
\langle x \rangle & \sim \langle y \rangle & \\
0 & \\
\end{align*}
\]

(this is a case $\alpha > 1$)

If $B_1/C_1 \not\cong C_2$ in $e_3$, then $B_2 = B_3 = \cdots = 0$. Because, $B_2 = xB_1$ for some $x$ in $eRe$. Since $\Delta(B_1) = \Delta$, $x = x_1 + j$, where $x_1B_1 = B_1$ and $j \in eJ e$. Then $B_2 \subseteq B_1 + jB_1$. If $jB_1 \subseteq B_1, B_2 = 0$. $jB_1 \subseteq ejf \subseteq ej^2$ and $|B_1| > |jB_1|$. Hence $jB_1 = C_2$ provided $jB_1 \subseteq B_1$, which implies that $B_1/C_1 \cong C_2$. Therefore, if $B_1/C_1 \not\cong C_2$, $jB_1 \subseteq B_1$, so $B_2 = 0$.

References