Shibata, K.
Osaka J. Math.
10 (1973), 33-42

A NOTE ON THE FORMAL GROUP LAW OF UNORIENTED COBORDISM THEORY

KATSUYUKI SHIBATA

(Received May, 9, 1972)

Introduction

This is a continuation of the author's previous work [6] on the cobordism generators defined by J.M. Boardman in [1]. Previously we have used the Landweber-Novikov operations to calculate the coefficients z_{2i} and z_{4i+1} of a primitive element

$$P=W_1+z_2W_1^3+z_4W_1^5+z_5W_1^6+z_6W_1^7+z_7W_1^8+\cdots$$

in $\mathfrak{N}^*(BO(1))$.

This time we use the Steenrod-tom Dieck operations in the unoriented cobordism theory ([2], [8]) to deduce that the coefficient z_{i-1} for the "canonical primitive element" P_0 is represented by the "iterated Dold manifold" $(R_1)^a(P_{2b})$ for $i=2^a(2b+1)$, where $R_1(M)=S^1\times(M\times M)/a\times T$ (Theorem 3.2).

In other words, let $L=Z_2[e_{i-1}\colon i\neq 2^k]$ be the Lazard ring of characteristic 2 and $F(x, y)=g^{-1}(g(x)+g(y))$ with $g(x)=\sum_{i\geqslant 1}e_{i-1}x^i(e_0=1,\ e_{2^k-1}=0)$ be the universal formal group law. Then the canonical ring isomorphism of Quillen [5] $\varphi\colon L\to\mathfrak{N}^*$ sends the generator e_{i-1} to $[(R_1)^a(P_{2b})]$ for $i=2^a(2b+1)$.

We also study the behaviour of the Dold-tom Dieck homomorphism $R_j\colon \mathfrak{N}_*\to\mathfrak{N}_{2*+j}$ defined by $R_j([M])=[S^j\times(M\times M)/a\times T]$. In particular, we present the following product formula (Lemma 2.2);

$$R_j(xy)=\sum_{j\geqslant k+m\geqslant 0}(\sum\prod_{i\geqslant 0}[P_{2n_i}]^{2^i})R_k(x)R_m(y)\,.$$

In the final section, we examine the relation between the algebra structure of $\mathfrak{N}_*(BO(1))\cong\mathfrak{N}_*(Z_2)$ and the coalgebra structure of $\mathfrak{N}^*(BO(1))$. As an application, we obtain the following formulas for the Smith homomorphism Δ ([3]);

$$\Delta([S^m, a]\cdot[S^n, a])=\sum_{i,j\geqslant 0}a_{i,j}\Delta^i[S^m, a]\Delta^j[S^n, a]$$
$$=(\Delta[S^m, a])\,[S^n, a]+[S^m, a]\,(\Delta[S^n, a])+[P_2](\Delta[S^m, a]\Delta^2[S^n, a]$$
$$+\Delta^2[S^m, a]\Delta[S^n, a])+\cdots,\text{ and}$$
$$\Delta^{2k}([S^m, a]\cdot x)=[S^m, a]\cdot\Delta^{2k}(x)$$

for $2^k > m \geqslant 0$ (Corollary 4.3). The former equation would be an answer to a question of J.C. Su [7] on the relation between Δ and the multiplication in $\mathfrak{N}_*(Z_2)$. The latter formula for $k \leqslant 3$ was first proved by Uchida [9] utilizing the multiplicative structures of S^1, S^3 and S^7.

In the appendix, we state brief comments on the unrestricted bordism ring of involution $I_*(Z_2)$ ([3], IV 28). We define the "switching involution" homomorphism $S\colon \mathfrak{N}_* \to I_{2*}(Z_2)$, which is a ring monomorphism with a left inverse. We see, by definition, that $R_j = K_j \circ S$ with K_j the $\mathfrak{N}_*$-homomorphism studied by Conner-Floyd in [4], and thus give a proof for the well-definedness of the Dold-tom Dieck homomorphism R_j.

The author would like to acknowledge stimulating conversation with Professor Tammo tom Dieck. The author also wishes to express his gratitude to Professor Larry Siebenmann for cordial and constant encouragement.

1. Formal group law in the unoriented cobordism theory

As in [6], let

$$\mu^*\colon \mathfrak{N}^*(BO(1)) \to \mathfrak{N}^*(BO(1)) \underset{\mathfrak{N}^*}{\otimes} \mathfrak{N}^*(BO(1))$$

be the comultiplication defined by the H-space map.

The cobordism first Stiefel-Whitney class W_1 is mapped by μ^* to a formal power series

$$\mu^*(W_1) = W_1 \otimes 1 + 1 \otimes W_1 + \sum_{i,j \geqslant 1} a_{i,j} (W_1)^i \otimes (W_1)^j \tag{1.1}$$

$$(a_{i,j} = a_{j,i} \in \mathfrak{N}_{i+j-1}).$$

The formal power series defined by these coefficients

$$F(x, y) = x + y + \sum_{i,j \geqslant 1} a_{i,j} x^i y^j \tag{1.2}$$

is a commutative formal group law [5]; it satisfies the following properties

(1.3) (1) $F(x, 0) = 0$,

(2) $F(F(x, y), z) = F(x, F(y, z))$,

(3) $F(x, y) = F(y, x)$.

The following lemma explains the relation of primitive elements in $\mathfrak{N}^1(BO(1))$ to the formal group law $F(x, y)$ of (1.2).

Lemma 1.4. *An element $g(W_1) = W_1 + \sum_{i \geqslant 2} z_{i-1} W_1^i$ of $\mathfrak{N}^1(BO(1))$ is primitive if and only if $F(x, y) = g^{-1}(g(x) + g(y))$, where $g^{-1}(x)$ is the inverse of $g(x)$; $g(g^{-1}(x)) = g^{-1}(g(x)) = x$.*

Proof. If $g(W_1)$ is primitive, then

$$F(W_1\otimes 1,\ 1\otimes W_1)=\mu^*{}_1W_1=\mu^*g^{-1}(g(W_1))=g^{-1}(\mu^*g(W_1))$$
$$=g^{-1}(g(W_1)\otimes 1+1\otimes g(W_1))=g^{-1}(g(W_1\otimes 1)+g(1\otimes W_1))\,.$$

Conversely, if $F(x, y)=g^{-1}(g(x)+g(y))$, then

$$\mu^*g(W_1)=g(\mu^*W_1)=g(F(W_1\otimes 1,\ 1\otimes W_1))=g(W_1\otimes 1)+g(1\otimes W_1)$$
$$=g(W_1)\otimes 1+1\otimes g(W_1)\,.$$

Lemma 1.5. *Concerning the coefficients of the formal group law* (1.2), *we have the following formulas for every integer* $k\geqslant 1$.

(1) $a_{1,\,2k-1}=0$.

(2) $\sum_{k\geqslant j\geqslant 0} a_{1,2j}[P_{2(k-j)}]=0$.

(3) $\sum_{\substack{k\geqslant j\geqslant 0\\ k-j:\,\mathrm{even}}} a_{1,2j}[P_{k-j}]^2=[P_{2k}]$.

In the above formulas, P_i denotes the real projective space of dimension i.

Proof. Putting $m=1$ in (3.4) of [2] (p. 190), we obtain

$$[H(1, n)]=\sum_{i,j\geqslant 0} a_{i,j}[P_{1-i}]\,[P_{n-j}]\,,$$

where $H(1, n)$ is Milnor's hypersurface in $P_1\times P_n$. But $[P_{2i-1}]=0$ and $[H(1, n)]=0$ for every $n\geqslant 1$ ([1]). So $\sum_{\substack{j\geqslant 0\\ n-j:\,\mathrm{even}}} a_{1,j}[P_{n-j}]=0$. Letting $n=2k-1$, we have $\sum_{k\geqslant j\geqslant 1} a_{1,2j-1}\,[P_{2(k-j)}]=0$ and part (1) follows by induction on k. Analogously letting $n=2k\geqslant 2$, part (2) follows. Now, from part (2).

$$a_{1,2k}+[P_{2k}]=\sum_{k>j>0} a_{1,2j}[P_{2(k-j)}]\,.$$

So

$$[P_{2k}]=a_{1,2k}+\sum_{k>j>0}\Big(\sum_{j>m\geqslant 0} a_{1,2m}[P_{2(j-m)}]\Big)\,[P_{2(k-j)}]$$
$$=a_{1,2k}+\sum_{k-2\geqslant j\geqslant 0} a_{1,2j}\,\Big(\sum_{k-j-1\geqslant i\geqslant 1}[P_{2i}]\,[P_{2(k-i-j)}]\Big)\,.$$

This yields part (3) since $\mathfrak{N}^*$ is a Z_2-vector space.

2. Steenrod-tom Dieck operations

T. tom Dieck has defined in [8] the stable cohomology operations

$$R^i\colon \mathfrak{N}^*(X)\to\mathfrak{N}^{*+i}(X)\qquad(-\infty<i<\infty)$$

such that

(2.1) (a) For $x\in\mathfrak{N}^i(X)$, $R^i(x)=x^2$.

(b) For $x\in\mathfrak{N}^i(X)$ and $j>i$, $R^j(x)=0$.

(c) $R^k(xy)=\sum_{i=-\infty}^{\infty} R^i(x)R^{k-i}(y)$.

(d) For the natural transformation $\mu=\mathfrak{N}^*(\ )\to H^*(\ ;Z_2)$, it holds that $\mu\circ R^k=S_q{}^k\circ\mu$, ($S_q{}^k$: Steenrod operation, $S_q{}^k=0$ for $K<0$).

On the other hand, tom Dieck has also defined in [8] the following mapping $R_j\colon \mathfrak{N}_*\to\mathfrak{N}_{2*+j}$ for $j\geqslant 0$; for a closed differentiable manifold M, let $R_j(M)$ be the orbit space of the free involution $(S^j\times(M\times M),\ a\times T)$, where a is the antipodal involution and T is the switching map. It was proved in [8] that, if M is bordant to M', then $R_j(M)$ is bordant to $R_j(M')$ and that this construction yields consequently a mapping of the bordism set $R_j\colon \mathfrak{N}_*\to\mathfrak{N}_{2*+j}$.

The mappings R_j are expressed by the operations on $\mathfrak{N}^*(pt)$

$$R^i\colon \mathfrak{N}^*(pt)\to\mathfrak{N}^{*+i}(pt)$$

and vice versa as in the following lemma. (Recall that we are always identifying $\mathfrak{N}_i$ with $\mathfrak{N}^{-i}$ via the Atiyah-Poincaré duality.)

Lemma 2.2.

(1) *For* $x\in\mathfrak{N}_m$, $R_i(x)=\sum_{i\geqslant j\geqslant 0}[P_{i-j}]R^{-m-j}(x)$, *and consequently,*

(2) $R_j(x+y)=R_j(x)+R_j(y)$.

(3) $R_j(xy)=\sum_{j\geqslant k+m\geqslant 0}(\sum\prod_{i\geqslant 0}[P_{2n_i}]^{2^i})R_k(x)R_m(y)$,

where the latter summation runs through all the sequences of non-negative integers $(n_0, n_1, \cdots, n_i, \cdots)$ *such that* $\sum_{i\geqslant 0}2^{i+1}n_i=j-(k+m)$.

Proof. Part (1) follows easily from (14.1) of [8]. Since the R^i are stable cohomology operations, they are additive and so part (2) follows from (1). For $x\in\mathfrak{N}_m$ and $y\in\mathfrak{N}_n$,

$$R_j(xy)=\sum_{j\geqslant 2i\geqslant 0}[P_{2i}]\Big(\sum_{k+q=j-2i}R^{-m-k}(x)R^{-n-q}(y)\Big) \tag{2.3}$$

by part (1) and 2.1 (c). On the other hand,

$$\begin{aligned}&\sum_{j\geqslant 2i\geqslant 0}[P_{2i}]\Big(\sum_{k+q=j-2i}R_k(x)R_q(y)\Big)\\ &=\sum_{a,b}[P_a][P_b]\Big(\sum_{j\geqslant 2i\geqslant 0}[P_{2i}]\Big(\sum_{s+t=(j-a-b)-2i}R^{-m-s}(x)R^{-n-t}(y)\Big)\Big) \qquad \text{by (1)}\\ &=\sum_{a,b}[P_{2a}][P_{2b}]R_{j-2(a+b)}(xy) \qquad \text{by (2.3)}\\ &=\sum_{a\geqslant 0}[P_{2a}]^2R_{j-4a}(xy)\,.\end{aligned}$$

So $R_j(xy)=\sum_{j\geqslant 2i\geqslant 0}[P_{2i}]\Big(\sum_{k+m=j-2i}R_k(x)R_m(y)\Big)+\sum_{a\geqslant 1}[P_{2a}]^2R_{j-4a}(xy)$.

Substituting repeatedly the latter part of the right hand side, we obtain part (3).

REMARK 2.4. In 2.4 below, we give a complete description of the mapping R_1 with respect to the "canonical ring generators" of $\mathfrak{N}^*$. This would be a partial answer to a question of tom Dieck ([8]) on the behaviour of the mappings

R_j.

Corollary 2.5. *Let* $1=[pt]\in\mathfrak{N}^*$ *be the unit element. Then, for* $-\infty<j<\infty$,

$$R^j(1)=\begin{cases}1 & (j=0)\\ 0 & (j\neq 0).\end{cases}$$

Proof. For $j\geqslant 0$, the assertion is clear by 2.1 (a) and (b). So let $j=-i$ ($i<0$). Then $R_i(1)=[P_i]$ by definition. On the other hand, by 2.2 (1),

$$R_i(1)=[P_i]+R^{-i}(1)-\sum_{i>j>0}[P_{i-j}]R^{-j}(1).$$

So $R^{-i}(1)=\sum_{i>j>0}[P_{i-j}]R^{-j}(1)$ and the assertion follows by induction on i. (Of course, this result can also be obtained directly from the definition of the operation R^j.)

Corollary 2.6. *Let* $P=W_1+\sum_{i\geqslant 2}z_{i-1}W_1{}^i$ *be a primitive element in* $\mathfrak{N}^1(BO(1))$. *Then, for every integer* $j(-\infty<j\leqslant 1)$, $R^j(P)$ *is also primitive.*

Proof. $\mu^*R^j(P)=R^j(\mu^*(P))=R^j(P\times 1+1\times P)$

$$=\sum_{i=-\infty}^{\infty}(R^i(P)\times R^{j-i}(1)+R^i(1)\times R^{j-i}(P)) \qquad \text{by 2.1 (c)}$$

$$=R^j(P)\times 1+1\times R^j(P) \qquad \text{by 2.5.}$$

Lemma 2.7. *Let* $X=\sum_{i\geqslant 1}x_{i-1}W_1{}^i$ *be an arbitrary element of* $\mathfrak{N}^1(BO(1))$. *Then*

$$R^0(X)=\sum_{i\geqslant 1}\Big(\sum_{2k+j=1}(x_{2k})^2a_{1,2j}\Big)W_1{}^{2i+1}+\sum_{i\geqslant 0}R_1(x_{i-1})W_1{}^{2i},$$

where the $a_{1,2j}$ *are the coefficients in* (1.1).

Proof. $R^0(\sum_{i\geqslant 1}x_{i-1}W_1{}^i)=\sum_{i\geqslant 1}R^0(x_{i-1}W_1{}^i)$

$$=\sum_{i\geqslant 1}\sum_{j=-\infty}^{\infty}R^{-j}(x_{i-1})R^j(W_1{}^i) \qquad \text{by 2.1 (c)}$$

$$=\sum_{i\geqslant 1}\sum_{i\geqslant j\geqslant i-1}R^{-j}(x_{i-1})R^j(W_1{}^i) \qquad \text{by 2.1 (b)}$$

$$=\sum_{i\geqslant 1}\{(x_{i-1})^2R^{i-1}(W_1{}^i)+R^{-i}(x_{i-1})W_1{}^{2i}\} \qquad \text{by 2.1 (a)}$$

$$=\sum_{i\geqslant 1}(x_{i-1})^2(i)W_1{}^{2(i-1)}R^0(W_1)+\sum_{i\geqslant 1}R^{-i}(x_{i-1})W_1{}^{2i}$$

$$=\sum_{i\geqslant 1}(x_{2i})^2W_1{}^{4i}R^0(W_1)+\sum_{i\geqslant 1}R^{-i}(x_{i-1})W_1{}^{2i}.$$

It was observed in [2] (p. 141) that $R^0(W_1)=W_1+\sum_{j\geqslant 1}a_{1,j}W_1{}^{j+1}(=\sum_{j\geqslant 0}a_{1,2j}W_1{}^{2j+1}$ by 1.5(1)), and $R^{-i}(x_{i-1})=R_1(x_{i-1})$ by 2.2 (1). Therefore the lemma follows.

3. Determination of Boardman's generators

Let $P_0 = W_1 + \sum_{i\geqslant 2} z_{i-1} W_1{}^i \in \mathfrak{N}^1(BO(1))$ be the (unique) primitive element such that $z_{2^k-1}=0$ $(k\geqslant 1)$ (see [1], [6] Introduction).
Then we have

Lemma 3.1 $R^0(P_0)=P_0$.

Proof. By 2.6, $R^0(P_0)$ is primitive, and by 2.7, $R^0(P_0)$ is of the form $W_1 + \sum_{i\geqslant 2} x_{i-1} W_1{}^i$, with $x_{2^k-1} = R_1(x_{2^{k-1}-1})=0$ $(k\geqslant 1)$. So, by the uniqueness of such a primitive element ([1]), the lemma follows.

Theorem 3.2 *The coefficient z_{i-1} of the canonical primitive element $P_0 = W_1 + \sum_{i\geqslant 2} z_{i-1} W_1{}^i \in \mathfrak{N}^1(BO(1))$ with $z_{2^k-1}=0$ $(k\geqslant 1)$ is the cobordism class of the "iterated Dold manifold" $(R_1)^a(P_{2b}) = R_1(\cdots(R_1(P_{2b}))\cdots)$ for $i=2^a(2b+1)$ $(a\geqslant 0$, $b\geqslant 1)$.*

Proof. We prove by induction on $a\geqslant 0$, using 3.1 and 2.7.
(1) In case $a=0$. By 3.1 and 2.7, we have

$$z_{2b} = \sum_{2k+j=b} (z_{2k})^2 a_{1,2j}\,.$$

So $z_2 = (z_0)^2\, a_{1,2} = [P_2]$ by 1.5 (2), and inductively on b we can deduce, by 1.5 (3), that $z_{2b} = \sum_{2k+j=b} [P_{2k}]^2\, a_{1,2j} = [P_{2b}]$.
(2) If we suppose that the theorem holds for $a-1\geqslant 0$, then for $i=2^a(2b+1)$, 3.1 and 2.7 imply that $z_{i-1} = R_1(z_{j-1})$ with $j=2^{a-1}(2b+1)$. So, by induction hypothesis, $z_{i-1} = R_1([(R_1)^{a-1}(P_{2b})]) = [(R_1)^a(P_{2b})]$ as desired.

Corollary 3.4.
(1) *The cobordism class $[(R_1)^a(P_{2b})]$ can be taken as a ring generator of $\mathfrak{N}_*$ in dimension $2^a(2b+1)-1$.*
(2) *Denoting $[(R_1)^a(P_{2b})]$ by $X(a, b)$, an additive basis for $\mathfrak{N}_*$ is given by $\{\prod_{a\geqslant 0, b\geqslant 1} X(a, b)^{2\lambda(a,b)+\varepsilon(a,b)};\ \lambda(a, b)\geqslant 0,\ 1\geqslant \varepsilon(a, b)\geqslant 0,\ \lambda(a, b)=\varepsilon(a, b)=0$ except for a finite number of pairs $(a, b)\}$.*
(3) *With respect to this basis, the additive homomorphism*

$$R_1\colon \mathfrak{N}_* \to \mathfrak{N}_{2*+1}$$

is determined by the following formula;

$$\begin{aligned} &R_1\Big(\prod_{a\geqslant 0, b\geqslant 1} X(a, b)^{2\lambda(a,b)+\varepsilon(a,b)}\Big) \\ &= \sum_{a,b} \Big\{\varepsilon(a, b)\Big(\prod_{(c,d)\neq(a,b),(a+1,b)} X(c, d)^{4\lambda(c,d)+2\varepsilon(c,d)}\Big) \\ &\qquad \cdot X(a, b)^{4\lambda(a,b)} X(a+1, b)^{4\lambda(a+1,b)+2\varepsilon(a+1,b)+1}\Big\}\,. \end{aligned}$$

Proof. Part (1) and (2) are the consequence of the fact that the coefficients z_{i-1} of the primitive element P_0 are indecomposable in $\mathfrak{N}_*$ ([1]). Part (3) follows from 2.2 (3) and the definition of the $X(a, b)$.

4. Bordism algebra of free involutions

In this section we consider the relation between the algebra $\mathfrak{N}_*(BO(1))$ with the multiplication

$$\mu_*: \mathfrak{N}_*(BO(1))\underset{\mathfrak{N}_*}{\otimes}\mathfrak{N}_*(BO(1)) \to \mathfrak{N}_*(BO(1))$$

and the coalgebra $\mathfrak{N}^*(BO(1))$ with the comultiplication

$$\mu^*: \mathfrak{N}^*(BO(1)) \to \mathfrak{N}^*(BO(1))\underset{\mathfrak{N}^*}{\otimes}\mathfrak{N}^*(BO(1)$$

via the cap product ([2] p. 186)

$$\cap : \tilde{\mathfrak{N}}^i(X)\otimes\tilde{\mathfrak{N}}_n(X) \to \tilde{\mathfrak{N}}_{n-i}(X)\,.$$

Let η_n: $P_n\to BO(1)$ be a classifying map of the canonical line bundle over P_n, and denote by $\{n\}$ the singular bordism class $[P_n, \eta_n]\in\mathfrak{N}_n(BO(1))$. It is well-known that $\mathfrak{N}_*(BO(1))$ is a free $\mathfrak{N}_*$-module with basis $\{\{0\}, \{1\},\cdots, \{n\}, \cdots\}$.

Let $\alpha_k(m, n)\in\mathfrak{N}_{m+n-k}$ be the element such that

$$\{m\}\cdot\{n\}=\sum_k\alpha_k(m, n)\{k\}\,.$$

It is equivalent to define $[S^m, a]\cdot[S^n, a]=\sum\alpha_k(m, n)[S^k, a]$ in $\mathfrak{N}_*(Z_2)$ ([9]).

Theorem 4.1.

(1) $\alpha_{k+1}(m, n)=\sum_{i,j\geqslant 0} a_{i,j}\alpha_k(m-i, n-j)$, *where the* $a_{i,j}$ *are the coefficients in* (1.1).

(2) $\sum_{i\geqslant 0} z_{i-1}\alpha_{k+i}(m, n)=\sum_{i\geqslant 0} z_{i-1}(\alpha_k(m-i, n)+\alpha_k(m, n-i))$, *where the* z_{i-1} *are the coefficients of a primitive element* P (Uchida [9])).

(3) $\sum_{i\geqslant 0}\alpha_{2i+1}(m, n)[P_{2i}]=[H(m, n)]$, *where* $H(m, n)$ *denotes Milnor's hypersurface in* $P_m\times P_n$.

Proof. The proof of [2] XIII (3.3) shows that

$$\begin{aligned}W_1\cap\mu_*(\{m\}\otimes\{n\})&=W_1\cap\Big(\sum_{k+1}\alpha_{k+1}(m, n)\{k+1\}\Big)=\sum_k\alpha_{k+1}(m, n)\{k\}\\&=\mu_*(\mu^*W_1\cap\{m\}\otimes\{n\})=\mu_*\Big(\sum_{i,j\geqslant 0}a_{i,j}W_1{}^i\otimes W_1{}^j\Big)\cap\{m\}\otimes\{n\}\Big)\\&=\sum_{i,j\geqslant 0}a_{i,j}\mu_*(\{m-i\}\otimes\{n-j\})=\sum_{i,j\geqslant 0}a_{i,j}\Big(\sum_k\alpha_k(m-i, n-j)\{k\}\Big)\\&=\sum_k\Big(\sum_{i,j\geqslant 0}a_{i,j}\alpha_k(m-i, n-j)\Big)\{k\}\,.\end{aligned}$$

Comparing the coefficient of $\{k\}$, part (1) follows. Analogously,

$$P\cap\mu_*(\{m\}\otimes\{n\})=\sum_k(\sum_i z_{i-1}\alpha_{k+i}(m,n))\{k\}$$
$$=\mu_*(\mu^*(P)\cap\{m\}\otimes\{n\})=\mu_*((P\otimes 1+1\otimes P)\cap\{m\}\otimes\{n\})$$
$$=\sum_k(\sum_i z_{i-1}\alpha_k(m-i,n)+\alpha_k(m,n-i))\{k\}\ .$$

Part (3) follows from the proof of [2], XII (3.3).

Corollary 4.2. *In $\mathfrak{N}_*(Z_2)$, the following multiplicative relations hold.*

(1) $[S^1, a][S^{2n}, a]=\sum_{i\geqslant 0}a_{1,2i}[S^{2n-2i+1}, a]$, *where the $a_{1,2j}$ are determined by the formula* 1.5 (2).

$$[S^{2n+1}, a][S^{2n+1}, a]=0\ .$$

(Uchida [9])

(2) $$[S^2, a][S^{2n}, a]=\sum_{i\geqslant 0}a_{2,2i}[S^{2n-2i+1}, a]$$
$$+\sum_{i\geqslant 0}(\alpha_0(2, 2i-2)+\varepsilon(n+1-i)(a_{1,i})^2)[S^{2n+2-2i}, a]\ ,$$
$$[S^2, a][S^{2n-1}, a]=\sum_{i\geqslant 0}\{\alpha_0(2, 2i-2)+\varepsilon(n-i)(a_{1,i})^2+a_{2,2i-1}\}[S^{2n-2i+1}, a]\ ,$$

where $\varepsilon(n-i)=0$ $(n-i$: *even*), $=1$ $(n-i$: *odd*), *and*

$$\alpha_0(2, 2i)=[P_2][P_{2i}]+\varepsilon(i)\sum_{j\geqslant 0}[P_{4j+2}](a_{1,i-2j})^2$$
$$+\sum_{j\geqslant 1}[P_{2j}]\alpha_0(2, 2i-2j)\ \textit{with}\ \alpha_0(2, 0)=0\ .$$

Proof. Letting $m=1$ in 4.1 (1), we have

$$\alpha_{k+1}(1, n)=\alpha_k(1, n-1)+a_{1,n-k}\ .$$

This yields, by induction on k, the former part of (1) and $[S^1, a][S^{2n+1}, a]=0$. Together with 5.1 (2), this in turn gives $[S^1, a](\sum_{j\geqslant 0}[P_{2j}][S^{2n-2j}, a])=\sum_{k\geqslant 0}(\sum_{i+j=k}[P_{2j}]a_{1,2i})[S^{2n-2k+1}, a]=[S^{2n+1}, a]$. So $[S^{2m+1}, a][S^{2n+1}, a]=[S^1, a][S^{2n+1}, a](\sum_{j\geqslant 0}[P_{2j}][S^{2m-2j}, a])=0$.

Analogously, letting $m=2$ in 4.1 (2), part (2) follows.

Corollary 4.3. *Concerning the Smith homomorphism Δ, we have the following formulas.*

(1) $$\Delta([S^m, a]\cdot[S^n, a])=\sum_{i,j\geqslant 0}a_{i,j}\Delta^i[S^m, a]\cdot\Delta^j[S^n, a]$$
$$=(\Delta[S^m, a])[S^n, a]+[S^m, a](\Delta[S^n, a])$$
$$+[P_2](\Delta[S^m, \mathrm{a}]\Delta^2[S^n, a]+\Delta^2[S^m, a]\Delta[S^n, a])+\cdots$$

(2) $\Delta^{2k}([S^m, a]\cdot x)=[S^m, a]\cdot\Delta^{2k}(x)$ for $2^k>m\geqslant 0$.

(3) $\Delta^{2k}([S^{2k}, a]\cdot x)=[S^{2k}, a]\cdot\Delta^{2k}(x)+x+\sum_{j\geqslant 1}(a_{1,2j})^{2k}\Delta^{2^{k+1}j}(x)$.

Proof. Part (1) is a paraphrase of 4.1 (1).

Substituting repeatedly the second factor in the right side of 4.1 (1), we obtain

$$\begin{aligned}\alpha_{s+2^k}(m, n) &= \sum \prod_{1\leqslant q\leqslant 2^k} a_{i_q, j_q})\alpha_s(m-\sum_q i_q,\ n-\sum_q j_q)\\ &= \sum_{i,j\geqslant 0} (a_{i,j})^{2k}\alpha_s(m-2^k i,\ \mathrm{n}-2^k j)\\ &= \alpha_s(m,\ n-2^k)+\sum_{i\geqslant 1, j\geqslant 0}(a_{i,j})2^{2^k}\alpha_s(m-2^k i,\ n-2^k j)\,.\end{aligned}$$

This yields part (2) and (3).

Appendix. Unrestricted bordism algebra of involutions

In this appendix, we consider the unrestricted bordism module of all involutions (admitting fixed point sets). The basic notations are found in Conner-Floyd [3], IV 28.

The unrestricted bordism group of involution $I_*(Z_2)$ has an $\mathfrak{N}_*$-algebra structure via the cartesian product. The direct sum $\sum_m \mathfrak{N}_*(BO(m))$ also admits a multiplicative structure by the formula $[M, \xi]\cdot[N, \eta]=[M\times N, p_1{}^*\xi\oplus p_2{}^*\eta]$.

Lemma 1. *There is the well-defined ring homomorphism*

$$\tau\colon \mathfrak{N}_* \to \sum_m \mathfrak{N}_*(BO(m))$$

defined by $\tau[M]=[M, \tau_M]$, where τ_M denotes the tangent bundle of M.

Proof. Let W be a manifold giving the bordant relation of M to N; $\partial W= M\cup N$. Then $\partial(W, \tau_W)=(M, \tau_M\oplus 1)\cup(N, \tau_N\oplus 1)$. So we have

$$(i_{n,n+1})_*[M, \tau_M]=(i_{n,n+1})_*[N, \tau_N]$$

where $i_{n,n+1}\colon BO(n)\to BO(n+1)$ is the canonical map (up to homotopy). But $(i_{n,n+1})_*\colon \mathfrak{N}_n(BO(n))\to\mathfrak{N}_n(BO(n+1))$ is a monomorphism ([3], 26.3). So $[M, \tau_M] =[N, \tau_N]$. The assertion that τ is a ring homomorphism is clear from the definitions.

Corollary 2. *There is the ring homomorphism*

$$S\colon \mathfrak{N}_* \to I_{2*}(Z_2)$$

defined by $S([M])=[M\times M, T]$, where $T(x, y)=(y, x)$.

Proof. Consider the ring monomorphism $i_*\colon I_*(Z_2)\to\sum_m\mathfrak{N}_*(BO(m))$ of [3] (28.1). By the definition of i_* and the proof of [3] (24.3), $i_*([M\times M, T])=[M, \tau_M]$ and $i_*([N\times N, \tau_N])=[N, \tau_N]$. Therefore by the preceding lemma, $[M\times M, T]=[N\times N, T]$ if $[M]=[N]$. Next we show that S is additive. $S([M]+[N]) =[(M\cup N)\times(M\cup N), T]=S([M])+S([N])+[(M\times N)\cup(N\times M), T]$. Since any free involution bords in $I_*(Z_2)$, $[(M\times N)\cup(N\times M),\ T]=0$ and S is additive. The multiplicativity of S is clear by difinition.

Corollary 3. *$R_j=K_j\circ S$, i.e. the following diagram commutes*

$$\begin{array}{ccc} & I_*(Z_2) & \\ S\nearrow & & \searrow K_j \\ \mathfrak{N}_* & \xrightarrow[R_j]{\qquad} & \mathfrak{N}_*\,, \end{array}$$

where R_j is the Dold-tom Dieck homomorphism of (2.2) *and K_j is the $\mathfrak{N}_*$-homomorphism defined by $K_j([M,\ \mu])=[S^j\times M/a\times\mu]$* (Conner-Floyd [4]).

The proof is obvious from the definitions.

Corollary 4. *As a ring, $I_*(Z_2)$ contains the polynomial subalgebra $Z_2[S(z_{i-1})$: $i-1\neq 2^k-1]$ as a direct summand.*

Proof. Let ε: $\sum_m\mathfrak{N}_*(BO(m))\to\mathfrak{N}_*$ be the augmentation homomorphism induced by the constant map. Then $\varepsilon\circ i_*\circ S=\mathrm{id}$: $\mathfrak{N}_*\to\mathfrak{N}_*$ and ε, i_* and S are all ring homomorphisms. So the corollary follows.

OSAKA UNIVERSITY AND UNIVERSITÉ DE PARIS (ORSAY)

References

[1] J.M. Boardman: Stable Homotopy Theory, Ch. IV, (mimeographed) Warwick Univ., 1967.

[2] T. Bröcker-T. tom Dieck: Kobordismentheorie, Springer-Verlag, 1970.

[3] P.E. Conner-E.E. Floyd: Differentiable Periodic Maps, Springer-Verlag, 1964.

[4] ――――: *Fibering within a cobordism class*, Michigan Math. J. **12** (1965), 33–47.

[5] D. Quillen: *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[6] K. Shibata: *On Boardman's generating sets of the unoriented bordism theory*, Osaka J. Math. **8** (1971), 219–232.

[7] J.C. Su: *A note on the bordism algebra of involutions*, Michigan Math. J. **12** (1965), 25–31.

[8] T. tom Dieck: *Steenrod Operationen in Kobordismen-Theorie*, Math. Z, **107** (1968), 380–401.

[9] F. Uchida: *Bordism algebra of involutions*, Proc. Japan Acad. **46** (1970), 615–619.