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QUOTIENTS OF REAL ALGEBRAIC *G* VARIETIES AND ALGEBRAIC REALIZATION PROBLEMS

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1. Introduction

Let G be a compact Lie group. A *real algebraic* G variety in an orthogonal representation Ω is the common zeros of polynomials $p_1,\cdots,p_m:\Omega\to\boldsymbol{R}$, which is invariant under the action of G on Ω. In this case we also say that G acts algebraically on V. There is a more obvious definition of algebraic actions of algebraic groups on algebraic varieties via algebraic automorphisms. Namely, since any compact Lie group has a unique real algebraic variety structure we can define an algebraic action of G on a real algebraic variety V as a G action whose action map $\theta: G\times V\to V$ is a regular map between real algebraic varieties. Remember that a map $f: V\subset\boldsymbol{R}^n\to W\subset\boldsymbol{R}^m$ between two real algebraic varieties is regular if f can be extended to a polynomial map $F:\boldsymbol{R}^n\to\boldsymbol{R}^m$. The above two definitions of real algebraic G variety are equivalent, see [3] or [8].

In smooth transformation group theory it is well known that the orbit space of a smooth manifold with a free action of a compact Lie group is a smooth manifold. Ozan proves an algebraic analogue of this for odd order group actions, [11]. In fact Ozan proves, in particular, that if an odd order group acts algebraically and freely on a non-singular irreducible real algebraic variety, then its orbit space is also a non-singular irreducible real algebraic variety. Before Ozan, Procesi and Schwarz [12] prove that the orbit space of real representation space of an odd order group has a real algebraic variety structure. In section 2 of this paper we extend Ozan's result to get the following theorem.

Theorem A. *Let G be a compact Lie group, and let H be an odd order group. Let X be a real algebraic $G\times H$ variety. Then the orbit space X/H has a real algebraic G variety structure. Moreover if the action of H is free and X is non-singular, then X/H is also non-singular.*

Theorem A is applied to algebraic realizations of closed smooth G manifolds

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with one orbit type and smooth G vector bundles over them. A smooth closed G manifold M is said to be *algebraically realized* if it is G diffeomorphic to a non-singular real algebraic G variety V. A set $\mathscr{F}$ of G vector bundles ξ over M is said to be *algebraically realized* if M is algebraically realized by V with a G diffeomorphism $\phi: M \to V$ and there exists a set $\mathscr{F}'$ of strongly algebraic G vector bundles over V such that each $\xi \in \mathscr{F}$ is G isomorphic to $\phi^*\eta$ for some $\eta \in \mathscr{F}'$. A strong algebraic G vector bundle is defined in Section 3. The questions we are interested in here are whether a given closed smooth G manifold is algebraically realized, and if the case is true, whether any set of G vector bundles over a closed smooth G manifold can be algebraically realized. The first question is called the manifold realization problem, and the second one is called the bundle realization problem. In section 3 we use Theorem A to solve the manifold realization problem for closed G manifolds with one orbit type:

Theorem B. *Let G be a compact Lie group acting smoothly on a closed manifold M with one orbit type. Then M can be algebraically realized.*

In section 4, using similar technique, we can partially solve the bundle realization problem over closed G manifolds with one orbit type:

Theorem C. *Let G be a compact Lie group acting smoothly on a closed manifold M with one orbit type. Let G/H be the unique orbit type, and let $K := N/H$ where N is the normalizer of H in G. If K is an odd order group, then the set of all G vector bundles over M can be algebraically realized.*

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2. Quotients of algebraic $G \times H$ varieties by H

Let V be a real algebraic G variety. The orbit space V/G has, in general, a semialgebraic set structure. However Ozan showed that if G is an odd order group and V is irreducible, then the orbit space V/G has a real algebraic variety structure. Moreover if the action is free and V is non-singular, then V/G is also non-singular, see [13].

In this section we extend Ozan's result to the quotient space V/H of real algebraic $G \times H$ variety V, which is not necessarily irreducible, where G is a compact Lie group and H an odd order group. The following theorem is the main result of this section. For simplicity we identify H (respectively G) with the subgoup $0 \times H$ (respectively $G \times 0$) of $G \times H$.

Theorem A. *Let G be a compact Lie group, and let H be an odd order* *group. Let X be a real algebraic $G\times H$ variety. Then the quotient space X/H has a real algebraic G variety structure. Moreover if the action of H is free and X is non-singular, then X/H is also non-singular.*

Let G and H be compact Lie groups. Let Ω be an orthogonal representation of $G\times H$. Let $\boldsymbol{R}[\Omega]$ be the $\boldsymbol{R}$-algebra of polynomial functions defined on Ω. This algebra has the induced action of $G\times H$ from the linear action of $G\times H$ on Ω defined by $k\cdot f=f\circ k^{-1}$ for $f\in\boldsymbol{R}[\Omega]$ and $k\in G\times H$. The H-fixed point set $\boldsymbol{R}[\Omega]^H$ is the subalgebra of H-invariant polynomials. By a theorem of Hilbert and Hurewitz [15, Ch 8, section 14] the subalgebra $\boldsymbol{R}[\Omega]^H$ is finitely generated.

Let X be a real algebraic $G\times H$ variety in an orthogonal representation Ω. Let $\mathscr{I}(X)$ denote the ideal of polynomials on Ω which vanish on X. Then the ring $\boldsymbol{R}[X]$ of polynomial functions on X is defined to be $\boldsymbol{R}[\Omega]/\mathscr{I}(X)$. This ring is an $\boldsymbol{R}$-algebra with the induced $G\times H$ action from the $G\times H$ action on $\boldsymbol{R}[\Omega]$.

Lemma 2.1. *The subalgebra $\boldsymbol{R}[X]^H$ of H invariant polynomial functions on X is finitely generated.*

Proof. Let $i:X\subsetneq\Omega$ be the inclusion, and let $i^*:\boldsymbol{R}[\Omega]\to\boldsymbol{R}[X]$ be the corresponding algebra homomorphism. If we restrict i^* to $\boldsymbol{R}[\Omega]^H$, then clearly its image $i^*(\boldsymbol{R}[\Omega]^H)$ is contained in $\boldsymbol{R}[X]^H$. Since $\boldsymbol{R}[\Omega]^H$ is finitely generated it is enough to show that $i^*:\boldsymbol{R}[\Omega]^H\to\boldsymbol{R}[X]^H$ is surjective. For $f\in\boldsymbol{R}[X]^H$ we can consider that f is a polynomial $\Omega\to\boldsymbol{R}$ which is H-invariant on X, i.e. $f(hx)=f(x)$ for all $x\in X$ and $h\in H$. Define $\bar{f}:\Omega\to\boldsymbol{R}$ by $f(x)=\int_H f(hx)dh$, where dh is the Haar measure of H. Then $\bar{f}$ is a polynomial function which is H-invariant on Ω and $\bar{f}=f$ on X. Namely $i^*(\bar{f})=f\in\boldsymbol{R}[X]^H$. This shows that $i^*:\boldsymbol{R}[\Omega]^H\to\boldsymbol{R}[X]^H$ is surjective, and hence $\boldsymbol{R}[X]^H$ is finitely generated. □

Let $p_1,\cdots,p_d$ generate $\boldsymbol{R}[X]^H$, and let us consider the regular map

$$p=(p_1,\cdots,p_d):X\to\boldsymbol{R}^d.$$

Let Z be the real algebraic variety in $\boldsymbol{R}^d$ defined by the polynomial relations of $p_1,\cdots,p_d$. Since p is constant on H-orbits of X the map p factors through the quotient space X/H. Let $\bar{p}:X/H\to Z$ be the induced map such that $p=\bar{p}\circ\pi$ where $\pi:X\to X/H$ is the quotient map. In general, the map $\bar{p}:X/H\to Z$ is not surjective but is a homemorphism onto its image, see [13].

We now complexify the above argument. For a real algebraic variety V its complexification V_C is the complex Zariski closure of V, namely the smallest complex algebraic variety which contains V. Since every compact Lie group K has a unique real algebraic variety structure we can consider its complexification K_C. Then K_C is a complex reductive algebraic group with K as a maximal compact subgroup, see [14]. Note that if K is a finite group, then $K_{\boldsymbol{C}}=K$.

Let K be a compact Lie group. Let V be a real algebraic K variety in an orthogonal representation Ω. Let $\theta: K\times V \to V \subset \Omega$ be the algebraic action map. Then θ is a regular (i.e. polynomial) map. Consider the complexification $\theta_{\boldsymbol{C}}:(K\times V)_{\boldsymbol{C}} \to \Omega_{\boldsymbol{C}}=\Omega\otimes_{\boldsymbol{R}}\boldsymbol{C}$, where $\theta_{\boldsymbol{C}}$ is the same polynomial as θ viewed as a complex polynomial map. In Zariski topology $(K\times V)_{\boldsymbol{C}}$ is the closure of $K\times V$ and the regular map $\theta_{\boldsymbol{C}}$ is a continuous function. Therefore $\theta_{\boldsymbol{C}}((K\times V)_{\boldsymbol{C}})$ is contained in the Zariski closure of V which is $V_{\boldsymbol{C}}$. We know that $(K\times V)_{\boldsymbol{C}} \subset K_{\boldsymbol{C}}\times V_{\boldsymbol{C}}$ because $(K\times V)_{\boldsymbol{C}}$ is the smallest complex algebraic variety containing $K\times V$. On the other hand

$$\boldsymbol{C}[(K\times V)_{\boldsymbol{C}}] \cong \boldsymbol{R}[K\times V]\otimes_{\boldsymbol{R}}\boldsymbol{C} \cong (\boldsymbol{R}[K]\otimes_{\boldsymbol{R}}\boldsymbol{C})\otimes_{\boldsymbol{C}}(\boldsymbol{R}[V]\otimes_{\boldsymbol{R}}\boldsymbol{C}) \cong \boldsymbol{C}[K_{\boldsymbol{C}}\times V_{\boldsymbol{C}}].$$

Thus $(K\times V)_{\boldsymbol{C}}=K_{\boldsymbol{C}}\times V_{\boldsymbol{C}}$.

Let X be a real algebraic $G\times H$ variety, and let Z be the variety as defined in the paragraph after Lemma 2.1. Then $\boldsymbol{C}[X_{\boldsymbol{C}}] \cong \boldsymbol{R}[X]\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Since X is a real algebraic $G\times H$ variety $X_{\boldsymbol{C}}$ is a complex algebraic $G_{\boldsymbol{C}}\times H_{\boldsymbol{C}}$ variety. As in the real case the $\boldsymbol{C}$-algebra $\boldsymbol{C}[X_{\boldsymbol{C}}]$ of complex polynomial functions on $X_{\boldsymbol{C}}$ has the induced action of $G_{\boldsymbol{C}}\times H_{\boldsymbol{C}}$. Let $\boldsymbol{C}[X_{\boldsymbol{C}}]^{H_{\boldsymbol{C}}}$ be the $H_{\boldsymbol{C}}$-invariant polynomials. Then $\boldsymbol{C}[X_{\boldsymbol{C}}]^{H_{\boldsymbol{C}}} \cong \boldsymbol{C}[X_{\boldsymbol{C}}]^{H} \cong \boldsymbol{R}[X]^{H}\otimes_{\boldsymbol{R}}\boldsymbol{C}$, where the first isomorphism follows because H is Zariski dense in $H_{\boldsymbol{C}}$. Therefore the regular map $p: X \to \boldsymbol{R}^d$ naturally induces the complex regular map $p_{\boldsymbol{C}}: X_{\boldsymbol{C}} \to \boldsymbol{C}^d$ where $p_{\boldsymbol{C}}=(p_{1_{\boldsymbol{C}}},\cdots,p_{d_{\boldsymbol{C}}})$ is the same polynomial map as p viewed as a complex polynomial map. The complex variety in $\boldsymbol{C}^d$ defined by the polynomial relations of $p_{1_{\boldsymbol{C}}},\cdots,p_{d_{\boldsymbol{C}}}$ is obviously the Zariski closure $Z_{\boldsymbol{C}}$ of Z. Such constructed variety $Z_{\boldsymbol{C}}$ is called an *algebraic quotient* of $X_{\boldsymbol{C}}$ by $H_{\boldsymbol{C}}$. The following lemma is well known, see [14].

Lemma 2.2. *The map $p_{\boldsymbol{C}}: X_{\boldsymbol{C}} \to \boldsymbol{C}^d$ maps $X_{\boldsymbol{C}}$ onto $Z_{\boldsymbol{C}}$, and separates $H_{\boldsymbol{C}}$-orbits of $X_{\boldsymbol{C}}$.*

Lemma 2.3. *The algebraic action of $G_{\boldsymbol{C}}\times H_{\boldsymbol{C}}$ on $X_{\boldsymbol{C}}$, which is the complexification of real algebraic action of $G\times H$ on X, induces an algebraic action of $G_{\boldsymbol{C}}$ on $Z_{\boldsymbol{C}}$. Moreover this action restricts to a real algebraic action of G on Z.*

Proof. Define an action of $G_{\boldsymbol{C}}$ on $Z_{\boldsymbol{C}}$ as follows: Let $\Phi:(G_{\boldsymbol{C}}\times H_{\boldsymbol{C}})\times X_{\boldsymbol{C}} \to X_{\boldsymbol{C}}$ be the algebraic action map, and let $\Phi_{|}: G_{\boldsymbol{C}}\times X_{\boldsymbol{C}} \to X_{\boldsymbol{C}}$ be the restriction of Φ to $G_{\boldsymbol{C}}\times X_{\boldsymbol{C}}$. Since $Z_{\boldsymbol{C}}$ is the algebraic quotient of $X_{\boldsymbol{C}}$ by $H_{\boldsymbol{C}}$ the map $p_{\boldsymbol{C}}: X_{\boldsymbol{C}} \to Z_{\boldsymbol{C}}$ satisfies the follwing universal property, see (3.5) of [14] or p123 of [6]:

If $\phi: X_{\boldsymbol{C}} \to V$ is a regular map between complex algebraic varieties which is constant on $H_{\boldsymbol{C}}$-orbits, then ϕ is the composition $\psi\circ p_{\boldsymbol{C}}$ where $\psi: Z_{\boldsymbol{C}} \to V$ is a regular map between complex algebraic varieties.

We may consider $G_{\boldsymbol{C}}\times X_{\boldsymbol{C}}$ as an algebraic $H_{\boldsymbol{C}}$-variety where $H_{\boldsymbol{C}}$ acts trivially on $G_{\boldsymbol{C}}$, and acts on $X_{\boldsymbol{C}}$ via Φ. Then $\mathrm{Id}_{G_{\boldsymbol{C}}}\times p_{\boldsymbol{C}}: G_{\boldsymbol{C}}\times X_{\boldsymbol{C}} \to G_{\boldsymbol{C}}\times Z_{\boldsymbol{C}}$ is an algebraic quotient map, and thus above universal property is satisfied. Now consider the composition

$$p_{\boldsymbol{C}} \circ \Phi_| : G_{\boldsymbol{C}} \times X_{\boldsymbol{C}} \to X_{\boldsymbol{C}} \to Z_{\boldsymbol{C}}.$$

Since $p_{\boldsymbol{C}} \circ \Phi_|$ is a regular map we can apply the above universal property to find a regular map $\theta : G_{\boldsymbol{C}} \times Z_{\boldsymbol{C}} \to Z_{\boldsymbol{C}}$ such that $\phi_{\boldsymbol{C}} \circ \Phi_| = \theta \circ (\mathrm{Id} \times p_{\boldsymbol{C}})$. We claim that θ defines an algebraic action of $G_{\boldsymbol{C}}$ on $Z_{\boldsymbol{C}}$. To do this we have to show that

(1) $\theta(e,z) = z$ for all $z \in Z_{\boldsymbol{C}}$, and
(2) $\theta(g,\theta(h,z)) = \theta(gh,z)$ for $g,h \in G$ and $z \in Z_{\boldsymbol{C}}$.

Let $x \in p^{-1}(z)$. Then

$$\begin{aligned}\theta(e,z) &= \theta \circ (\mathrm{Id} \times p_{\boldsymbol{C}})(e,x)\\ &= p_{\boldsymbol{C}} \circ \Phi_|(e,x)\\ &= p_{\boldsymbol{C}}(x) = z.\end{aligned}$$

For $g,h \in G$ and $z \in Z_{\boldsymbol{C}}$

$$\begin{aligned}\theta(gh,z) &= \theta \circ (\mathrm{Id} \times p_{\boldsymbol{C}})(gh,x)\\ &= p_{\boldsymbol{C}}((gh)x).\end{aligned}$$

On the other hand

$$\begin{aligned}\theta(g,\theta(h,z)) &= \theta(g,\theta \circ (\mathrm{Id} \times p_{\boldsymbol{C}})(h,x))\\ &= \theta(g,p_{\boldsymbol{C}}(hx))\\ &= \theta \circ (\mathrm{Id} \times p_{\boldsymbol{C}})(g,hx)\\ &= p_{\boldsymbol{C}}(g(h(x)))\\ &= p_{\boldsymbol{C}}((gh)x)\\ &= \theta(gh,z).\end{aligned}$$

This proves that the map θ is actually an action map. Therefore if we take the real part of $\theta : G_{\boldsymbol{C}} \times Z_{\boldsymbol{C}} \to Z_{\boldsymbol{C}}$, then it defines a real algebraic action of G on Z. □

By Lemma 2.2 the map $p_{\boldsymbol{C}} : X_{\boldsymbol{C}} \to Z_{\boldsymbol{C}}$ is surjective, but as we have mentioned before $p : X \to Z$ is not surjective in general. Next lemma gives a sufficient condition for $p : X \to Z$ to be surjective.

Lemma 2.4. *If H is an odd order group, then the map $p : X \to Z$ is surjective and $\bar{p} : X/H \to Z$ is a G homeomorphism. Therefore the quotient space X/H can be given a real algebraic G variety structure by Z.*

Proof. Let $Z_0:=p(X)$. Suppose there exists a point $x\in Z-Z_0$. Note that since H is a finite group $H_C=H$. Since $p_C: X_C\to Z_C$ is surjective the preimage $p_C^{-1}(x)$ of the point x is non-empty and consists of at most $|H|$ points by Lemma 2.2. Since $x\in Z-Z_0$ none of the points in the preimage are contained in the real part $X_C\cap\Omega$ of X_C because $X_C\cap\Omega=X$ and X is mapped onto Z_0.

On the other hand p_C is a polynomial with real coefficients, and X_C is defined by real polynomials because $\mathscr{I}(X_C)=\mathscr{I}(X)\otimes_{\boldsymbol{R}}C$. Therefore if $a\in X_C$ then its complex conjegate $\bar{a}\in X_C$, and if $p_C(a)$ is real then both a and $\bar{a}$ are mapped to the same point by p_C. This implies that the cardinality of the preimage $p_C^{-1}(x)$ is an even number. On the other hand since the map p_C separates orbits, the cardinality of $p_C^{-1}(x)$ is the same as the order of a quotient group of H. This is a contradiction because $|H|$ is odd the order of any quotient group of H is odd. This proves that $Z_0=Z$. Thus Lemma 2.3 implies that X/H can be endowed with a real algebraic G variety structure by Z.

Proof of Theorem A. From Lemma 2.4 the quotient space X/H has a real algebraic G variety structure. In fact the real algebraic G variety Z is the desired variety structure on X/H. It remains to prove that if X is nonsingular and H acts freely, then Z is nonsingular.

For a complex algebraic varieties non-singularity at a point x is equivalent to smoothness around x, see [10]. Also note that a real algebraic variety V is nonsingular at x if and only if the complexification X_C is non-singular at x.

Let $X_{C_{(1)}}$ denote the set of points of the principal isotropy type, and let $Z_{C_{(1)}}$ denote the image $p_C(X_{C_{(1)}})\subset Z_C$. Then $Z_{C_{(1)}}$ is an open smooth manifold of dimension $2n$, where n is the complex dinension of the variety Z_C, [9 III.2.4].

Suppose Z is singular at $z\in Z$. Then Z_C is singular at z. Therefore Z_C is not a smooth manifold around z. On the other hand since H acts freely on X it is clear that $p(X)=Z$ is contained in the smooth manifold $Z_{C_{(1)}}$. This is a contradiction. Therefore Z is nonsingular. □

By a similar but easier argument we can show that every compact homogeneous space G/H of compact Lie group G and a closed subgroup H has a non-singular real algebraic variety structure, see [4, p54].

3. Algebraic Realization of close smooth G manifolds.

The main result of this section is the following theorem.

Theorem B. *Let G be a compact Lie group acting smoothly on a closed manifold M with one orbit type. Then M can be algebraically realized.*

The rest of the section is devoted to the proof of Theorem B.

Let M be a smooth closed G manifold with the unique orbit type G/H. Let

$K:=N/H$, where N is the normalizer of H. From 2.5.11 and 6.2.5 of [2] or 4.8 of [7] there is a G diffeomorphism

$$(G/H)\times_K M^H \to M$$
$$[gH,x] \mapsto g(x).$$

We consider two cases.

Case 1. $|K| \neq odd$. Note that the induced action of K on M^H is free because there is only one orbit type. Since $|K| \neq$ odd and K acts freely on M^H Proposition 4.1 of [3] implies that M^H bounds K equivariantly. Namely, there exists a smooth K manifold W with $\partial W = M^H$. Then $(G/H)\times_K W$ is a smooth G manifold (G acts on G/H as a left translation) with $\partial((G/H)\times_K W) = (G/H)\times_K \partial(W) = (G/H)\times_K M^H \cong M$. Therefore we have proved that M is a G equivariant boundary.

Case 2. $|K| = odd$. In this case it is proved in [3] that every smooth closed K manifold is K equivariantly cobordant to a non-singular real algebraic K variety. Therefore there exists a K manifold W with $\partial W = M^H \amalg Z$ where Z is a nonsingular real algebraic K variety. Here $\amalg$ denotes disjoint union.

Consider the manifold $G/H \times W$. This is a $G\times K$ manifold with the following action. The action of G is the left translation on G/H, and the action of K is $kH\cdot(gH,w) = (gk^{-1}H,kw)$. Note that the action of K on $G/H\times W$ is free. The orbit space $G/H\times_K W$ of the K action on $G/H\times W$ is a smooth G manifold with $\partial(G/H\times_K W) = G/H\times_K M^H \amalg G/H\times_K Z$.

Since every orbit G/H has a canonical non-singular real algebraic G variety structure, $G/H\times Z$ is a non-singular real algebraic $G\times K$ variety with free K action. Thus Theorem A implies that the quotient space $G/H\times_K Z$ is a nonsingular real algebraic G variety.

In both cases $M \cong (G/H\times_K M^H)$ is G equivariantly cobordant to a non-singular algebraic G variety V including the case $V=\emptyset$. Now Theorem B follows from the following theorem.

Theorem 3.1. ([4]) *A smooth closed G manifold M is algebraically realized if and only if it is G equivariantly cobordant to a non-singular real algebraic G variety.*

4. Algebraic Realization of G Vector Bundles.

A *strongly algebraic* G vector bundle ξ over a non-singular real algebraic G variety V is a G vector bundle whose equivariant classifying map $\mu_\xi: V \to G_{\boldsymbol{R}}(\Xi,k)$ is an equivariant entire rational map, i.e., if $V \subset \boldsymbol{R}^n$ and $G_{\boldsymbol{R}}(\Xi,k) \subset \boldsymbol{R}^m$, then there exist polynomials $P: \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $Q: \boldsymbol{R}^n \to \boldsymbol{R}$ with $Q^{-1}(0) \cap V = \emptyset$ such that $\mu_\xi = P/Q$

on V. Remember that a set $\mathscr{F}$ of G vector bundles over a closed smooth G manifold M is *algebraically realized* if there are a non-singular real algebraic G variety V, a G diffeomorphism $\phi: M \to V$, and a set $\mathscr{F}'$ of strongly algebraic G vector bundles over V such that for each $\xi \in \mathscr{F}$ there exists $\eta \in \mathscr{F}'$ such that ξ and $\phi^*\eta$ are G isomorphic, or equivalently an equivariant classifying map $\mu_\xi: M \to G_{\mathbf{R}}(\Xi,k)$ of ξ is G homotopic to $\mu_\eta \circ \phi$ where μ_η is an equivariant classifying map of η.

The question we are interested in here is whether any set of G vector bundles over a closed smooth G manifold is algebraically realized. This bundle realization problem is treated in [5] and we refer the reader to the cited paper for details on the subject. One of the fundamental result of [5] is the following theorem 4.1 which reduces the bundle realization problem to a non-oriented equivariant bordism theoretic problem. For this we need some terminology. Let $f: M^n \to Y$ be a G map from a closed smooth G manifold to a G space Y. Let $g: N^n \to Y$ be another G map. They are equivalent if they are corbodant, i.e., there exist a smooth G manifold W^{n+1} with $\partial W = M \quad N$ and a G map $F: W \to Y$ such that $F|_M = f$ and $F|_N = g$. The collection of all equvalent classes of pairs (M,f) forms an abelian group with addition induced from disjoint union. This group is called the *(non-oriented) G equivariant bordism group* of Y, and is denoted by $\mathscr{N}_n^G(Y)$. The class of the pair (M,f) is denoted by $[M,f]$. The identity element of the bordism group is represented by a pair (M,f) which is an equivariant boundary, i.e., there exists a smooth G manifold W and a smooth G map $F: W \to Y$ such that $\partial W = M$ and $F|_M = f$.

Let Y be a non-singular real algebraic G variety. An equivariant bordism class $[M,f] \in \mathscr{N}_*^G(Y)$ is said to be *algebraic* if $[M,f] = [V,g]$ where V is a non-singular real algebraic G variety and $g: V \to Y$ is an enitre rational G map including the case when (M,f) is an equivariant boundary. A pair (M,f) of a closed smooth G manifold and a smooth G map $f: M \to Y$ is said to be *algebraically realized* if there are a non-singular real algebraic G variety V, an entire rational G map $g: V \to Y$ and a G diffeomorphism $\phi: V \to M$ such that $f \circ \phi$ and g are G homotopic.

The following theorem gives a necessary and sufficient condition for a pair (M,f) to be algebraically realized.

Theorem 4.1. ([5]) *Let G be a compact Lie group and Y a non-singular real algebraic G variety. Let M be a closed smooth G manifold and $f: M \to Y$ a smooth G map. Then (M,f) is algebraically realized if and only if its bordism class $[M,f]$ is algebraic.* □

Another needed result from [5] is the following.

Lemma 4.2. *Let G be an odd order group acting freely on a closed smooth manifold M. Let Y be a non-singular real algebraic G variety such that Y^L has* totally algebraic homology for every subgroup $L \subset G$. *Then for a smooth G map $f: M \to Y$ the bordism class $[M,f] \in \mathcal{N}^G_*(Y)$ is algebraic. In fact, every $[M,f]$ is represented by $g: Z \to Y$ where Z is a non-singular real algebraic G variety and g is a G-regular map.*

We do not give the definition of totally algebraic homology because it is not an essential concept in this paper. We refer the reader to [1] or [5] for details.

Proof of Lemma 4.2. All others are proved in [5] except for the last sentence. For the last claim we can examine the proof in [5], and show that the algebraic representative (Z,g) can be chosen so that g is a G-regular map instead of an entire rational G map. This can actually be done so. Here, however, instead of doing so, we prove the last claim by proving a generalized version. The last claim follows immediately from the following proposition. $\square$

Proposition 4.3. *Let G be a compact Lie group. Let M be a closed smooth G manifold and Y a non-singular real algebraic G variety. Let $f: M \to Y$ be a smooth G-map. If $[M,f] \in \mathcal{N}^G_*(Y)$* is algebraic, then *$[M,f]$ can be represented by a G-regular map $g': Z' \to Y$, where Z' is a nonsingular real algebraic G variety.*

Proof. Since $[M,f] \in \mathcal{N}^G_*(Y)$ is algebraic there exists a nonsingular real algebraic G variety Z and a G-entire rational map $g: Z \to Y$ which represents the bordism class $[M,f]$. Now consider the graph

$$\Gamma(g) = \{(x, g(x)) \in Z \times Y | x \in Z\}$$

and the projection map $\pi_2: \Gamma(g) \to Y$, $\pi_2(x, g(x)) = g(x)$. Then it is elementary to see that $\Gamma(g)$ is a nonsingular real algebraic G variety, and π_2 is a G-regular map. Moreover, $(\Gamma(g), \pi_2)$ is clearly G-cobordant to (Z,g). Therefore $(Z',g') :=$ $(\Gamma(g), \pi_2)$ is a desired representative of the bordism class $[M,f]$. $\square$

From now on we assume that M is a closed smooth G manifold with one orbit type, and let G/H be the unique orbit type of M. As noted in section 3 there is a G diffeomorphism $G/H \times_K M^H \to M$ is defined by $[gH, x] \mapsto g(x)$. Here $(G/H) \times_K M^H$ is the orbit space of $(G/H) \times M^H$ by the K action $kH \cdot (gH, m)$ $= (gk^{-1}H, km)$. Here $K = N/H$ and N is the normalizer of H in G. Note that any G-equivariant map $f: G/H \times_K M \to Y$ is of the form $\mathrm{Ind}\, h$ for the K-equivariant map $h = f^H$. Here $\mathrm{Ind}\, h$ is defined by $\mathrm{Ind}\, h[gH, m] = g \cdot h(m)$ for $gH \in G/H$ and $m \in M^H$.

Lemma 4.4. *Let H and K be as above. Assume that Z is a non-singular real algebraic K variety and $h: Z \to Y^H \subset Y$ a K equivariant regular map. If K is an odd order group, then $G/H \times_K Z$ has a non-singular real algebraic G variety structure such that* $\mathrm{Ind}\, h$ *is a G equivariant regular map.*

Proof. Consider the space $G/H \times Z$ with $G \times K$ action defined as follows : the action of G is the left multiplication on G/H, and the action of K is defined by $kH \cdot (gH, z) = (gk^{-1}H, kz)$. Since every orbit G/H has a canonical non-singular real algebraic G variety structure $G/H \times Z$ is non-singular real algebraic $G \times K$ variety with free K action. Since $|K| = \text{odd}$ Theorem B implies that $G/H \times_K Z$ is non-singular real algebraic G variety. It remains to show that $\operatorname{Ind} h$ is a regular map. Let $\theta : G \times Y \to Y$ be the algebraic action map of G on Y. Let $\Phi : G \times Z \to Y$ be the map defined by $\Phi(g,z) = \theta(g, h(z))$ for $g \in G$ and $z \in Z$. Then Φ is clearly a regular map. Let $H \times K$ act on $G \times Z$ as follows: H acts on G by the right multiplication, trivially on Z, and K acts on $G \times Z$ by $k(g,z) = (gk^{-1}, kz)$ for $k \in K$, $g \in G$, and $z \in Z$. Let $p : G \times Z \to (G \times Z)/(H \times K) = G/H \times_K Z$ be the orbit map. We may assume that $G/H \times_K Z$ is a real algebraic G variety and p is a G-regular map. It is clear that Φ is constant on $H \times K$ orbits of $G \times Z$. Thus Φ factors through $G/H \times_K Z$ and $\Phi = \operatorname{Ind} h \circ p$. We now complexify the above argument. We note that the ring of polynomial functions of the algebraic quotient of $(G \times Z)_{\boldsymbol{C}}$ by the action of $(H \times K)_{\boldsymbol{C}}$ is isomorphic to $\boldsymbol{C}[(G \times Z)_{\boldsymbol{C}}]^{(H \times K)_{\boldsymbol{C}}}$ which is isomorphic to $\boldsymbol{C}[(G \times Z)_{\boldsymbol{C}}]^{H \times K}$ because $H \times K$ is Zariski dense in $(H \times K)_{\boldsymbol{C}}$. On the other hand

$$\boldsymbol{C}[(G \times Z)_{\boldsymbol{C}}]^{H \times K} \cong \boldsymbol{R}[G \times Z]^{H \times K} \otimes_{\boldsymbol{R}} \boldsymbol{C} \cong \boldsymbol{R}[G/H \times_K Z] \otimes_{\boldsymbol{R}} \boldsymbol{C} \cong \boldsymbol{C}[(G/H \times_K Z)_{\boldsymbol{C}}].$$

Thus $(G/H \times_K Z)_{\boldsymbol{C}}$ can be identified with the algebraic quotient of $(G \times Z)_{\boldsymbol{C}}$ by the action of $(H \times K)_{\boldsymbol{C}}$. As in the proof of Lemma 2.3 the universal property of algebraic quotients implies that there is a complex regular map $\rho : (G/H \times_K Z)_{\boldsymbol{C}} \to Y_{\boldsymbol{C}}$ such that $\Phi_{\boldsymbol{C}} = \rho \circ p_{\boldsymbol{C}}$. Therefore the restriction of ρ to the real part, which is in fact $\operatorname{Ind} h$, is a regular map. □

The following theorem is the main result of this section.

Theorem C. *Let G be a compact Lie group acting smoothly on a colsed manifold M with one orbit type. Let G/H be the unique orbit type, and let $K := N/H$ where N is the normalizer of H in G. If K is an odd order group, then the set of all G vector bundles over M can be algebraically realized.*

Proof. By Proposition 2.13 of [5] algebraic realization of the set of all G vector bundles is equivalent to algebraic realization of arbitrary finite set of G vector bundles. Therefore it is enough to realize a given finite collection $\mathscr{F} = \{\xi_i | i = 1, \cdots, n\}$ of G vector bundles algebraically. Let $\mu_i : M \to G_{\boldsymbol{R}}(\Xi_i, k_i)$ be equivariant classifying maps of ξ_i for $i = 1, \cdots, n$. Set $\mu := \Pi_{i=1}^n \mu_i : M \to G(\mathscr{F})$ where $G(\mathscr{F}) := \Pi_{i=1}^n G_{\boldsymbol{R}}(\Xi_i, k_i)$. Then $\mu = \operatorname{Ind} h$ where $h = \mu^H : M^H \to G(\mathscr{F})^H$. The pair (M^H, h) defines an element of the bordism group $\mathscr{N}_*^K(G(\mathscr{F})^H)$. It is proved in [5] that $(G(\mathscr{F})^H)^L$ has totally algebraic homology for every subgroup $L \subset K$. By Lemma 4.2 there exist a smooth K manifold W with $\partial W = M^H \amalg Z$ and a smooth

K map $F: W \to G(\mathscr{F})^H$ such that Z is a non-singular real algebraic K variety, $F|_{M^H} = h$, and $F|_Z = \psi$ is a regular K map. Consider the manifold $G/H \times W$. This is a $G \times K$ manifold with the following action: the action of G is the left multiplication on G/H, and the action of K is defined by $kH \cdot (gH, w) = (gk^{-1}H, kw)$. Therefore the orbit space $G/H \times_K W$ of the K action on $G/H \times W$ is a smooth G manifold with $\partial(G/H \times_K W) = (G/H \times_K M^H) \quad (G/H \times_K Z)$. Moreover the G equivariant map $\mathrm{Ind}\, F: G/H \times_K W \to G(\mathscr{F})$ is well defined. By the remark after Theorem B $G/H \times_K M^H$ is G diffeomorphic to M. Therefore if we identify M with $G/H \times_K M^H$, then $\mu: M \to G(\mathscr{F})$ is identified with $\mathrm{Ind}\, h$ which is equal to $\mathrm{Ind}\, F|_{G/H \times_K M}$. This is one end of the cobordism. On the other end of the cobordism we have $G/H \times_K Z$ which is a non-singular real algebraic G variety by Theorem A and a G map $\mathrm{Ind}\, F|_{G/H \times_K Z} = \mathrm{Ind}\, \psi$ which is regular, thus an entire rational G map by Lemma 4.4. This shows that the bordism class $[M, \mu]$ is algebraically realized. Therefore by Theorem 4.1 (M, μ) is algebraically realized, say by (V, v). Let $p_i: G(\mathscr{F}) \to G_{\mathbf{R}}(\Xi_i, k_i)$ be the projection. Then the set of G vector bundles corresponding to the classifying map $p_i \circ v$ over V realizes $\mathscr{F}$ algebraically. This proves the theorem. □

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