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ALGORITHMS WITH MEDIANT CONVERGENTS AND THEIR METRICAL THEORY

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0 Introduction

Let $x\in(0, 1)$ be an irrational number and $x=[0: a_1, a_2, \cdots]$ be the continued fraction expansion of x. The principal convergents p_n/q_n of x are obtained by so called continued fraction transformation S as follows: let S be a transformation on $X=[0, 1)$ such that

$$Sx = \begin{cases} \dfrac{1}{x}-\left[\dfrac{1}{x}\right] & \text{if} \quad x\in(0, 1) \\ 0 & \text{if} \quad x=0 \end{cases}$$

and put $a_n(x)=\left[\dfrac{1}{S^{n-1}x}\right]$, then the principal convergents p_n/q_n, $n=1, 2, \cdots$ of α are given by

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

We know in [1] and [5] that the transformation S has an invariant measure ν with density

$$d\nu = \frac{1}{\log 2} \frac{dx}{(1+x)}$$

and that the natural extension $\bar{S}$ of S on $\bar{X}=[0, 1)\times[0, 1)$ given by

$$\bar{S}(x, y) = \left(\frac{1}{x}-\left[\frac{1}{x}\right], \frac{1}{[1/x]+y}\right)$$

has an invariant measure $\bar{\nu}$ with density

$$d\bar{\nu} = \frac{1}{\log 2} \frac{dx\,dy}{(1+xy)^2},$$

and that the dynamical systems (X, S, ν) and $(\bar{X}, \bar{S}, \bar{\nu})$ are ergodic.

As an application of Birkhoff's ergodic theorem, we obtain several metrical results.

Theorem. *For almost* $x\in[0, 1)$,

(1) $\lim_{n\to\infty}\frac{1}{n}\log q_n=\frac{\pi^2}{12\log 2}$,

(2) $\lim_{n\to\infty}-\frac{1}{n}\log\left|x-\frac{p_n}{q_n}\right|=\frac{\pi^2}{6\log 2}$,

(3) $\lim_{N\to\infty}\frac{1}{N}\sharp\{n\,|\,0\leqq n\leqq N,\ q_n|q_nx-p_n|<\lambda\}$

$$=\begin{cases}\dfrac{\lambda}{\log 2} & \textit{for}\quad 0\leqq\lambda<1/2\\[2mm] \dfrac{-\lambda+\log 2\lambda+1}{\log 2} & \textit{for}\quad 1/2\leqq\lambda<1\,,\end{cases}$$

(4) $\lim_{N\to\infty}\frac{\sharp\{(q, p)\,|\,q|qx-p|<\lambda,\ (q, p)=1,\ q<N\}}{\log N}=\frac{\pi^2}{12}\lambda$ *for* $0<\lambda<1/2$.

Remark. The first proof of the statement (1) and (2) is given by Kinchine, and the proof from ergodic theoretical standpoint is given by C. Ryll-Nardzewski in [7]. The statement (3) is obtained from the ergodicity of the natural extension of S (see [2] and [5]). The number theoretical proof of statement (4) is given by P. Erdös for "any" $\lambda>0$ in [4], and an ergodic theoretical proof for $0<\lambda<1/2$ is found in [5].

In this paper, an algorithm T which induces the mediant convergents $\left\{\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}\,\middle|\,k=1, \cdots, a_{n+1}-1,\ n=1, 2\cdots\right\}$ of x is proposed as follows: let T be a transformation on X such that

$$Tx=\begin{cases}\dfrac{x}{1-x} & \text{if}\quad x\in I_0=[0, 1/2)\\[2mm] \dfrac{1-x}{x} & \text{if}\quad x\in I_1=[1/2, 1]\,,\end{cases}$$

and put

$$\varepsilon_n(x)=\begin{cases}0 & \text{if}\quad T^{n-1}x\in I_0\\ 1 & \text{if}\quad T^{n-1}x\in I_1\,.\end{cases}$$

Let us define the matrices

$$A_0=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\quad\text{and}\quad A_1=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

and

$$\begin{pmatrix}r_n & s_n\\ t_n & u_n\end{pmatrix}=A_{\varepsilon_1}A_{\varepsilon_2}\cdots A_{\varepsilon_n}\,.$$

Then the convergents w_n/v_n, $n=1, 2, \cdots$, where $v_n=r_n+s_n$ and $w_n=t_n+u_n$, are not only principal convergents of x but also mediant convergents of x. However, the mediant convergents transformation T has only a σ-finite but infinite invariant measure μ with density $d\mu=dx/x$, and so the ergodic theorem is not useful to observe the limit distribution. Therefore a modified algorithm T_1, which is constructed by the jump transformation from T, is provided as follows:

$$T_1 x = \begin{cases} \dfrac{1-x}{x} & \text{if } x\in[1/2, 1) \\ \dfrac{x}{1-x} & \text{if } x\in[1/3, 1/2) \\ \dfrac{x}{1-(k-2)x} & \text{if } x\in[1/(k+1), 1/k) \quad (k\geqq 3)\,. \end{cases}$$

We see in Theorem 2.1 the algorithm T_1 generates the approximation fractions $w_n^{(1)}/v_n^{(1)}$ of x, $n=1, 2,\cdots$, which is not only the principal convergents but also the first mediant convergents $\dfrac{p_n+p_{n-1}}{q_n+q_{n-1}}$ and the last mediant convergents $\dfrac{p_n-p_{n-1}}{q_n-q_{n-1}}$.

We see also the transformation T_1 has a finite invariant measure μ_1 with density

$$d\mu_1 = \begin{cases} \dfrac{1}{2\log 2}\dfrac{dx}{1+x} & \text{if } x\in[0, 1/3) \\ \dfrac{1}{2\log 2}\dfrac{dx}{x} & \text{if } x\in[1/3, 1)\,, \end{cases}$$

and the dynamical system is ergodic.

By constructing of natural wxtension of T_1 and applying ergodic theorem, we obtain the metrical results.

Result. *For almost all* $x\in[0, 1)$,

(1) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\log v_n^{(1)}=\frac{\pi^2}{24\log 2}$,

(2) $\displaystyle\lim_{n\to\infty}-\frac{1}{n}\log\left|x-\frac{w_n^{(1)}}{v_n^{(1)}}\right|=\frac{\pi^2}{12\log 2}$,

(3) $$\lim_{N\to\infty}\frac{\sharp\{n\,|\,v_n^{(1)}|v_n^{(1)}x-w_n^{(1)}|<\lambda,\ 1\leqq n\leqq N\}}{N} = \begin{cases} \dfrac{\lambda}{2\log 2} & \textit{for } \lambda\leqq 1 \\ \dfrac{2-\lambda+2\log\lambda}{2\log 2} & \textit{for } 1\leqq\lambda<2, \end{cases}$$

(4) $\displaystyle\lim_{N\to\infty}\frac{\sharp\{(q, p)\,|\,q|qx-p|<\lambda,\ (q, p)=1,\ q<N\}}{\log N}=\frac{\pi^2}{12}\lambda$ for $0<\lambda<1$.

1 Mediant convergent transformation

In this section an algorithm which induces mediant convergents is proposed. Let $X=[0, 1]$ and let the map T be defined on X by

$$(1,1) \qquad Tx = \begin{cases} \dfrac{x}{1-x}, & \text{if } x\in I_0 \\ \dfrac{1-x}{x}, & \text{if } x\in I_1, \end{cases}$$

where $I_0=[0, 1/2]$ and $I_1=[1/2,1]$ (see figure 1).

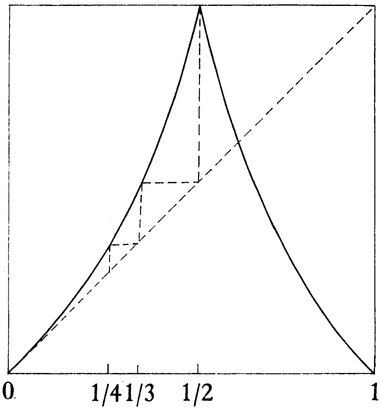

figure 1

We denote the inverse branches of T by

$$(1,2) \qquad V_0(x)=\frac{x}{x+1} \quad \text{and} \quad V_1(x)=\frac{1}{x+1}.$$

All inverse branches are modular transformations. So we use the following matrix representations for them:

$$(1,3) \qquad A_0=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\left(=\frac{0+1\cdot x}{1+1\cdot x}\right) \quad \text{and} \quad A_1=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\left(=\frac{1+0\cdot x}{1+1\cdot x}\right).$$

For irrational $x\in(0, 1)$ put

$$(1,4) \qquad \varepsilon_n=\varepsilon_n(x)=\begin{cases} 0, & \text{if } T^{n-1}x\in I_0 \\ 1, & \text{if } T^{n-1}x\in I_1 \end{cases}$$

and

$$(1,5) \qquad \begin{pmatrix}r_n & s_n\\ t_n & u_n\end{pmatrix}=\begin{pmatrix}r_n(x) & s_n(x)\\ t_n(x) & u_n(x)\end{pmatrix}=A_{\varepsilon_1(x)}A_{\varepsilon_2(x)}\cdots A_{\varepsilon_n(x)}.$$

Then we obtain the following.

Proposition 1.1. *For any irrational $x \in X$ we have*

$$x = \frac{t_n(x) + T^n x \cdot u_n(x)}{r_n(x) + T^n x \cdot s_n(x)} \tag{1,6}$$

Proof. Let $X_{\varepsilon_1 \cdots \varepsilon_n}$ be a cylinder set of rank n, that is,

$$X_{\varepsilon_1 \cdots \varepsilon_n} = \{x;\ T^{k-1}x \in I_{\varepsilon_k}\ 1 \leq k \leq n\}.$$

Then T^n is a bijective map from $X_{\varepsilon_1 \cdots \varepsilon_n}$ to I, and the matrix representation of the inverse branch of T^n restricted to $X_{\varepsilon_1 \cdots \varepsilon_n}$ is $\begin{pmatrix} r_n & s_n \\ t_n & u_n \end{pmatrix}$.

Let S be the simple continued fraction transformation:

$$Sx = \frac{1}{x} - k, \qquad \text{if} \quad x \in \left[\frac{1}{k+1}, \frac{1}{k}\right) \ (k \geq 1) \tag{1,7}$$

We denote the inverse branches of S by

$$W_k(x) = \frac{1}{x+k} \tag{1,8}$$

and the associated matrices by

$$C_k = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \ \left(= \frac{1 + 0 \cdot x}{k + 1 \cdot x}\right). \tag{1,9}$$

For each irrational $x \in (0, 1)$ put

$$a_n = a_n(x) = k, \qquad \text{if} \quad S^{n-1}x \in \left[\frac{1}{k+1}, \frac{1}{k}\right),$$

and

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n(x) & 1 \\ 1 & 0 \end{pmatrix} \tag{1,10}$$

where

$$\begin{pmatrix} q_0 & q_{-1} \\ p_0 & p_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The following formula is well known: For any irratoinal $x \in (0, 1)$

$$x = \frac{p_n + S^n x \cdot p_{n-1}}{q_n + S^n x \cdot q_{n-1}} \qquad (n \geq 1). \tag{1,11}$$

The relation between T and S is given by

$$Sx = T^k x, \qquad \text{if} \quad x \in \left[\frac{1}{k+1}, \frac{1}{k}\right). \tag{1,12}$$

If $x\in\left[\frac{1}{k+1}, \frac{1}{k}\right)$, then $(\varepsilon_1(x), \cdots, \varepsilon_k(x))=(0, 0, \cdots, 0, 1)$. Therefore the inverse map W_k of S is represented by

$$W_k = V_{\varepsilon_1(x)} V_{\varepsilon_2(x)} \cdots V_{\varepsilon_k(x)}$$

, that is,

$$\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1,13}$$

Lemma 1.1. *Put* $j=j(n: x)=\#\{k; \varepsilon_k(x)=1, k\leq n\}$ and $\underline{1}=\underline{1}(n; x)=\max\{k; \varepsilon_k(x)=1, k\leq n\}$ *where* $\underline{1}=\underline{1}(n; x)=0$ if $\{k; \varepsilon_k(x)=1, k\leq n\}=\phi$. *Then, for any irrational* $x\in(0, 1)$

$$\begin{pmatrix} r_n(x) & s_n(x) \\ t_n(x) & u_n(x) \end{pmatrix} = \begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{pmatrix} \begin{pmatrix} 1 & n-\underline{1} \\ 0 & 1 \end{pmatrix} \qquad (n\geq 1) \tag{1,14}$$

Proof. If $j=0$, then

$$\begin{pmatrix} r_n(x) & s_n(x) \\ t_n(x) & u_n(x) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} q_0 & q_{-1} \\ p_0 & p_{-1} \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

If $j\geq 1$, then $S^j x=T^{a_1+\cdots+a_j}x$ and $T^n x=T^{n-\underline{1}}(T^{\underline{1}}x)=T^{n-\underline{1}}(S^j x)$. Therefore, by (1,13), the representation of the inverse branch of T^n is

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n-\underline{1} \\ 0 & 1 \end{pmatrix}.$$

The fraction $\frac{p_n}{q_n}$ is called the n-th principal convergent of x and the fractions $\left\{\frac{\lambda\cdot p_n+p_{n-1}}{\lambda\cdot q_n+q_{n-1}}: \lambda=1, 2, \cdots, a_{n+1}-1\right\}$ are called mediant convergents of $\frac{p_n}{q_n}$.

Theorem 1.1. *Put* $v_n(x)=r_n(x)+s_n(x)$ *and* $w_n(x)=t_n(x)+u_n(x)$. *Then for any* irrational $x\in(0, 1)$

$$\left\{\frac{w_n}{v_n}: n\geqq 1\right\} = \bigcup_{k=0}^{\infty} \left\{\frac{\lambda\cdot p_k+p_{k-1}}{\lambda\cdot q_k+q_{k-1}}: \lambda=1, 2, \cdots, a_{k+1}\right\}.$$

Proof. Put

$$\bar{1} = \bar{1}(n: x) = \min_k \{k: \varepsilon_k(x)=1, n<k\}$$

$$\underline{1} = \underline{1}(n: x) = \max_k \{k; \varepsilon_k(x)=1, k\leqq n\}.$$

Then from (1,12) we have

$$\bar{1}-\underline{1}=a_{j+1}(x)\,.$$

By the lemma 1.1.

$$\begin{pmatrix} r_n+s_n \\ t_n+u_n \end{pmatrix} = \begin{pmatrix} (n-\underline{1}+1)q_j+q_{j-1} \\ (n-\underline{1}+1)p_j+p_{j-1} \end{pmatrix}.$$

Putting $\lambda=n-\underline{1}+1$, we have $1\leq\lambda\leq a_{j+1}$ and so we obtain the result.

We now call a fraction

$$\frac{w_n}{v_n}=\frac{t_n(x)+u_n(x)}{r_n(x)+s_n(x)}$$

the *n-th mediant convergent* of x, and the algorithm (X, T) the *mediant convergent transformation.* We prepare some formulae concerning the approximation.

Proposition 1.2. *For any irrational* $x\in(0, 1)$

$$\left|x-\frac{w_n(x)}{v_n(x)}\right|=\frac{1-T^n x}{v_n^2(x)\left\{\frac{r_n}{v_n}(1-T^n x)+T^n x\right\}}. \tag{1,15}$$

In particular,

$$\left|x-\frac{w_n(x)}{v_n(x)}\right| \quad \textit{and} \quad |v_n(x)\cdot x-w_n(x)|$$

converge to 0 *as* $n\to\infty$.

Proof. By proposition 1.1. and since $r_n u_n - s_n t_n=\pm 1$, we have

$$\begin{aligned}\left|x-\frac{w_n}{v_n}\right| &= \left|\frac{t_n+T^n x\cdot u_n}{r_n+T^n x\cdot s_n}-\frac{t_n+u_n}{r_n+s_n}\right| \\ &= \frac{1-T^n x}{v_n(r_n+T^n x\cdot s_n)}\,.\end{aligned}$$

From $r_n\nearrow\infty$ and the definition of v_n, w_n, we obtain the proposition.

For any $0-1$ sequence $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, let φ_{ε_i} be the affine transformation of the (ξ_{i-1}, η_{i-1})-plane into the (ξ_i, η_i)-plane such that

$$\varphi_{\varepsilon_i}\colon \begin{pmatrix} \xi_{i-1} \\ \eta_{i-1} \end{pmatrix} = A_{\varepsilon_i}\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}.$$

Then we have

Proposition 1.3. *For any irrational* $x\in(0, 1)$

$$|x\cdot\xi_0-\eta_0|=g(x)g(Tx)\cdots g(T^{n-1}x)|T^n x\cdot\xi_n-\eta_n|. \tag{1,16}$$

In particular,

$$|x\cdot v_n-w_n|=g(x)g(Tx)\cdots g(T^{n-1}x)(1-T^n x) \tag{1,17}$$

where

$$g(x)=\begin{cases} 1-x\,, & \text{if } x\in I_0 \\ x\,, & \text{if } x\in I_1. \end{cases}$$

Proof. By $\varphi_{\varepsilon_1(x)}$ the linear form $x\xi_0-\eta_0$ is transformed into the following linear form:

$$x\cdot\xi_0-\eta_0=\begin{cases} (1-x)(Tx\cdot\xi_1-\eta_1)\,, & \text{if } x\in I_0 \\ -x(Tx\cdot\xi_1-\eta_1)\,, & \text{if } x\in I_1. \end{cases}$$

This shows that formula (1,16) is valid for $n=1$. The general case is obtained by induction. Using the relation

$$\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}=A_{\varepsilon_1(x)}\cdots A_{\varepsilon_n(x)}\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix}=\begin{pmatrix} r_n & s_n\\ t_n & u_n\end{pmatrix}\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix}.$$

We obtain (1,17) by putting $(\xi_n, \eta_n)=(1, 1)$.

It is well known that the simple continued fraction transformato transformation (X, S) has the invariant measure ν with density $d\nu=\dfrac{1}{\log 2}\dfrac{dx}{1+x}$ and that the dynamical system (X, S, ν) is ergodic. The following was proved in [3] and [6].

Theorem. *The mediant convergent transformation* (X, T) *has a* σ*-finite invariant measure* μ:

$$d\mu=\frac{dx}{x}$$

and the dynamical system (X, T, μ) *is ergodic.*

This can also be seen by using a suitable jump transformation [9].

Here we introduce the natural extension of (X, T). We will see afterwards that the natural extension is useful for number theoretical considerations.

Let $\bar{X}=[0, 1]\times[0, 1]$ and let the map $\bar{T}$ be defined on $\bar{X}$ by

$$(1,18)\qquad \bar{T}(x,y)=\begin{cases}\left(\dfrac{x}{1-x},\ \dfrac{y}{1+y}\right), & \text{if}\quad x\in I_0\\[2mm] \left(\dfrac{1-x}{x},\ \dfrac{1}{1+y}\right), & \text{if}\quad x\in I_1\end{cases}$$
$$=\begin{cases}(Tx,\ V_0y), & \text{if}\quad x\in I_0\\ (Tx,\ V_0y), & \text{if}\quad x\in I_1\,.\end{cases}$$

Then we see the map $\bar{T}$ is one to one and onto.

Theorem 1.3. *Let $\bar{\mu}$ be the measure on $\bar{X}$ given by*

$$(1,19)\qquad d\bar{\mu}=\frac{dxdy}{(x+y-xy)^2}\,.$$

Then $\bar{\mu}$ is a σ-finite invariant measure for $\bar{T}$, and the natural extension $(\bar{X}, \bar{T}, \bar{\mu})$ is ergodic.

Proof. The Jacobian $J(\bar{T})$ of $\bar{T}$ is

$$J(\bar{T})=\begin{cases}\dfrac{1}{(1-x)^2}\cdot\dfrac{1}{(1+y)^2}, & \text{if}\quad x\in I_0\\[2mm] \dfrac{1}{x^2}\cdot\dfrac{1}{(1+y)^2}, & \text{if}\quad x\in I_1.\end{cases}$$

Putting $k(x, y)=\dfrac{1}{(x+y-xy)^2}$, it is not difficult to see that the following equation holds:

$$k(\bar{T}(x, y))J(\bar{T})=k(x, y)\,.$$

Hence $\bar{\mu}$ is an invariant measure for $\bar{T}$. The ergodicity of $(\bar{X}, \bar{T}, \bar{\mu})$ is due to [6].

Sub-lemma. *Let $(\varepsilon_1, \cdots, \varepsilon_n)$ be a $0-1$ sequence. Put*

$$\begin{pmatrix} r_n & s_n\\ t_n & u_n\end{pmatrix}=A_{\varepsilon_1}\cdots A_{\varepsilon_n}$$

and

$$(1,20)\qquad \begin{pmatrix} r'_n & s'_n\\ t'_n & u'_n\end{pmatrix}=A_{\varepsilon_n}\cdots A_{\varepsilon_1}.$$

Then

$$(1,21)\qquad t'_n+u'_n=r_n \quad \textit{and} \quad r'_n+s'_n=r_n+s_n\,.$$

The proof is easily obtained by induction.

Fundamental-lemma.

$$\bar{T}^n(x, 1) = \left(T^n x, \frac{r_n}{r_n + s_n}\right).$$

Proof. By the definition of $\bar{T}$ and notation (1,20), we have

$$\bar{T}^n(x, y) = \left(T^n x, \frac{t'_n + u'_n y}{r'_n + s'_n y}\right).$$

In particular,

$$\bar{T}^n(x, 1) = \left(T^n x, \frac{r_n}{r_n + s_n}\right) \qquad \text{(sub-lemma)}.$$

We know the following basic properties:

(1) If $q|qx-p|<1/2$ and $(q, p)=1$, then $\frac{p}{q}$ is a principal convergent of x, i.e., there exists k such that $\frac{p}{q}=\frac{p_k}{q_k}$ (Legendre's theorem [8]).

(2) If $q|qx-p|\leq 1$ and $(q, p)=1$, then $\frac{p}{q}$ si a principal or a mediant convergent of x.

Conversly, for all irrational x

$$(3) \qquad q_n|q_n x-p_n|<1 \qquad \text{for all} \quad n\geq 1.$$

For the mediant convergents $\frac{w_n}{v_n}$, the values $v_n|v_n\cdot x-w_n|$ are unbounded in general. In fact, put

$$(1,22) \qquad f(x, y) = \frac{1-x}{y(1-x)+x} \qquad \text{on} \quad \bar{X}.$$

Then from proposition 1.2. and the fundamental lemma we have

$$(1,23) \qquad v_n|v_n\cdot x-w_n| = f(\bar{T}^n(x, 1)) \qquad n\geq 1.$$

This suggests that the values $v_n|v_n\cdot x-w_n|$ are unbounded for some x.

Let $D_\lambda(\lambda>0)$ be the subset of $\bar{X}$ defined by

$$D_\lambda = \{(x, y)\in\bar{X}; f(x, y)\leq\lambda\}.$$

Then we have

Proposition 1.4. *For any irrational* $x\in(0, 1)$

$$v_n|v_n\cdot x-w_n|\leq\lambda \quad \textit{iff} \quad \bar{T}^n(x, 1)\in D_\lambda.$$

2 Nearest mediant convergent transformation

In this section another algorithm which will be called nearest mediant convergents transformation is proposed.

Let $X=[0, 1]$ and let the map T_1 be defined on X by

$$(2,1)\qquad T_1x=\begin{cases}\dfrac{1-x}{x}, & \text{if } x\in J_1\\[2mm] \dfrac{x}{1-x}, & \text{if } x\in J_2\\[2mm] \dfrac{x}{1-(k-2)x}, & \text{if } x\in J_k \quad (k\geq 3)\end{cases}$$

where

$$J_n=\left[\frac{1}{n+1}, \frac{1}{n}\right)$$

(see figure 2).

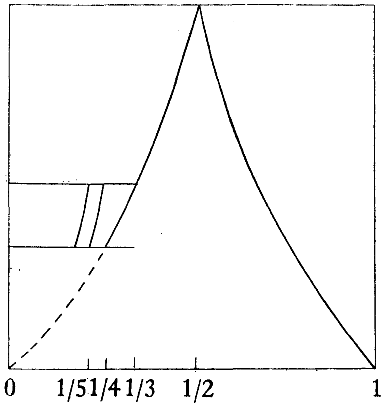

figure 2

The relations between the maps T, S and T_1 are as follows:

$$(2,2)\qquad T_1x=\begin{cases}Tx, & \text{if } x\in J_1\cup J_2\\ T^{k-2}x, & \text{if } x\in \bigcup_{k\geqq 3} J_k\end{cases}$$

and

$$(2,3)\qquad Sx=\begin{cases}T_1x, & \text{if } x\in J_1\\ T_1^2x, & \text{if } x\in J_2\\ T_1^3x, & \text{if } x\in \bigcup_{k\geqq 3} J_k .\end{cases}$$

We denote the inverse branches of T_1 by

$$Z_1(x) = \frac{1}{1+x} \qquad x \in [0, 1]$$

(2,4)
$$Z_2(x) = \frac{x}{1+x} \qquad x \in [1/2, 1]$$

and

$$Z_k(x) = \frac{x}{1+(k-2)x} \qquad x \in [1/3, 1/2] \quad (k \geq 3)$$

and their associated matrices by

(2,5)
$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 1 & k-2 \\ 0 & 1 \end{pmatrix}.$$

Then, the relations (2,2) and (2,3) have the representations:

(2,6)
$$B_k = \begin{cases} A_1, & \text{if } k=1 \\ A_0, & \text{if } k=2 \\ \underbrace{A_0 \cdots A_0}_{k-2}, & \text{if } k \geq 3 \end{cases}$$

and

(2,7)
$$C_k = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} = \begin{cases} B_1, & \text{if } k=1 \\ B_2 B_1, & \text{if } k=2 \\ B_k B_2 B_1, & \text{if } k \geq 3. \end{cases}$$

Put $\delta_n = \delta_n(x) = k$, if $T_1^{n-1} x \in J_k$. Then the sequences of digits δ_n have the following Markov property:

(2,8)
$$\begin{array}{lll} \text{if } \delta_i \geq 3, & \text{then } \delta_{i+1} = 2 \\ \text{if } \delta_i = 2, & \text{then } \delta_{i+1} = 1. \\ \text{if } \delta_i = 1, & \text{then there is no restriction on } \delta_{i+1}. \end{array}$$

Let the 2×2 matrix $\begin{pmatrix} r_n^{(1)} & s_n^{(1)} \\ t_n^{(1)} & u_n^{(1)} \end{pmatrix}$ be defined by

(2,9)
$$\begin{pmatrix} r_n^{(1)} & s_n^{(1)} \\ t_n^{(1)} & u_n^{(1)} \end{pmatrix} = B_{\delta_1(x)} B_{\delta_2(x)} \cdots B_{\delta_n(x)}.$$

Then we have the following.

Proposition 2.1. *For any irrational* $x \in (0, 1)$

(2,10)
$$x = \frac{t_n^{(1)} + u_n^{(1)} \cdot T_1^n x}{r_n^{(1)} + s_n^{(1)} \cdot T_1^n x}$$

The proof is easily obtained by using the identity:

$$x = Z_{\delta_1(x)}(Z_{\delta_2(x)} \cdots Z_{\delta_n(x)}(T_1^n x)) .$$

Sub-lemma

(i) *If* $x \in J_k$ $(k \geq 3)$, *then* (1) $a_1(x) = \delta_1(x) = k$, $\delta_2(x) = 2$, $\delta_3(x) = 1$

(2) $T_1^3 x = Sx$

(3) $B_{\delta_1(x)} B_{\delta_2(x)} B_{\delta_3(x)} = \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix}$

(ii) *If* $x \in J_2$, *then* (1) $a_1(x) = \delta_1(x) = 2$, $\delta_2(x) = 1$

(2) $T_1^2 x = Sx$

(3) $B_{\delta_1(x)} B_{\delta_2(x)} = \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix}$

(iii) *If* $x \in J_1$, *then* (1) $a_1(x) = \delta_1(x) = 1$

(2) $T_1 x = Sx$

(3) $B_{\delta_1(x)} = \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix}.$

Lemma 2.1. *Let* $j = j(n: x) = {}^\sharp\{k: \delta_k(x) = 1, k \leq n\}$ *and* $l = l(n: x) = \max\{k: \delta_k(x) = 1, k \leq n\}$. *Then the matrix* $\begin{pmatrix} r_n^{(1)} & s_n^{(1)} \\ t_n^{(1)} & u_n^{(1)} \end{pmatrix}$ *has one of the following forms*:

$$\begin{pmatrix} r_n^{(1)} & s_n^{(1)} \\ t_n^{(1)} & u_n^{(1)} \end{pmatrix} = \begin{cases} \begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{pmatrix}, & \text{if } n = l \\ \begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{if } n - l = 1 \text{ and } S^j x \in J_2 \\ \begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{pmatrix} \begin{pmatrix} 1 & a_{j+1} - 2 \\ 0 & 1 \end{pmatrix}, & \text{if } n - l = 1 \text{ and } S^j x \in J_k \\ \begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{pmatrix} \begin{pmatrix} 1 & a_{j+1} - 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{if } n - l = 2 \text{ and } S^j x \in J_k \end{cases}$$

Proof. From the sublemma we have

$$B_{\delta_1(x)} B_{\delta_2(x)} \cdots B_{\delta_n(x)} = \begin{cases} \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix} B_{\delta_4(x)} \cdots B_{\delta_n(x)}, & \text{if } x \in J_k \\ \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix} B_{\delta_3(x)} \cdots B_{\delta_n(x)}, & \text{if } x \in J_2 \\ \begin{pmatrix} a_1(x) & 1 \\ 1 & 0 \end{pmatrix} B_{\delta_2(x)} \cdots B_{\delta_n(x)}, & \text{if } x \in J_1 . \end{cases}$$

Repeating this procedure with x replaced by Sx, S^2x, $\cdots$, S^jx and so on, we obtain the lemma. For $j = 0$, lemma 2.1. is also valid.

Theorem 2.1. *Put* $v_n^{(1)}=r_n^{(1)}(x)+s_n^{(1)}(x)$ *and* $w_n^{(1)}=t_n^{(1)}(x)+u_n^{(1)}(x)$. *Then for any irrational* $x\in(0, 1)$

$$\left\{\frac{w_n^{(1)}}{v_n^{(1)}};\, n\geq 1\right\}=\bigcup_{k=1}^{\infty}\left\{\frac{p_k-p_{k-1}}{q_k-q_{k-1}},\ \frac{p_k}{q_k},\ \frac{p_k+p_{k-1}}{q_k+q_{k-1}}\right\}.$$

Proof. By lemma 2.1.

$$\frac{w_n^{(1)}}{v_n^{(1)}}=\begin{cases}\dfrac{p_j+p_{j-1}}{q_j+q_{j-1}}, & \text{if } n=l\\[2mm] \dfrac{2p_j+p_{j-1}}{2q_j+q_{j-1}}=\dfrac{p_{j+1}}{q_{j+1}}, & \text{if } n-l=1 \text{ and } S^jx\in J_2,\\[2mm] \dfrac{(a_{j+1}-1)p_j+p_{j-1}}{(a_{j+1}-1)q_j+q_{j-1}}=\dfrac{p_{j+1}-p_j}{q_{j+1}-q_j}, & \text{if } n-l=1 \text{ and } S^jx\in J_k\\[2mm] \dfrac{p_{j+1}}{q_{j+1}}, & \text{if } n-l=2 \text{ and } S^jx\in J_k\end{cases}$$

Therefore for any $n\geq 1$.

$$\frac{w_n^{(1)}}{v_n^{(1)}}\in\bigcup_{k=1}^{\infty}\left\{\frac{p_k-p_{k-1}}{q_k-q_{k-1}},\ \frac{p_k}{q_k},\ \frac{p_k+p_{k-1}}{q_k+q_{k-1}}\right\}.$$

Conversely, from (2,7), for any $\dfrac{p_k+p_{k-1}}{q_k+q_{k-1}}$ there exists n such that

$$\begin{pmatrix} q_k & q_{k-1}\\ p_k & p_{k-1}\end{pmatrix}=\begin{pmatrix} r_n^{(1)} & s_n^{(1)}\\ t_n^{(1)} & u_n^{(1)}\end{pmatrix}.$$

Therefore

$$\frac{q_k+q_{k-1}}{p_k+p_{k-1}}=\frac{w_n^{(1)}}{v_n^{(1)}}.$$

Similary, for any $\dfrac{p_k}{q_k}$, if $a_k\geqq 2$ then there exists an n such that

$$\begin{pmatrix} q_{k-1} & q_{k-2}\\ p_{k-1} & p_{k-2}\end{pmatrix}\begin{pmatrix} 1 & a_k-2\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix} r_n^{(1)} & s_n^{(1)}\\ t_n^{(1)} & u_n^{(1)}\end{pmatrix},$$

if $a_k=1$ then there exists an n such that

$$\begin{pmatrix} q_{k-1} & q_{k-2}\\ p_{k-1} & p_{k-2}\end{pmatrix}=\begin{pmatrix} r_n^{(1)} & s_n^{(1)}\\ t_n^{(1)} & u_n^{(1)}\end{pmatrix}.$$

Therefore

$$\frac{p_k}{q_k}=\frac{w_n^{(1)}}{v_n^{(1)}}.$$

Finally, for any $\frac{p_k-p_{k-1}}{q_k-q_{k-1}}\left(\neq\frac{p_{k-2}}{q_{k-2}}\right)$, there exists an n such that

$$\begin{pmatrix} q_{k-1} & q_{k-2} \\ p_{k-1} & p_{k-2} \end{pmatrix}\begin{pmatrix} 1 & a_k-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_n^{(1)} & s_n^{(1)} \\ t_n^{(1)} & u_n^{(1)} \end{pmatrix},$$

hence

$$\frac{p_k-p_{k-1}}{q_k-q_{k-1}} = \frac{w_n^{(1)}}{v_n^{(1)}}.$$

We now call the fraction $\frac{w_n^{(1)}}{v_n^{(1)}}=\frac{t_n^{(1)}(x)+u_n^{(1)}(x)}{r_n^{(1)}(x)+s_n^{(1)}(x)}$ the n-th *nearest mediant convergent* of x, and the algorithm (X, T_1) the *nearest mediant convergent* transformation. We prepare also some formula concering the approximation.

Proposition 2.2. *For any irrational* $x\in(0, 1)$

$$(2,11)\qquad \left|x-\frac{w_n^{(1)}(x)}{v_n^{(1)}(x)}\right| = \frac{1-T_1^n x}{(v_n^{(1)})^2\left\{\frac{r_n^{(1)}}{v_n^{(1)}}(1-T_1^n x)+T_1^n x\right\}}$$

The proof is the same as for proposition 1.2.

Proposition 2.3. *For any irrational* $x\in(0, 1)$

$$(2,12)\qquad |x\cdot v_n^{(1)}(x)-w_n^{(1)}(x)| = g_1(x)g_1(T_1x)\cdots g_1(T_1^{n-1}x)(1-T_1^n x)$$

where

$$g_1(x) = \begin{cases} x, & \text{if } x\in J_1 \\ 1-x, & \text{if } x\in J_2 \\ 1-(k-2)x, & \text{if } x\in J_k \quad (k\geq 3). \end{cases}$$

Proof. The proof is similar to that of Proposition 1.3. In fact, for each sequence $(\delta_1(x), \cdots, \delta_n(x))$, we consider the affine transformations φ_{δ_i} from (ξ_{i-1}, η_{i-1})-plane to the (ξ_i, η_i)-plane defined by

$$\varphi_{\delta_i}\colon \begin{pmatrix} \xi_{i-1} \\ \eta_{i-1} \end{pmatrix} = B_{\delta_i}\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}.$$

The absolute value of the linear form $x\cdot\xi_0-\eta_0$ is transformed in the following way:

$$|x\cdot\xi_0-\eta_0| = g_1(x)\,|T_1x\cdot\xi_1-\eta_1|.$$

Therefore, we have

$$(2,13)\qquad |x\cdot\xi_0-\eta_0| = g_1(x)\cdots g_1(T_1^{n-1}x)\,|T_1^n x\cdot\xi_n-\eta_n| \qquad (n\geq 1).$$

On the other hand we know

$$\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix} = B_{\delta_1(x)} \cdots B_{\delta_n(x)} \begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix} = \begin{pmatrix} r_n^{(1)} & s_n^{(1)} \\ t_n^{(1)} & u_n^{(1)} \end{pmatrix}\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix}.$$

Hence, we obtain the result by putting the value $(\xi_n, \eta_n)=(1,1)$ into (2,13).

Now, we introduce the natural extension of (X, T_1). Let R be the subset of $\bar{X}$ such that

$$\begin{aligned} R &= \{(x, y)\in\bar{X};\ x\geq 1/3 \text{ or } (x\leq 1/3 \text{ and } y\geq 1/2)\} \\ &= J_1\times I \cup J_2\times I \cup (\bigcup_{k\geqq 3} J_k\times I_1) \end{aligned}$$

where $I=[0, 1]$ and $I_1=[1/2, 1]$, and let the map $\bar{T}_1$ be defined on R by

$$\bar{T}_1(x, y) = \begin{cases} \left(\dfrac{1-x}{x}, \dfrac{1}{1+y}\right), & \text{if } (x, y)\in J_1\times I \\ \left(\dfrac{x}{1-x}, \dfrac{y}{1+y}\right), & \text{if } (x, y)\in J_2\times I \\ \left(\dfrac{x}{1-(k-2)x}, \dfrac{y}{1+(k-2)y}\right), & \text{if } (x, y)\in J_k\times I_1 \end{cases}$$

In other words,

$$= \begin{cases} (T_1x, Z_1(y)), & \text{if } (x, y)\in J_1\times I \\ (T_1x, Z_2(y)), & \text{if } (x, y)\in J_2\times I \\ (T_1x, Z_k(y)), & \text{if } (x, y)\in J_k\times I_1 . \end{cases}$$

Theorem 2.1. *The transformation $(R, \bar{T}_1)$ is the induced map of $(\bar{X}, \bar{T})$ on R. Therefore, the transformation $(R, \bar{T}_1)$ has an invariant probability measure $\bar{\mu}_R$ with density*

$$d\bar{\mu}_R = \frac{1}{2\log 2}\cdot\frac{dxdy}{(x+y-xy)^2}.$$

Moreover, the dynamical system $(R, \bar{T}_1, \bar{\mu}_R)$ is ergodic.

Proof. From the definition (2,1), we can easily see that

$$\begin{aligned} \bar{T}(J_1\times I) &= I\times[1/2, 1] \\ \bar{T}(J_2\times I) &= J_1\times[0, 1/2] \end{aligned}$$

and for $k\geq 3$

$$\bar{T}^{k-2}(J_k\times I) = J_2\times\left[\frac{1}{k}, \frac{1}{k-1}\right)$$

and

$$\bar{T}^j(J_k\times I)\cap R=\phi \qquad (1\leq j<k-2).$$

(see figure 3).

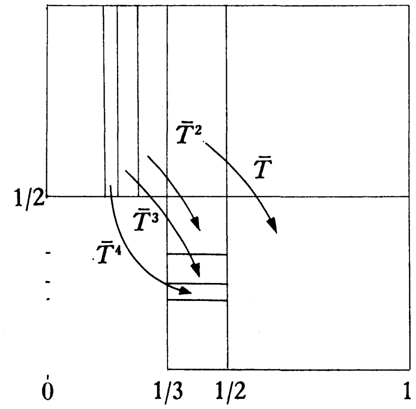

figure 3

Therefore let $\bar{T}_R$ be the induced automorphism of $\bar{T}$ on R, then

$$\bar{T}_R(x, y) = \bar{T}_1(x, y) .$$

Hence by proposition 2.1. the invariant measure $\bar{\mu}_R$ is given by

$$d\bar{\mu}_R = \frac{1}{2\log 2} \cdot \frac{dxdy}{(x+y-xy)^2}$$

where $2 \log 2$ is a normalizing constant. The ergodicity of the dynamical system $(R, \bar{T}_1, \bar{\mu}_R)$ follows from the ergodicity of $(\bar{X}, \bar{T}, \bar{\mu})$.

Taking the marginal distribution we have

Corollary 2.1. *The transformation* (X, T_1) *has an invariant measure* μ_1:

$$d\mu_1 = \begin{cases} \dfrac{1}{2\log 2} \cdot \dfrac{dx}{1+x}, & \text{if } x \in [0, 1/3] \\ \dfrac{1}{2\log 2} \cdot \dfrac{dx}{x}, & \text{if } x \in [1/3, 1] \end{cases}$$

and the dynamical system (X, T_1, μ_1) *is ergodic.*

Corollary 2.3.

$$\bar{T}_1^n(x, 1) = \left(T_1^n x, \frac{r_n^{(1)}}{r_n^{(1)}+s_n^{(1)}}\right)$$

Proof. There exists an $m=m(n, x, 1)$ such that $\bar{T}_1^n(x, 1)=\bar{T}^m(x, 1)$ and so $r_m=r_n^{(1)}$ and $s_m=s_n^{(1)}$. Therefore we obtain the result, from the fundamental lemma in § 1.

Corollary 2.3.

(i) *Let a fraction* $\frac{p}{q}$ *satisfies* $q|q\cdot x-p|<1$ *and* $(q, p)=1$. *Then there exists a* k *such that* $\frac{p}{q}=\frac{w_k^{(1)}}{v_k^{(1)}}$ (Fatou).

(ii) *If* $\frac{w_n^{(1)}}{v_n^{(1)}}$ *is the n-th convergent of x, then* $v_n^{(1)}|v_n^{(1)}\cdot x-w_n^{(1)}|\leq 2$.

Proof. To prove (i), note that by property 1.2. and theorem 1.1., there exists n such that $\frac{p}{q}=\frac{w_n}{v_n}$, that is, $\frac{w_n}{v_n}$ is the n-th mediant convergent and satisfies

$$v_n|v_n x-w_n|\leq 1 .$$

From proposition 1.4. this is equivalent to

$$\bar{T}^n(x, 1)\in D_1 .$$

Since D_1 is a subset of R, there exists k such that

$$\bar{T}_R^k(x, 1)=\bar{T}^n(x, 1)$$

, in other words, $\bar{T}_R^k(x, 1)=\bar{T}_1^k(x, 1)$. Therefore

$$\frac{w_n}{v_n}=\frac{w_k^{(1)}}{v_k^{(1)}} .$$

Part (ii) can be seen as follows. By proposition 2.1. and corollary 2.2.

$$v_n^{(1)}|v_n^{(1)}\cdot x-w_n^{(1)}|=f(\bar{T}_1^k(x, 1)) .$$

On the other hand, $\bar{T}_1^n(x, 1)\in R$ and $D_2\supset R$. Therefore

$$v_n^{(1)}|v_n^{(1)}\cdot x-w_n^{(1)}|\leq 2 .$$

3. Some metrical results

In this section we prove Erdös' theorem for $0<\lambda\leqq 1$ by using the ergodic theorem.

Proposition 3.1. *For almost all* $x\in(0, 1)$

$$\lim_{n\to\infty}\frac{1}{n}\log v_n^{(1)}(x)=\frac{\pi^2}{24\log 2} . \tag{3,1}$$

Proof. Let $\{a_k(x); k\leqq N\}$ be a sequence of digits with respect to a simple continued fraction. Put

$$n(m)={}^\sharp\{k; a_k(x)=1, k\leq m\}+2^\sharp\{k; a_k(x)=2, k\leq m\}+3^\sharp\{k; a_k(x)\geq 3, k\leq m\} .$$

Then, from theorem 2.1. we have

$$v^{(1)}_{n(m)} = q_m \qquad \text{for all} \quad m \geq 1 .$$

By using the ergodic theorem for the dynamical system (X, S, ν), we know ([1]) that for almost all $x \in (0, 1)$

(1) $\lim\limits_{m\to\infty} \dfrac{n(m)}{m} = \nu(J_1) + 2\nu(J_2) + 3\nu(\bigcup\limits_{k\geq 3} J_k) = 2$

and

(2) $\lim\limits_{m\to\infty} \dfrac{1}{m} \log q_m = \dfrac{\pi^2}{12 \log 2}.$

Therefore,

$$\lim_{m\to\infty} \frac{1}{n(m)} \log v^{(1)}_{n(m)} = \lim_{m\to\infty} \frac{m}{n(m)} \frac{1}{m} \log q_m = \frac{\pi^2}{24 \log 2} .$$

Noting that $mn(m+1) - n(m) \leq 3$ and $v^{(1)}_{n+1} > v^{(1)}_n$, we get the result.

Proposition 3.3. *For almost all* $x \in (0, 1)$

(i) $\lim\limits_{n\to\infty} -\dfrac{1}{n} \log | v^{(1)}_n \cdot x - w^{(1)}_n | = \dfrac{\pi^2}{24 \log 2}$

and

(ii) $\lim\limits_{n\to\infty} -\dfrac{1}{n} \log \left| x - \dfrac{w^{(1)}_n}{v^{(1)}_n} \right| = \dfrac{\pi^2}{12 \log 2}$

Proof. From proposition 2.2. we have

$$-\frac{1}{n} \log | v^{(1)}_n \cdot x - w^{(1)}_n | = \frac{1}{n} \log v^{(1)}_n - \frac{1}{n} \log f(\bar{T}^n_1(x, 1)) .$$

We show that for almost all $x \in (0, 1)$

$$\lim_{n\to\infty} \frac{1}{n} \log f(\bar{T}^n_1(x, 1)) = 0 . \tag{3,2}$$

From (1,21) and theorem 2.2. we have

$$\bar{\mu}_1(f(\bar{T}^n_1(x, y)) > \eta) = \bar{\mu}_1(f(x, y) < \eta) = \frac{\eta}{2 \log 2}$$

for $0 < \eta \leq 1$.

Therefore, we see that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \bar{\mu}_1 \{ f(\bar{T}^n_1(x, y)) < e^{-n\varepsilon} \} < +\infty .$$

Hence, by using the Borel-Cantelli lemma,

$$^{\#}\left\{ n ; -\frac{1}{n} \log f(\bar{T}^n(x, y) > \varepsilon \right\} < +\infty$$

for almost all (x, y), that is,

$$\lim_{n\to\infty}\frac{1}{n}\log f(\bar{T}_1^n(x, y))=0 \qquad \text{for} \quad \text{a.a. } (x, y). \tag{3,3}$$

Note that the following inequality holds:

$$|f(x, y)-f(x, y')|\leq c|y-y'|$$

where

$$c=\frac{1}{\min\limits_{(x,y)\in E}\left(y+\dfrac{x}{1-x}\right)}.$$

In particular, remarking that from the definition of $\bar{T}_1$

$$\bar{T}_1^n(x, y)=\left(T_1^n x, \frac{t_n'^{(1)}+y\cdot u_n'^{(1)}}{r_n'^{(1)}+y\cdot s_n'^{(1)}}\right),$$

we have from sublemma in § 1

$$|f(\bar{T}_1^n(x, y))-f(\bar{T}_1^n(x, 1))|\leq c\left|\frac{t_n'^{(1)}+yu_n'^{(1)}}{r_n'^{(1)}+ys_n'^{(1)}}-\frac{t_n'^{(1)}+u_n'^{(1)}}{r_n'^{(1)}+s_n'^{(1)}}\right|$$

$$<\frac{c}{r_n'^{(1)}+s_n'^{(1)}}=\frac{c}{v_n^{(1)}}.$$

Therefore, from proposition 3.1. there exists $0<\eta<1$ such that

$$|f(\bar{T}_1^n(x, y))-f(\bar{T}_1^n(x, 1))|<c\cdot\eta^n,$$

and so (3,3) imply (3,2). This completes the proof of (i). Part (ii) is obtained from

$$-\frac{1}{n}\log\left|x-\frac{w_n^{(1)}}{v_n^{(1)}}\right|=2\frac{1}{n}\log v_n^{(1)}-\frac{1}{n}\log f(\bar{T}_1^n(x, 1)).$$

Theorem 3.1. *For almost all $x\in(0, 1)$*

$$\lim_{N\to\infty}\frac{\#\{n;\, v_n^{(1)}|v_n^{(1)}\cdot x-w_n^{(1)}|\leqq\lambda,\ 1\leq n\leq N\}}{N}=\begin{cases}\dfrac{\lambda}{2\log 2} & \textit{for} \quad 0\leqq\lambda<1\\[2ex] \dfrac{2-\lambda+2\log\lambda}{2\log 2} & \textit{for} \quad 1\leqq\lambda<2.\end{cases}$$

Proof. From proposition 1.4. we get

$$\frac{\#\{n;\, v_n^{(1)}|v_n^{(1)}\cdot x-w_n^{(1)}|\leqq\lambda,\ 1\leq n\leq N\}}{N}=\frac{\sum_{n=1}^{N}\chi_\lambda(\bar{T}_1^n(x, 1))}{N},$$

where χ_λ is the indicator function of the set D_λ.

On the other hand, it is clear from the ergodic theorem that

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\chi_\lambda(\bar{T}_1^n(x, y))}{N}=\bar{\mu}_1(D_\lambda)$$

for almostl al (x, y).
Note that

$$\begin{aligned}&\{(x, y): \chi_\lambda(\bar{T}_1^n(x, y))\neq\chi_\lambda(\bar{T}_1^n(x, 1))\}\\&\quad\subset\{(x, y): \lambda-c\eta^n<f(\bar{T}_1^n(x, y))<\lambda+c\eta^n\}\end{aligned}$$

where c and η are the same constants as in the proof of proposition 3.2. Therefore, we have

$$\bar{\mu}_1\{\chi_\lambda(\bar{T}_1^n(x, y))\neq\chi_\lambda(\bar{T}_1^n(x, 1))\}<\frac{c\eta^n}{\log 2}.$$

Hence, by using the Borel-Cantelli lemma, for almost all (x, y)

$$\sharp\{n; \chi_\lambda(\bar{T}_1^n(x, y))\neq\chi_\lambda(\bar{T}_1^n(x, 1))\}<\infty.$$

By easy calculation for $\bar{\mu}_1(D_\lambda)$, we obtain the conclusion.

Theorem 3·3· *For* $1\geq\lambda\geq 0$

$$\lim_{N\to\infty}\frac{\sharp\{(q, p); q|qx-p|<\lambda, (q, p)=1, q\leq N\}}{\log N}=\lambda\frac{12}{\pi^2}$$

for almost all x.

Proof. If $v_{n-1}^{(1)}\leq N<v_n^{(1)}$, then by corollary 2.3.

$$\begin{aligned}&\sharp\{(q, p); q|qx-p|<\lambda,\ (q, p)=1,\ q\leq N\}\\&\geqq\sharp\{(v_k^{(1)}, w_k^{(1)}); v_k^{(1)}|v_k^{(1)}x-w_k^{(1)}|<\lambda,\ k\leq n-1\}.\end{aligned}$$

Hence, by theorem 3.1. and proposition 3.1.

$$\begin{aligned}&\lim_{N\to\infty}\frac{\sharp\{(q, p); q|qx-p|<\lambda,\ (q, p)=1,\ q\leq N\}}{\log N}\\&\geqq\lim_{n\to\infty}\frac{\sharp\{k; v_k^{(1)}v|x_k^{(1)}-w_k^{(1)}|<\lambda,\ 1\leq k<n-1\}}{\log v_n^{(1)}}\\&=\lambda\cdot\frac{12}{\pi^2}\qquad\text{for almost all } x.\end{aligned}$$

Replacing $v_n^{(1)}$ by $v_{n-1}^{(1)}$ we obtain the reverse inequality.

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