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THE LIFTED FUTAKI INVARIANTS AND
THE SPIN\textsuperscript{C}–DIRAC OPERATORS

Dedicated to the memory of Professor Masahisa Adachi

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1. Introduction

The Futaki invariant \( f \) which is a Lie algebra homomorphism (cf. [6]) is naturally lifted to a Lie group homomorphism \( F \) by virtue of the result in [10]. In [11], we obtained a formula to calculate \( 2^{*+1}F \) and showed that \( F \) can be non-trivial even when no nonzero holomorphic vector field exists. Our purpose in this paper is to refine the formula in [11] so that we can calculate \( F \) itself (Theorem 2.10). When \( M \) is a Kähler surface with \( c_1(M)>0 \), the group of holomorphic automorphisms of \( M \) (for generic complex structures) are classified (cf. [14]) and, using Theorem 2.10 and the results in [18], [19], we can show that \( F \) vanishes if and only if \( M \) admits a Kähler-Einstein metric (Theorem 3.6). Moreover we show that \( F \) vanishes for some Kähler manifolds which are shown recently to admit a Kähler-Einstein metric (cf. [16]). Futaki conjectured that \( F \) as well as \( f \) is an obstruction to the existence of Kähler-Einstein metrics on a compact Kähler manifold with \( c_1(M)>0 \). We might take the results obtained in this paper to encourage the Futaki’s conjecture.

Now let \( M \) be a compact \( n \)-dimensional complex manifold. A Kähler metric \( h \) is called a Kähler-Einstein (which is abbreviated to K-E hereafter) metric if there exists a real constant \( k \) such that

\[
\rho(h) = k \omega(h)
\]

where \( \rho(h) \) is the Ricci form of \( h \) and \( \omega(h) \) is the fundamental 2-form of \( h \). Note that the first Chern class \( c_1(M) \) has a definite sign (namely, \( c_1(M)>0 \), \( c_1(M)=0 \) or \( c_1(M)<0 \) according to \( k>0 \), \( k=0 \) or \( k<0 \)) if \( M \) admits a K-E metric because \( c_1(M) \) is represented by \( \rho(h) \). The converse is true when \( c_1(M)=0 \) or \( c_1(M)<0 \).

**Theorem 1.1.** ([3], [21]) *Let \( M \) be a Kähler manifold with \( c_1(M)=0 \) or \( <0 \). Then \( M \) admits a K-E metric.*
So the problem is whether $M$ admits a K-E metric if $c_1(M) > 0$.

Now let $A(M)$ be the Lie group of all holomorphic automorphisms of $M$ and $H(M)$ its Lie algebra consisting of all holomorphic vector fields on $M$. When $c_1(M) > 0$ and $H(M) \neq \{0\}$, there exists an obstruction to the existence of K-E metrics called the Futaki invariant (cf. see [6]). The Futaki invariant $f : H(M) \to \mathbb{C}$ can be expressed as follows:

\[(1.2)\quad f(X) = \frac{(n+1)i}{2\pi} \int_M \text{div}_h(X) \rho(h)^n\]

for any $X \in H(M)$ where $h$ is any Kähler metric on $M$ and $\text{div}_h$ is the divergence with respect to $h$. It is shown [6], [10] that $f(X)$ is determined only by the complex structure of $M$ and is independent of the choice of $h$ and that $f$ is a Lie algebra homomorphism. ($\mathbb{C}$ is regarded as a trivial Lie algebra.) If $h$ is a K-E metric, the right term of (1.2) is equal to

\[f(X) = \frac{(n+1)i}{2\pi} \int_M \text{div}_h(X) \omega(h)^n\]

which equals to 0 by the divergence formula. Since $f(X)$ is independent of the choice of $h$, the following result can be deduced.

**Theorem 1.3.** [6] _If $M$ admits a K-E metric, then $f(X) = 0$ for any $X \in H(M)$. _

When $H(M) = \{0\}$, there is no known obstruction to the existence of K-E metrics, and it is not known whether there exists an example of $M$ such that $c_1(M) > 0$, $H(M) = \{0\}$ but $M$ does not admit any K-E metric.

On the other hand, by virtue of the result in [10], $f$ can naturally be lifted to a group homomorphism $F : A(M) \to \mathbb{C}/\mathbb{Z}$ as follows.

**Definition 1.4.** Fix any $g \in A(M)$. Let $M_g$ denote the mapping torus $M_g = M \times [0,1] / \sim$ where $(p,0) \sim (g(p),1)$. Let $\mathcal{F}_g$ denote the holomorphic foliation defined by the $[0,1]$-directed vector field. Then, by definition,

\[(1.5)\quad F(g) = Sc_1^{n+1}(v(\mathcal{F}_g))[M_g] \in \mathbb{C}/\mathbb{Z}\]

where $[M_g]$ is the fundamental cycle of $M_g$ and

\[Sc_1^{n+1}(v(\mathcal{F}_g)) \in H^{2n+1}(M_g; \mathbb{C}/\mathbb{Z})\]

is the Simons character of the first Chern class $c_1$ to the power $n+1$ for the normal bundle $v(\mathcal{F}_g)$ with respect to any Bott connection. (For details, see [10], [17].)
Then, it is shown [7] that $F: A(M) \to C/Z$ is a Lie group homomorphism where $C/Z$ is regarded as an additive group, and the following holds.

**Theorem 1.6.** [10] We have $F(\exp X) = f(X) \mod Z$ for any $X \in H(M)$. In particular, we have $F_\ast = f$.

Though it immediately follows from Theorem 1.3 and Theorem 1.6 that $F|_{A_0(M)}$ (where $A_0(M)$ denotes the identity component of $A(M)$) is an obstruction to the existence of K-E metrics on $M$, it is not known whether $F$ itself is an obstruction to the existence of K-E metrics on $M$ or not. If the Futaki's conjecture turns out to be true, $F$ may become the unique obstruction which is valid even when $H(M) = \{0\}$.

**Remark 1.7.** In [9], $f$ is lifted to a group homomorphism $\det \circ \phi: A(M) \to C^\ast (\simeq C/Z)$. A multiple of $f$ gives rise to a power of the lifting. Theorem 1.6 implies that $f$ is normalized so as to satisfy the integrability condition that $f(X)$ is an integer for any $X \in H(M)$ such that $\exp X = 1$.

2. A calculation formula for $F$

Let $M$ be a compact $n$-dimensional complex manifold and $M_g$ the mapping torus for $g \in A(M)$ defined as in Definition 1.4. In [11], we showed that $2^{n+1}F$ is equal to the eta invariant of the signature operator on $M_g$. In this section, we shall show a similar formula by using the spin$^c$-Dirac operators.

Now fix an element $g \in A(M)$ which we assume has a finite order $p \geq 2$. (Note that $A(M)$ itself is a finite group if $c_1(M) > 0$ and $H(M) = \{0\}$.) We may assume that $g$ preserves the Hermitian metric $h$ on $M$. Then the Hermitian connection $\nabla^M$ of the holomorphic tangent bundle $TM$, which is uniquely determined under the conditions that the connection form of $\nabla^M$ is of type $(1,0)$ and that $\nabla^M$ preserves $h$, is necessarily $g$-invariant.

Let $X = M \times D^2$, $Y = \partial X = M \times S^1$ be spin$^c$-manifolds with the spin$^c$-structures defined by the $U(n)$-structure of $M$ and the trivial spin$^c$-structures of $D^2$, $S^1$, respectively. Then the cyclic group $K = Z_p = \langle g \rangle$ acts on $(X, Y)$ as follows:

$$g(m, re^{i\theta}) = (g(m), re^{i\theta + 2\pi/p})$$

for $(m, re^{i\theta}) \in X = M \times D^2; 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. Note that $Y/K = M_g$, $(TM \times S^1)/K = \nu(\mathcal{F}_g)$ and that $\nabla^M$ naturally defines a Bott connection $\nabla^\mathcal{F}$ of $\nu(\mathcal{F}_g)$. On the other hand, we give a rotationally symmetric Hermitian metric on the complex manifold $D^2$ such that it is a product metric of $S^1 \times [0,\delta)$ near the boundary $\partial D^2 = S^1$. Then the complex structures and the Hermitian metrics on $M$, $D^2$ define a $K$-invariant complex structure and a $K$-invariant Hermitian metric on $X$. Let $\nabla^X$ be the $K$-invariant Hermitian connection of $TX$. Then $\nabla^X|_Y$ descends to
a Hermitian connection $\nabla^{X/K}$ of $T(X/K)|_{M_g}$ and it can be shown

$$T(X/K)|_{M_g} = (TX|_{Y})/K = v(F_g) \oplus \epsilon$$

where $\epsilon$ denotes the trivial complex line bundle of all $F_g$-directed vectors and $\nabla^{X/K}$ splits as

$$\nabla^{X/K} = \nabla^F \oplus \nabla^0$$

where $\nabla^0$ denotes the globally flat connection of $\epsilon$.

Now, since $M_g$ is a stably almost complex manifold, it follows from the result of Morita[15] that there exists a compact $(2n+2)$-dimensional almost complex manifold $W$ such that $\partial W = M_g$ and $W = X/K$ near $M_g$ as an almost complex manifold with a Hermitian metric. Then we have the following lemma by the same arguments as in the proof of Theorem 3.7 in [11].

**Lemma 2.1.** We have $F(g) = \int_Y c_1(TW)^{g+1}$ where $c_1(TW)$ is the first Chern form of $TW$ with respect to a unitary connection $\nabla^W$ of $TW$ (namely, $\nabla^W$ preserves the metric and the almost complex structure on $TW$) which coincides with $\nabla^{X/K}$ near $M_g$.

Now, let $\xi$ be the virtual complex vector bundle over $M$ defined by

$$\xi = \bigotimes^{n+1}(K^{-1}_M - \epsilon)$$

where $K^{-1}_M$ is the anticanonical bundle of $M$ and $\epsilon$ is the trivial complex line bundle over $M$. Set $\xi_X = q^*\xi$ and $\xi_Y = q^*\xi$ where $q_X : X = M \times D^2 \to M$ and $q_Y : Y = M \times S^1 \to M$ are the canonical projections. $\xi_X$ and $\xi_Y$ are virtual vector bundles with unitary connections with respect to the metrics and the connections naturally defined by the Hermitian metric and the Hermitian connection of $TM$. Using the spin^c-structures, the metrics and the connections of $TX$ and $TY$, we can define the spin^c-Dirac operators (or Dolbeault operators)

$$D_X : \Gamma(E^+_X \otimes \xi_X) \to \Gamma(E^-_X \otimes \xi_X)$$

$$D_Y : \Gamma(E_Y \otimes \xi_Y) \to \Gamma(E_Y \otimes \xi_Y)$$

where $E^+_X$ denote the half spinor bundles over $X$ and $E_Y = E^+_X|_Y = E^-_X|_Y$ is the spinor bundle over $Y$. (For details of spin^c-Dirac operators and Dolbeault operators on almost complex manifolds, see [12],[13].) Since the metric and the connection of $TX$ is $K$-invariant and is product near $\partial X = Y$, $D_X$ and $D_Y$ are $K$-invariant and $D_X$ can be expressed as

$$D_X = \sigma \left( \frac{\partial}{\partial u} + D_Y \right)$$

on the collar $Y \times [0,\delta) \subset X$ where $u$ is the coordinate of $[0,\delta)$ and $\sigma$ is a bundle
Theorem 2.4. [1] We have

$$\text{Index}(D_X) = \int_X Ch(\xi_X)Td(X) - \frac{1}{2}(\eta_Y + h_Y)$$

where $\text{Index}(D_X)$ is the index of $D_X$ with a certain global boundary condition, $Ch(\xi_X)$ is the Chern character form of $\xi_X$ with the unitary connection, $Td(X)$ is the Todd form of $(TX, \nabla_X)$, $\eta_Y$ is the eta invariant of $D_Y$ and $h_Y = \dim(\text{Ker } D_Y)$.

Now, let $\xi_g = \xi_Y/K$ be a virtual vector bundle over $M_g$ with a unitary connection. Then, since $D_Y$ is $K$-invariant, $D_Y$ naturally defines a differential operator $D_g$, which is the $\xi_g$-valued spin$^c$-Dirac operator on $M_g = Y/K$. Our first result is the following.

Theorem 2.5. We have

$$F(g) = -\eta_g \pmod{Z}$$

where $\eta_g$ is the eta invariant of $D_g$.

Proof. Set

$$\xi_w = \otimes^{n+1}(\wedge^{n+1}TW - 1)$$

where $\varepsilon$ denotes the trivial complex line bundle over $W$. Note that $\wedge^{n+1}TW$ is also a complex line bundle over $W$. The unitary connection $\nabla_w$ of $TW$ naturally defines a unitary connection of $\xi_w$. Then the spin$^c$-Dirac operator

$$D_w : \Gamma(E_w \otimes \xi_w) \to \Gamma(E_w \otimes \xi_w)$$

is defined similarly as in (2.2). It can be seen that $\xi_w|_{M_g} = \xi_g$ and, similarly as in (2.3), $D_w$ can be expressed as

$$D_w = \sigma \left( \frac{\partial}{\partial u} + D_g \right)$$

on the collar $M_g \times [0, \delta) \subset W$. Hence it follows from the Atiyah-Patodi-Singer's theorem (cf. Theorem 2.4) that

$$\int_W Ch(\xi_w)Td(W) = \frac{1}{2}(\eta_g + h_g) \pmod{Z}$$

where $Ch(\xi_w)$ is the Chern character form of $\xi_w$, $Td(W)$ is the Todd form of $TW$. 


and \( h_g = \dim(\ker D_g) \). Since
\[
Ch(\xi_W) = \{ Ch(\wedge^{n+1}TW) - 1 \}^{n+1} = \{ c_1(TW) \}^{n+1}
\]
and the leading term of \( Td(W) \) is equal to 1, it follows from Lemma 2.1 that
\[
F(g) = \frac{1}{2} (n_g + h_g).
\]
Therefore the theorem follows from Lemma 2.6 below.

**Lemma 2.6.** We have \( \frac{1}{2} h_g = 0 \mod Z \).

Proof. Since the spin\(^c\)(2\(n\)+1)-structure of \( M_g \) comes from the natural U(\(n\))-structure of \( M_g \), the spinor bundle \( E_g = E_\gamma/K \) on \( M_g \) splits into \( E_g = E_g^+ \oplus E_g^- \) and \( D_g \) splits into \( D_g = D_g^+ \oplus D_g^- \) where
\[
D_g^+ : \Gamma(E_g^+ \otimes \xi_g) \to \Gamma(E_g^- \otimes \xi_g)
\]
and
\[
D_g^- = (D_g^+)^* : \Gamma(E_g^- \otimes \xi_g) \to \Gamma(E_g^+ \otimes \xi_g).
\]
Hence we have
\[
h_g = \dim(\ker D_g) = \dim(\ker D_g^+) + \dim(\ker D_g^-).
\]
On the other hand, since the dimension of \( M_g \) is odd, it follows that
\[
\text{Index}(D_g^+) = \dim(\ker D_g^+) - \dim(\ker (D_g^+)^*) = 0.
\]
Therefore we have
\[
\dim(\ker D_g^-) = \dim(\ker (D_g^+)^*) = \dim(\ker D_g^+)
\]
and hence we have
\[
\frac{1}{2} h_g = \dim(\ker D_g^+) \in Z.
\]
This completes the proof.

Now, let \( \Omega(k) \subset X \) be the fixed point set of \( g^k \in K \) (1 \( \leq k \leq p - 1 \)) which is the disjoint union of compact connected complex submanifolds \( N \). Note that the fixed point set \( \Omega(k) \subset X \) of the \( g^k \)-action on \( X \) coincides with the fixed point set \( \Omega(k) \subset M = M \times \{0\} \subset X \) of the \( g^k \)-action on \( M \). Let \( \nu(N,X) \), \( \nu(N,M) \) be the normal bundle of \( N \) in \( X \), \( M \), respectively. Then \( \nu(N,M) \) is decomposed into the direct sum of subbundles
\[
(2.7) \quad \nu(N,M) = \bigoplus_j \nu(N,\theta_j)
\]
where $g^k$ acts on $v(N, \theta_j)$ via multiplication by $e^{i\theta_j}$.

**Definition 2.8.** We define the characteristic class $\mathcal{Y}(v(N, \theta_j))$ by

$$\mathcal{Y}(v(N, \theta_j)) = \prod_{k=1}^{r} \frac{1}{1 - e^{-x_k - i\theta_j}} \in H^{*}(N; \mathbb{C}) \quad (r = \text{rank}(v(N, \theta_j)))$$

where $\prod_k(1 + x_k)$ equals to the total Chern class of $v(N, \theta_j)$.

Since $v(N, X)$ is decomposed into the direct sum

$$v(N, X) = N \oplus \mathcal{E}$$

and $g^k$ acts on the trivial complex line bundle $\mathcal{E}$ over $N$ via multiplication by $e^{2\pi i k/p}$, the following lemma can be deduced from Theorem 1.2 in [5]. (See also Lemma 3.5.4 in [12] and (4.6) in [2].)

**Lemma 2.9.** Fix any $g^k (1 \leq k \leq p - 1)$. Suppose that $g^k$ acts on $K_{M}^{-1}|_{N}$ via multiplication by $e^{i\phi(k)}$. Then we have

$$\text{Index}(D_X, g^k) = \sum_{N \in \tau(k)} \frac{1}{1 - e^{-2\pi i k/p}} (e^{i\phi(k)} \text{Ch}(K_{M}^{-1}|_{N}) - 1)^{r+1} Td(N) \prod_{f} \mathcal{Y}(v(N, \theta_j))[N]$$

$$- \frac{1}{2} \left( \eta_Y(g^k) + \text{Tr}(g^k|_{\ker D_Y}) \right)$$

where $\text{Index}(D_X, g^k)$ is the index of $D_X$ with the global boundary condition in Theorem 2.4 evaluated at $g^k$, namely,

$$\text{Index}(D_X, g^k) = \text{Tr}(g^k|_{\ker D_X}) - \text{Tr}(g^k|_{\text{coker } D_X})$$

(Note that $\text{Index}(D_X, 1) = \text{Index}(D_X)$, $\text{Ch}(K_{M}^{-1}|_{N})$ is the Chern character of $K_{M}^{-1}|_{N}$, $Td(N)$ is the Todd class of $TN$, $[N]$ is the fundamental cycle of $N$ and $\eta_Y(g^k)$ is the eta invariant of $D_Y$ evaluated at $g^k$ (cf. [5])). (Note that $\eta_Y(1)$ is equal to $\eta_Y$ in Theorem 2.4.)

Using Lemma 2.9 and the fact that

$$\text{Ch}(K_{M}^{-1}|_{N}) = e^{c_1(K_{M}^{-1}|_{\mathcal{X}})} = e^{c_1(TM|_{\mathcal{X}})} = e^{c_1(N)} + c_1(v(N, M))$$

where $c_1(N)$ is the first Chern class of $TN$, we can obtain the following theorem.

**Theorem 2.10.** In the notation of the above lemma, we have

$$F(g) = \frac{1}{p} \sum_{k=1}^{p-1} \sum_{\tau(k) \in N} \frac{1}{1 - e^{2\pi i k/p}} (e^{c_1(N)} + c_1(v(N, M)) + i\phi(k) - 1)^{r+1} Td(N) \prod_{f} \mathcal{Y}(v(N, \theta_j))[N].$$
Proof. Similarly as in (3.6) in [5], we have

$$\frac{1}{2} \eta_g = \frac{1}{p} \sum_{k=1}^{p-1} \eta_v(g^k).$$

Hence it follows from Theorem 2.4, Theorem 2.5 and Lemma 2.9 that

$$F(g) = \frac{1}{p} \sum_{k=1}^{p-1} \sum_{N \in \mathbb{N}(k)} \frac{1}{e^{-2\pi ik/p}} (e^{\zeta(N) + \zeta_i(v(N,M)) + i\eta(k)} - 1)^{p+1} Td(N) \prod_j \psi(v(N,\theta_j))[N]$$

$$+ \frac{1}{p} \int_X \frac{Ch(\xi_T) Td(X)}{\eta_v} \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{p-1} \sum_{k=1}^{p-1} \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{2}$$

mod $Z$. Here it follows from the same arguments as in Lemma 3.11 in [11] that

$$\int_X Ch(\xi_T) Td(X) = 0$$

and from Lemma 2.11 below that

$$\sum_{k=1}^{p} \text{Index} (D_X, g^k) = 0 \mod p.$$ 

Therefore it suffices to show that

$$\sum_{k=1}^{p-1} \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{2}$$

Now, since the spin^{(2n+1)}-structure of $Y = M \times S^1$ comes from the U(n)-structure of $M$, the spinor bundle $E_Y$ splits into $E_Y = E_Y^+ \oplus E_Y^-$ and $D_Y$ splits into $D_Y = D_Y^+ \oplus D_Y^-$ where

$$D_Y^+ : \Gamma(E_Y^+ \otimes \xi_Y) \to \Gamma(E_Y^+ \otimes \xi_Y)$$

$$D_Y^- = (D_Y^+)^* : \Gamma(E_Y^- \otimes \xi_Y) \to \Gamma(E_Y^- \otimes \xi_Y)$$

as in Lemma 2.6. Since $g^k (1 \leq k \leq p-1)$ acts freely on $Y$, it follows from the fixed point formula that

$$\text{Index} (D_Y^+, g^k) = \text{Tr}(g^k|_{\text{Ker} D_Y^+}) - \text{Tr}(g^k|_{\text{Ker} D_Y^-}) = 0$$

for any $1 \leq k \leq p-1$. Moreover, since the dimension of $Y$ is odd, it follows as in Lemma 2.6 that

$$\text{Index} (D_Y^+) = \text{Tr}(g^p|_{\text{Ker} D_Y^+}) - \text{Tr}(g^p|_{\text{Ker} D_Y^-}) = 0.$$

Hence it follows from Lemma 2.11 below that
This completes the proof.

**Lemma 2.11.** For any finite dimensional $\mathbb{Z}_p$-module $V$ where $\mathbb{Z}_p = \langle g \rangle$, we have

$$\sum_{k=1}^{p} \frac{1}{2} \text{Tr}(g^k|_{\text{Ker}D_y}) = 0 \mod p.$$  

Proof. Apply the next (2.12) to the eigenvalues $\lambda_j (1 \leq j \leq \dim V)$ of $g|_V$.

(2.12) 

$$\lambda^p = 1 \Rightarrow \sum_{k=1}^{p} \lambda^k = 0 \mod p.$$  

3. $F$ of Kähler surfaces with positive first Chern class

It is an immediate consequence of Theorem 1.6 and a known fact for $f$ (cf. [8, p100]) that $F$ does not vanish for the blowing-up of $CP^2$ at one or two points. Here, however, we compute $F$ of those complex manifolds as examples of Theorem 2.10. First, let $M$ be the surface obtained from $CP^2$ by blowing up one point $[1:0:0]$ where $[z_0:z_1:z_2]$ is the homogeneous coordinate on $CP^2$. Let $g$ be an element of $A(M)$ which is naturally induced by the element of $A(CP^2) = PGL(3; \mathbb{C})$ represented by

$$\begin{pmatrix} 1 \\ \alpha \\ \alpha \end{pmatrix}$$

where $\alpha = e^{2\pi i/p}$ for an integer $p \geq 2$. Then the fixed point set $\Omega(k) \subset M$ of $g^k$-action $(1 \leq k \leq p - 1)$ is independent of $k$ and is equal to the disjoint union of the exceptional divisor $E$ over $[1:0:0]$ and the hyperplane $H$ defined by $z_0 = 0$. Here the normal bundle $v(E,M)$ is equal to the tautological line bundle $J$ and the normal bundle $v(H,M)$ is equal to its dual $J^*$. $g^k$ acts on $J$ via multiplication by $\alpha^k$ and on $J^*$ via multiplication by $\alpha^{-k}$. Let

$$u \in H^2(E) = H^2(CP^1) = \mathbb{Z}, \quad v \in H^2(H) = H^2(CP^1) = \mathbb{Z}$$

be positive generators such that $u[E] = 1$ and $v[H] = 1$ where $[E], [H]$ denote the
fundamental cycles. Then we have \( c_1(E) = 2u \) and \( c_1(H) = 2v \) and hence we have
\[
T_\delta(E) = 1 + u, \quad T_\delta(H) = 1 + v.
\]
Furthermore, since
\[
c_1(\nu(E, M)) = c_1(J) = -u, \quad c_1(\nu(H, M)) = c_1(J*) = v,
\]
we have, by setting \( \theta = 2\pi k/p \),
\[
\nu'(\nu(E, \theta)) = \frac{1}{1 - \alpha^{-k}} \frac{\alpha^{-k}}{(1 - \alpha^{-k})^2} U,
\]
\[
\nu'(\nu(H, \theta)) = \frac{1}{1 - \alpha^k} \frac{\alpha^k}{(1 - \alpha^k)^2} v.
\]
Thus it follows from Theorem 2.10 that
\[
F(g) = \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \alpha^{-k}} (\alpha^k e^\theta - 1)^3 (1 + u) \left( \frac{1}{1 - \alpha^{-k}} + \frac{\alpha^{-k}}{(1 - \alpha^{-k})^2} U \right)[E]
\]
\[
+ \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \alpha^{-k}} (\alpha^{-k} e^{3\theta} - 1)^3 (1 + v) \left( \frac{1}{1 - \alpha^k} - \frac{\alpha^k}{(1 - \alpha^k)^2} v \right)[H]
\]
\[
= \frac{1}{p} \sum_{k=1}^{p-1} \{ \alpha^{2k}(\alpha^k - 1) + 4\alpha^{3k}\} U[E]
\]
\[
+ \frac{1}{p} \sum_{k=1}^{p-1} \{ \alpha^{-k}(1 - \alpha^{-k}) + (2\alpha^{-k} - 10\alpha^{-2k}) v \} [H]
\]
\[
= \frac{1}{p} \sum_{k=1}^{p-1} (4\alpha^{2k} + 2\alpha^{-k} - 10\alpha^{-2k})
\]
Now it follows from (2.12) that
\[
\sum_{k=1}^{p-1} \alpha^{jk} = -1 \quad \text{mod.} p \quad \text{for any integer} \ j.
\]
Hence it follows that
\[
F(g) = \frac{1}{p} (-4 - 2 + 10) = \frac{4}{p} \quad \text{mod.} Z.
\]
In particular, \( F(g) \neq 0 \) if \( p \neq 2, 4 \).
Secondly, let \( M \) be the surface obtained from \( CP^2 \) by blowing up two points \([1:0:0], [0:1:0]\) and \( \pi : M \to CP^2 \) the canonical projection. Let \( g \) be an element of \( A(M) \) which is naturally induced by the element of \( A(CP^2) \) represented by
\[
\begin{pmatrix}
1 \\
\alpha \\
\alpha^2
\end{pmatrix}
\]

where \(\alpha = e^{2\pi i/p}\) for an odd integer \(p \geq 3\). Then the fixed point set \(\Omega(k) \subset M\) of \(g^k\)-action \((1 \leq k \leq p - 1)\) is independent of \(k\) and is equal to the disjoint union of five points \(p_1, p_2, p_3, p_4, p_5\) where \(p_1 = \pi^{-1}([0:0:1]), p_2 \in \pi^{-1}([1:0:0])\) is the point in \(M\) defined by the line: \(z_1 = 0\) through the point \([1:0:0]\) in \(CP^2\), \(p_3 \in \pi^{-1}([1:0:0])\) is the point in \(M\) defined by the line: \(z_2 = 0\) through the point \([1:0:0]\) in \(CP^2\), \(p_4 \in \pi^{-1}([0:1:0])\) is the point in \(M\) defined by the line: \(z_0 = 0\) through the point \([0:1:0]\) in \(CP^2\) and \(p_5 \in \pi^{-1}([0:1:0])\) is the point in \(M\) defined by the line: \(z_2 = 0\) through the point \([0:1:0]\) in \(CP^2\).

Let \(T_j = g|_{T_pM}\) denote the transformation of the tangent space \(T_pM\) induced by \(g\). Then we can see that

\[
T_1 = \begin{pmatrix}
\alpha^{-2} \\
\alpha^{-1} \\
\alpha^2
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
\alpha^{-1} \\
\alpha^2
\end{pmatrix}, \quad T_3 = \begin{pmatrix}
\alpha \\
\alpha
\end{pmatrix}, \quad T_4 = \begin{pmatrix}
\alpha^{-2} \\
\alpha
\end{pmatrix}, \quad T_5 = \begin{pmatrix}
\alpha^{-1} \\
\alpha^2
\end{pmatrix}
\]

Now it follows from Theorem 2.10 that

\[
F(g) = \frac{1}{p} \sum_{k=1}^{p-1} \sum_{j=1}^{5} \frac{1}{1-\alpha^{-k}(\alpha^{(j)k}\alpha^{(j)k} - 1)} \frac{1}{1-\alpha^{-r(j)k}1-\alpha^{-s(j)k}}
\]

where \(\alpha^{(j)}, \alpha^{(j)}\) are the eigenvalues of \(T_j\). Hence, by setting \(\alpha^k = \beta\), we have

\[
F(g) = \frac{1}{p} \sum_{k=1}^{p-1} P(\beta)
\]

where

\[
P(\beta) = \frac{1}{1-\beta^{-1}(\beta^{-3} - 1)^3} \frac{1}{1-\beta^2} \frac{1}{1-\beta}
+ \frac{1}{1-\beta^{-1}(\beta - 1)^3} \frac{1}{1-\beta^{-1}} \frac{1}{1-\beta^{-2}}
+ \frac{1}{1-\beta^{-1}(\beta^2 - 1)^3} \frac{1}{1-\beta^{-1}} \frac{1}{1-\beta^{-1}}
+ \frac{1}{1-\beta^{-1}(\beta^{-1} - 1)^3} \frac{1}{1-\beta^{-1}} \frac{1}{1-\beta^{-1}}
+ \frac{1}{1-\beta^{-1}(\beta - 1)^3} \frac{1}{1-\beta^{-1}} \frac{1}{1-\beta^{-2}}
\]
where \( Q(\beta) \) is a polynomial of \( \beta \) and \( R \in \mathbb{C} \). Here we can see that \( Q(1) = -8 \) and \( R = 4 \). Hence it follows from (2.12) that

\[
\sum_{k=1}^{p-1} Q(\beta) = 8 \mod p.
\]

Therefore it follows that

\[
F(g) = \frac{1}{p} \left( 8 + \sum_{k=1}^{p-1} \frac{4}{\beta + 1} \right)
= \frac{1}{p} \left( 8 + \sum_{k=1}^{p-1} \frac{2 - 2i \tan \frac{\pi k}{p}}{\beta + 1} \right)
= \frac{1}{p} \left( 2p + 6 \right) = \frac{6}{p} \mod \mathbb{Z}.
\]

Thus \( F(g) \neq 0 \) if \( p \neq 3 \).

REMARK 3.2. Let \( g_1, g_2, g_3, \tau \) be the elements of \( A(M) \) which are naturally induced by

\[
\begin{pmatrix}
\alpha & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & \alpha \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]

respectively. Then it follows immediately from (3.1) that

\[
F(g_2) + 2F(g_3) = \frac{6}{p} \mod \mathbb{Z}.
\]

Moreover it is clear that

\[
F(g_1) = F(\tau^{-1} g_1 \tau) = F(g_2),
F(g_1) + F(g_2) + F(g_3) = F(1) = 0.
\]

Using (3.3) and (3.4), we can obtain that

\[
F(g_1) = F(g_2) = \frac{-2}{p}, \quad F(g_3) = \frac{4}{p} \text{ if } p \neq 0 \mod 3.
\]
Now, let $M$ be a 2-dimensional Kähler manifold with $c_1(M) > 0$, which is classified as one of $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, or $\mathbb{CP}^2(m)$ where $\mathbb{CP}^2(m)$ denotes the surface obtained from $\mathbb{CP}^2$ by blowing up $m$-points ($1 \leq m \leq 8$) in general position. (cf. [4, p.321]) Note that the complex structure of $\mathbb{CP}^2(m)$ ($5 \leq m \leq 8$) depends on the position of the $m$-points. When $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2$, $M$ clearly admits a K-E metric. When $M = \mathbb{CP}^2(1)$ or $\mathbb{CP}^2(2)$, as was seen in this section, there exists $g \in A_\theta(M)$ such that $F(g) \neq 0$ and hence $M$ does not admit any K-E metric. (cf. Theorem 1.3 and Theorem 1.6.) When $M = \mathbb{CP}^2(m)$ ($3 \leq m \leq 8$), Tian-Yau [18],[19] proved recently that $M$ admits a K-E metric. Here we have the following.

**Theorem 3.6.** Let $M$ be a Kähler surface with $c_1(M) > 0$. Assume that the complex structure is generic in the sense of [14] when $M = \mathbb{CP}^2(m)$ ($5 \leq m \leq 8$). Then $F$ does not vanish if and only if $M = \mathbb{CP}^2(1)$ or $\mathbb{CP}^2(2)$.

Proof. When $M = \mathbb{CP}^2$, $F(g) = 0$ for any $g \in A(M)$ because $A(M)$ is connected and $f(\lambda) = 0$ for any $\lambda \in H(M)$. (cf. Theorem 1.3 and Theorem 1.6) When $M = \mathbb{CP}^2(1)$ or $\mathbb{CP}^2(2)$, as was seen in this section, there exists $g \in A_\theta(M)$ such that $F(g) \neq 0$. When $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ or $\mathbb{CP}^2(3)$, $F(g) = 0$ for any $g \in A_\theta(M)$ because $f(\lambda) = 0$ for any $\lambda \in H(M)$ (cf. [8, p100]). Now we can see that $A(\mathbb{CP}^1 \times \mathbb{CP}^1)/A_\theta(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is isomorphic to $\mathbb{Z}_2$ and it follows from the Theorem in [14] that

\[
A(\mathbb{CP}^2(3)) = A_\theta(\mathbb{CP}^2(3)) \cdot D(12),
\]

(D(12) denotes the dihedral group of order 12.)

\[
A(\mathbb{CP}^2(4)) = \text{symmetric group } S(5), \quad A(\mathbb{CP}^2(5)) = \oplus^4 \mathbb{Z}_2,
\]

\[
A(\mathbb{CP}^2(6)) = \{1\}, \quad A(\mathbb{CP}^2(7)) = \mathbb{Z}_2, \quad A(\mathbb{CP}^2(8)) = \mathbb{Z}_2.
\]

Hence it suffices to show that

(3.7) \quad $F(g) = 0$ if the dimension of $M$ is 2 and the order of $g \in A(M)$ is 2.

Now fix any $g \in A(M)$ of order 2. Let $\Omega \subset M$ be the fixed point set of $g$, which consists of $q$-points $p_1, p_2, \ldots, p_q$ and $r$-curves $D_1, D_2, \ldots, D_r$. Then it follows from Theorem 2.10 that

(3.8) \quad $F(g) = \frac{1}{4} \left\{ \sum_{s=1}^{q} \Phi(p_s) + \sum_{t=1}^{r} \Psi(D_t) \right\}$

where

$\Phi(p_s) = (e^{s_1} + e^{(p_s,M)} + \psi - 1)^{s+1} Td(p_s) \mathcal{V}(\varphi(p_s\pi)) [p_s]$ \hspace{1cm}

and

$\Psi(D_t) = (e^{s_1} + e^{(D_t,M)} + \psi - 1)^{s+1} Td(D_t) \mathcal{V}(\varphi(D_t\pi)) [D_t]$. 


Now it is clear that \( c_1(p_s) = c_1(v(p_s, M)) = 0 \) and we have \( e^{\phi} = 1 \) because \( g \) acts on \( K_M^{1} \mid_{p_s} \) via multiplication by 1. Hence it follows that \( \Phi(p_s) = 0 \) for any \( 1 \leq s \leq q \). On the other hand, let \( a, b \) denote \( c_1(D_s), c_1(v(D_s, M)) \), respectively. Then, we have \( e^{\phi} = -1 \) because \( g \) acts on \( K_M^{1} \mid_{D_s} \) via multiplication by \(-1\) and moreover we have

\[
\begin{align*}
&c^{\varepsilon_1}(D_s) + c_1(v(D_s, M)) = 1 + (a + b) \\
&Td(D_s) = 1 + \frac{1}{2}a \\
&\gamma(v(D_s, \pi)) = \frac{1}{1 + e^b} = \frac{1}{2} + \frac{1}{4}b.
\end{align*}
\]

Hence it follows that

\[
\Psi(D_s) = (-1 + (1)(a + b) - 1)^3(1 + \frac{1}{2}a(\frac{1}{2} + \frac{1}{4}b)[D_s] = 0 \mod.4 \quad (1 \leq t \leq r).
\]

Thus it follows from (3.8) that \( F(g) = 0 \).

This completes the proof.

4. Other examples and some remarks

Now let \( M \subset \mathbb{C}P^{n+r} \) be a complete intersection of degree \((d_1, d_2, \ldots, d_r)\) defined by the simultaneous equations

\[
\begin{align*}
a_{10}z_1^{d_1} + a_{11}z_1^{d_1} + \cdots + a_{1n+r}z_1^{d_1} &= 0 \\
a_{20}z_1^{d_2} + a_{21}z_1^{d_2} + \cdots + a_{2n+r}z_1^{d_2} &= 0 \\
\vdots \\
a_{r0}z_1^{d_r} + a_{r1}z_1^{d_r} + \cdots + a_{rn+r}z_1^{d_r} &= 0
\end{align*}
\]

Assume that \( \{d_1, d_2, \ldots, d_r\} \) has the greatest common divisor \( p \geq 2 \). Assume moreover that \( a_{j0} \neq 0 \) for some \( j \) and that \( N = M \cap \{z_0 = 0\} \subset \mathbb{C}P^{n+r-1} \) defined by

\[
\begin{align*}
a_{11}z_1^{d_1} + \cdots + a_{1n+r}z_1^{d_1} &= 0 \\
a_{21}z_1^{d_2} + \cdots + a_{2n+r}z_1^{d_2} &= 0 \\
\vdots \\
a_{r1}z_1^{d_r} + \cdots + a_{rn+r}z_1^{d_r} &= 0
\end{align*}
\]

is also a complete intersection in \( \mathbb{C}P^{n+r-1} \). Then \( Z_p = \langle g \rangle \) acts on \( M \) by

\[
g \cdot [z_0 : z_1 : \cdots : z_{n+r}] = [\alpha z_0 : z_1 : \cdots : z_{n+r}] \quad \text{where} \quad \alpha = e^{2\pi i/p}.
\]
**Theorem 4.1.** \( F(g) = 0 \) for any \( n, r \) and any \((d_1, d_2, \cdots, d_r)\).

**Proof.** The fixed point set \( \Omega \subset M \) of \( g^k \)-action \((1 \leq k \leq p - 1)\) is the hypersurface \( N = M \cap \{z_0 = 0\} \) in \( M \). Let \( L \) be the hyperplane bundle of \( CP^{n+r-1} \), which is the dual bundle of the tautological bundle of \( CP^{n+r-1} \). Set
\[
x = c_1(L|_N) \in H^2(N).
\]
Then \( x^{-1}[N] = D \) and \( c_1(N) = (n + r - d)x \) where \( D = d_1 + d_2 + \cdots + d_r \). Now, since
\[
TCP^{n+r-1}|_N = TN \oplus \bigoplus_{j=1}^r \mathcal{O}(L|_N),
\]
it follows that
\[
Td(N) = \left( \frac{x}{1 - e^{-x}} \right)^{n+r} \prod_{j=1}^r \frac{1 - e^{-d_jx}}{d_jx}.
\]
Moreover, since \( TM|_N = TN \oplus (L|_N) \) and \( g^k \) acts on \( L|_N \) via multiplication by \( x^k \), it follows that
\[
e^{c_1(N) + c_1(\nu(N,M)) + i\omega(k)} = x^k e^{(n + r + 1 - d)x},
\]
\[
\nu(N, \theta) = \frac{1}{1 - x^{-k}e^{-x}}.
\]
Hence it follows from Theorem 2.10 that
\[
F(g) = \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - x^{-k}} \left( x^k e^{(n + r + 1 - d)x} - 1 \right)^{n+1} \left( \frac{x}{1 - e^{-x}} \right)^{n+r} \left( \prod_{j=1}^r \frac{1 - e^{-d_jx}}{d_jx} \right) \frac{1}{1 - x^{-k}e^{-x}}[N].
\]
Thus we have
\[
F(g) = D \sum_{k=1}^{p-1} C(k)
\]
where \( C(k) \) denotes the \( x^{-1} \)-coefficient of
\[
\frac{1}{1 - x^{-k}} \left( x^k e^{(n + r + 1 - d)x} - 1 \right)^{n+1} \left( \frac{x}{1 - e^{-x}} \right)^{n+r} \left( \prod_{j=1}^r \frac{1 - e^{-d_jx}}{d_jx} \right) \frac{1}{1 - x^{-k}e^{-x}} \in C[[x]].
\]
Now, \( x^{-1} \)-coefficient of
\[
D \left( x^k e^{(n + r + 1 - d)x} - 1 \right)^{n+1} \left( \frac{x}{1 - e^{-x}} \right)^{n+r} \left( \prod_{j=1}^r \frac{1 - e^{-d_jx}}{d_jx} \right) \frac{1}{1 - x^{-k}e^{-x}}
\]
\[ = x^{-1} - \text{coefficient of} \]
\[ \frac{\alpha^k e^x \left\{ \alpha^k e^{(n+r+1-d)x} - 1 \right\}^{n+1}}{\alpha^k e^x - 1} \left( \frac{1}{1 - e^{-x}} \right)^{n+r} \prod_{j=1}^{r} \left( 1 - e^{-d_j x} \right) \]
\[ = \frac{1}{2\pi i} \oint_{C(z)} \frac{\alpha^k e^z \left\{ \alpha^k (e^z)^{n+r+1-d} - 1 \right\}^{n+1}}{\alpha^k e^z - 1} \left( \frac{e^z}{e^z - 1} \right)^{n+r} \prod_{j=1}^{r} \left( \frac{(e^{d_j})}{(e^{d_j} - 1)} \right)^{e^z} dz \]

(where \( C(z) \) is a sufficiently small counterclockwise loop around the origin)

\[ = \frac{1}{2\pi i} \oint_{C(u)} \frac{\alpha^k (\alpha^{u+1})^{n+r+1-d} - 1}{\alpha^k (u+1) - 1} \frac{(u+1)^{n+r}}{u^{n+r}} \prod_{j=1}^{r} u(d_j + h_f(u)) du \]

(via the substitution \( u = e^z - 1 \), where \( C(u) \) is a counterclockwise loop around the origin)

\[ = u^{-1} - \text{coefficient of} \]
\[ \frac{\alpha^k (\alpha^{u+1})^{n+r+1-d} - 1}{\alpha^k (u+1) - 1} \frac{(u+1)^{n+r-d}}{(u+1)^{n+r}} \prod_{j=1}^{r} (d_j + h_f(u)) \]

(where \( h_f(u) \) is an integral polynomial of order \( \geq 1 \) in \( u \))

\[ = u^{n-1} - \text{coefficient of} \]
\[ \frac{\alpha^k (\alpha^{u+1})^{n+r+1-d} - 1}{\alpha^k (u+1) - 1} \frac{(u+1)^{n+r-d}}{(u+1)^{n+r}} \prod_{j=1}^{r} (d_j + h_f(u)) \]

Set

\[ P(u) = (u+1)^{n+r-d} \prod_{j=1}^{r} (d_j + h_f(u)) \]
\[ Q(u) = \sum_{k=1}^{r} \frac{1}{\alpha^{k} \left( \frac{1}{1 - \alpha^{-k}} \right)^{n+1}} \]

Then it follows from the calculation above that it suffices to show that the \( u^{n-1} \)-coefficient of \( P(u)Q(u) \) is 0 mod.\( p \). Note that \( P(u) \), \( Q(u) \) can be expanded to convergent power series around \( u = 0 \). Note moreover that \( P^{(s)}(0) \) is an integral multiple of \( s! \) because \( P(u) \) can be expanded to a convergent power series with integral coefficients.

Now set

\[ \Phi(x, u) = \left( x(u+1)^{n+r+1-d} - 1 \right)^{n+1}. \]
Then we can see that, for any integer $s$ with $0 \leq s \leq n + 1$,

\[(4.2) \quad \frac{\partial^s}{\partial u^s} \Phi|_{u=0} = s! \phi_s(x)(x-1)^{n+1-s} \text{ for some integral polynomial } \phi_s.
\]

Actually it is clear that

\[\frac{\partial^s}{\partial u^s} \Phi|_{u=0} = \mu_s(x)(x-1)^{n+1-s}\]

for some integral polynomial $\mu_s$. On the other hand, since $\Phi$ can be expanded to a convergent power series of $u$ around $u=0$ whose coefficients are integral polynomials of $x$, it follows that

\[\frac{\partial^s}{\partial u^s} \Phi|_{u=0} = s! \nu_s(x)\]

for some integral polynomial $\nu_s$. Hence it follows that

\[(4.3) \quad \mu_s(x)(x-1)^{n+1-s} = s! \nu_s(x).
\]

Since the top order term of $(x-1)^{n+1-s}$ is equal to 1, it follows from (4.3) that

\[\mu_s(x) = s! \phi_s(x) \text{ for some integral polynomial } \phi_s,
\]

which implies (4.2).

Now, for $m \leq n - 1$, we have

\[Q^{(m)}(0) = \sum_{k=1}^{p-1} (x^k)^2 \sum_{s=0}^{m} \left( m \atop s \right) \left( x^s (u+1) - 1 \right)^{-m-s} (\mu_s(x-1)^{n+1-s})(0)
\]

\[= \sum_{k=1}^{p-1} (x^k)^2 \sum_{s=0}^{m} \left( m \atop s \right) (-1)^{m-s}(m-s)! (x^k)^{m-s}(x^k-1)^{-m-s} s! \phi_s(x^k)(x^k-1)^{n+1-s}
\]

\[= m! \sum_{k=1}^{p-1} \sum_{s=0}^{m} (-1)^{m-s}(x^k)^2 + m-s \phi_s(x^k)(x^k-1)^{n-1-s}.
\]

Hence it follows from the fact (See (2.12).)

\[\sum_{k=1}^{p-1} \Psi(x^k) = -\Psi(1) \mod p \text{ for any integral polynomial } \Psi
\]

that $Q^{(m)}(0)$ is an integral multiple of $p \cdot m!$ if $m \leq n-2$ and is equal to an integral multiple of $(n-1)!$ if $m = n-1$. Therefore it follows that

\[
\frac{1}{(n-1)!} (PQ)^{(n-1)}(0)
\]
$$= \frac{1}{(n-1)!}\{P(0)Q^{(n-1)}(0) + \sum_{m=0}^{n-2} \binom{n-1}{m}P^{n-1-m}(0)Q^{m}(0)\}$$
$$= P(0)\frac{Q^{(n-1)}(0)}{(n-1)!} + \sum_{m=0}^{n-2} \frac{P^{n-1-m}(0)}{(n-1)!} \frac{Q^{m}(0)}{m!}$$
is equal to 0 mod.\(p\) because \(P(0)\) is equal to \(d_1 d_2 \cdots d_r\), which is an integral multiple of \(p\). Thus it follows that

**u**^{-1}-coefficient of \(P(u)Q(u) = 0\) mod.\(p\).

This completes the proof.

**REMARK 4.4.** Let \(M\) be the Fermat cubic surface

\[M : z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0 \quad \text{in} \quad \mathbb{CP}^3\]

and

\[g \cdot [z_0 : z_1 : z_2 : z_3] = [e^{2\pi i/3}z_0 : z_1 : z_2 : z_3].\]

Then \(A(M)\) is a finite group generated by \(g\) and the transposition of coordinates whose order is 2. Hence it follows from Theorem 4.1 and (3.7) that

\[F(g) = 0 \quad \text{for any} \quad g \in A(M).\]

Note that the Fermat cubic surface is isomorphic to the six points blowing-up of \(\mathbb{CP}^2\) with non-generic complex structure in the sense in section 3.

**REMARK 4.5.** In [16] certain kinds of complete intersections including the case that \(r = 1, \frac{n+1}{2} \leq d_1 \leq n + 1\) are shown to admit K-E metrics, and no example of a complete intersection which does not admit any K-E metric is known.

**REMARK 4.6.** Using the \(\otimes^{n+1}(TM - \varepsilon^n)\)-valued spin\(^c\)-Dirac operators (where \(\varepsilon^n\) denotes the trivial bundle \(M \times \mathbb{C}^n\)) instead of the \(\otimes^{n+1}(K_M^{-1} - \varepsilon)\)-valued spin\(^c\)-Dirac operators, we can obtain a formula similar to Theorem 2.10.

**REMARK 4.7.** We can see that the lifted Futaki invariant \(F\) is interpreted as a "holonomy" of a \(\otimes^{n+1}(TM - \varepsilon^n)\)-valued spin\(^c\)-Dirac operator (cf. [20]).

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