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THE LIFTED FUTAKI INVARIANTS AND THE SPIN^C–DIRAC OPERATORS

Dedicated to the memory of Professor Masahisa Adachi

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1. Introduction

The Futaki invariant f which is a Lie algebra homomorphism (cf. [6]) is naturally lifted to a Lie group homomorphism F by virtue of the result in [10]. In [11], we obtained a formula to calculate $2^{n+1}F$ and showed that F can be non-trivial even when no nonzero holomorphic vector field exists. Our purpose in this paper is to refine the formula in [11] so that we can calculate F itself (Theorem 2.10). When M is a Kähler surface with $c_1(M)>0$, the group of holomorphic automorphisms of M (for generic complex structures) are classified (cf. [14]) and, using Theorem 2.10 and the results in [18], [19], we can show that F vanishes if and only if M admits a Kähler-Einstein metric (Theorem 3.6). Moreover we show that F vanishes for some Kähler manifolds which are shown recently to admit a Kähler-Einstein metric (cf. [16]). Futaki conjectured that F as well as f is an obstruction to the existence of Kähler-Einstein metrics on a compact Kähler manifold with $c_1(M)>0$. We might take the results obtained in this paper to encourage the Futaki's conjecture.

Now let M be a compact n-dimensional complex manifold. A Kähler metric h is called a Kähler-Einstein (which is abbreviated to K-E hereafter) metric if there exists a real constant k such that

$$\rho(h)=k\omega(h)$$

where $\rho(h)$ is the Ricci form of h and $\omega(h)$ is the fundamental 2-form of h. Note that the first Chern class $c_1(M)$ has a definite sign (namely, $c_1(M)>0$, $c_1(M)=0$ or $c_1(M)<0$ according to $k>0$, $k=0$ or $k<0$) if M admits a K-E metric because $c_1(M)$ is represented by $\rho(h)$. The converse is true when $c_1(M)=0$ or $c_1(M)<0$.

Theorem 1.1. ([3], [21]) *Let M be a Kähler manifold with $c_1(M)=0$ or <0. Then M admits a K-E metric.*

So the preblem is whether M admits a K-E metric if $c_1(M)>0$.

Now let $A(M)$ be the Lie group of all holomorphic automorphisms of M and $H(M)$ its Lie algebra consisting of all holomorphic vector fields on M. When $c_1(M)>0$ and $H(M)\neq\{0\}$, there exists an obstruction to the existence of K-E metrics called the Futaki invariant (cf. see [6]). The Futaki invariant $f: H(M)\to \boldsymbol{C}$ can be expressed as follows:

$$f(X)=\frac{(n+1)i}{2\pi}\int_M \operatorname{div}_h(X)\rho(h)^n \tag{1.2}$$

for any $X\in H(M)$ where h is any Kähler metric on M and div_h is the divergence with respect to h. It is shown [6], [10] that $f(X)$ is determined only by the complex structure of M and is independent of the choice of h and that f is a Lie algebra homomorphism. ($\boldsymbol{C}$ is regarded as a trivial Lie algebra.) If h is a K-E metric, the right term of (1.2) is equal to

$$f(X)=\frac{(n+1)i}{2\pi}k^n\int_M \operatorname{div}_h(X)\omega(h)^n$$

which equals to 0 by the divergence formula. Since $f(X)$ is independent of the choice of h, the following result can be deduced.

Theorem 1.3. [6] *If M admits a K-E metric, then $f(X)=0$ for any $X\in H(M)$.*

When $H(M)=\{0\}$, there is no known obstruction to the existence of K-E metrics, and it is not known whether there exists an example of M such that $c_1(M)>0$, $H(M)=\{0\}$ but M does not admit any K-E metric.

On the other hand, by virtue of the result in [10], f can naturally be lifted to a group homomorphism $F: A(M)\to \boldsymbol{C}/\boldsymbol{Z}$ as follows.

DEFINITION 1.4. Fix any $g\in A(M)$. Let M_g denote the mapping torus $M_g=M\times[0,1]/\sim$ where $(p,0)\sim(g(p),1)$. Let $\mathscr{F}_g$ denote the holomorphic foliation defined by the $[0,1]$-directed vector field. Then, by definition,

$$F(g)=Sc_1^{n+1}(\nu(\mathscr{F}_g))[M_g]\in \boldsymbol{C}/\boldsymbol{Z} \tag{1.5}$$

where $[M_g]$ is the fundamental cycle of M_g and

$$Sc_1^{n+1}(\nu(\mathscr{F}_g))\in \mathrm{H}^{2n+1}(M_g;\boldsymbol{C}/\boldsymbol{Z})$$

is the Simons character of the first Chern class c_1 to the power $n+1$ for the normal bundle $\nu(\mathscr{F}_g)$ with respect to any Bott connection. (For details, see [10], [17].)

Then, it is shown [7] that $F: A(M) \to \boldsymbol{C}/\boldsymbol{Z}$ is a Lie group homomorphism where $\boldsymbol{C}/\boldsymbol{Z}$ is regarded as an additive group, and the following holds.

Theorem 1.6. [10] *We have $F(\exp X) = f(X) \bmod . \boldsymbol{Z}$ for any $X \in H(M)$. In particular, we have $F_* = f$.*

Though it immediately follows from Theorem 1.3 and Theorem 1.6 that $F|_{A_0(M)}$ (where $A_0(M)$ denotes the identity component of $A(M)$) is an obstruction to the existence of K-E metrics on M, it is not known whether F itself is an obstruction to the existence of K-E metrics on M or not. If the Futaki's conjecture turns out to be true, F may become the unique obstruction which is valid even when $H(M) = \{0\}$.

REMARK 1.7. In [9], f is lifted to a group homomorphism $\det \circ \phi : A(M) \to \boldsymbol{C}^*$ ($\simeq \boldsymbol{C}/\boldsymbol{Z}$). A multiple of f gives rise to a power of the lifting. Theorem 1.6 implies that f is normalized so as to satisfy the integrability condition that $f(X)$ is an integer for any $X \in H(M)$ such that $\exp X = 1$.

2. A calculation formula for F

Let M be a compact n-dimensional complex manifold and M_g the mapping torus for $g \in A(M)$ defined as in Definition 1.4. In [11], we showed that $2^{n+1}F$ is equal to the eta invariant of the signature operator on M_g. In this section, we shall show a similar formula by using the spin^c-Dirac operators.

Now fix an element $g \in A(M)$ which we assume has a finite order $p \geq 2$. (Note that $A(M)$ itself is a finite group if $c_1(M) > 0$ and $H(M) = \{0\}$.) We may assume that g preserves the Hermitian metric h on M. Then the Hermitian connection ∇^M of the holomorphic tangent bundle TM, which is uniquely determined under the conditions that the connection form of ∇^M is of type (1,0) and that ∇^M preserves h, is necessarily g-invariant.

Let $X = M \times D^2$, $Y = \partial X = M \times S^1$ be spin^c-manifolds with the spin^c-structures defined by the U(n)-structure of M and the trivial spin^c-structures of D^2, S^1, respective-ly. Then the cyclic group $K = \boldsymbol{Z}_p = \langle g \rangle$ acts on (X, Y) as follows:

$$g(m, re^{i\theta}) = (g(m), re^{i(\theta + 2\pi/p)})$$

for $(m, re^{i\theta}) \in X = M \times D^2; 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$. Note that $Y/K = M_g$, $(TM \times S^1)/K = \nu(\mathscr{F}_g)$ and that ∇^M naturally defines a Bott connection $\nabla^{\mathscr{F}}$ of $\nu(\mathscr{F}_g)$. On the other hand, we give a rotationally symmetric Hermitian metric on the complex manifold D^2 such that it is a product metric of $S^1 \times [0, \delta)$ near the boundary $\partial D^2 = S^1$. Then the complex structures and the Hermitian metrics on M, D^2 define a K-invariant complex structure and a K-invariant Hermitian metric on X. Let ∇^X be the K-invariant Hermitian connection of TX. Then $\nabla^X|_Y$ descends to a Hermitian connection $\nabla^{X/K}$ of $T(X/K)|_{M_g}$ and it can be shown

$$T(X/K)|_{M_g}=(TX|_Y)/K=\nu(\mathscr{F}_g)\oplus\varepsilon$$

where ε denotes the trivial complex line bundle of all $\mathscr{F}_g$-directed vectors and $\nabla^{X/K}$ splits as

$$\nabla^{X/K}=\nabla^{\mathscr{F}}\oplus\nabla^0$$

where ∇^0 denotes the globally flat connection of ε.

Now, since M_g is a stably almost complex manifold, it follows from the result of Morita[15] that there exists a compact $(2n+2)$-dimensional almost complex manifold W such that $\partial W=M_g$ and $W=X/K$ near M_g as an almost complex manifold with a Hermitian metric. Then we have the following lemma by the same arguments as in the proof of Theorem 3.7 in [11].

Lemma 2.1. *We have $F(g)=\int_W c_1(TW)^{n+1}$ where $c_1(TW)$ is the first Chern form of TW with repect to a unitary connection ∇^W of TW (namely, ∇^W preserves the metric and the almost complex structure on TW) which coincides with $\nabla^{X/K}$ near M_g.*

Now, let ξ be the virtual complex vector bundle over M defined by

$$\xi=\otimes^{n+1}(K_M^{-1}-\varepsilon)$$

where K_M^{-1} is the anticanonical bundle of M and ε is the trivial complex line bundle over M. Set $\xi_X=q_X^*\xi$ and $\xi_Y=q_Y^*\xi$ where $q_X:X=M\times D^2\to M$ and $q_Y: Y=M\times S^1\to M$ are the canonical projections. ξ_X and ξ_Y are virtual vector bundles with unitary connections with respect to the metrics and the connections naturally defined by the Hermitian metric and the Hermitian connection of TM. Using the spin^c-structures, the metrics and the connections of TX and TY, we can define the spin^c-Dirac operators (or Dolbeault operators)

$$\begin{aligned} D_X&:\Gamma(E_X^+\otimes\xi_X)\to\Gamma(E_X^-\otimes\xi_X)\\ D_Y&:\Gamma(E_Y\otimes\xi_Y)\to\Gamma(E_Y\otimes\xi_Y)\end{aligned}\tag{2.2}$$

where $E_X^\pm$ denote the half spinor bundles over X and $E_Y=E_X^+|_Y=E_X^-|_Y$ is the spinor bundle over Y. (For details of spin^c-Dirac operators and Dolbeault operators on almost complex manifolds, see [12],[13].) Since the metric and the connection of TX is K-invariant and is product near $\partial X=Y$, D_X and D_Y are K-invariant and D_X can be expressed as

$$D_X=\sigma\left(\frac{\partial}{\partial u}+D_Y\right)\tag{2.3}$$

on the collar $Y\times[0,\delta)\subset X$ where u is the coordinate of $[0,\delta)$ and σ is a bundle isomorphism.

Theorem 2.4. [1] *We have*

$$\mathrm{Index}(D_X)=\int_X Ch(\xi_X)Td(X)-\frac{1}{2}(\eta_Y+h_Y)$$

where $\mathrm{Index}(D_X)$ *is the index of* D_X *with a certain global boundary condition,* $Ch(\xi_X)$ *is the Chern character form of* ξ_X *with the unitary connection,* $Td(X)$ *is the Todd form of* (TX,∇^X), η_Y *is the eta invariant of* D_Y *and* $h_Y=\dim(\mathrm{Ker}\,D_Y)$.

Now, let $\xi_g=\xi_Y/K$ be a virtual vector bundle over M_g with a unitary connection. Then, since D_Y is K-invariant, D_Y naturally defines a differential operator D_g, which is the ξ_g-valued spin^c-Dirac operator on $M_g=Y/K$. Our first result is the following.

Theorem 2.5. *We have*

$$F(g)=\frac{1}{2}\eta_g \quad (\mathrm{mod}.\boldsymbol{Z})$$

where η_g *is the eta invariant of* D_g.

Proof. Set

$$\xi_W=\otimes^{n+1}(\wedge^{n+1}TW-\varepsilon)$$

where ε denotes the trivial complex line bundle over W. Note that $\wedge^{n+1}TW$ is also a complex line bundle over W. The unitary connection ∇^W of TW naturally defines a unitary connection of ξ_W. Then the spin^c-Dirac operator

$$D_W:\Gamma(E_W^+\otimes\xi_W)\to\Gamma(E_W^-\otimes\xi_W)$$

is defined similarly as in (2.2). It can be seen that $\xi_W|_{M_g}=\xi_g$ and, similarly as in (2.3), D_W can be expressed as

$$D_W=\sigma\left(\frac{\partial}{\partial u}+D_g\right)$$

on the collar $M_g\times[0,\delta)\subset W$. Hence it follows from the Atiyah-Patodi-Singer's theorem (cf. Theorem 2.4) that

$$\int_W Ch(\xi_W)Td(W)=\frac{1}{2}(\eta_g+h_g) \quad (\mathrm{mod}.\boldsymbol{Z})$$

where $Ch(\xi_W)$ is the Chern character form of ξ_W, $Td(W)$ is the Todd form of TW

and $h_g = \dim(\mathrm{Ker}\, D_g)$. Since

$$Ch(\xi_W) = \{Ch(\wedge^{n+1} TW) - 1\}^{n+1} = \{c_1(TW)\}^{n+1}$$

and the leading term of $Td(W)$ is equal to 1, it follows from Lemma 2.1 that

$$F(g) = \frac{1}{2}(\eta_g + h_g).$$

Therefore the theorem follows from Lemma 2.6 below.

Lemma 2.6. *We have* $\frac{1}{2} h_g = 0$ mod.$\boldsymbol{Z}$.

Proof. Since the spin$^c(2n+1)$-structure of M_g comes from the natural $\mathrm{U}(n)$-structure of M_g, the spinor bundle $E_g = E_Y/K$ on M_g splits into $E_g = E_g^+ \oplus E_g^-$ and D_g splits into $D_g = D_g^+ \oplus D_g^-$ where

$$D_g^+ : \Gamma(E_g^+ \otimes \xi_g) \to \Gamma(E_g^- \otimes \xi_g)$$
$$D_g^- = (D_g^+)^* : \Gamma(E_g^- \otimes \xi_g) \to \Gamma(E_g^+ \otimes \xi_g)$$

Hence we have

$$h_g = \dim(\mathrm{Ker} D_g) = \dim(\mathrm{Ker}\, D_g^+) + \dim(\mathrm{Ker} D_g^-).$$

On the other hand, since the dimension of M_g is odd, it follows that

$$\mathrm{Index}(D_g^+) = \dim(\mathrm{Ker}\, D_g^+) - \dim(\mathrm{Ker}(\mathrm{D}_g^+)^*) = 0.$$

Therefore we have

$$\dim(\mathrm{Ker}\, D_g^-) = \dim(\mathrm{Ker}(D_g^+)^*) = \dim(\mathrm{Ker} D_g^+)$$

and hence we have

$$\frac{1}{2} h_g = \dim(\mathrm{Ker}\, D_g^+) \in \boldsymbol{Z}.$$

This completes the proof.

Now, let $\Omega(k) \subset X$ be the fixed point set of $g^k \in K$ $(1 \leq k \leq p-1)$ which is the disjoint union of compact connected complex submanifolds N. Note that the fixed point set $\Omega(k) \subset X$ of the g^k-action on X coincides with the fixed point set $\Omega(k) \subset M = M \times \{0\} \subset X$ of the g^k-action on M. Let $\nu(N,X)$, $\nu(N,M)$ be the normal bundle of N in X, M, respectively. Then $\nu(N,M)$ is decomposed into the direct sum of subbundles

$$\nu(N,M) = \oplus_j \nu(N,\theta_j) \tag{2.7}$$

where g^k acts on $\nu(N,\theta_j)$ via multiplication by $e^{i\theta_j}$.

DEFINITION 2.8. We define the characteristic class $\mathscr{V}(\nu(N,\theta_j))$ by

$$\mathscr{V}(\nu(N,\theta_j))=\prod_{k=1}^{r}\frac{1}{1-e^{-x_k-i\theta_j}}\in \mathrm{H}^{**}(N;\boldsymbol{C})\quad (r=\mathrm{rank}(\nu(N,\theta_j)))$$

where $\Pi_k(1+x_k)$ equals to the total Chern class of $\nu(N,\theta_j)$.

Since $\nu(N,X)$ is decomposed into the direct sum

$$\nu(N,X)=\nu(N,M)\oplus\varepsilon$$

and g^k acts on the trivial complex line bundle ε over N via multiplication by $e^{2\pi ik/p}$, the following lemma can be deduced from Theorem 1.2 in [5]. (See also Lemma 3.5.4 in [12] and (4.6) in [2].)

Lemma 2.9. *Fix any* g^k $(1\leq k\leq p-1)$. *Suppose that* g^k *acts on* $K_M^{-1}|_N$ *via multiplication by* $e^{i\varphi(k)}$. *Then we have*

$$\mathrm{Index}\,(D_X,g^k)=\sum_{N\subset\Omega(k)}\frac{1}{1-e^{-2\pi ik/p}}(e^{i\varphi(k)}Ch(K_M^{-1}|_N)-1)^{n+1}Td(N)\prod_j\mathscr{V}(\nu(N,\theta_j))[N]$$
$$-\frac{1}{2}(\eta_Y(g^k)+\mathrm{Tr}(g^k|_{\mathrm{Ker}\,D_Y}))$$

where $\mathrm{Index}\,(D_X,g^k)$ *is the index of* D_X *with the global boundary condition in Theorem* 2.4 *evaluated at* g^k, *namely,*

$$\mathrm{Index}\,(D_X,g^k)=\mathrm{Tr}(g^k|_{\mathrm{Ker}\,D_X})-\mathrm{Tr}(g^k|_{\mathrm{Coker}\,D_X})$$

(*Note that* $\mathrm{Index}\,(D_X,1)=\mathrm{Index}(D_X)$), $Ch(K_M^{-1}|_N)$ *is the Chern character of* $K_M^{-1}|_N$, $Td(N)$ *is the Todd class of* TN, $[N]$ *is the fundamental cycle of* N *and* $\eta_Y(g^k)$ *is the eta invariant of* D_Y *evaluated at* g^k (*cf.* [5]). (*Note that* $\eta_Y(1)$ *is equal to* η_Y *in Theorem* 2.4.)

Using Lemma 2.9 and the fact that

$$Ch(K_M^{-1}|_N)=e^{c_1(K_M^{-1}|_N)}=e^{c_1(TM|_N)}=e^{c_1(N)+c_1(\nu(N,M))}$$

where $c_1(N)$ is the first Chern class of TN, we can obtain the following theorem.

Theorem 2.10. *In the notation of the above lemma, we have*

$$F(g)=\frac{1}{p}\sum_{k=1}^{p-1}\sum_{N\subset\Omega(k)}\frac{1}{1-e^{-2\pi ik/p}}(e^{c_1(N)+c_1(\nu(N,M))+i\varphi(k)}-1)^{n+1}Td(N)\prod_j\mathscr{V}(\nu(N,\theta_j))[N]\,.$$

Proof. Similarly as in (3.6) in [5], we have

$$\frac{1}{2}\eta_g=\frac{1}{p}\sum_{k=1}^{p}\frac{1}{2}\eta_Y(g^k).$$

Hence it follows from Theorem 2.4, Theorem 2.5 and Lemma 2.9 that

$$F(g)=\frac{1}{p}\sum_{k=1}^{p-1}\sum_{N\subset\Omega(k)}\frac{1}{1-e^{-2\pi ik/p}}(e^{c_1(N)+c_1(\nu(N,M))+i\varphi(k)}-1)^{n+1}Td(N)\prod_j \mathscr{V}(\nu(N,\theta_j))[N]$$
$$+\frac{1}{p}\int_X Ch(\xi_X)Td(X)-\frac{1}{p}\sum_{k=1}^{p}\frac{1}{2}\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y})-\frac{1}{p}\sum_{k=1}^{p}\operatorname{Index}(D_X,g^k)$$

mod.$\boldsymbol{Z}$. Here it follows from the same arguments as in Lemma 3.11 in [11] that

$$\int_X Ch(\xi_X)Td(X)=0$$

and from Lemma 2.11 below that

$$\sum_{k=1}^{p}\operatorname{Index}(D_X,g^k)=0 \quad \text{mod.}p.$$

Therefore it suffices to show that

$$\sum_{k=1}^{p}\frac{1}{2}\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y})=0 \quad \text{mod.}p.$$

Now, since the spin$^c(2n+1)$-structure of $Y=M\times S^1$ comes from the U(n)-structure of M, the spinor bundle E_Y splits into $E_Y=E_Y^+\oplus E_Y^-$ and D_Y splits into $D_Y=D_Y^+\oplus D_Y^-$ where

$$D_Y^+:\Gamma(E_Y^+\otimes\xi_Y)\to\Gamma(E_Y^-\otimes\xi_Y)$$
$$D_Y^-=(D_Y^+)^*:\Gamma(E_Y^-\otimes\xi_Y)\to\Gamma(E_Y^+\otimes\xi_Y)$$

as in Lemma 2.6. Since g^k $(1\le k\le p-1)$ acts freely on Y, it follows from the fixed point formula that

$$\operatorname{Index}(D_Y^+,g^k)=\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y^+})-\operatorname{Tr}(g^k|_{\operatorname{Ker}(D_Y^+)^*})=0$$

for any $1\le k\le p-1$. Moreover, since the dimension of Y is odd, it follows as in Lemma 2.6 that

$$\operatorname{Index}(D_Y^+)=\operatorname{Tr}(g^p|_{\operatorname{Ker}D_Y^+})-\operatorname{Tr}(g^p|_{\operatorname{Ker}(D_Y^+)^*})=0.$$

Hence it follows from Lemma 2.11 below that

$$\sum_{k=1}^{p}\frac{1}{2}\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y})=\sum_{k=1}^{p}\frac{1}{2}\{\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y^+})+\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y^-})\}$$

$$=\sum_{k=1}^{p}\frac{1}{2}\{\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y^+})+\operatorname{Tr}(g^k|_{\operatorname{Ker}(D_Y^+)^*})\}$$

$$=\sum_{k=1}^{p}\operatorname{Tr}(g^k|_{\operatorname{Ker}D_Y^+})=0 \quad \operatorname{mod}.p.$$

This completes the proof.

Lemma 2.11. *For any finite dimensional $\boldsymbol{Z}_p$-module V where $\boldsymbol{Z}_p=\langle g\rangle$, we have*

$$\sum_{k=1}^{p}\operatorname{Tr}(g^k|_V)=0 \quad \operatorname{mod}.p.$$

Proof. Apply the next (2.12) to the eigenvalues $\lambda_j\,(1\le j\le \dim V)$ of $g|_V$.

$$\lambda^p=1\Rightarrow\sum_{k=1}^{p}\lambda^k=0 \quad \operatorname{mod}.p. \tag{2.12}$$

3. F of Kähler surfaces with positive first Chern class

It is an immediate consequence of Theorem 1.6 and a known fact for f (cf. [8, p100]) that F does not vanish for the blowing-up of $\boldsymbol{CP}^2$ at one or two points. Here, however, we compute F of those complex manifolds as examples of Theorem 2.10. First, let M be the surface obtained from $\boldsymbol{CP}^2$ by blowing up one point $[1:0:0]$ where $[z_0:z_1:z_2]$ is the homogeneous coordinate on $\boldsymbol{CP}^2$. Let g be an element of $A(M)$ which is naturally induced by the element of $A(\boldsymbol{CP}^2)=PGL(3;\boldsymbol{C})$ represented by

$$\begin{pmatrix}1&&\\&\alpha&\\&&\alpha\end{pmatrix}$$

where $\alpha=e^{2\pi i/p}$ for an integer $p\ge 2$. Then the fixed point set $\Omega(k)\subset M$ of g^k-action $(1\le k\le p-1)$ is independent of k and is equal to the disjoint union of the exceptional divisor E over $[1:0:0]$ and the hyperplane H defined by $z_0=0$. Here the normal bundle $\nu(E,M)$ is equal to the tautological line bundle J and the normal bundle $\nu(H,M)$ is equal to its dual J^*. g^k acts on J via multiplication by α^k and on J^* via multiplication by α^{-k}. Let

$$u\in \mathrm{H}^2(E)=\mathrm{H}^2(\boldsymbol{CP}^1)=\boldsymbol{Z},\quad v\in \mathrm{H}^2(H)=\mathrm{H}^2(\boldsymbol{CP}^1)=\boldsymbol{Z}$$

be positive generators such that $u[E]=1$ and $v[H]=1$ where $[E]$, $[H]$ denote the fundamental cycles. Then we have $c_1(E)=2u$ and $c_1(H)=2v$ and hence we have

$Td(E)=1+u$, $Td(H)=1+v$. Furthermore, since

$$c_1(\nu(E,M))=c_1(J)=-u, \quad c_1(\nu(H,M))=c_1(J^*)=v,$$

we have, by setting $\theta=2\pi k/p$,

$$\mathscr{V}(\nu(E,\theta))=\frac{1}{1-\alpha^{-k}}+\frac{\alpha^{-k}}{(1-\alpha^{-k})^2}u,$$

$$\mathscr{V}(\nu(H,-\theta))=\frac{1}{1-\alpha^{k}}-\frac{\alpha^{k}}{(1-\alpha^{k})^2}v.$$

Thus it follows from Theorem 2.10 that

$$\begin{aligned}
F(g)=&\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^{-k}}(\alpha^k e^u-1)^3(1+u)\left(\frac{1}{1-\alpha^{-k}}+\frac{\alpha^{-k}}{(1-\alpha^{-k})^2}u\right)[E]\\
&+\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^{-k}}(\alpha^{-k}e^{3v}-1)^3(1+v)\left(\frac{1}{1-\alpha^{k}}-\frac{\alpha^{k}}{(1-\alpha^{k})^2}v\right)[H]\\
=&\frac{1}{p}\sum_{k=1}^{p-1}\{\alpha^{2k}(\alpha^k-1)+4\alpha^{3k}u\}[E]\\
&+\frac{1}{p}\sum_{k=1}^{p-1}\{\alpha^{-k}(1-\alpha^{-k})+(2\alpha^{-k}-10\alpha^{-2k})v\}[H]\\
=&\frac{1}{p}\sum_{k=1}^{p-1}(4\alpha^{3k}+2\alpha^{-k}-10\alpha^{-2k})
\end{aligned}$$

Now it follows from (2.12) that

$$\sum_{k=1}^{p-1}\alpha^{jk}=-1 \quad \text{mod.} p \quad \text{for any integer } j.$$

Hence it follows that

$$F(g)=\frac{1}{p}(-4-2+10)=\frac{4}{p} \quad \text{mod.} \boldsymbol{Z}.$$

In particular, $F(g)\neq 0$ if $p\neq 2,4$.

Secondly, let M be the surface obtained from $\boldsymbol{CP}^2$ by blowing up two points $[1:0:0]$, $[0:1:0]$ and $\pi: M\to \boldsymbol{CP}^2$ the canonical projection. Let g be an element of $A(M)$ which is naturally induced by the element of $A(\boldsymbol{CP}^2)$ represented by

$$\begin{pmatrix} 1 & & \\ & \alpha & \\ & & \alpha^2 \end{pmatrix}$$

where $\alpha = e^{2\pi i/p}$ for an odd integer $p \geq 3$. Then the fixed point set $\Omega(k) \subset M$ of g^k-action $(1 \leq k \leq p-1)$ is independent of k and is equal to the disjoint union of five points p_1, p_2, p_3, p_4, p_5 where $p_1 = \pi^{-1}([0:0:1])$, $p_2 \in \pi^{-1}([1:0:0])$ is the point in M defined by the line: $z_1 = 0$ through the point $[1:0:0]$ in $\boldsymbol{CP}^2$, $p_3 \in \pi^{-1}([1:0:0])$ is the point in M defined by the line: $z_2 = 0$ through the point $[1:0:0]$ in $\boldsymbol{CP}^2$, $p_4 \in \pi^{-1}([0:1:0])$ is the point in M defined by the line: $z_0 = 0$ through the point $[0:1:0]$ in $\boldsymbol{CP}^2$ and $p_5 \in \pi^{-1}([0:1:0])$ is the point in M defined by the line: $z_2 = 0$ through the point $[0:1:0]$ in $\boldsymbol{CP}^2$. Let $T_j = g|_{T_{p_j}M}$ denote the transformation of the tangent space $T_{p_j}M$ induced by g. Then we can see that

$$T_1 = \begin{pmatrix} \alpha^{-2} & \\ & \alpha^{-1} \end{pmatrix}, \quad T_2 = \begin{pmatrix} \alpha^{-1} & \\ & \alpha^2 \end{pmatrix}, \quad T_3 = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix},$$
$$T_4 = \begin{pmatrix} \alpha^{-2} & \\ & \alpha \end{pmatrix}, \quad T_5 = \begin{pmatrix} \alpha^{-1} & \\ & \alpha^2 \end{pmatrix}.$$

Now it follows from Theorem 2.10 that

$$F(g) = \frac{1}{p} \sum_{k=1}^{p-1} \sum_{j=1}^{5} \frac{1}{1-\alpha^{-k}} (\alpha^{r(j)k} \alpha^{s(j)k} - 1)^3 \frac{1}{1-\alpha^{-r(j)k}} \frac{1}{1-\alpha^{-s(j)k}}$$

where $\alpha^{r(j)}$, $\alpha^{s(j)}$ are the eigenvalues of T_j. Hence, by setting $\alpha^k = \beta$, we have

$$F(g) = \frac{1}{p} \sum_{k=1}^{p-1} P(\beta)$$

where

$$\begin{aligned} P(\beta) = & \frac{1}{1-\beta^{-1}} (\beta^{-3} - 1)^3 \frac{1}{1-\beta^2} \frac{1}{1-\beta} \\ & + \frac{1}{1-\beta^{-1}} (\beta - 1)^3 \frac{1}{1-\beta} \frac{1}{1-\beta^{-2}} \\ & + \frac{1}{1-\beta^{-1}} (\beta^2 - 1)^3 \frac{1}{1-\beta^{-1}} \frac{1}{1-\beta^{-1}} \\ & + \frac{1}{1-\beta^{-1}} (\beta^{-1} - 1)^3 \frac{1}{1-\beta^2} \frac{1}{1-\beta^{-1}} \\ & + \frac{1}{1-\beta^{-1}} (\beta - 1)^3 \frac{1}{1-\beta} \frac{1}{1-\beta^{-2}} \end{aligned}$$

$$=\frac{-\beta^{p-8}(\beta^2+\beta+1)^3-\beta^3+\beta^3(\beta+1)^4+\beta^{p-1}-\beta^3}{\beta+1}$$

$$=Q(\beta)+\frac{R}{\beta+1}$$

where $Q(\beta)$ is a polynomial of β and $R\in\boldsymbol{C}$. Here we can see that $Q(1)=-8$ and $R=4$. Hence it follows from (2.12) that

$$\sum_{k=1}^{p-1} Q(\beta)=8 \quad \text{mod}.p.$$

Therefore it follows that

$$F(g)=\frac{1}{p}\left(8+\sum_{k=1}^{p-1}\frac{4}{\beta+1}\right) \tag{3.1}$$

$$=\frac{1}{p}\left(8+\sum_{k=1}^{p-1}(2-2i\tan\frac{\pi k}{p})\right)$$

$$=\frac{1}{p}(2p+6)=\frac{6}{p} \quad \text{mod}.\boldsymbol{Z}.$$

Thus $F(g)\neq 0$ if $p\neq 3$.

Remark 3.2. Let g_1, g_2, g_3, τ be the elements of $A(M)$ which are naturally induced by

$$\begin{pmatrix}\alpha & & \\ & 1 & \\ & & 1\end{pmatrix},\ \begin{pmatrix}1 & & \\ & \alpha & \\ & & 1\end{pmatrix},\ \begin{pmatrix}1 & & \\ & 1 & \\ & & \alpha\end{pmatrix},\ \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

respectively. Then it follows immediately from (3.1) that

$$F(g_2)+2F(g_3)=\frac{6}{p} \quad (\text{mod}.\boldsymbol{Z}). \tag{3.3}$$

Moreover it is clear that

$$\begin{aligned} &F(g_1)=F(\tau^{-1}g_1\tau)=F(g_2),\\ &F(g_1)+F(g_2)+F(g_3)=F(1)=0. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we can obtain that

$$F(g_1)=F(g_2)=-\frac{2}{p},\quad F(g_3)=\frac{4}{p} \quad \text{if } p\neq 0 \quad \text{mod}.3. \tag{3.5}$$

Now, let M be a 2-dimensional Kähler manifold with $c_1(M)>0$, which is classified as one of $M=\boldsymbol{CP}^1\times\boldsymbol{CP}^1$, $\boldsymbol{CP}^2$ or $\boldsymbol{CP}^2(m)$ where $\boldsymbol{CP}^2(m)$ denotes the surface obtained from $\boldsymbol{CP}^2$ by blowing up m-points $(1\leq m\leq 8)$ in general position. (cf. [4, p.321]) Note that the complex structure of $\boldsymbol{CP}^2(m)$ $(5\leq m\leq 8)$ depends on the position of the m-points. When $M=\boldsymbol{CP}^1\times\boldsymbol{CP}^1$ or $\boldsymbol{CP}^2$, M clearly admits a K-E metric. When $M=\boldsymbol{CP}^2(1)$ or $\boldsymbol{CP}^2(2)$, as was seen in this section, there exists $g\in A_0(M)$ such that $F(g)\neq 0$ and hence M does not admit any K-E metric. (cf. Theorem 1.3 and Theorem 1.6.) When $M=\boldsymbol{CP}^2(m)$ $(3\leq m\leq 8)$, Tian-Yau [18],[19] proved recently that M admits a K-E metric. Here we have the following.

Theorem 3.6. *Let M be a Kähler surface with $c_1(M)>0$. Assume that the complex structure is generic in the sense of* [14] *when $M=\boldsymbol{CP}^2(m)$ $(5\leq m\leq 8)$. Then F does not vanish if and only if $M=\boldsymbol{CP}^2(1)$ or $\boldsymbol{CP}^2(2)$.*

Proof. When $M=\boldsymbol{CP}^2$, $F(g)=0$ for any $g\in A(M)$ because $A(M)$ is connected and $f(X)=0$ for any $X\in H(M)$. (cf. Theorem 1.3 and Theorem 1.6) When $M=\boldsymbol{CP}^2(1)$ or $\boldsymbol{CP}^2(2)$, as was seen in this section, there exists $g\in A_0(M)$ such that $F(g)\neq 0$. When $M=\boldsymbol{CP}^1\times\boldsymbol{CP}^1$ or $\boldsymbol{CP}^2(3)$, $F(g)=0$ for any $g\in A_0(M)$ because $f(X)=0$ for any $X\in H(M)$ (cf. [8, p100]). Now we can see that $A(\boldsymbol{CP}^1\times\boldsymbol{CP}^1)/A_0(\boldsymbol{CP}^1\times\boldsymbol{CP}^1)$ is isomorphic to $\boldsymbol{Z}_2$ and it follows from the Theorem in [14] that

$A(\boldsymbol{CP}^2(3))=A_0(\boldsymbol{CP}^2(3))\cdot D(12)$,

($D(12)$ denotes the dihedral group of order 12.)

$A(\boldsymbol{CP}^2(4))=$ symmetric group $S(5)$, $A(\boldsymbol{CP}^2(5))=\oplus^4\boldsymbol{Z}_2$,

$A(\boldsymbol{CP}^2(6))=\{1\}$, $A(\boldsymbol{CP}^2(7))=\boldsymbol{Z}_2$, $A(\boldsymbol{CP}^2(8))=\boldsymbol{Z}_2$.

Hence it suffices to show that

(3.7) $F(g)=0$ if the dimension of M is 2 and the order of $g\in A(M)$ is 2.

Now fix any $g\in A(M)$ of order 2. Let $\Omega\subset M$ be the fixed point set of g, which consists of q-points $p_1, p_2, \cdots, p_q$ and r-curves $D_1, D_2, \cdots, D_r$. Then it follows from Theorem 2.10 that

$$F(g)=\frac{1}{4}\{\sum_{s=1}^{q}\Phi(p_s)+\sum_{t=1}^{r}\Psi(D_t)\} \tag{3.8}$$

where

$$\Phi(p_s)=(e^{c_1(p_s)+c_1(\nu(p_s,M))+i\varphi}-1)^{n+1}Td(p_s)\mathscr{V}(\nu(p_s,\pi))[p_s]$$

and

$$\Psi(D_t)=(e^{c_1(D_t)+c_1(\nu(D_t,M))+i\psi}-1)^{n+1}Td(D_t)\mathscr{V}(\nu(D_t,\pi))[D_t].$$

Now it is clear that $c_1(p_s)=c_1(v(p_s,M))=0$ and we have $e^{i\varphi}=1$ because g acts on $K_M^{-1}|_{p_s}$ via multiplication by 1. Hence it follows that $\Phi(p_s)=0$ for any $1\leq s\leq q$. On the other hand, let a, b denote $c_1(D_t)$, $c_1(v(D_t,M))$, respectively. Then, we have $e^{i\psi}=-1$ because g acts on $K_M^{-1}|_{D_t}$ via multiplication by -1 and moreover we have

$$e^{c_1(D_t)+c_1(v(D_t,M))}=1+(a+b)$$

$$Td(D_t)=1+\frac{1}{2}a$$

$$\mathscr{V}(v(D_t,\pi))=\frac{1}{1+e^{-b}}=\frac{1}{2}+\frac{1}{4}b.$$

Hence it follows that

$$\begin{aligned}\Psi(D_t)&=(-1+(-1)(a+b)-1)^3(1+\frac{1}{2}a)(\frac{1}{2}+\frac{1}{4}b)[D_t]\\&=-8(a+b)[D_t]=0 \quad \text{mod.}4 \quad (1\leq t\leq r).\end{aligned}$$

Thus it follows from (3.8) that $F(g)=0$.
This completes the proof.

4. Other examples and some remarks

Now let $M\subset \boldsymbol{CP}^{n+r}$ be a complete intersection of degree $(d_1,d_2,\cdots,d_r)$ defined by the simultaneous equations

$$\begin{aligned}&a_{10}z_0^{d_1}+a_{11}z_1^{d_1}+\cdots+a_{1\,n+r}z_{n+r}^{d_1}=0\\&a_{20}z_0^{d_2}+a_{21}z_1^{d_2}+\cdots+a_{2\,n+r}z_{n+r}^{d_2}=0\\&\cdots\cdots\\&a_{r0}z_0^{d_r}+a_{r1}z_1^{d_r}+\cdots+a_{r\,n+r}z_{n+r}^{d_r}=0\end{aligned}$$

Assume that $\{d_1,d_2,\cdots,d_r\}$ has the greatest common divisor $p\geq 2$. Assume moreover that $a_{j0}\neq 0$ for some j and that $N=M\cap\{z_0=0\}\subset \boldsymbol{CP}^{n+r-1}$ defined by

$$\begin{aligned}&a_{11}z_1^{d_1}+\cdots+a_{1\,n+r}z_{n+r}^{d_1}=0\\&a_{21}z_1^{d_2}+\cdots+a_{2\,n+r}z_{n+r}^{d_2}=0\\&\cdots\cdots\\&a_{r1}z_1^{d_r}+\cdots+a_{r\,n+r}z_{n+r}^{d_r}=0\end{aligned}$$

is also a complete intersection in $\boldsymbol{CP}^{n+r-1}$. Then $\boldsymbol{Z}_p=\langle g\rangle$ acts on M by

$$g\cdot[z_0:z_1:\cdots:z_{n+r}]=[\alpha z_0:z_1:\cdots:z_{n+r}] \quad \text{where } \alpha=e^{2\pi i/p}.$$

Theorem 4.1. *$F(g)=0$ for any n, r and any $(d_1,d_2,\cdots,d_r)$.*

Proof. The fixed point set $\Omega\subset M$ of g^k-action $(1\leq k\leq p-1)$ is the hypersurface $N=M\cap\{z_0=0\}$ in M. Let L be the hyperplane bundle of $\boldsymbol{CP}^{n+r-1}$, which is the dual bundle of the tautological bundle of $\boldsymbol{CP}^{n+r-1}$. Set

$$x=c_1(L|_N)\in \mathrm{H}^2(N).$$

Then $x^{n-1}[N]=D$ and $c_1(N)=(n+r-d)x$ where $D=d_1d_2\cdots d_r$ and $d=d_1+d_2+\cdots+d_r$. Now, since

$$T\boldsymbol{CP}^{n+r-1}|_N=TN\oplus\oplus_{j=1}^r\otimes^{d_j}(L|_N),$$

it follows that

$$Td(N)=\left(\frac{x}{1-e^{-x}}\right)^{n+r}\prod_{j=1}^r\frac{1-e^{-d_jx}}{d_jx}.$$

Moreover, since $TM|_N=TN\oplus(L|_N)$ and g^k acts on $L|_N$ via multiplication by α^k, it follows that

$$e^{c_1(N)+c_1(\nu(N,M))+i\varphi(k)}=\alpha^k e^{(n+r+1-d)x},$$

$$\mathscr{V}(\nu(N,\theta_j))=\frac{1}{1-\alpha^{-k}e^{-x}}.$$

Hence it follows from Theorem 2.10 that

$$F(g)=\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^{-k}}\{\alpha^k e^{(n+r+1-d)x}-1\}^{n+1}\left(\frac{x}{1-e^{-x}}\right)^{n+r}\left(\prod_{j=1}^r\frac{1-e^{-d_jx}}{d_jx}\right)\frac{1}{1-\alpha^{-k}e^{-x}}[N].$$

Thus we have

$$F(g)=\frac{D}{p}\sum_{k=1}^{p-1}C(k)$$

where $C(k)$ denotes the x^{n-1}-coefficient of

$$\frac{1}{1-\alpha^{-k}}\{\alpha^k e^{(n+r+1-d)x}-1\}^{n+1}\left(\frac{x}{1-e^{-x}}\right)^{n+r}\left(\prod_{j=1}^r\frac{1-e^{-d_jx}}{d_jx}\right)\frac{1}{1-\alpha^{-k}e^{-x}}\in\boldsymbol{C}[[x]].$$

Now,

x^{n-1}-coefficient of

$$D\{\alpha^k e^{(n+r+1-d)x}-1\}^{n+1}\left(\frac{x}{1-e^{-x}}\right)^{n+r}\left(\prod_{j=1}^r\frac{1-e^{-d_jx}}{d_jx}\right)\frac{1}{1-\alpha^{-k}e^{-x}}$$

$=x^{-1}$-coefficient of

$$\frac{\alpha^k e^x\{\alpha^k e^{(n+r+1-d)x}-1\}^{n+1}}{\alpha^k e^x-1}\left(\frac{1}{1-e^{-x}}\right)^{n+r}\prod_{j=1}^{r}(1-e^{-d_j x})$$

$$=\frac{1}{2\pi i}\oint_{C(z)}\frac{\alpha^k\{\alpha^k(e^z)^{n+r+1-d}-1\}^{n+1}}{\alpha^k e^z-1}\left(\frac{e^z}{e^z-1}\right)^{n+r}\left(\prod_{j=1}^{r}\frac{(e^z)^{d_j}-1}{(e^z)^{d_j}}\right)e^z dz$$

(where $C(z)$ is a sufficiently small counterclockwise loop around the origin)

$$=\frac{1}{2\pi i}\oint_{C(u)}\frac{\alpha^k\{\alpha^k(u+1)^{n+r+1-d}-1\}^{n+1}}{\alpha^k(u+1)-1}\frac{(u+1)^{n+r}}{u^{n+r}}\prod_{j=1}^{r}\frac{(u+1)^{d_j}-1}{(u+1)^{d_j}}du$$

(via the substitution $u=e^z-1$, where $C(u)$ is a counterclockwise loop around the origin)

$=u^{-1}$-coefficient of

$$\frac{\alpha^k\{\alpha^k(u+1)^{n+r+1-d}-1\}^{n+1}}{\alpha^k(u+1)-1}\frac{(u+1)^{n+r-d}}{u^{n+r}}\prod_{j=1}^{r}u(d_j+h_j(u))$$

(where $h_j(u)$ is an integral polynomial of order ≥ 1 in u)

$=u^{n-1}$-coefficient of

$$\frac{\alpha^k\{\alpha^k(u+1)^{n+r+1-d}-1\}^{n+1}}{\alpha^k(u+1)-1}(u+1)^{n+r-d}\prod_{j=1}^{r}(d_j+h_j(u)).$$

Set

$$P(u)=(u+1)^{n+r-d}\prod_{j=1}^{r}(d_j+h_j(u))$$

$$Q(u)=\sum_{k=1}^{p-1}\frac{1}{1-\alpha^{-k}}\frac{\alpha^k\{\alpha^k(u+1)^{n+r+1-d}-1\}^{n+1}}{\alpha^k(u+1)-1}.$$

Then it follows from the calculation above that it suffices to show that the u^{n-1}-coefficient of $P(u)Q(u)$ is 0 mod.p. Note that $P(u)$, $Q(u)$ can be expanded to convergent power series around $u=0$. Note moreover that $P^{(s)}(0)$ is an integral multiple of $s!$ because $P(u)$ can be expanded to a convergent power series with integral coefficients.

Now set

$$\Phi(x,u)=\{x(u+1)^{n+r+1-d}-1\}^{n+1}.$$

Then we can see that, for any integer s with $0 \leq s \leq n+1$,

$$\frac{\partial^s}{\partial u^s}\Phi|_{u=0} = s!\phi_s(x)(x-1)^{n+1-s} \text{ for some integral polynomial } \phi_s. \tag{4.2}$$

Actually it is clear that

$$\frac{\partial^s}{\partial u^s}\Phi|_{u=0} = \mu_s(x)(x-1)^{n+1-s}$$

for some integral polynomial μ_s. On the other hand, since Φ can be expanded to a convergent power series of u around $u=0$ whose coefficients are integral polynomials of x, it follows that

$$\frac{\partial^s}{\partial u^s}\Phi|_{u=0} = s!\nu_s(x)$$

for some integral polynomial ν_s. Hence it follows that

$$\mu_s(x)(x-1)^{n+1-s} = s!\nu_s(x). \tag{4.3}$$

Since the top order term of $(x-1)^{n+1-s}$ is equal to 1, it follows from (4.3) that

$$\mu_s(x) = s!\phi_s(x) \quad \text{for some integral polynomial } \phi_s,$$

which implies (4.2).

Now, for $m \leq n-1$, we have

$$\begin{aligned} Q^{(m)}(0) &= \sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1} \sum_{s=0}^{m} \binom{m}{s} (\{\alpha^k(u+1)-1\}^{-1})^{(m-s)}(0)(\{\alpha^k(u+1)^{n+r+1-d}-1\}^{n+1})^{(s)}(0) \\ &= \sum_{k=1}^{p-1} \frac{(\alpha^k)^2}{\alpha^k - 1} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s}(m-s)!(\alpha^k)^{m-s}(\alpha^k-1)^{-m+s-1} s!\phi_s(\alpha^k)(\alpha^k-1)^{n+1-s} \\ &= m! \sum_{k=1}^{p-1} \sum_{s=0}^{m} (-1)^{m-s}(\alpha^k)^{2+m-s}\phi_s(\alpha^k)(\alpha^k-1)^{n-1-m}. \end{aligned}$$

Hence it follows from the fact (See (2.12).)

$$\sum_{k=1}^{p-1} \Psi(\alpha^k) = -\Psi(1) \quad \text{mod.} p \quad \text{for any integral polynomial } \Psi$$

that $Q^{(m)}(0)$ is an integral multiple of $p \cdot m!$ if $m \leq n-2$ and is equal to an integral multiple of $(n-1)!$ if $m=n-1$. Therefore it follows that

$$\frac{1}{(n-1)!}(PQ)^{(n-1)}(0)$$

$$=\frac{1}{(n-1)!}\{P(0)Q^{(n-1)}(0)+\sum_{m=0}^{n-2}\binom{n-1}{m}P^{(n-1-m)}(0)Q^{(m)}(0)\}$$

$$=P(0)\frac{Q^{(n-1)}(0)}{(n-1)!}+\sum_{m=0}^{n-2}\frac{P^{(n-1-m)}(0)}{(n-1-m)!}\frac{Q^{(m)}(0)}{m!}$$

is equal to 0 mod.p because $P(0)$ is equal to $d_1 d_2 \cdots d_r$ which is an integral multiple of p. Thus it follows that

$$u^{n-1}\text{-coefficient of } P(u)Q(u)=0 \quad \text{mod.}p.$$

This completes the proof.

Remark 4.4. Let M be the Fermat cubic surface

$$M: z_0^3+z_1^3+z_2^3+z_3^3=0 \quad \text{in } \boldsymbol{CP}^3$$

and

$$g\cdot[z_0:z_1:z_2:z_3]=[e^{2\pi i/3}z_0:z_1:z_2:z_3].$$

Then $A(M)$ is a finite group generated by g and the transposition of coordinates whose order is 2. Hence it follows from Theorem 4.1 and (3.7) that

$$F(g)=0 \quad \text{for any } g\in A(M).$$

Note that the Fermat cubic surface is isomorphic to the six points blowing-up of $\boldsymbol{CP}^2$ with non-generic complex structure in the sense in section 3.

Remark 4.5. In [16] certain kinds of complete intersections including the case that $r=1, \frac{n+1}{2}\le d_1 \le n+1$ are shown to admit K-E metrics, and no example of a complete intersection which does not admit any K-E metric is known.

Remark 4.6. Using the $\otimes^{n+1}(TM-\varepsilon^n)$-valued spin^c-Dirac operators (where ε^n denotes the trivial bundle $M\times \boldsymbol{C}^n$) instead of the $\otimes^{n+1}(K_M^{-1}-\varepsilon)$-valued spin^c-Dirac operators, we can obtain a formula similar to Theorem 2.10.

Remark 4.7. We can see that the lifted Futaki invariant F is interpreted as a "holonomy" of a $\otimes^{n+1}(TM-\varepsilon^n)$-valued spin^c-Dirac operator (cf. [20]).

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