



Title	Moishezon fourfolds homeomorphic to $Q^4$ c
Author(s)	Nakamura, Iku
Citation	Osaka Journal of Mathematics. 1994, 31(4), p. 787-829
Version Type	VoR
URL	<a href="https://doi.org/10.18910/6835">https://doi.org/10.18910/6835</a>
rights	
Note	

*The University of Osaka Institutional Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

## MOISHEZON FOURFOLDS HOMEOMORPHIC TO $Q_C^3$

IKU NAKAMURA

(Received March 30, 1993)

### Introduction

In general, there are many different complex manifolds having the same underlying topological or differentiable structure. However there are a few exceptional cases where we can expect that homeomorphy to a given compact complex manifold implies analytic isomorphism to it, for instance, an irreducible compact Hermitian symmetric space. Among irreducible Hermitian symmetric spaces, the complex projective space  $P_C^n$  and a smooth hyperquadric  $Q_C^n$  in  $P_C^{n+1}$  seem to be nice exceptions which we can handle with algebraic methods. In [15] we studied the complex projective space  $P_C^n$ , while in the present article we study a smooth hyperquadric  $Q_C^n$  in  $P_C^{n+1}$  in the same way as in [15]. A goal we have in mind is the following

**Conjecture  $MQ_n$ .** *Any Moishezon complex manifold homeomorphic to  $Q_C^n$  is isomorphic to  $Q_C^n$ .*

The conjecture has been solved partially by Brieskorn [3] under the assumption that the manifold in question is *Kählerian* and odd-dimensional. In the even-dimensional *Kählerian* case, there still remains a possibility of manifolds of general type. Recently Kollár [7] and the author [13] solved Conjecture  $MQ_3$  in the affirmative, each supplementing the other. Peternell [16][17] also asserts the same consequence. See [7, 5.3.6].

**Theorem 1.** *Any Moishezon threefold homeomorphic to  $Q_C^3$  is isomorphic to  $Q_C^3$ .*

The main purpose of the present article is to give a partial solution to the above conjecture  $MQ_4$  in dimension 4. We prove,

**Theorem 2.** *Let  $X$  be a Moishezon fourfold homeomorphic to  $Q_C^4$ ,*

and  $L$  a line bundle on  $X$  with  $L^4 = 2$ . Assume  $h^0(X, O_X(L)) \geq 5$ . Then  $X$  is isomorphic to  $Q_C^4$ .

**Corollary 3.** Any global deformation of  $Q_C^4$  is isomorphic to  $Q_C^4$ .

It is easy to see that any complex analytic (global) deformation of  $Q_C^n$  is Moishezon. However it is possible that there appears a *non-projective* or a *non-Kählerian* Moishezon manifold of dimension  $n \geq 3$  as a global deformation of a *projective* or a *Kählerian* manifold (Hironaka [6]). This is one of the reasons why we consider a possibly *non-projective* or a possibly *non-Kählerian* Moishezon manifold as in Theorem 1 and Theorem 2. We easily derive Corollary 3 from Theorem 2. In fact, any global deformation of  $Q_C^4$  not only in any complex analytic family but also in any differentiable family is isomorphic to  $Q_C^4$ .

Now we give an outline of our proof of Theorem 2. Let  $X$  be a Moishezon fourfold homeomorphic to  $Q_C^4$ . Then we have a unique line bundle  $L$  on  $X$  such that  $\text{Pic } X \simeq \mathbb{Z}L$ ,  $c_1(X) = 4c_1(L)$ , and  $L^4 = 2$ . Let  $m := h^0(X, O_X(L)) - 1 \geq 4$ . We consider the rational map  $h: X \rightarrow \mathbb{P}_C^m$  associated to the linear system  $|L|$ . Let  $W$  be the closure of the image  $h(X \setminus \text{Bs } |L|)$ . Let  $d = \deg W$ . Then  $d \geq m + 1 - \dim W$ . Since  $\text{Pic } X \simeq \mathbb{Z}L$ , we have  $\dim W \geq 2$ . Let  $\tau$  be a complete intersection  $D \cap D'$  for general  $D$  and  $D' \in |L|$ . Then  $\tau$  is connected, pure two-dimensional and Gorenstein.

Assume first  $\dim W = 2$ . Then we have reduced irreducible components  $Z_i$  ( $1 \leq i \leq d$ ) of  $\tau$  outside  $B := \text{Bs } |L|$ . We note that  $d \geq m - 1 \geq 3$ . Each  $Z_i$  is nonsingular outside  $B$  by Bertini's theorem. Let  $v_i: Y_i \rightarrow Z_i$  be the normalization of  $Z_i$ ,  $f_i: S_i \rightarrow Y_i$  the minimal resolution of  $Y_i$  and  $g_i := v_i \cdot f_i$ . We see that  $K_{S_i} = -2g_i^*(L) - A_i$  for some effective divisor  $A_i$  with  $\text{supp}(A_i) \subset g_i^{-1}(B)$ . Since  $g_i^*(L)$  is effective by  $m \geq 2$ ,  $S_i \simeq \mathbb{P}_C^2$  or  $S_i$  is ruled.

If  $S_i \simeq \mathbb{P}_C^2$ , then  $S_i \simeq Y_i \simeq Z_i$ ,  $g_i^*(L) = A_i \in |O_{\mathbb{P}^2}(1)|$ . If moreover  $Z_i \cap Z_j \neq \emptyset$  for  $i \neq j$ , then  $d = 2$ , which contradicts  $d \geq 3$ . If  $Z_i \cap Z_j = \emptyset$  for  $i \neq j$ , then  $W$  turns out to be a cone over a smooth variety of minimal degree by the Del Pezzo-Bertini classification [5]. Any such  $W$  has a reducible or noreduced hyperplane section for  $d \geq 3$ , which contradicts  $\text{Pic } X \simeq \mathbb{Z}L$ . If  $S_i$  is ruled, then we can derive a contradiction similarly.

Similarly we can disprove  $\dim W = 3$ . Consequently  $\dim W = 4$ . Bertini's theorem shows that a scheme-theoretic complete (not necessarily proper) intersection  $l$  of general 3 members of  $|L|$  is pure one-dimensional and irreducible nonsingular outside  $B$ . We infer from  $c_1(X) = 4c_1(L)$  that  $l$  has a rational curve  $C$  with  $LC = 2$  as an irreducible component outside  $B$ . Then applying (in sections 3 and 4) the same argument as,

in fact simpler than, in [15], we can study the morphism  $h$  in detail. Subsequently we see that  $h^0(X, L) = 6$  and that  $h$  is an isomorphism of  $X$  onto a smooth hyperquadric  $Q_c^4$  in  $P_c^5$ .

The article is organized as follows. In sections one and two, we study a scheme-theoretic complete intersection  $l_V$  of general  $(n-1)$ -members in  $|L|$  along reduced curve-components.

In sections 3 and 4, we study Moishezon manifolds of dimension  $n$  with the second Betti number  $b_2(X)$  equal to one, and with  $c_1(X) = nc_1(L)$  for some line bundle  $L$  on  $X$ . In section 5, we prove Theorem 2 by applying the results in the previous sections.

NOTATION. The notation is indexed at the end of the article.

ACKNOWLEDGEMENT. The author would like to express his hearty gratitude to A. Fujiki, F. Hidaka and I. Shimada for their advices during the preparation of the article. Fujiki kindly showed simpler proofs for some of our proofs in the first version of the article.

## 1. A complete intersection $l_V$ (1) —local structure—

(1.1) BASIC ASSUMPTIONS. Let  $X$  be a complete nonsingular algebraic variety of dimension  $n$ ,  $L$  a line bundle on  $X$ . We assume

$$(1.1.1) \quad c_1(X) = dc_1(L) \quad \text{for some integer } d,$$

$$(1.1.2) \quad h^0(X, L) \geq n.$$

Let  $V$  be an  $(n-1)$ -dimensional subspace of  $H^0(X, L)$ ,  $l := l_V$  a scheme-theoretic complete intersection associated with  $V$ . This means that the ideal  $I_l$  of  $O_X$  defining  $l$  is defined by  $I_l = \sum_{s \in V} s O_X$ .

We say that  $C$  is a *reduced curve-component* of  $l$  if  $C$  is an irreducible one-dimensional component of  $l$  along which  $l$  is reduced generically. We assume that

$$(1.1.3) \quad l \text{ has a reduced curve-component outside } B \quad (:= \text{Bs } |L|).$$

(1.2) TORSION SHEAVES  $Q_C$ ,  $Q'_C$  AND  $Q''_C$ . Let  $C$  a reduced curve-component of  $l_V$ , and  $I_C$  the ideal sheaf of  $O_X$  defining  $C$  with  $\sqrt{I_C} = I_C$ . Let  $v: \tilde{C} \rightarrow C$  be the normalization of  $C$ . Then we have a natural exact sequence

$$(1.2.1) \quad 0 \rightarrow (I_l/I_l^2) \otimes O_{\tilde{C}} (\simeq O_{\tilde{C}}(-v^*L)^{\oplus(n-1)}) \xrightarrow{\phi_C} [(I_C/I_C^2) \otimes O_{\tilde{C}}] \rightarrow Q_C \rightarrow 0$$

where  $Q_C := \text{Coker } \phi_C$  and  $[J] := J/O_{\tilde{C}}$ -torsions for an  $O_{\tilde{C}}$ -module  $J$ . We also have a natural exact sequence

$$(1.2.2) \quad 0 \rightarrow Q''_C \rightarrow \Omega_C^1 \otimes O_{\tilde{C}} \xrightarrow{\eta} \Omega_C^1 \rightarrow Q'_C \rightarrow 0$$

where  $Q'_C = \text{Coker } \eta$ ,  $Q''_C = \text{Ker } \eta$ .

We define  $l(F) := \dim_C F$  for a torsion  $O_{\tilde{C}}$ -module  $F$  and  $l(F, p) := l(F_p)$  for a stalk  $F_p$  of  $F$  at  $p$ . See (1.10) and Theorem 1.11. Then by [15, §1 and 2], we have,

**Lemma 1.3.** *Under the notation and assumptions in (1.1), let  $C$  be a reduced curve-component of  $l_V$ . Let  $Q_C^{(0)} := Q_C$ ,  $Q_C^{(1)} := Q'_C$  and  $Q_C^{(2)} := Q''_C$ . Then,*

$$(1.3.1) \quad (d-n+1)LC + c_1(\Omega_C^1) + l(Q_C) + l(Q''_C) - l(Q'_C) = 0.$$

$$(1.3.2) \quad l(Q_C^{(v)}) = \sum_{p \in C} l(Q_C^{(v)}, p) \quad (v=0, 1, 2).$$

$$(1.3.3) \quad l(Q_C^{(v)}, p) \geq 0, \quad l(Q''_C, p) \geq l(Q'_C, p) \quad \text{for any } p \in C.$$

$$(1.3.4) \quad l(Q_C, p) = 0 \quad \text{if and only if } (C, p) \simeq (l, p).$$

$$(1.3.5) \quad l(Q''_C, p) \geq l(Q'_C, p) \quad \text{for any } p \in C, \text{ where equality holds if and only if } (C, p) \text{ is irreducible nonsingular. If } (C, p) \text{ is nonsingular, then } l(Q''_C, p) = l(Q'_C, p) = 0.$$

$$(1.3.6) \quad \text{If } (C, p) \text{ is irreducible and singular, then } l(Q''_C, p) \geq l(Q'_C, p) + 2.$$

$$(1.3.7) \quad \text{Assume that } (C, p) \text{ is reducible. Let } (C_\lambda, p) \ (\lambda \in \Lambda) \text{ be all the irreducible components of } (C, p), \text{ and } \Lambda_{ns} \text{ (resp. } \Lambda_s) \text{ the subset of } \Lambda \text{ consisting of all } \lambda \text{ with } (C_\lambda, p) \text{ nonsingular (resp. singular). Then } l(Q''_C, p) \geq l(Q'_C, p) + 2\#(\Lambda_s) + \#(\Lambda_{ns}).$$

The proof of (1.3.5)–(1.3.7) is partially based on the following Lemma 1.4. See [15, §2] for the details.

**Lemma 1.4.** *Assume that  $(C, p)$  is irreducible and singular. Let  $x_1, \dots, x_n$  be a local coordinate system of  $(X, p)$ . We (may) assume that the normalization  $v: \tilde{C} \rightarrow C$  ( $\subset X$ ) is locally given by*

$$\begin{aligned} x_1 &= t^m \\ x_j &= f_j(t) = t^{m_j} g_j(t), \quad g_j(0) \neq 0 \quad (2 \leq j \leq s) \\ x_j &= 0 \quad (s+1 \leq j \leq n) \end{aligned}$$

for some  $f_j, g_j \in O_{\bar{C}}$ , where  $2 \leq m < m_1 < m_2 < m_3 < \dots < m_s$ , none of  $m_j$  and none of  $m_j - m_k$  is an integral multiple of  $m$ , while  $s$  is the embedding dimension of  $(C, p)$ . Let  $q$  be the unique positive integer such that  $m \leq qm < m_2 < (q+1)m$ . Then

$$(1.4.1) \quad l(Q'_C, p) = m - 1.$$

$$(1.4.2) \quad l(Q''_C, p) \geq \min(2qm, m_3) + m - m_2 \geq m + 1.$$

Proof. See [15, (2.3)] for the details. We recall the proof only for the later use. By the proof of [15, (2.3)],  $l(Q'_C, p) = l(\Omega_{\bar{C}, \bar{p}}^1 / \Omega_{C, p}^1 \otimes O_{\bar{C}, \bar{p}}) = m - 1$ . Let

$$e_j = dx_j \otimes 1 \in \Omega_X^1 \otimes O_{\bar{C}}, \quad \bar{e}_j = dx_j \otimes 1 \in \Omega_C^1 \otimes O_{\bar{C}}.$$

Then the element  $\sigma_j = (f'_j(t)/mt^{m-1})\bar{e}_1 - \bar{e}_j$  is contained in  $Q''_C$ .

Since  $Q''_C$  is a torsion sheaf, we (can) choose the minimal integer  $N \geq 0$  such that  $t^N \sigma_2 = 0$ . By definition  $l(Q''_C, p) \geq N$ . The condition  $t^N \sigma_2 = 0$  means that there exist some  $F_i \in \mathcal{C}[[t]]$  and  $\varphi_i \in I_C$  ( $1 \leq i \leq l$ ) such that

$$(1.4.3) \quad t^N((f'_2(t)/mt^{m-1})e_1 - e_2) = \sum_{j=1}^s \left( \sum_{i=1}^l F_i(t) v^*(\partial \varphi_i / \partial x_j) \right) e_j.$$

The coefficient of  $e_1$  in the right hand side of (1.4.3) is equal to  $\sum_{i=1}^l F_i(t) v^*(\partial \varphi_i / \partial x_1)$ . Take any element  $\varphi \in I_C \cap \mathcal{C}[[x_1, \dots, x_s]]$  ( $\subset m_p^2$ ) with its expansion given by

$$\varphi = \sum_{i_1 + \dots + i_s \geq 2} a_{i_1 \dots i_s} x_1^{i_1} \dots x_s^{i_s}.$$

Then  $\varphi \in I_{C, p}$  implies that  $a_{j0 \dots 0} = 0$  ( $1 \leq j \leq 2q$ ),  $a_{j10 \dots 0} = 0$  ( $1 \leq j \leq q$ ). Hence

$$\partial \varphi / \partial x_1 = (2q+1)ax_1^{2q} + (q+1)bx_1^q x_2 + cx_2^2 + dx_3 + \dots$$

$$\deg v^*(\partial \varphi / \partial x_1) \geq \min(2qm, qm + m_2, 2m_2, m_3) = \min(2qm, m_3)$$

for some constants  $a, b, c$  and  $d$ . Hence  $\deg t^{N-m+1} f'_2(t) \geq \min(2qm, m_3) \geq m_2 + 1$ , which completes the proof of (1.4.2). q.e.d.

(1.5) THREE CASES. Let  $C$  be a reduced curve-component of  $l, p$  a point of  $C$ . Assume  $d = n$ ,  $LC \geq 0$  and that  $(C, p)$  is singular. Then by (1.3.1) we have,

$$l(Q_C) + l(Q''_C) - l(Q'_C) \leq -c_1(\Omega_C^1) \leq 2.$$

If  $(C, p)$  is irreducible, then  $l(Q_C) = 0$ , and  $l(Q''_C) - l(Q'_C) = 2$ . If  $(C, p)$  is reducible,  $(C, p)$  has by (1.3.7) exactly two irreducible components, and any irreducible component of it is nonsingular. Thus we have only to consider the following three cases:

$$\text{CASE A.} \quad l(Q_C, p) = 1, \quad l(Q''_C, p) = l(Q'_C, p) = 0.$$

$$\text{CASE B.} \quad l(Q_C, p) = 0, \quad l(Q''_C, p) - l(Q'_C, p) = 2.$$

$$\text{CASE C.} \quad l(Q_C, p) = 2, \quad l(Q''_C, p) = l(Q'_C, p) = 0.$$

**Lemma 1.6** (CASE A). *Assume that  $l(Q_C, p) = 1$ ,  $l(Q''_C, p) = l(Q'_C, p) = 0$ . Then  $(l, p)$  has two irreducible components  $(C, p)$  and  $(C', p)$ , and there exists a local parameter system  $x_1, \dots, x_n$  such that*

$$I_{l,p} = (x_1, \dots, x_{n-2}, x_{n-1}x_n),$$

$$I_{C,p} = (x_1, \dots, x_{n-1}), \quad I_{C',p} = (x_1, \dots, x_{n-2}, x_n).$$

*Proof.* By (1.3.5), the germ  $(C, p)$  is nonsingular, so that we can choose local parameters  $x_1, \dots, x_{n-1}$  such that  $I_{C,p} = (x_1, \dots, x_{n-1})$ . The condition  $l(Q_C, p) = 1$  implies that we may assume  $x_i \in I_{l,p}$  ( $1 \leq i \leq n-2$ ). Moreover we can choose an  $(n-1)$ -th generator  $f_{n-1}$  of  $I_{l,p}$  such that  $f_{n-1} \bmod I_C^2$  has a single zero at  $p$  as a local section of  $I_C/I_C^2$ . Therefore by choosing an  $n$ -th local coordinate  $x_n$  at  $p$  suitably, we may assume  $I_{l,p} = (x_1, \dots, x_{n-2}, x_{n-1}x_n)$ . It follows that  $l$  has another irreducible component  $(C', p)$  as above. q.e.d.

**Lemma 1.7** (CASE B). *Assume that  $l(Q_C, p) = 0$ ,  $l(Q''_C, p) = l(Q'_C, p) + 2$ . Then there exists a local parameter system  $x_1, \dots, x_n$  such that one of the following is true.*

$$(1.7.1) \quad I_{l,p} = I_{C,p} = (x_1, \dots, x_{n-2}, x_{n-1}^3 - x_n^2),$$

$$(1.7.2) \quad I_{l,p} = I_{C,p} = (x_1, \dots, x_{n-2}, x_{n-1}x_n).$$

*Proof.* By  $l(Q_C, p) = 0$ , we have  $I_{l,p} + I_{C,p}^2 = I_{C,p}$ . By Nakayama's lemma we see  $I_{l,p} = I_{C,p}$ , whence  $(l, p) \simeq (C, p)$ . There are two subcases CASE B-1 and CASE B-2 according as  $(C, p)$  is irreducible or not.

CASE B-1. Assume that  $(C, p)$  is irreducible. We use the same notation as in Lemma 1.4. Then by the proof of Lemma 1.4

$$l(Q'_C, p) = m - 1, \quad l(Q''_C, p) \geq N \geq m + 1.$$

Consequently we have

$$(1.7.3) \quad N = l(Q''_{C,p}) = m + 1 \geq 3.$$

Moreover by the proof of Lemma 1.4, we see that there exists  $\varphi \in I_{C,p}$  such that

$$(1.7.4) \quad N - m + m_2 = \deg v^*(\partial\varphi/\partial x_1) = \min(2qm, m_3).$$

CASE B-1-1. First we consider the case where  $2qm \leq m_3$ . Then  $N - m + m_2 = 2qm$ . Since  $N = m + 1$  and  $qm < m_2 < (q + 1)m$ , we have  $q = 1$ ,  $m_2 = 2m - 1$ , and  $m_3 \geq 2m + 1$ . In view of the proof of Lemma 1.4 the expansions of  $\varphi$  and  $\partial\varphi/\partial x_1$  are given by

$$\begin{aligned} \varphi &= ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_1x_3 + ex_2^2 + \cdots, \\ \partial\varphi/\partial x_1 &= 3ax_1^2 + 2bx_1x_2 + cx_2^2 + dx_3 + \cdots. \end{aligned}$$

Since  $m_3 \geq 2m + 1$ ,  $v^*x_1^2$  is the unique monomial term of degree  $2m$  in the right hand side of  $\partial\varphi/\partial x_1$ . Since  $\deg v^*\partial\varphi/\partial x_1 = 2qm = 2m$  by (1.7.4), we have  $a \neq 0$ . On the other hand since  $v^*\varphi = 0$ , there is another nontrivial term of degree  $3m$  besides  $x_1^3$  in the right hand side of  $\varphi$ , which is just  $ex_2^2$  by the choice of  $m_j$  in Lemma 1.4. Therefore  $e \neq 0$ . Hence we have  $\varphi = x_1^3 - x_2^2 + \cdots$  by modifying  $x_2$  and  $x_3$  by constant multiples. Therefore  $3m = 2m_2 = 4m - 2$ ,  $m = 2$  and  $m_2 = 3$ . It follows that the normalization  $v: \tilde{C} \rightarrow C$  is given by

$$x_1 = t^2, \quad x_2 = t^3g_2(t)$$

for some holomorphic  $g_2(t)$  with  $g_2(0) = 1$ . Then there exist  $g_{21}(x_1)$  and  $g_{22}(x_1) \in m_p$  such that

$$v^*x_2 = t^3 + g_{21}(t^2) + t^3g_{22}(t^2) = t^3(1 + v^*g_{22}(x_1)) + v^*g_{21}(x_1).$$

By taking  $x'_2 = (x_2 - g_{21}(x_1))(1 + g_{22}(x_1))^{-1}$  instead of  $x_2$ , the normalization  $v: \tilde{C} \rightarrow C$  is given by

$$x_1 = t^2, \quad x_2 = t^3.$$

Since any monomial  $t^n$  ( $n \geq 4$ ) is a product of  $t^2$  and  $t^3$ , we may assume  $x_j = 0$  ( $j \geq 3$ ), so that the embedding dimension of  $(C, p)$  is equal to 2. Thus we see that

$$I_{l,p} = I_{C,p} = (x_1^3 - x_2^2, x_3, x_4, \cdots, x_n),$$



$$l(Q''_{C,p})=3, \quad l(Q'_{C,p})=1.$$

CASE B-1-2. Next we consider the case where  $2qm > m_3$ . By (1.7.3) and (1.7.4), we see  $N=m+1$ ,  $m_3=m_2+1$ . Moreover there exists  $\varphi \in I_{C,p} \cap m_p^2$  such that  $\deg v^*(\partial\varphi/\partial x_1)=m_3$ . Since  $\varphi \in I_{C,p}$  has at least two monomial terms of minimum degree in  $t$  by the condition  $v^*\varphi=0$ , we see

$$\varphi = ax_1x_3 + bx_2^2 + \cdots,$$

where  $a \neq 0$ ,  $b \neq 0$ , and  $m+m_3=2m_2$  by the choice of  $m_j$  in Lemma 1.4. It follows that  $m_2=m+1$ ,  $m_3=m+2$ . Hence  $\varphi$  has exactly two monomial terms  $x_1x_3$  and  $x_2^2$  of minimum degree. We may assume  $\varphi = x_1x_3 - x_2^2 + (\text{higher terms})$  by choosing  $x_2, x_3$  suitably. Since  $\varphi_j \in I_{C,p}$  ( $1 \leq j \leq l$ ) in the right hand side of (1.4.3), by the above argument we can write  $\varphi_j = c_j\varphi + \varphi_j^*$  where  $c_j$  is a constant and  $\varphi_j^*$  has no monomial terms  $x_1x_3$  and  $x_2^2$ . Let  $c = \sum_{i=1}^l c_i F_i(0)$ . Then we have

$$\deg v^*(\partial\varphi_j^*/\partial x_1) \geq m_3 + 1 \geq m + 3, \quad \deg v^*(\partial\varphi_j^*/\partial x_3) \geq \min(2m, m_2) \geq m + 1.$$

It follows from (1.7.3) that the coefficient of  $e_1$  (resp.  $e_3$ ) in the right hand side of (1.4.3) starts with  $ct^{m+2}$  (resp.  $-ct^m$ ), where  $N=m+1$  implies that  $c \neq 0$ . However the coefficient of  $e_3$  in the left hand side of (1.4.3) is equal to 0, which implies  $c=0$ , a contradiction.

CASE B-2. Assume that  $(C,p)$  is reducible. Let  $(C_\lambda, p)$  ( $\lambda \in \Lambda$ ) be all the irreducible components of  $(C,p)$ . In view of (1.3.7),  $\Lambda = \Lambda_{\text{ns}}$ ,  $\#(\Lambda)=2$ . Let  $\Lambda = \{0, 1\}$ . Then for  $\lambda=0, 1$ , we have

$$(1.7.5) \quad l(\text{Ker}(\Omega_C^1 \otimes O_{C_\lambda} \rightarrow \Omega_{C_\lambda}^1)) = 1$$

by the proof of (1.3.7) [15, §2]. We choose a local coordinate system  $x_1, \dots, x_n$  at  $p$  such that  $I_{C_0,p} = (x_1, \dots, x_{n-1})$ . The normalization  $v: \tilde{C}_0 \rightarrow C_0$  is clearly given by

$$x_i = 0 \quad (1 \leq i \leq n-1), \quad x_n = t.$$

Let  $S := O_{C_0} \{d\varphi; \varphi \in I_C, v^* \frac{\partial\varphi}{\partial x_n} = 0\}$ . Then we have

$$(1.7.6) \quad \text{Ker}(\Omega_C^1 \otimes O_{C_0} \rightarrow \Omega_{C_0}^1) \simeq O_{C_0} dx_1 + \cdots + O_{C_0} dx_{n-1} / S.$$

By (1.7.5) we may assume that  $x_i \in I_{C,p}$  ( $1 \leq i \leq n-2$ ) and that the right hand side of (1.7.6) is generated by  $dx_{n-1}$ . Moreover there exists  $\varphi \in I_{C,p}$  such that

$$v^*(\partial\varphi/\partial x_{n-1})=t, \quad v^*(\partial\varphi/\partial x_i)=0 \quad (1 \leq i \leq n-2 \text{ or } i=n).$$

It follows that  $\varphi = x_{n-1}x_n \bmod (x_1, \dots, x_{n-1})^2$ . Therefore by choosing  $\varphi \in I_{C,p} \bmod (x_1, \dots, x_{n-2})$  and  $x_n \bmod (x_1, \dots, x_{n-1})$  suitably we may assume  $\varphi = x_{n-1}x_n \in I_{C,p}$ . Then we have

$$I_{C,p} = (x_1, \dots, x_{n-2}, x_{n-1}x_n).$$

In fact, let  $J = (x_1, \dots, x_{n-2}, x_{n-1}x_n)$  and  $O_{C'} := O_{X,p}/J$ . Then  $(C', p)$  is a reduced subvariety of  $(X, p)$  with two irreducible components  $C_0$  and  $C_1$  at  $p$ , so that  $\text{supp}(C') = \text{supp}(C)$ . Hence  $l(I_{C,p}/J)$  is finite. Therefore  $I_{C,p} \supseteq J = \sqrt{J} = \sqrt{I_{C,p}} \supseteq I_{C,p}$ , whence  $J = I_{C,p}$  and  $(C, p) \simeq (C', p)$ . Thus  $(C, p)$  has two irreducible components  $C_0$  and  $C_1$  defined by

$$I_{C_0,p} = (x_1, \dots, x_{n-2}, x_{n-1}), \quad I_{C_1,p} = (x_1, \dots, x_{n-2}, x_n).$$

q.e.d.

(1.8) EXAMPLE. Let  $s$  be an integer  $\geq 1$ . Consider a germ  $(C, p)$  defined by

$$I_{C,p} = (x_1x_3 - x_2^2, x_1^{s+2} - x_2x_3^s, x_1^{2s+3} - x_3^{2s+1}).$$

The normalization  $v: (\tilde{C}, p) \rightarrow (C, p)$  is given by

$$x_1 = t^{2s+1}, \quad x_2 = t^{2s+2}, \quad x_3 = t^{2s+3}.$$

Let  $e_i^* := dx_i \otimes 1 \in Q''_{C,p}$ . The torsion sheaf  $Q''_{C,p}$  is generated by two elements

$$\sigma_1 := t^2e_1^* - 2te_2^* + e_3^*, \quad \sigma_2 := (2s+2)te_1^* - (2s+1)e_2^*$$

where  $t^{2s}\sigma_1 = 0$ ,  $t^{2s^2+3s}\sigma_2 = 0$ . Thus we have  $l(Q''_{C,p}) = 2s^2 + 5s$  and  $l(Q'_{C,p}) = 2s$ . Compare the proof of Lemma 1.7 CASE B-1-2.

**Lemma 1.9** (GENERIC CASE IN CASE C). *Assume that  $(l, p)$  is sufficiently general and that  $(l, p)$  is reduced, nonsingular and pure one-dimensional outside  $B := B_s|L|$ . Let  $C$  be a movable reduced curve-component of  $l$ . If  $l(Q_C, p) = 2$ ,  $l(Q''_{C,p}) = l(Q'_{C,p}) = 0$ , then there exists a local coordinate system  $x_1, \dots, x_n$  at  $p$  such that*

$$I_{C,p} = (x_1, \dots, x_{n-1}),$$

and that one of the following is true.

$$(1.9.1) \quad I_{l,p} = (x_1, \dots, x_{n-3}, x_{n-2}, x_{n-1}x_n^2), \quad I_{B,p} = (x_1, \dots, x_{n-3}, x_{n-2}, x_n^2).$$

$$(1.9.2) \quad I_{l,p} = (x_1, \dots, x_{n-3}, x_{n-2}, (x_n^2 - x_{n-1}^s)x_{n-1}), \\ I_{B,p} = (x_1, \dots, x_{n-3}, x_{n-2}, x_n^2 - x_{n-1}^s) \quad (s \geq 2).$$

$$(1.9.3) \quad I_{l,p} = (x_1, \dots, x_{n-3}, x_{n-2}x_n, x_{n-1}x_n), \quad I_{B,p} = (x_1, \dots, x_{n-3}, x_n).$$

$$(1.9.4) \quad I_{l,p} = (x_1, \dots, x_{n-3}, x_{n-2}x_n, (x_n + a(x))x_{n-1})$$

for some  $a(x) (\neq 0) \in \mathcal{C}[[x_{n-2}, x_{n-1}]] \cap m_p$ . The germ  $(l, p)$  has at least 3 irreducible components. Among them, there are at most 4 movable components of  $(l, p)$ .

(1.9.4.1) If  $(l, p)$  has exactly 3 irreducible components, then  $a(x_{n-2}, x_{n-1}) = x_{n-2}^{e_{n-2}} x_{n-1}^{e_{n-1}-1}$  for some  $e_{n-2} \geq 0$ ,  $e_{n-1} \geq 0$  with  $e_{n-2} + e_{n-1} \geq 1$ .

(1.9.4.2) If  $(l, p)$  has exactly 4 movable components  $(C_j, p)$  ( $0 \leq j \leq 3$ ) with  $C_0 = C$ , then  $a(x_{n-2}, x_{n-1}) = x_{n-2} + x_{n-1}$  so that  $(l, p)$  has no fixed components, and  $I_{B,p} = (x_1, \dots, x_{n-3}, x_{n-2}, x_{n-1}, x_n)$ .

Proof. Since  $(C, p)$  is nonsingular by (1.3.5), we can choose a local coordinate system  $x_1, \dots, x_n$  at  $p$  such that

$$I_{C,p} = (x_1, \dots, x_{n-1}), \quad I_{l,p} = (x_1, \dots, x_{n-3}, \varphi_{n-2}, \varphi_{n-1})$$

for some  $\varphi_j \in I_{l,p}$ . Since  $l(Q_C, p) = 2$ , we may assume by choosing  $\varphi_{n-2}$  and  $\varphi_{n-1}$  suitably that one of the following is true;

$$\text{CASE C-1.} \quad \varphi_{n-2} = x_{n-2}, \quad \varphi_{n-1} = x_n^2 x_{n-1}$$

$$\text{CASE C-2.} \quad \varphi_{n-2} = x_{n-2}, \quad \varphi_{n-1} = (x_n^2 - x_{n-1}^s)x_{n-1} \quad (s \geq 2)$$

$$\text{CASE C-3.} \quad \varphi_{n-2} = x_n x_{n-2}, \quad \varphi_{n-1} = x_n x_{n-1}$$

$$\text{Case C-4.} \quad \varphi_{n-2} = x_n x_{n-2}, \quad \varphi_{n-1} = (x_n + \psi(x_{n-2}, x_{n-1}, x_n))x_{n-1}$$

where  $\psi (\neq 0) \in \mathcal{C}[[x_{n-2}, x_{n-1}, x_n]] \cap m_p$ .

CASE C-1. Since any nonreduced component of  $(l, p)$  is contained in  $B_{\text{red}}$ ,  $B_{\text{red}}$  passes through  $p$  and (1.9.1) follows.

CASE C-2. If  $s (\geq 3)$  is odd, then  $x_n^2 - x_{n-1}^s$  is irreducible. Hence  $(l, p)$  has an irreducible component  $(C', p)$  besides  $(C, p)$  defined by

$$I_{C',p} = (x_1, \dots, x_{n-2}, x_n^2 - x_{n-1}^s).$$

Since  $(C', p)$  is singular,  $(C', p)$  does not belong to the same algebraic

family as a movable  $(C, p)$ , whence  $(C', p)$  is contained in  $B_{\text{red}}$ . Thus (1.9.2) for  $s$  odd follows.

If  $s=2q$  ( $\geq 4$ ) is even, then we have, in addition to  $(C, p)$ , two irreducible components  $(C', p)$  and  $(C'', p)$  of  $(l, p)$ . Although  $(C, p)$  and  $(C', p)$  (or  $(C'', p)$ ) intersect transversally,  $(C', p)$  and  $(C'', p)$  have a contact. That is,  $I_{C', p} + I_{C'', p} \neq m_p$ , where  $m_p$  is the maximal ideal of  $O_{X, p}$ . Hence none of  $(C', p)$  and  $(C'', p)$  belongs to the same algebraic family as  $(C, p)$ , whence both  $(C', p)$  and  $(C'', p)$  are contained in  $B_{\text{red}}$ . Therefore (1.9.2) follows.

CASE C-3. In this case,  $B_{\text{red}}$  passes through  $p$  and  $(B, p)$  is a surface defined by  $I_{B, p} = (x_1, \dots, x_{n-3}, x_n)$ . Therefore (1.9.3) follows.

CASE C-4. By modifying  $x_n$  by a suitable unit, and by deleting some multiples of  $x_1, \dots, x_{n-3}$  from  $\varphi_j$ , we may assume

$$\varphi_{n-2} = x_n x_{n-2}, \quad \varphi_{n-1} = (x_n + a(x_{n-2}, x_{n-1}))x_{n-1}$$

for some  $a(x_{n-2}, x_{n-1}) \in \mathcal{C}[[x_{n-2}, x_{n-1}]] \cap m_p$ . We have 3 components  $C_j$  ( $0 \leq j \leq 2$ ),  $C_0 = C$  and the rest  $C'$  defined by

$$\begin{aligned} I_{C_0, p} &= (x_1, \dots, x_{n-3}, x_{n-2}, x_{n-1}) \\ I_{C_1, p} &= (x_1, \dots, x_{n-3}, x_{n-2}, x_n + a(0, x_{n-1})) \\ I_{C_2, p} &= (x_1, \dots, x_{n-3}, x_n, x_{n-1}) \\ I_{C', p} &= (x_1, \dots, x_{n-3}, x_n, a(x_{n-2}, x_{n-1})). \end{aligned}$$

where  $C'$  can be reducible or nonreduced, and it may contain  $C_1$  and  $C_2$ .

Let  $C'_{\text{red}} = C_3 + \dots + C_m$  and let  $C'' = C_3 + \dots + C_d$  ( $d \leq m$ ) be the union of movable components of  $C'$ . Any movable component of  $(l, p)$  is algebraically equivalent to  $(C, p)$ , any  $(C_i, p)$  ( $3 \leq i \leq d$ ) is nonsingular, so that there exists  $(a_i, b_i) \in \mathcal{C}^2 \setminus (0, 0)$  such that

$$I_{C_i, p} = (x_1, \dots, x_{n-3}, x_n, a_i x_{n-2} + b_i x_{n-1} + (\text{higher terms})).$$

We see  $C_j \neq C_1, C_2$  for  $3 \leq j \leq d$  and that  $\text{emb.dim}(C_2 + C_j + C_k) = 2$  for  $3 \leq j < k \leq m$ .

**Claim 1.9.5.** *Let  $C_j$  be an irreducible component of  $C'$ . Then*

$$(1.9.5.1) \quad \text{emb.dim}(C_0 + C_1 + C_j) = 2 \quad \text{if and only if } C_j = C_1.$$

$$(1.9.5.2) \quad \text{emb.dim}(C_0 + C_2 + C_j) = 2 \quad \text{if and only if } C_j = C_2.$$

Proof. We may assume  $n=3$  without loss of generality. We let  $I_j = I_{C_j}$  and  $x := x_1, y = x_2, z = x_3$ . We let

$$I_0 = (x, y), \quad I_1 = (x, z + a(x, y)), \quad I_2 = (y, z), \quad I_j = (z, h(x, y))$$

where  $a(x, y)$  is divisible by  $h(x, y)$  in  $O_{X,p}$ .

First we prove (1.9.5.1). let  $u := z + a(x, y)$ . Then we have  $I_1 = (x, u)$ ,  $I_j = (u, h)$  and  $I_0 \cap I_1 = (x, yu)$ . Therefore  $\text{emb.dim}(C_0 + C_1 + C_j) = 2$  implies the existence of an element  $b \in O_{X,p}$  such that  $x + byu \in I_j$ . Hence  $x$  is divisible by  $h$  in  $O_{X,p}$ , so that  $h = x$  up to a unit multiple. Therefore  $I_1 = I_j$ . The converse is obviously true.

(1.9.5.2) is proved similarly. In fact,  $I_0 \cap I_2 = (y, xz)$ . Hence  $\text{emb.dim}(C_0 + C_2 + C_j) = 2$  implies that  $y$  is divisible by  $h$  in  $O_{X,p}$ , so that  $I_2 = I_j$ . q.e.d.

Now we go back to the proof of Lemma 1.9 CASE C-4.

CASE C-4-1. We consider the case where  $C_2$  is a movable component of  $l$ . Assume  $d \geq 3$  and  $m \geq 4$ . Any movable component of  $l$  belongs to one and the same algebraic family, whence  $\text{emb.dim}(C_2 + C_3 + C_4) = \text{emb.dim}(C_0 + C_2 + C_4) = 3$  by (1.9.5.2), a contradiction. Hence  $d \leq 3$ . Moreover if  $d = 3$ , then  $m = 3$  and  $C' = C'' = C_3$ .

Assume  $d = 3$ . We let  $a(x_{n-2}, x_{n-1}) = a_3 x_{n-2} + b_3 x_{n-1} + (\text{higher terms})$ . If  $a_3 = 0$ , then  $C_2$  and  $C_3$  have a contact, while if  $b_3 = 0$ , then  $C_1$  and  $C_3$  have a contact, which contradicts transversal intersection of  $C_0, C_1$  and  $C_2$  in either case. Hence  $a_3 \neq 0, b_3 \neq 0$ . We may choose  $a_3 = b_3 = 1$  and  $a(x) = x_{n-2} + x_{n-1}$  by multiplying  $x_{n-2}$  and  $x_{n-1}$  by some units. This proves (1.9.4.2) in this case.

CASE C-4-2. We consider the case where  $C_2$  is fixed. If  $d \geq 4$ , then  $\text{emb.dim}(C_2 + C_3 + C_4) = \text{emb.dim}(C_0 + C_2 + C_3) = 3$  by Claim 1.9.5, a contradiction. Hence we have  $d \leq 3$ . Therefore  $l$  has at most 3 movable irreducible components.

The remaining assertions of (1.9.4) are easy to prove. q.e.d.

### Appendix. Local invariants $e(Q_V^{(y)}, B_U)$

(1.10) NOTATION. Let  $C$  be an irreducible curve,  $v: \tilde{C} \rightarrow C$  the normalization,  $F$  a torsion  $O_{\tilde{C}}$ -module,  $p$  (resp.  $q$ ) a point of  $C$  (resp.  $\tilde{C}$ ). Then we define  $e(F, q)$ ,  $l(F, p)$  and  $l(F)$  as follows,

$$e(F, q) = l(F_q) = \dim_{\mathbb{C}} F_q, \quad l(F, p) = \sum_{q \text{ above } p} l(F_q).$$

Then we recall

**Theorem 1.11** [15, (2.6)].

(1.11.1)  $l(Q'_C, p) = \sum_{q \text{ above } p} e(Q'_C, q)$ ,  $l(Q''_C, p) = \sum_{q \text{ above } p} e(Q''_C, q)$  for any  $p \in C$ .

(1.11.2) If  $(C, p)$  is irreducible, then  $e(Q''_C, q) \geq e(Q'_C, q)$  for a unique point  $q$  above  $p$ . Equality holds if and only if  $(C, p)$  is nonsingular. If  $(C, p)$  is singular, then  $e(Q''_C, q) \geq e(Q'_C, q) + 2$ .

(1.11.3) Under the same notation and assumptions in (1.3.7), let  $q$  be a unique point of the normalization  $\tilde{C}_\lambda$  of  $C_\lambda$  above  $p$ . Then

(1.11.3.1)  $e(Q''_C, q) \geq 1$ ,  $e(Q'_C, q) = 0$  for  $\lambda \in \Lambda_{ns}$ ,

(1.11.3.2)  $e(Q''_C, q) \geq e(Q''_\lambda, q) \geq e(Q'_\lambda, q) + 2$  for  $\lambda \in \Lambda_s$ ,

(1.12) TORSION SHEAVES  $Q'_V$  AND  $Q''_V$ . Let  $Z$  be an irreducible reduced algebraic variety,  $v: Y \rightarrow Z$  the normalization. Let  $U = Y \setminus \text{Sing } Y$ ,  $V = v(U)$ . Then we have an exact sequence

$$(1.12.1) \quad 0 \rightarrow Q''_V \rightarrow v^* \Omega_Z^1 \otimes \mathcal{O}_U \xrightarrow{\phi} \Omega_U^1 \rightarrow Q'_V \rightarrow 0$$

where  $Q''_V := \text{Ker } \phi$  and  $Q'_V := \text{Coker } \phi$ . Now take an arbitrary prime Weil divisor  $B$  of  $Y$  (resp.  $\bar{B}$  of  $Z$ ) with  $v(B) = \bar{B}$ . We define  $e(F, B)$  (resp.  $e(F, \bar{B})$ ) to be the length of a torsion sheaf  $F$  at a generic point of  $B$  (resp.  $\bar{B}$ ) as a  $k(B)$ -module (resp. as a  $k(\bar{B})$ -module).

Let  $B_U := B \cap U$  and  $\bar{B}_V := \bar{B} \cap V$ . Then we have

$$(1.12.2) \quad e(Q'_V, B_U) = \inf_{C, q} e(Q'_C, q), \quad e(Q''_V, B_U) = \inf_{C, q} e(Q''_C, q)$$

where  $p$  ranges over  $\bar{B}_V$ ,  $C$  is a local curve of  $V$  intersecting  $\bar{B}_V$  transversally at  $p$ , and  $q$  is a point of  $B_U$  above  $p$ .

By Theorem 1.11 we have

$$(1.12.3) \quad e(Q''_V, B_U) \geq e(Q'_V, B_U).$$

(1.13) A TORSION SHEAF  $Q_V$ . Let  $X$  be a smooth algebraic variety of dimension  $n$ ,  $D_i$  a reduced irreducible divisor of  $X$  ( $1 \leq i \leq m$ ). Assume that the scheme-theoretic complete intersection  $\tau = D_1 \cap \cdots \cap D_m$  has an irreducible component  $Z = Z_{\text{red}}$  of dimension  $n - m$  along which  $\tau$  is reduced generically. Let  $v: Y \rightarrow Z$  be the normalization of  $Z$ ,  $U = Y \setminus \text{Sing}$

$Y$ , and  $V := v(U)$ . Let  $I_{D_i}$  (resp.  $I$ ) be the ideal sheaf of  $O_X$  defining  $D_i$  (resp.  $Z$ ) and let  $I_\tau = I_{D_1} + \cdots + I_{D_m}$ . So we note  $\sqrt{I_{D_i}} = I_{D_i}$  and  $\sqrt{I} = I$ .

Then we have an exact sequence

$$(1.13.1) \quad 0 \rightarrow \bigoplus_{i=1}^m O_U(-v^*D_i) \rightarrow [v^*(I/I^2) \otimes O_U] \rightarrow Q_V \rightarrow 0$$

where  $[F \otimes O_U] := F/O_U$ -torsions in  $F$ . If  $Z$  intersects one of the irreducible components of  $\tau$  other than  $Z$  along a prime Weil divisor  $\bar{B}$  of  $V$ , then

$$(1.13.2) \quad e(Q_V, B_U) \geq 1$$

for any prime Weil divisor  $B$  above  $\bar{B}$ .

Moreover by (1.12.1) and (1.13.1), we have

**Theorem 1.14** [15, (2.A)]. *Under the notation in (1.12) and (1.13), let  $i: U \rightarrow Y$  be the inclusion map, and let  $\Delta := \sum_B (e(Q_V, B_U) + e(Q'_V, B_U) - e(Q'_V, B_U))B$ . Then  $\Delta$  is an effective divisor of  $Y$  and we have*

$$K_Y := i_*(K_U) \simeq v^*K_X + \sum_{i=1}^m v^*C_i - \Delta.$$

(1.15) **REMARK.** If  $Z$  is singular along a prime Weil divisor  $\bar{B}$ , then by Theorem 1.11  $e(Q'_V, B_U) \geq e(Q'_V, B_U) + 1$  for any prime Weil divisor  $B$  of  $Y$  lying over  $\bar{B}$ . If  $Z$  intersects one of the irreducible components of  $\tau$  other than  $Z$  along a prime Weil divisor  $\bar{B}$ , then by the definition  $e(Q_V, B_U) \geq 1$  for any prime Weil divisor  $B$  lying over  $\bar{B}$ . Thus we see that  $\text{supp}(v_*\Delta)$  is the union of all the Weil divisors of  $Z$  whose supports are contained in either  $\text{Sing } Z$  or one of the irreducible components of  $\tau$  other than  $Z$ . See [15, (2.A)] for the detail.

## 2. A complete intersection $l_V$ (2) —global structure—

**Lemma 2.1.** *Assume  $d=n$ , and  $h^0(X, L) \geq n$ . Let  $l$  be a scheme-theoretic complete intersection of  $(n-1)$ -members of  $|L|$  and  $B := Bs |L|$ . Assume that  $l$  has a reduced curve-component  $C$  outside  $B$  with  $LC \geq 1$ . Then one of the following cases occurs.*

$$(2.1.1) \quad LC = 2, C \simeq \mathbf{P}^1, N_{C/X} \simeq O_C(2)^{\oplus(n-1)}, C \text{ is a connected component of } l,$$

(2.1.2)  $LC=1$ ,  $C \simeq \mathbf{P}^1$ ,  $N_{C/X} \simeq O_C \oplus O_C(1)^{\oplus(n-2)}$ , and  $C$  intersects  $B$  at a point  $p$  transversally, where

$$\begin{aligned} I_{l,p} &= (x_1, \dots, x_{n-2}, x_{n-1}x_n), \\ I_{C,p} &= (x_1, \dots, x_{n-2}, x_{n-1}), \\ I_{B,p} &= (x_1, \dots, x_{n-2}, x_n) \end{aligned}$$

by choosing a suitable local coordinate  $x_1, \dots, x_n$  at  $p$ .

(2.1.3) There is another component  $C_1$  of such that  $C_i \simeq \mathbf{P}^1$ ,  $C = C_0$ ,  $LC_i = 1$ ,  $N_{C_i/X} \simeq O_{C_i} \oplus O_{C_i}(1)^{\oplus(n-2)}$  ( $i=0,1$ ). The components  $C_0$  and  $C_1$  intersect transversally at a point  $p$  where

$$\begin{aligned} I_{l,p} &= (x_1, \dots, x_{n-2}, x_{n-1}x_n), \\ I_{C_0,p} &= (x_1, \dots, x_{n-2}, x_{n-1}), \\ I_{C_1,p} &= (x_1, \dots, x_{n-2}, x_n), \\ I_{B,p} &= (x_1, \dots, x_{n-2}, x_{n-1}, x_n) \end{aligned}$$

in terms of suitable coordinates at  $p$ .

(2.1.4) There is a chain of  $m+1$  ( $\geq 2$ ) smooth rational curves  $C_i$  ( $0 \leq i \leq m$ ) such that

$$C = C_0, \quad LC_0 = LC_m = 1, \quad LC_i = 0 \quad (1 \leq i \leq m-1)$$

$$N_{C_i/X} \simeq \begin{cases} O_{C_i} \oplus O_{C_i}(1)^{\oplus(n-2)} & (i=0, m) \\ O_{C_i}(-2) \oplus O_{C_i}^{\oplus(n-2)} \quad \text{or} \quad O_{C_i}(-1)^{\oplus 2} \oplus O_{C_i}^{\oplus(n-3)} & (1 \leq i \leq m-1). \end{cases}$$

The curves  $C_j$  and  $C_i$  ( $j < i$ ) intersect nowhere unless  $j = i-1$ , while  $C_{i-1}$  and  $C_i$  intersect transversally at a point  $p_i$  where

$$\begin{aligned} I_{l,p_i} &= (x_1, \dots, x_{n-2}, x_{n-1}x_n), \\ I_{C_{i-1},p_i} &= (x_1, \dots, x_{n-2}, x_{n-1}), \\ I_{C_i,p_i} &= (x_1, \dots, x_{n-2}, x_n) \end{aligned}$$

in terms of suitable local coordinates at  $p_i$ . Moreover  $C_0 + \dots + C_m$  is a connected component of  $l$  with  $C_i \cap B_{\text{red}} = \emptyset$  ( $1 \leq i \leq m-1$ ).

NOTE. (2.1.1)–(2.1.3) are known to exist for (not necessarily complete) general linear systems. However there are no examples of (2.1.4) except for  $m=1$  or  $2$ ,  $n \geq 3$ . We also note that (2.1.3) and (2.1.4) with  $m=1$  are distinguished by the condition that  $C_0 \cap C_1$  is a base point



or not.

Proof. By (1.3.6), we have  $l(Q'')=l(Q')$  or  $l(Q'')\geq l(Q')+2$ . Hence there are two cases by (1.3.1).

CASE 1.  $LC=2$ ,  $C\simeq \mathbf{P}^1$ ,  $l(Q_C)=l(Q'')=l(Q')=0$ .

CASE 2.  $LC=1$ ,  $\tilde{C}\simeq \mathbf{P}^1$ ,  $l(Q_C)=1$ ,  $l(Q'')=l(Q')=0$ .

CASE 1. In this case  $C$  is nonsingular by (1.3.5).  $\phi_C$  in (1.2.1) is an isomorphism by  $Q_C=0$ , so that  $I_C=I_l$  along  $C$  by Nakayama's lemma. This implies that  $C$  is a connected component of  $l$ . It is clear that  $N_{C/X}=(I_C/I_C^2)^\vee \simeq O_C(2)^{\oplus(n-1)}$ .

CASE 2. In this case  $C$  is nonsingular by (1.3.5). Consider the homomorphism  $\phi_C$

$$\phi_C: O_C(-L)^{\oplus(n-1)} (\simeq (I_l/I_l^2) \otimes O_C) \rightarrow I_C/I_C^2.$$

In view of  $l(Q_C)=1$ , there is a unique point  $p$  of  $C$  such that  $l(\text{Coker } \phi_{C,p})=1$ . By Lemma 1.6, we can choose a local coordinate system  $x_1, \dots, x_n$  of  $X$  at  $p$  such that

$$I_{l,p}=(x_1, \dots, x_{n-2}, x_{n-1}x_n), \quad I_{C,p}=(x_1, \dots, x_{n-1}).$$

It is easy to see  $N_{C/X} \simeq O_C \oplus O_C(1)^{\oplus(n-2)}$ . Therefore we have another irreducible component  $C_1$  of  $l$  whose defining ideal  $I_{C_1,p}$  is given by

$$I_{C_1,p}=(x_1, \dots, x_{n-2}, x_n).$$

In particular,  $l$  is generically reduced along  $C_1$ . Then there are two subcases  $C_1 \subset B_{\text{red}}$  or  $C_1 \not\subset B_{\text{red}}$ .

CASE 2-1. If  $C_1 \subset B_{\text{red}}$ , then (2.1.2) is true.

CASE 2-2. If  $C_1 \not\subset B_{\text{red}}$ , then  $LC_1 \geq 0$ . If  $LC_1 \geq 2$ , then by CASE 1 above, we see that  $LC_1=2$  and  $C_1$  is a connected component of  $l$ , which is absurd. If  $LC_1=1$ , then  $C_1 \simeq \mathbf{P}^1$  and by the same argument as above,  $p$  is a unique point of  $C_1$  such that  $\text{Coker } \phi_{C_1,p} \neq 0$ . The union of  $C$  and  $C_1$  is a connected component of  $l$ .

If  $LC_1=0$ , then  $C_1 \simeq \mathbf{P}^1$ ,  $C_1 \cap B_{\text{red}} = \emptyset$  and  $l(Q_{C_1})=2$ ,  $l(Q''_C)=l(Q'_C)=0$  by (1.3.6). Hence there is another point  $p_2$  of  $C_1$  ( $p_2 \neq p$ ) such that  $\text{Coker } \phi_{C_1,p_2} \neq 0$ . In fact, by (1.3.1) we have  $l(Q_{C_1,p})=l(Q_{C_1,p_2})=1$ . Therefore by Lemma 1.6, we can choose a local coordinate system  $y_1, \dots, y_n$

at  $p_2$  such that

$$I_{l,p_2} = (y_1, \dots, y_{n-2}, y_{n-1}y_n), \quad I_{C_1,p_2} = (y_1, \dots, y_{n-2}, y_{n-1})$$

Hence we have the third reduced curve-component  $C_2$  of  $l$  with  $I_{C_2,p_2} = (y_1, \dots, y_{n-2}, y_n)$ . Since  $C_1 \not\subset B_{\text{red}}$ , we see  $C_2 \not\subset B_{\text{red}}$ . As before  $C_2 \simeq \mathbf{P}^1$  and  $LC_2 = 0$  or  $1$ . If  $LC_2 = 1$ , then  $C_0 (:= C) + C_1 + C_2$  is a connected component of  $l$ .

If  $LC_2 = 0$ , then by repeating the same argument, we eventually obtain a chain of rational curves  $C_0, C_1, \dots, C_m$  with  $LC_0 = LC_m = 1$ ,  $LC_i = 0$  ( $1 \leq i \leq m-1$ ) such that  $C_{i-1}$  and  $C_i$  intersect transversally at a point  $p_i$  ( $1 \leq i \leq m$ ),  $p_1 := p$  and  $C_0 + C_1 + \dots + C_m$  is a connected component of  $l$  with  $C_i \cap B_{\text{red}} = \emptyset$  ( $1 \leq i \leq m-1$ ). We also see that

$$N_{C_i/X} \simeq O_{C_i} \oplus O_{C_i}(1)^{\oplus(n-2)} \quad (i=0, m)$$

$$N_{C_i/X} \simeq O_{C_i}(-2) \oplus O_{C_i}^{\oplus(n-2)} \text{ or } O_{C_i}(-1)^{\oplus 2} \oplus O_{C_i}^{\oplus(n-3)} \quad (1 \leq i \leq m-1).$$

q.e.d.

**Proposition 2.2.** *Assume  $h^0(X, L) \geq n$  and let  $l$  be a scheme-theoretic complete intersection of general  $(n-1)$ -members of  $|L|$ . Assume moreover that  $l$  has a reduced curve-component  $C$  not contained in  $B_{\text{red}}$  with  $LC = 0$ . Then  $C$  is a nonsingular elliptic curve with  $C \cap B_{\text{red}} = \emptyset$ .*

*Proof.* If  $C \cap B_{\text{red}} \neq \emptyset$ , then  $C$  is contained in  $B_{\text{red}}$  by  $LC = 0$ , which is absurd. If  $l$  is general enough, then by Bertini's theorem,  $\text{Sing } l$  is contained in  $B_{\text{red}}$ . Hence  $l$  is nonsingular along  $C$ , whence  $C$  is nonsingular and it is a connected component of  $l$ . q.e.d.

### 3. Moishezon manifolds with $c_1(X) = nc_1(L)$ and $b_2 = 1$ (1)

The purpose of this and the next sections is to prove:

**Proposition 3.1.** *Let  $X$  be a Moishezon manifold of dimension  $n$  ( $\geq 3$ ) with  $b_2 = 1$ , and  $L$  a line bundle on  $X$ . Assume that  $c_1(X) = nc_1(L)$  and  $h^0(X, L) \geq n+1$ . If a scheme-theoretic complete intersection  $l$  of general  $(n-1)$ -members of  $|L|$  has an irreducible curve-component  $C$  with  $LC \geq 2$  outside  $\text{Bs } |L|$ , then  $X \simeq \mathcal{Q}^n$ .*

In this section we prove Proposition 3.1 assuming  $h^0(X, L) \geq n+2$ . Our proof of Proposition 3.1 in this section is completed in (3.8). In the next section we disprove  $h^0(X, L) = n+1$ .

**Lemma 3.2.** *Let  $m = h^0(X, L) - 1$  ( $\geq n + 1$ ), and let  $B := \text{Bs } |L|$  be the scheme-theoretic base locus of  $|L|$ , and  $h: X \rightarrow \mathbf{P}^m$  the rational map associated with  $|L|$ . Then  $m = n + 1$  and  $h$  is a birational map of  $X$  onto a hyperquadric  $W$  in  $\mathbf{P}^{n+1}$ .*

*Proof.* STEP 1. Let  $W$  be the closure of  $h(X \setminus B)$ , and  $d = \deg W$ . Then  $d \geq m + 1 - \dim W$ . By the assumption  $\dim W \geq n - 1$ . Hence by choosing general  $(n - 2)$ -members  $D_i \in |L|$  ( $1 \leq i \leq n - 2$ ), we have reduced irreducible components  $Z_i$  ( $1 \leq i \leq e$ ) of  $\tau := D_1 \cap \cdots \cap D_{n-2}$  outside  $B$ . Each  $Z_i$  is nonsingular outside  $B$  by Bertini's theorem. Let  $v_i: Y_i \rightarrow Z_i$  be the normalization of  $Z_i$ ,  $f_i: S_i \rightarrow Y_i$  the minimal resolution of  $Y_i$ ,  $g_i = v_i \cdot f_i$ . Let  $Z = Z_1$ ,  $Y = Y_1$ ,  $S = S_1$ ,  $f = f_1$ ,  $v = v_1$ , and  $g = g_1$ . Then there exist by Theorem 1.14 an effective Weil divisor  $\Delta$  on  $Y$ , effective Cartier divisors  $E$  and  $G$  on  $S$  with no components in common such that the canonical sheaves  $K_Y$  and  $K_S$  are given by

$$K_Y = \mathcal{O}_Y(v^*(K_X + (n - 2)L) - \Delta), \quad K_S = \mathcal{O}_S(g^*(K_X + (n - 2)L) - E - G)$$

where  $f_*(E) = \Delta$ ,  $f_*(G) = 0$  and  $E$  is finite over  $\Delta$ . Let  $\Sigma := f^{-1}(\Delta) \cup g^{-1}(\text{Sing } Z)$ . Then  $\Sigma$  contains  $\text{supp}(E + G)$  and  $g|_{S \setminus \Sigma}$  is an isomorphism. We also note that the base locus  $\text{Bs } g^*|L|$  contains  $\text{supp}(E + G)$  if  $D_i$ 's are sufficiently general. Since  $h^0(X, L) \geq n$  and  $Z \not\subset B$ ,  $g^*L$  is effective. Since  $c_1(S) = 2c_1(g^*L) + c_1(E + G)$  and  $S$  is projective, we have  $P_m(S) = 0$ . Therefore  $S \simeq \mathbf{P}^2$  or  $S$  is ruled, that is,  $S$  has a morphism onto an algebraic curve with general fiber  $\simeq \mathbf{P}^1$ . Since any  $Z_i$  is algebraically equivalent to each other,  $S_i \simeq \mathbf{P}^2$  for any  $i$  or  $S_i$  is ruled for any  $i$ .

STEP 2. Assume  $S \simeq \mathbf{P}^2$ . Then we have  $G = 0$  and  $S \simeq Y$ . Let  $H \in g^*|L|$ . Then  $K_S = -2H - E$ . Since  $K_{\mathbf{P}^2}$  is indivisible by 2, we have  $E \neq 0$  and  $H = E \in |O_{\mathbf{P}^2}(1)|$  in view of  $E_{\text{red}} \subset H_{\text{red}}$ . This shows that  $(D_1 \cap \cdots \cap D_{n-1})_{\text{red}} \subset B$  for any  $D_{n-1}$ , which contradicts the assumption that general  $l$  contains a curve-component outside  $B$ .

STEP 3. By STEP 2,  $S$  has a morphism  $\pi: S \rightarrow T$  onto an algebraic curve  $T$  with  $F$  ( $\simeq \mathbf{P}^1$ ) a general fiber of  $\pi$ . Let  $H \in g^*|L|$ , and let  $M$  (resp.  $N$ ) be the movable part (resp. the fixed part) of  $H$  in  $g^*|L|$ . Since  $F \simeq \mathbf{P}^1$ , we have

$$-2 = K_S F + F^2 = K_S F = -2HF - (E + G)F.$$

Since  $E_{\text{red}} + G_{\text{red}} \subset H_{\text{red}}$ , we have  $HF = 1$ ,  $EF = GF = 0$ . Therefore there exists a unique irreducible component  $\Gamma$  of  $H$  with  $\Gamma F = 1$  and  $\Gamma \not\subset E + G$ . If  $\Gamma \subset N$ , then  $MF = 0$ , whence  $M$  is a sum of general

fibers. Choose  $D_{n-1} \in |L|$  such that  $H = g^*D_{n-1}$ . Then  $g(F)$  is a general movable component of  $l := D_1 \cap \cdots \cap D_{n-1}$  with  $(Lg_*(F))_X = (g^*LF)_S = 1$ . This contradicts the assumption that there exists a component  $C$  of  $l$  with  $LC \geq 2$ . Hence  $\Gamma \subset M$ . Consequently  $\Gamma^2 \geq 0$ , and  $M = \Gamma$ ,  $NF = HF - MF = 0$ .

STEP 4. Assume  $\dim W = n - 1$ . Then  $\tau$  is smooth and irreducible outside  $B$  for general  $D_i$  by Bertini's theorem. Since  $M$  is irreducible, we have  $\deg W = d = 1 \geq m - n + 2 \geq 3$ , a contradiction. Hence  $\dim W = n$ . Moreover  $\Gamma^2 > 0$ . In fact, if  $\Gamma^2 = 0$ , then we have  $\Gamma \cap \Gamma' = \emptyset$  for any general  $\Gamma' \in |\Gamma|$ , whence  $Z \setminus B$  is mapped onto a curve by  $h$ . Hence  $\dim W = n - 1$ , a contradiction.

We also have,

$$K_S \Gamma + \Gamma^2 = -\Gamma^2 - (2N + E + G)\Gamma \leq -1.$$

Therefore the inclusion  $E_{\text{red}} + G_{\text{red}} \subset N_{\text{red}}$  shows that  $\Gamma \simeq \mathbf{P}^1$ ,  $\Gamma^2 = 2$ ,  $K_S \Gamma = -4$ ,  $N\Gamma = E\Gamma = G\Gamma = 0$  and  $g^*(L)\Gamma = H\Gamma = (\Gamma + N)\Gamma = 2$ .

Since  $\dim W = n$ ,  $C := g(\Gamma)$  is an irreducible component of a general complete intersection  $l := D_1 \cap \cdots \cap D_{n-1}$  outside  $B$ . Clearly  $(LC)_X = (g^*(L)\Gamma)_S = 2$ , while we have an obvious relation

$$(g^*(L)\Gamma)_S = \deg(h \cdot g)|_{\Gamma} \deg W + \deg \text{Bs } g^*|L|_{\Gamma}.$$

Since  $g^*(L)\Gamma \geq d \geq m - n + 1 \geq 2$  by the assumption  $m \geq n + 1$ , we have  $d = \deg W = 2$ ,  $\deg(h \cdot g)|_{\Gamma} = 1$ ,  $m = n + 1$  and  $\text{Bs } g^*|L|_{\Gamma} = \emptyset$ .

STEP 5. STEP 4 shows that  $C (=g(\Gamma))$  is the unique irreducible component of  $l$  outside  $B$ .  $h(C)$  is an irreducible plane conic. Therefore  $h(C) \simeq \mathbf{P}^1$ ,  $C \simeq \mathbf{P}^1$ ,  $\deg h(C) = 2$  and  $|L|_C = |L_C|$ . Moreover  $\deg(h|_C) = 1$  and  $\text{Bs } |L|_C = \emptyset$  are clear from STEP 4. For  $D_i$  general, we have  $\deg h = \deg(h|_C) = 1$ . This completes the proof of Lemma 3.2. q.e.d.

**Lemma 3.3.**  $\text{Pic } X \simeq \mathbf{Z}L$ .

Proof. First we prove  $H^1(X, \mathcal{O}_X) = 0$ . Assume the contrary. Then since  $X$  is Moishezon, we have a nontrivial Albanese map  $\text{alb}: X \rightarrow \text{Alb}(X)$  where  $\text{Alb}(X)$  is projective. Since  $b_2 = 1$ , some multiple of  $L$  is a multiple of the pull back of an ample line bundle on  $\text{Alb}(X)$ . Therefore the morphism  $\text{alb}$  is generically finite, whence we have a nontrivial holomorphic two form on  $X$ . This contradicts  $b_2 = 1$ .

Now we consider an exact sequence

$$0 \rightarrow H^1(X, \mathcal{O}_X)(=0) \rightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X).$$

Since  $b_2=1$ ,  $\text{Coker } c_1$  is finite. As  $H^2(X, \mathcal{O}_X)$  is a  $\mathbb{C}$ -vector space, it has no torsions. Hence  $\text{Coker } c_1=0$ . Therefore  $\text{Pic } X := H^1(X, \mathcal{O}_X^*) \simeq H^2(X, \mathbb{Z})$ .

Next we prove  $\text{Tor } H^2(X, \mathbb{Z})=0$ . Assume the contrary. Then we have an unramified covering  $\rho: \tilde{X} \rightarrow X$  such that  $\deg \rho = \# \text{Tor } H^2(X, \mathbb{Z}) \geq 2$  and  $\rho^* \text{Tor } H^2(X, \mathbb{Z})=0$ . Letting  $\tilde{L} := \rho^* L$ , we have  $c_1(\tilde{X}) = nc_1(\tilde{L})$ , and  $h^0(\tilde{X}, \tilde{L}) \geq n+2$ . By Lemma 3.2, we have  $\dim W = n$ ,  $\deg W = 2$ ,  $\deg h = 1$  and  $h^0(X, L) = n+2$ . Its proof makes no use of  $b_2=1$ , whence we have the same conclusion for  $\tilde{X}$ . Let  $\tilde{h}$  be the rational map associated with  $|\tilde{L}|$ ,  $\tilde{B} := \rho^{-1}(B)$ , and  $\tilde{W}$  the closure of  $\tilde{h}(\tilde{X} \setminus \tilde{B})$ . Then we have  $\dim \tilde{W} = n$ ,  $\deg \tilde{W} = 2$ ,  $\deg \tilde{h} = 1$ , and  $h^0(\tilde{X}, \tilde{L}) = n+2$ . It follows that  $h^0(\tilde{X}, \tilde{L}) = h^0(X, L)$ ,  $\tilde{h} = h \cdot \rho$ ,  $\tilde{W} \simeq W$  and  $\deg \rho = 1$ . This is a contradiction. Hence  $\text{Tor } H^2(X, \mathbb{Z})=0$ , and  $\text{Pic } X \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$ .

Finally we prove  $\text{Pic } X \simeq \mathbb{Z}L$ . Choose a generator  $L_0$  of  $\text{Pic } X$  so that  $L = aL_0$  for some integer  $a \geq 1$ . Now we recall the proof of Lemma 3.2. Under the same notation as in Lemma 3.2, the surface  $S$  is ruled with  $F$  a fiber. Then  $(g^*LF)_S = (HF)_S = 1$ . We have  $(g^*LF)_S = a(g^*L_0F)_S$ , whence  $a=1$  and  $\text{Pic } X \simeq \mathbb{Z}L$ . q.e.d.

**Corollary 3.4.** *Let  $W$  be the closure of  $h(X \setminus B)$ . Then  $W$  is a normal hyperquadric with Hessian-rank  $\geq 5$ .*

*Proof.* If Hessian-rank  $W \leq 4$ , then  $W$  has a reducible or nonreduced hyperplane section, which contradicts Lemma 3.3. q.e.d.

(3.5) NOTATION. Let  $\hat{X}$  be the normalization of the closure in  $X \times W$  of the graph of  $h$ ,  $\hat{h}: \hat{X} \rightarrow W$  and  $\varphi: \hat{X} \rightarrow X$  the natural morphisms. Let  $\hat{B} = \varphi^{-1}(B)$  and  $\hat{B}^*$  be the minimal subvariety of  $\hat{X}$  containing  $\hat{B}$  such that  $\hat{h}$  is unramified on  $\hat{X} \setminus \hat{B}^*$ . Let  $B^* = \varphi(\hat{B}^*)$  and  $R^* = \hat{h}(\hat{B}^*)$ . We note that  $X \setminus B \simeq \hat{X} \setminus \hat{B}$ ,  $X \setminus B^* \simeq \hat{X} \setminus \hat{B}^* \simeq W \setminus R^*$  and therefore  $\hat{B}^* = \varphi^{-1}(B^*) = \hat{h}^{-1}(R^*)$ .

**Lemma 3.6.**  $\hat{B}^* = \hat{h}^{-1}(\text{Sing } W) \cup \hat{B}$ .

*Proof.* Let  $\mathcal{H} = H^0(X, L)$ . It is clear from  $\hat{X} \setminus \hat{B}^* \simeq W \setminus R^*$  that  $\text{Sing } W \subset R^*$  and  $\hat{h}^{-1}(\text{Sing } W) \subset \hat{B}^*$ . Assume that there exists a point  $\hat{p} \in \hat{B}^* \setminus (\hat{h}^{-1}(\text{Sing } W) \cup \hat{B})$ . Then  $q := \hat{h}(\hat{p})$  is a smooth point of  $W$  and  $\hat{h}^{-1}(q)$  is a connected subset of  $\hat{X}$  with  $\dim \hat{h}^{-1}(q) > 0$ . Let  $\hat{p} := \varphi(\hat{p}) \in B^* \setminus B$ . Then it follows that  $q = \hat{h}(\hat{p}) = h(\hat{p})$ . We infer from  $\hat{X} \setminus \hat{B} \simeq X \setminus B$  that

$\dim h^{-1}(q) > 0$  and that there exists a subset  $B'$  of  $B$  such that  $\varphi(\hat{h}^{-1}(q)) = h^{-1}(q) \cup B'$ . Since  $\hat{h}^{-1}(q)$  is connected, so is  $\varphi(\hat{h}^{-1}(q))$ .

Since  $q$  is a smooth point of  $W$ , we can choose a smooth conic  $l^*$  on  $W$  which is not contained in  $R^*$  and passes through  $q$ . Let  $V = V(l^*) \in \text{Grass}(n-1, \mathcal{H})$  be a linear subspace of  $\mathcal{H}$  corresponding to  $l^*$  with  $\dim V = n-1$ . Let  $l_V$  be the subvariety of  $X$  defined by  $I_{l_V} = \sum_{s \in V} s O_X$ . Then  $l_V$  passes through  $p$ , and it is one-dimensional and nonsingular outside  $B^*$ . Let  $C$  be the unique irreducible component of  $l_V$  outside  $B$  such that  $\overline{h(C \setminus B)} = l^*$ . Since  $LC = \deg(h|_C) \deg W + \deg \text{Bs } |L|_C$ , we have  $LC \geq 2$ , whence by Lemma 2.1  $C \simeq \mathbf{P}^1$ ,  $LC = 2$  and that  $C$  is a connected component of  $l_V$ . Let  $\hat{C}$  be the proper transform of  $C$  by  $\varphi^{-1}$ . Then since  $l^*$  passes through  $q$  and since  $\hat{h}^{-1}(q)$  is connected, the union  $\hat{C} \cup \hat{h}^{-1}(q)$  is a connected subset of  $\hat{X}$ , whence the union  $\varphi(\hat{C}) \cup \varphi(\hat{h}^{-1}(q)) = C \cup h^{-1}(q) \cup B'$  is a connected subset of  $l_V$ . This contradicts that  $C$  is a connected component of  $l_V$ . q.e.d.

**Lemma 3.7.**  $\hat{B}^* = \hat{h}^{-1}(\text{Sing } W)$ ,  $R^* = \text{Sing } W$ ,  $\hat{B} = \emptyset$ ,  $B = \emptyset$  and  $\hat{X} \simeq X$ .

*Proof.* Assume  $\hat{B}^* \neq \hat{h}^{-1}(\text{Sing } W)$ . Then  $R^* \neq \text{Sing } W$ . Then we can choose a smooth conic  $l^*$  on  $W$  which is not contained in  $R^*$  and meets  $R^* \setminus \text{Sing } W$ . Hence we can choose  $V' \in \text{Grass}(n-1, \mathcal{H})$  such that  $l_{V'}$  is pure one dimensional and irreducible nonsingular outside  $B^*$  and  $\overline{h(l_{V'} \setminus B^*)} = l^*$ . Let  $q$  be a point of  $(l^* \cap R^*) \setminus \text{Sing } W$ ,  $C$  the unique irreducible component of  $l_{V'}$  with  $\overline{h(C \setminus B^*)} = l^*$ . Note that  $\hat{h}^{-1}(q) \subset \hat{B}$  by Lemma 3.6, whence  $\varphi(\hat{h}^{-1}(q)) \subset B$ . Let  $\hat{C}$  be the proper transform of  $C$  by  $\varphi^{-1}$ . Then  $\hat{C} \cup \hat{h}^{-1}(q)$  is a connected subset of  $\hat{X}$ , whence  $C \cup \varphi(\hat{h}^{-1}(q))$  is a connected subset of  $l_{V'}$ . Since  $\varphi(\hat{h}^{-1}(q)) \subset B$ , this shows that  $B \cap C \neq \emptyset$ . Since  $\overline{h(C \setminus B^*)} = l^*$ , we have  $LC \geq 2$ . By Lemma 2.1, we have  $LC = 2$ , and  $B \cap C = \text{Bs } |L|_C = \emptyset$ , which is a contradiction. Hence  $\hat{B}^* = \hat{h}^{-1}(\text{Sing } W) \supset \hat{B}$  and  $R^* = \text{Sing } W$ .

Next we prove  $\hat{B} = \emptyset$ . Let  $p \in \hat{B}$ , and  $q = \hat{h}(p)$ . Since  $\hat{B} \subset \hat{h}^{-1}(\text{Sing } W)$ ,  $q$  is a singular point of  $W$ . A general (singular) conic  $l^*$  on  $W$  passing through  $q$  is a union of two lines. As before we choose  $V \in \text{Grass}(n-1, \mathcal{H})$  with  $\overline{h(l_V \setminus B^*)} = l^*$ . Let  $\sigma = C_0 + C_m$  be a minimal subcurve of  $l_V$  with  $\overline{h(\sigma \setminus B^*)} = l^*$ . We notice that Lemma 2.1 is true if we only assume that  $C$  is a reduced curve-component of  $l$  with  $LC \geq 1$ . We have  $LC_i = 1$  and  $C_i \cap B = \emptyset$  for  $i = 0, m$ . Hence only (2.1.4) is possible. By (2.1.4), the connected component of  $l_V$  containing  $C_0$  is a reducible curve  $C_0 + \dots + C_m$  disjoint from  $B$  with  $LC_i = 0$  ( $1 \leq i \leq m-1$ ). In the same manner as above we see that  $\sigma \cup \varphi(\hat{h}^{-1}(q))$  is a connected subset of  $l_V$  intersecting  $B$ , which contradicts that  $C_0 + \dots + C_m$  is a connected component of  $l_V$  disjoint from  $B$ . Hence  $\hat{B} = \emptyset$ ,  $B = \emptyset$  and  $\hat{X} \simeq X$ . q.e.d.

(3.8) PROOF OF PROPOSITION 3.1 UNDER THE ASSUMPTION  $h^0(X, L) \geq n+2$ . The birational map  $h$  is defined everywhere by Lemma 3.7. It is easy to see by using  $c_1(X) = nc_1(L)$  that  $X \simeq W$  if  $W$  is smooth. So we consider the case where  $W$  is singular. We prove that *this case is impossible*. We recall Hessian-rank  $W \geq 5$ . We note that a complete intersection of general  $(n-2)$  hyperplane sections of  $W$  passing through a singular point of  $W$  is a singular quadric surface  $Q$  ( $\simeq \bar{F}_2 := P(O_{P^1} \oplus O_{P^1}(2))$  with  $(-2)$ -curve contracted).

Let  $q \in W$  be a singular point of  $W$ . Let  $m_q$  be the maximal ideal of  $O_W$  defining  $q$ ,  $\Lambda := h^*(m_q O_W(1))$ , and let  $D_i \in \Lambda$  ( $1 \leq i \leq n-1$ ) be general members. Let  $\tau = D_1 \cap \cdots \cap D_{n-2}$  be a scheme-theoretic intersection of  $D_i$ ,  $Z$  the unique irreducible component of  $\tau$  mapped onto a singular quadric surface  $Q$  passing through  $q$ , where  $q$  is the unique singular point of  $Q$ . Keeping the same notation  $S, Y, g, E$  and  $G$  as in Lemma 3.2, we let  $H = g^*(D_{n-1})$ , and  $M$  (resp.  $N$ ) the movable part (resp. the fixed part) of  $H$  for  $D_{n-1} \in \Lambda$ . We see  $S \not\cong P^2$  as in Lemma 3.2 CASE 1. We note that  $\dim h^{-1}(q) \geq 1$ .

Hence  $S$  is a ruled surface with  $F$  a general fiber of the ruling  $\pi$ . It follows that

$$2 = -K_S F = (2H + E + G)F.$$

Since  $E_{\text{red}} + G_{\text{red}} \subset H_{\text{red}}$ , we see that  $HF=1$ ,  $EF=GF=0$  and that there is a unique irreducible component  $\Gamma$  of  $H$  with  $\Gamma F=1$ . Assume that  $\Gamma \subset M$ . Then  $M=\Gamma$  for general  $D_{n-1}$  because  $MF=\Gamma F=1$ . However  $g(M)$  is by Lemma 3.7 a complete intersection of  $(n-1)$ -hyperplane sections of  $W$  passing through the point  $q$ . Hence it is a singular conic, that is, a union of two lines, which contradicts the irreducibility of  $M$ . Hence  $\Gamma \subset N$ . Since  $B=\emptyset$ ,  $l := D_1 \cap \cdots \cap D_{n-1}$  has by Lemma 2.1 a connected component  $C_0 + \cdots + C_m$  of type (2.1.4) for general  $D_i$ , where  $g(C_i)$  ( $i=0, m$ ) is a line passing through  $q$ . Note that Lemma 2.1 is true only if  $C$  ( $C_0$  or  $C_m$  in this case) is a reduced curve component of  $l$  but even if  $D_i$  is not general. Since  $g(C_0)$  and  $g(C_m)$  are algebraically equivalent as lines on  $W$  passing through  $q$ ,  $C_0$  and  $C_m$  intersect the same irreducible component, say  $C_1$ , of  $l$  for general  $D_{n-1}$ . Hence  $m=2$ . Since  $B=\emptyset$  and  $h$  is birational,  $l$  is connected so that  $l_{\text{red}} = C_0 + C_2 + h^{-1}(q)_{\text{red}}$ . As  $h(C_1)=q$  and  $h^{-1}(q)$  is a connected subset of  $l$ , we have

$$l \simeq C_0 + C_1 + C_2, \quad C_0 + C_2 \simeq M, \quad h^{-1}(q)_{\text{red}} \simeq C_1 \simeq \Gamma \simeq P^1.$$

In particular,  $h^{-1}(q) \subset Z$ . Since  $\tau = h^{-1}(q) \cup Z$  by the choice of  $Q$ , this shows that  $\tau$  is irreducible and pure two-dimensional, hence Gorenstein. Since it is generically reduced, it is reduced everywhere, whence  $Z \simeq \tau$ .

Moreover by (2.1.4) we have  $I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n)$  at any point  $p$  of  $C_1$ , whence we may assume  $I_{\tau,p}=(x_1,\cdots,x_{n-2})$ . Hence  $\tau$  is smooth everywhere along  $h^{-1}(q)$ . Since  $\tau\backslash h^{-1}(q)\simeq Q\backslash\{q\}$ , we have  $\text{Sing } \tau \subset h^{-1}(q)$ . Thus  $\tau$  is smooth everywhere, so that  $S\simeq Y\simeq Z\simeq \tau$  and  $E=G=0$ . Hence we have  $H\Gamma=g^*(L)\Gamma=0$ ,  $K_S\Gamma=0$  and  $\Gamma^2=-2$ . We also have  $HF=1$ ,  $(H^2)_S=(L_Z^2)_Z=((h^*O_W(1))^n)_X=2$ . Since  $K_S=-2H$  and  $H^2=2$ ,  $S$  is relatively minimal. Hence  $\Gamma^2=-2$  implies that  $S\simeq P(\pi_*H)\simeq F_2$ . We also see that  $h^{-1}(\text{Sing } W)$  is a  $P^1$ -bundle over  $\text{Sing } W$ . We note that  $\dim \text{Sing } W=n+1-\text{Hessian-rank } W\leq n-4$ .

Let  $C=h^{-1}(q)$  for some  $q\in\text{Sing } W$ . Then by (2.1.4) we have

$$\chi(C, N_{C/X})=n-1+\deg N_{C/X}=n-3>\dim \text{Sing } W.$$

This shows that there exists an (at least)  $(n-3)$ -dimensional family of displacements  $C(t)$  of  $C$  in  $X$  [9, Proposition 3]. Since  $LC(t)=LC=0$ ,  $h(C(t))$  is a point, so that any general  $C(t)$  is not contained in  $h^{-1}(\text{Sing } W)$ . However by Lemma 3.7  $h$  is an isomorphism outside  $h^{-1}(\text{Sing } W)$ , a contradiction. Thus it is impossible that  $W$  is singular. q.e.d.

#### 4. Moishezon manifolds with $c_1(X)=nc_1(L)$ and $b_2=1$ (2)

The purpose of this section is to complete our proof of Proposition 3.1. In this section, we disprove the possibility of  $h^0(X,L)=n+1$ .

(4.1) NOTATION. In this section we always assume  $h^0(X,L)=n+1$ . We let  $B:=\text{Bs } |L|$  (resp.  $\text{Bs } |L|_C$ ) be the scheme-theoretic base locus of  $|L|$  (resp. that of the restriction  $|L|_C$  of  $|L|$  to  $C$ ), and  $h: X\rightarrow P^n$  the rational map associated with  $|L|$ ,  $W$  the closure of  $h(X\backslash B)$ . We notice that the same argument as in Lemma 3.2 shows  $\dim W=n$ , that is,  $W\simeq P^n$ . We define  $\hat{X}$ ,  $\hat{h}: \hat{X}\rightarrow W$ ,  $\varphi: \hat{X}\rightarrow X$ ,  $\hat{B}=\varphi^{-1}(B)$ ,  $\hat{B}^*\subset \hat{X}$ ,  $B^*=\varphi(\hat{B}^*)$  and  $R^*=\hat{h}(\hat{B}^*)$  in the same manner as in Lemma 3.6. Then we have  $\hat{B}^*=\varphi^{-1}(B^*)=\hat{h}^{-1}(R^*)$ ,  $X\backslash B\simeq \hat{X}\backslash \hat{B}$  and  $X\backslash B^*\simeq \hat{X}\backslash \hat{B}^*\simeq W\backslash R^*$ .

Let  $l^*$  be a line on  $W$  not contained in  $R^*$ ,  $\hat{l}(l^*)=\hat{h}^{-1}(l^*)$ ,  $\hat{C}(l^*)$  an irreducible component of  $\hat{h}^{-1}(l^*)$  mapped onto  $l^*$ ,  $C(l^*)=\varphi(\hat{C}(l^*))$ ,  $\sigma(l^*)=\varphi(\hat{h}^{-1}(l^*))$ . Let  $l(l^*)$  be a complete intersection of  $(n-1)$ -members of  $|L|$  corresponding to  $l^*$ . We keep the same notation in Lemmas 4.2–4.3.

**Lemma 4.2.** *Under the notation in (4.1), we have  $W\simeq P^n$  and  $C(l^*)\simeq P^1$ .*

(4.2.1)  *$l(l^*)$  is irreducible outside  $B$  for general  $l^*$ .*



(4.2.2)  $C := C(l^*)$  is the unique irreducible component of  $l(l^*)$  outside  $B$  for general  $l^*$ . Then one of the following is true.

$$(4.2.2.1) \quad \deg(h|_C) = 1, \quad LC = 2, \quad \deg Bs|_{L|_C} = 1.$$

$$(4.2.2.2) \quad \deg(h|_C) = 2, \quad LC = 2, \quad \deg Bs|_{L|_C} = 0.$$

(4.2.3) Let  $C'$  be an irreducible component of  $l(l^*)$  outside  $B$ . If  $LC' = 1$ , then  $Bs|_{L|_{C'}} = \emptyset$  (and  $l^*$  is not general).

Proof. We note that  $C(l^*) \simeq \mathbf{P}^1$  by Lemma 2.1. (4.2.1) is clear from Bertini's theorem. Next we prove (4.2.2). Let  $C := C(l^*)$ . The first assertion is clear. We prove  $LC = 2$ . If  $LC = 1$ , then by the equality

$$LC = \deg(h|_C) \deg W + \deg Bs|_{L|_C},$$

we have  $\deg(h|_C) = 1$  and  $Bs|_{L|_C} = \emptyset$ . The complete intersection  $l(l^*)$  is therefore smooth along  $C$ , so that  $l(l^*) \simeq C$  along  $C$ , which contradicts Lemma 2.1. Hence  $LC = 2$ . It follows that  $2 = \deg(h|_C) + \deg Bs|_{L|_C}$ . The rest is clear. q.e.d.

In view of (4.2.2), we have  $\deg h = \deg(h|_{C(l^*)})$  for general  $l^*$ . We disprove both the possibilities  $\deg h = 1$  and 2 respectively in Lemma 4.3 and Lemma 4.4.

**Lemma 4.3.**  $\deg h = 1$  is impossible.

We prove Lemma 4.3 in Claims 4.3.1–4.3.7 and (4.3.8).

Assume  $\deg h = 1$ . Keeping the notation in (4.1), we first prove

**Claim 4.3.1.**  $R^*$  is a hyperplane of  $W$  ( $\simeq \mathbf{P}^n$ ).

Proof. We keep the notation in (4.1) and Lemma 4.2. Assume first  $\dim R^* \leq n-2$ . Then there is a general line  $l^*$  on  $W$  such that  $l^* \cap R^* = \emptyset$  by Lemma 4.2. Let  $C = C(l^*)$ . Hence  $\hat{h}^{-1}(l^*) \cap \hat{B}^* = \emptyset$ , whence  $B \cap C = \emptyset$ . However we have  $\deg Bs|_{L|_C} = LC - \deg h = 1$ , a contradiction. Hence there is an irreducible component of  $R^*$  of dimension  $n-1$ . Assume that  $R^*$  is not a hyperplane. Then  $R^*$  has another irreducible component or  $R^*$  is a hypersurface of degree greater than one. Therefore there is a line  $l^*$  of  $W$  such that  $l^* \cap R^*$  contains two points  $q_i$  ( $i=1,2$ ). Then since  $\hat{h}^{-1}(l^*)$  is a connected set containing  $\hat{C}(l^*)$ ,  $\varphi(\hat{h}^{-1}(l^*))$  is a connected subset of  $l(l^*)$  containing  $C$ . Since  $C$  is a connected component of  $l(l^*)$  by Lemma 2.1, we have  $C = \varphi(\hat{h}^{-1}(l^*))$ . Hence  $p_i := \varphi(\hat{h}^{-1}(q_i))$  is a point of  $C$ . If  $p_i \notin B$ , then  $\hat{h}^{-1}(q_i) \cap \hat{B} = \emptyset$  so that  $\hat{h}^{-1}(q_i)$  is a point  $\hat{p}_i$  of  $\hat{X}$  by

the isomorphism  $\hat{X} \setminus \hat{B} \simeq X \setminus B$ . Since  $\hat{h}$  is birational,  $\hat{h}$  is unramified at  $\hat{p}_i$ , whence  $\hat{p}_i \notin \hat{B}^*$  and  $q_i \notin R^*$ , a contradiction. Hence  $p_i \in B$  ( $i=1,2$ ). However since  $\deg(\hat{h}|_{\hat{C}}) = \deg(h|_C) = 1$ , we have  $p_1 \neq p_2$ , which contradicts  $\deg \text{Bs}|L|_C = 1$ . Consequently  $R^*$  is a hyperplane. q.e.d.

**Claim 4.3.2.** *There exists a point  $p_0 \in X$  such that  $B = \{p_0\}$ ,  $\hat{X} \simeq Q_{p_0}(X)$  and  $\hat{B} = \varphi^{-1}(p_0) \simeq R^*$ .*

*Proof.* We first take and fix a point  $q_\infty$  of  $W \setminus R^*$ . Let  $p_\infty := h^{-1}(q_\infty)$  and  $\hat{p}_\infty := \hat{h}^{-1}(q_\infty)$ . For any point  $q \in R^*$  there exists a unique line  $l_q^*$  of  $W$  connecting  $q_\infty$  and  $q$ . Let  $l_q := l(l_q^*)$ ,  $\hat{l}_q := \hat{h}^{-1}(l_q^*)$ ,  $C_q := C(l_q^*)$  and  $\sigma_q := \varphi(\hat{l}_q)$ . Since  $\sigma_q$  is a connected subset of  $l_q$ , we have by Lemma 2.1  $\sigma_q = C_q \simeq \mathbf{P}^1$  for general  $q \in R^*$ . Take a general point  $q_0$  of  $R^*$ . Then by (4.2.1) and by the same argument as in Claim 4.3.1  $p_0 := \varphi(\hat{h}^{-1}(q_0))$  is a point of  $C_{q_0}$  with  $\{p_0\} = \text{Bs}|L|_{C_{q_0}}$  scheme-theoretically. Since  $C_{q_0}$  is a connected component of  $l_{q_0}$  by (2.1.1),  $p_0$  is an isolated point of  $B$  so that  $p_0 \in \sigma_q$  for any  $q$ , and  $\{p_0\} = \text{Bs}|L|_{C_q}$  for general  $q \in R^*$ .

Next we prove that  $\sigma_q$  is smooth at  $p_0$  for any  $q \in R^*$ . In fact,  $\sigma_q$  has a reduced curve component  $C_q$ . If  $LC_q = 2$ , then  $\sigma_q \simeq l_q \simeq C_q$  along  $C_q$  by Lemma 2.1, whence  $\sigma_q$  is smooth at  $p_0$ . If  $LC_q = 1$ , then  $B \cap C_q = \emptyset$  by (4.2.3), whence  $l_q$  along  $C_q$  is of type (2.1.4). Hence there exists a unique irreducible component  $C'_q$  of  $\sigma_q$  with  $LC'_q = 1$  containing  $p_0$ , where  $\sigma_q$  is smooth. Thus it turns out that  $\sigma_q$  is of type either (2.1.1) or (2.1.4) for any  $q \in R^*$ . We also see that  $\sigma_q$  is a connected component of  $l_q$  containing  $C_q$ . Moreover  $\sigma_q \cap B' = \emptyset$  for any  $q \in R^*$  where  $B' := B \setminus \{p_0\}$ .

Since  $\text{Bs}|L|_{C_q} = \{p_0\}$  for general  $q$ , there exist (general)  $n$ -members  $D_1, \dots, D_n$  of  $|L|$  and a closed subset  $A$  of  $X$  such that  $D_1 \cap \dots \cap D_n = p_0 + A$  and  $p_0 \notin A$ . Hence  $n$  equations defining  $D_i$  form a local coordinate at  $p_0$  so that after blowing up  $X$  at  $p_0$  we have a rational map of  $Q_{p_0}(X)$  onto  $W$  induced from  $h$ , which is a morphism near the exceptional set  $E := Q_{p_0}(p_0) \simeq \mathbf{P}^{n-1}$ . It follows that  $\hat{X} \simeq Q_{p_0}(X)$  near  $E$ . In what follows we view  $E$  as a divisor of  $\hat{X}$  by the above isomorphism. Then  $E = \varphi^{-1}(p_0)$ .

Next we prove that  $\hat{h}|_E$  is an isomorphism of  $E$  onto  $R^*$ . In fact, since  $\sigma_q$  is smooth at  $p_0$ ,  $\hat{l}_q$  intersects  $E$  at a unique point  $\hat{p}(q)$  with  $\varphi(\hat{p}(q)) = p_0$ . Since  $R^*$  is normal, this defines a morphism  $\hat{p}: R^* \rightarrow E$  such that  $\hat{h} \cdot \hat{p} = \text{id}_{R^*}$ . This shows that  $\hat{h}|_E$  is an isomorphism.

Finally we prove that  $\hat{B} = E$ . Assume the contrary. We define a closed subset  $\hat{B}'$  of  $\hat{X}$  by  $\hat{B}' := \varphi^{-1}(B')$ . As  $\hat{h}(\hat{B}^*) = R^*$ ,  $\hat{h}(\hat{B}')$  is a subset of  $R^*$ , from which we choose a point  $q$ . Since  $\hat{h}$  is birational,  $\hat{l}_q$  is a connected subset of  $\hat{X}$  intersecting  $\hat{B}'$ . Therefore  $\sigma_q \cap B' \neq \emptyset$ , a contradiction. Hence  $\hat{B} = E$ , whence  $B = \{p_0\}$ . Consequently  $\hat{X} \simeq Q_{p_0}(X)$ . q.e.d.

**Claim 4.3.3.** *Let  $P$  be a general plane of  $W$  passing through the point  $q_\infty$  in Claim 4.3.2, and  $Z(P) := \varphi(\hat{h}^{-1}(P))$ . Then  $Z(P) \simeq \mathbf{F}_0$  or  $\mathbf{F}_2$ .*

*Proof.* Let  $Z = Z(P)$ . First we note that  $Z \setminus \{p_0\}$  is smooth by Bertini's theorem. As was shown in the proof of Claim 4.3.2, there exist general  $n$ -members  $D_1, \dots, D_n$  of  $|L|$  such that  $D_1 \cap \dots \cap D_n = p_0 + A$ ,  $p_0 \notin A$  so that  $D_1 \cap \dots \cap D_{n-2}$  is smooth at  $p_0$ . This proves that  $Z$  is smooth at  $p_0$ . Since  $\sigma_q \simeq C_q \simeq \mathbf{P}^1$  is a member of  $|L_Z|$  for general  $q$ , we have  $(\sigma_q^2) = (L\sigma_q) = (LC_q)_X = 2$ . As we have  $K_Z \simeq -2L_Z$ ,  $Z$  is a smooth relatively minimal rational ruled surface, isomorphic to either  $\mathbf{F}_0$  or  $\mathbf{F}_2$ . q.e.d.

**Claim 4.3.4.** *Under the notation in Claim 4.3.2,*

(4.3.4.1)  $\sigma_q$  *is reduced for any*  $q \in R^*$ .

(4.3.4.2) Let  $\Delta := \{q \in R^*; \sigma_q \text{ is reducible}\}$ . *If a general*  $Z \simeq \mathbf{F}_0$  (resp.  $\mathbf{F}_2$ ), *then*  $\Delta$  *is a hypersurface of*  $R^*$  *with*  $\deg \Delta = 2$  (resp. 1).

*Proof.* First we prove (4.3.4.1). By the proof of Claim 4.3.2,  $\sigma_q$  is of type (2.1.1) or (2.1.4). In either case  $\sigma_q$  is reduced.

Next we prove (4.3.4.2). Assume that  $\sigma_q$  is reducible. Then by the proof of Claim 4.3.2,  $\sigma_q$  is of type (2.1.4), that is,  $\sigma_q \simeq C_q + C_{q,1} + \dots + C'_q$  with  $LC_q = LC'_q = 1$  and  $LC_{q,i} = 0$ , where we may assume that  $p_0 \in C_q$ ,  $h(C_q \setminus \{p_0\}) = \{q\}$  and  $h(C'_q) = l'_q$ . Since  $\sigma_q \in |L_Z|$ , we have

$$(C_q^2)_Z = (C_q'^2)_Z = (LC'_q)_X - 1 = 0, \quad (C_{q,i}^2)_Z = (LC_{q,i})_X - 2 = -2.$$

First we consider the case where  $Z \simeq \mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$ . We identify  $p_0 = (0, 0)$  and  $L_Z = \pi_1^* O_{\mathbf{P}^1}(1) \otimes \pi_2^* O_{\mathbf{P}^1}(1)$  via the isomorphism where  $\pi_i$  ( $i=1, 2$ ) is the  $i$ -th projection. Note that  $|L_Z| = |m_{p_0} L_Z|$ . The linear subsystem  $\{\sigma_q\}_{q \in P \cap R^*}$  coincides with  $|m_{p_\infty} m_{p_0} L_Z|$ . Since  $\sigma_q$  is irreducible for  $q$  general, there are no fibers of  $\pi_i$  ( $i=1, 2$ ) containing both  $p_0$  and  $p_\infty$ . Then by a direct computation we see that  $\sigma_q$  is reducible for exactly two (distinct) points of  $P \cap R^*$ . Thus  $\Delta$  contains a hypersurface  $\Delta_0$  of degree two in  $R^*$ . Similarly if  $Z \simeq \mathbf{F}_2$ , then  $\sigma_q$  is reducible for a unique point  $q$  of the line  $P \cap R^*$ , for which  $\sigma_q$  has exactly three components  $C_q$ ,  $C_{q,1}$ , and  $C'_q$  by the above proofs. Hence  $\Delta$  contains a hyperplane  $\Delta_0$  of  $R^*$ .

Finally we prove that  $\Delta = \Delta_0$ . Assume the contrary. Then choose a point  $q' \in \Delta \setminus \Delta_0$ . Then  $l_{q'} := D_1 \cap \dots \cap D_{n-1}$  is of type (2.1.4) with  $\sigma_{q'} \simeq C_{q'} + C_{q',1} + \dots + C'_{q'}$ . Let  $V \subset H^0(X, L)$  be an  $(n-1)$ -dimensional subspace defining  $l_{q'}$ . Then since  $\sigma_{q'}$  has at worst ordinary double singularities given in (2.1.4), we can choose a general  $(n-2)$ -dimensional subspace  $U$  of  $V$  such that the surface ( $=: Z$ ) defined as the common

zeroes of  $U$  is smooth. (This is clear from the form of the ideal defining  $\sigma_q$  in (2.1.4)). In other words, there exists a plane  $P$  of  $W$  such that  $Z := Z(P)$  is a smooth surface containing  $\sigma_q$ . Then by choosing  $U$  sufficiently general, we may assume that the line  $P \cap R^*$  intersects  $\Delta_0$  transversally. As we have seen above,  $Z \simeq F_0$  or  $F_2$ . If  $\deg \Delta_0 = 2$  (resp.  $\deg \Delta_0 = 1$ ), then  $\sigma_q$  is reducible for at least three (resp. two) distinct points of  $P \cap \Delta$ , a contradiction. Hence  $\Delta = \Delta_0$ . q.e.d.

**Claim 4.3.5.** *Under the notation in Claim 4.3.4,*

(4.3.5.1) *if  $\deg \Delta = 2$ , then  $\sigma_q$  has two irreducible components for any  $q \in \Delta \setminus \text{Sing } \Delta$ , while  $\sigma_q$  has three irreducible components for any  $q \in \text{Sing } \Delta$ .*

(4.3.5.2) *If  $\deg \Delta = 1$ , then  $\sigma_q$  has three irreducible components for any  $q \in \Delta$ .*

*Proof.* Let  $l^*$  be a line of  $R^*$ ,  $P := P(l^*)$  a plane of  $W$  spanned by  $l^*$  and  $q_\infty$ , and  $Z := Z(P) = \hat{h}^{-1}(P)$ . As we saw in the proof of Claim 4.3.4, we can choose, for any  $q \in \Delta$ , a general line  $l^* = l_q^*$  passing through  $q$  of  $R^*$  such that  $Z = Z(P(l^*))$  is a smooth surface.

Assume  $\deg \Delta = 2$ . For a smooth point  $q$  of  $\Delta$ , there exists a line  $l^*$  of  $R^*$  such that  $l^* \cap \Delta = q + q'$  for some point  $q' (\neq q)$ . Obviously  $q' \notin \text{Sing } \Delta$ . By the proof of Claim 4.3.4,  $Z \simeq F_0$ ,  $\sigma_q$  and  $\sigma_{q'}$  are the only reducible curves in  $|L|_Z$  passing through  $p_\infty$ . Hence  $\sigma_q$  as well as  $\sigma_{q'}$  has exactly two irreducible components. This proves the first part of (4.3.5.1).

If  $q$  is a singular point of  $\Delta$ , then there exists a general plane  $P$  with  $P \cap \Delta = \{2q\}$  such that  $Z = Z(P)$  is a smooth surface containing  $\sigma_q$ . (In fact, this is also clear from the form of the ideal defining  $\sigma_q$  in (2.1.4).) In the same way as in the proof of Claim 4.3.4, we see that  $K_Z \simeq -2L_Z$ ,  $(L_Z^2)_Z = (L_{\sigma_q})_X = 2$  and  $\sigma_q = C_q + C_{q,1} + \dots + C'_q$  with  $(C_q^2)_Z = (C_{q,i}^2)_Z = 0$ ,  $(C_{q,i}^2)_Z = -2$ . We note that  $\sigma_q \simeq C_q + C'_q$  is impossible. In fact, if so, then since  $P \cap \Delta = \{2q\}$ , there are no reducible members in  $|L|_Z$  other than  $\sigma_q$ . However if  $Z \simeq F_0$ , then there are two reducible members in  $|L|_Z$ , while if  $Z \simeq F_2$ , then there is a unique reducible member in  $|L|_Z$ , which however consists of 3 irreducible components. Hence  $\sigma_q \simeq C_q + C'_q$  is impossible. Therefore  $\sigma_q \simeq C_q + C_{q,1} + \dots + C'_q$ , whence  $Z \simeq F_2$ ,  $\sigma_q \simeq C_q + C_{q,1} + C'_q$ , which completes the proof of the second part of (4.3.5.1). (4.3.5.2) is proved similarly. q.e.d.

**Lemma 4.3.6.** *A general  $Z \simeq F_0$ , and  $\Delta$  is a smooth quadric hypersurface in  $R^*$ .*

*Proof.* First we note that  $K_Z \simeq -2L_Z$  for general  $Z = Z(P)$ . Let  $F$

be a fiber of the ruling of  $Z$ . Then  $(LF)_X = (L_Z F)_Z = -(K_Z F)_Z/2 = 1$ . This shows that  $\text{Pic } X/\text{Tor}(\text{Pic } X) \simeq \mathbb{Z}L$ .

Now we prove that  $\Delta$  is an irreducible hypersurface of degree two. Assume that  $\Delta$  contains a hyperplane  $\Delta'$ . By Claim 4.3.5,  $\sigma_q \simeq C_q + C'_q$  or  $\sigma_q \simeq C_{q,1} + C'_q$  for any  $q \in \Delta \setminus \text{Sing } \Delta$ , where we may assume that  $p_0 \in C_q$ ,  $p_\infty \in C'_q$ . Note that  $h(C_q \setminus \{p_0\}) = \{q\}$ ,  $h(C'_q) = l_q^*$ . Let

$$G := \overline{\bigcup_{q \in \Delta' \setminus \text{Sing } \Delta} C_q}, \quad G' := \overline{\bigcup_{q \in \Delta' \setminus \text{Sing } \Delta} C'_q}.$$

Then  $G$  and  $G'$  are (mutually distinct) divisors of  $X$ . In fact, if  $C_q \cap C_s \neq \{p_0\}$  for  $q, s \in \Delta \setminus \text{Sing } \Delta$ , then  $C_q = C_s$  by  $LC_q = LC_s = 1$  so that  $q = s$ . Hence  $\dim G = n - 1$ . Similarly  $\dim G' = n - 1$ . Meanwhile if for general  $q \in \Delta \setminus \text{Sing } \Delta$ , there exists  $s \in \Delta \setminus \text{Sing } \Delta$  such that  $C_q \cap C'_s \neq \emptyset$ , then  $C_q \cap \sigma_s$  contains at least two points, whence  $C_q \subset \sigma_s$  by  $LC_q = 1$ . Hence  $C_q = C_s$ , whence  $q = s$ . Therefore  $G \cap G'$  is at most  $(n - 2)$ -dimensional. Since  $L$  generates  $\text{Pic } X/\text{torsions}$ , we have  $c_1(G) = ac_1(L)$  and  $c_1(G') = a'c_1(L)$  for some positive integers  $a$  and  $a'$ . However  $G + G'$  is a subset of  $D := \bigcup_{q \in \Delta} \sigma_q$ , which is a member of  $|L|$ . It follows that  $a + a' \leq 1$ , a contradiction. Consequently  $\Delta$  is an irreducible hypersurface of degree two. In particular a general  $Z \simeq F_0$  by Claim 4.3.4. (There is another proof of  $Z \simeq F_0$  due to Fujiki.)

Next we prove that  $\Delta$  is smooth. Assume that  $\Delta$  is singular. By (4.3.5.1) we have  $\sigma_q \simeq C_q + C_{q,1} + C'_q$  with  $LC_{q,1} = 0$  for any  $q \in \text{Sing } \Delta$ . Conversely, given a rational curve  $C$  with  $LC = 0$ , we have a unique point  $q$  of  $\text{Sing } \Delta$  such that  $C = C_{q,1}$ . In fact, since  $p_0 \notin C$ , it suffices to set  $q := h(C)$ . It follows from Claim 4.3.5 that  $C \subset \sigma_q$ ,  $C = C_{q,1}$  and  $q \in \text{Sing } \Delta$ . Therefore  $C$  moves on  $X$  in an at most  $(n - 4)$ -dimensional family. On the other hand  $\chi(C, N_{C/X}) = n - 3$ , whence by [9, Proposition 3], there exists an at least  $(n - 3)$ -dimensional family of displacements  $C(t)$  of  $C$  in  $X$ , a contradiction. q.e.d.

**Claim 4.3.7.**  $\hat{X} \simeq Q_\Delta(W)$ , the monoidal transform of  $W$  with smooth center  $\Delta$ .

*Proof.* By Claim 4.3.5,  $\sigma_q \simeq C_q + C'_q$  for any  $q \in \Delta$ , where  $p_0 \in C_q$  and  $p_\infty \in C'_q$ . Let  $\hat{C}_q := \varphi^{-1}(C_q)$ ,  $\hat{\Delta} := \hat{h}^{-1}(\Delta)_{\text{red}}$ ,  $M := \varphi^*L - E$  and  $J(\hat{h}) := K_{\hat{X}} - \hat{h}^*K_W$ . Then  $\hat{\Delta} = \bigcup_{q \in \Delta} \hat{C}_q$  is a unique  $\hat{h}$ -exceptional divisor. Hence  $J(\hat{h}) = r\hat{\Delta}$  for a positive integer  $r$ .

First we prove that  $(\hat{\Delta} \hat{C}_q)_{\hat{X}} = -1$ . Since a general member of  $|L|$  intersects  $C_q$  ( $q \in \Delta$ ) transversally at  $p_0$ , we have  $M\hat{C}_q = 0$  and  $E\hat{C}_q = 1$ ,

whence  $M = \hat{h}^* O_W(1)$ . We also have

$$J(\hat{h}) = \varphi^* K_X + (n-1)E + (n+1)M = M - E = \varphi^* L - 2E,$$

whence  $r(\hat{\Delta} \hat{C}_q)_{\hat{X}} = (LC_q)_X - 2(E\hat{C}_q)_{\hat{X}} = -1$ . Hence  $r=1$  and  $(\hat{\Delta} \hat{C}_q)_{\hat{X}} = -1$ .

Next we prove that  $\hat{\Delta}$  is smooth. Let  $q \in \Delta$ . Then we can choose general  $(n-2)$ -hyperplanes  $H_1, \dots, H_{n-2}$  of  $W$  such that  $\Delta \cap H_1 \cap \dots \cap H_{n-2} = q + s$  for some  $s (\neq q)$ . Let  $P := H_1 \cap \dots \cap H_{n-2}$ . Then  $Z(P) \simeq F_0$  and  $Z(P) \cap \varphi(\hat{\Delta}) = C_q \cup C_s$ . Let  $M_i$  be the proper transform of  $H_i$ . It follows that  $\hat{\Delta} \cap M_1 \cap \dots \cap M_{n-2} = \hat{C}_q$  along  $\hat{C}_q \setminus E$ . Since  $\hat{C}_q$  is smooth, so is  $\hat{\Delta}$  along  $\hat{C}_q \setminus E$  for any  $q$ . Therefore  $\hat{\Delta} \setminus E$  is smooth. Meanwhile  $\hat{\Delta} \cap E \simeq \Delta$ , whence  $\hat{\Delta}$  is smooth along  $\hat{\Delta} \cap E$ . Hence  $\hat{\Delta}$  is smooth everywhere. Note that  $\varphi(\hat{\Delta}) \in |L|$ .

Since  $\hat{C}_q \cap \hat{C}_s = \emptyset$  for  $q \neq s$  ( $q, s \in \Delta$ ), this shows that  $\hat{X} \simeq Q_{\Delta}(W)$ . q.e.d.

(4.3.8) COMPLETION OF THE PROOF OF LEMMA 4.3. By Claim 4.3.2 and Claim 4.3.7,  $X$  is recovered from  $W (\simeq \mathbf{P}^n)$ ,  $R^*$  and  $\Delta$  as follows. By Claim 4.3.7,  $\hat{X}$  is the monoidal transform of  $W$  with  $\Delta$  center. Then  $E (\simeq \mathbf{P}^{n-1})$  is a proper transform of  $R^*$  with  $N_{E/\hat{X}} \simeq O_E(-1)$ . In fact,

$$\begin{aligned} N_{E/\hat{X}} &\simeq E_E \simeq (M - J(h))_E \\ &\simeq (\hat{h}|_E)^*(O_{R^*}(1) - \Delta) \\ &\simeq O_E(-1). \end{aligned}$$

Consequently we obtain  $X$  from  $\hat{X}$  by blowing down  $E$  to a smooth point  $p_0$  of  $X$ . Obviously  $X$  thus obtained is unique up to isomorphism. Hence we have  $X \simeq \mathcal{Q}^n$ , whence  $h^0(X, L) = n+2$ , a contradiction. This completes the proof of Lemma 4.3. q.e.d.

**Lemma 4.4.**  *$\deg h = 2$  is impossible.*

We prove Lemma 4.4 in Claims 4.4.1–4.4.4. We use the same notation as in (4.1) and Lemmas 4.2–4.3.

We assume  $\deg h = 2$ . We first prove

**Claim 4.4.1.** *Let  $l^*$  be a line of  $W (\simeq \mathbf{P}^n)$  not contained in  $R^*$ . Then*

$$(4.4.1.1) \quad \#(l^* \cap R^*) = 1 \text{ or } 2.$$

$$(4.4.1.2) \quad \hat{l}(l^*) := \hat{h}^{-1}(l^*) \text{ is a connected subset of } \hat{X}.$$

$$(4.4.1.3) \quad \sigma(l^*) := \varphi(\hat{l}(l^*)) \text{ is a connected subset of } X \text{ disjoint from } B.$$

Proof. First we prove (4.4.1.2). Let  $\sigma_0$  be a connected component of  $\sigma(l^*)$  mapped onto  $l^*$ , and  $\hat{\sigma}_0 := \varphi^{-1}(\sigma_0)$ . Then  $\sigma_0$  is of type (2.1.1) or (2.1.4) by Lemma 4.2.

If  $l^*$  is general, then  $\sigma(l^*)$  is irreducible outside  $B$  by Lemma 4.2, whence  $\sigma_0$  is of type (2.1.1). That is,  $\sigma_0$  is a rational curve  $C$  with  $LC=2$ . Hence by Lemma 4.2  $\sigma_0 \cap B = \emptyset$  and  $\sigma_0 \cap B^* = p_1 + p_2$  for some points  $p_i$  because  $\deg(h|_C) = \deg h = 2$ . Note that  $\hat{\sigma}_0 \simeq \sigma_0$  because  $\sigma_0 \cap B = \emptyset$ . Let  $q_i := h(p_i)$ . Since  $\deg \hat{h} = 2$ ,  $\hat{h}^{-1}(q_i)$  is a connected subset of  $\hat{l}(l^*)$ . Since  $\hat{l}(l^*) = \hat{\sigma}_0 \cup \hat{h}^{-1}(q_1) \cup \hat{h}^{-1}(q_2)$ ,  $\hat{l}(l^*)$  is connected. Since  $\hat{l}(l^*)$  is connected for general  $l^*$ , it is connected for any  $l^*$ . This proves (4.4.1.2).

Next we prove (4.4.1.3). By (4.4.1.2)  $\sigma(l^*)$  is connected. Hence  $\sigma(l^*)$  is of type (2.1.1) or (2.1.4). If  $\sigma(l^*)$  is of type (2.1.1), then it is a smooth rational curve  $C$  with  $LC=2$ . Since  $\deg h|_C = 2$ , we have  $C \cap B = \emptyset$  by Claim 4.2.2. If  $\sigma(l^*)$  is of type (2.1.4), then  $\sigma(l^*) \simeq C_0 + \cdots + C_m$  with  $LC_0 = LC_m = 1$ ,  $LC_i = 0$  ( $1 \leq i \leq m-1$ ). We prove  $\sigma(l^*) \cap B = \emptyset$ . If  $\sigma(l^*) \cap B \neq \emptyset$ , then we may assume  $C_m \cap B \neq \emptyset$ . Then  $h(C_j)$  ( $1 \leq j \leq m-1$ ) and  $h(C_m \setminus B)$  are one and the same point of  $W$ , while  $h|_{C_0}$  is unramified on  $C_0 \setminus C_1$  by  $LC_0 = 1$ . Since  $l^* \neq R^*$ , this contradicts  $\deg h = 2$ . Therefore  $\sigma(l^*) \cap B = \emptyset$ . This proves (4.4.1.3).

Finally we prove (4.4.1.1). If  $\sigma_0$  is of type (2.1.1), then  $l^* \cap R^*$  is two points by the above proof. If  $\sigma_0$  is of type (2.1.4), then  $l^* \cap R^*$  is one point. q.e.d.

**Claim 4.4.2.**  $B = \emptyset$ ,  $\hat{B} = \emptyset$  and  $R^*$  is an irreducible hyperquadric of  $W$ .

Proof. First we prove  $B = \emptyset$ . Assume  $B \neq \emptyset$ , whence  $\hat{B} \neq \emptyset$ . Then there is a line  $l^*$  not contained in  $R^*$  such that  $\hat{l}(l^*) \cap \hat{B} \neq \emptyset$ . Hence  $\sigma(l^*) \cap B \neq \emptyset$ , which contradicts (4.4.1.3). Hence  $B = \emptyset$ . Therefore  $X \simeq \hat{X}$ .

Next we prove that  $R^*$  is a hypersurface of degree two in  $W$ . Choose a general line  $l^*$  of  $W$  intersecting  $R^*$ . Then  $l(l^*)$  contains a rational curve  $C$  with  $LC=2$  by the assumption. By Lemma 4.2 we have  $B \cap C = \emptyset$ ,  $\dim |L|_C = 1$ . It follows that there exist exactly two points  $p_i \in C$  ( $i=1,2$ ) such that  $h|_C$  is unramified on  $C \setminus \{p_1, p_2\}$ . Note that  $p_i \notin B$ . Let  $q_i := h(p_i)$ . Then  $C \cap B^* = p_1 + p_2$  and  $l^* \cap R^* = q_1 + q_2$ . Hence  $R^*$  contains a hypersurface  $R_0^*$  of degree two in  $W$ . If  $R^* \neq R_0^*$ , then there exists a line  $l^*$  not contained in  $R^*$  but intersecting  $R^*$  at (at least) 3 distinct points. This contradicts (4.4.1.1).

Finally we prove that  $R^*$  is irreducible. Assume the contrary. Then  $\dim \text{Sing } R^* = n-2$ . We prove that  $C := h^{-1}(q)$  is a rational curve with  $LC=0$  for any  $q \in \text{Sing } R^*$ . For this purpose we choose a general point  $q_\infty \in W \setminus R^*$  and a line  $l^*$  connecting  $q_\infty$  and  $q$  with  $l^* \cap R^* = \{2q\}$ . Moreover we choose a general plane  $P$  containing  $l^*$  so that  $Z(P)$  is a smooth

surface. Choose point  $p_\infty$  with  $q_\infty = h(p_\infty)$ . In the same manner as in the proof of Claim 4.3.5,  $Z(P) \simeq \mathbf{F}_2$ . We have a unique reducible curve  $\sigma$  in  $|L_Z|$  with  $h(\sigma) = l^*$  passing through  $p_\infty$ , whence  $\sigma(l^*) = \sigma \simeq C_0 + C_1 + C_2$  with  $LC_i = 1$  ( $i=0,2$ ),  $LC_1 = 0$ . Therefore  $C_1 = C = h^{-1}(q)$ . Let  $G := h^{-1}(\text{Sing } R^*)$ . Obviously  $G$  is a divisor of  $X$  with  $h_*(G) = 0$ , which contradicts  $\text{Pic } X / \text{torsions} \simeq \mathbf{Z}L$ . This proves the irreducibility of  $R^*$ .  
q.e.d.

**Claim 4.4.3.**  $R^*$  is smooth.

Proof. By Claim 4.4.2, we have  $\text{Hessian-rank } R^* \geq 3$ . Assume  $4 \leq \text{Hessian-rank } R^* \leq n$  so that  $\dim \text{Sing } R^* \leq n-4$ . Then by the proof of Claim 4.4.2,  $C = h^{-1}(q)$  ( $q \in \text{Sing } R^*$ ) is a rational curve with  $LC = 0$ , whence  $\chi(N_{C/X}) = n-3 > \dim \text{Sing } R^*$  by Lemma 2.1. Then we derive a contradiction as in the proof of Claim 4.3.4. Therefore  $R^*$  is smooth or  $\text{Hessian-rank } R^* = 3$ .

Let  $r(w) = 0$  be the equation defining  $R^*$  in  $W$ .  $h^*R^*$  is a divisor with multiplicity 2 above a generic point of  $R^*$ . let  $h^*R^* = 2A + A'$  for some effective divisors  $A$  ( $\neq 0$ ) and  $A'$  with  $h_*(A') = 0$ . Since  $\text{Pic } X / \text{torsions} \simeq \mathbf{Z}L \simeq \mathbf{Z}h^*O_W(1)$ , and  $h^*(R^*) \in |2L|$ , we have  $A \in |L|$  and  $A' = 0$ . Hence  $h^*(R^*)$  is a divisor of  $X$  with multiplicity 2, whence we have an element  $\psi(x) \in H^0(X, L)$  such that  $(h^*r)(x) = \psi(x)^2$ . Let  $\mathbf{H}$  be the total space of the hyperplane bundle  $O_W(1)$  on  $W$  with fiber coordinate  $\zeta$ ,  $Y$  a hypersurface of  $\mathbf{H}$  defined by  $\zeta^2 = r(w)$ . Then using  $\psi(x)$ , we can define a natural morphism  $g$ , compatible with  $h$ , of  $X$  onto  $Y$  by  $g^*\zeta = \psi(x)$ . If  $\text{Hessian-rank } R^* = 3$ ,  $Y$  is isomorphic to a hyperquadric of  $\mathbf{P}^{n+1}$  with  $\text{Hessian-rank } Y = 4$ , whence it has a reducible hyperplane section. This contradicts  $\text{Pic } X / \text{torisons} \simeq \mathbf{Z}L$ . Consequently  $R^*$  is smooth. q.e.d.

(4.4.4) COMPLETION OF THE PROOFS OF LEMMA 4.4 AND PROPOSITION 3.1. By Claim 4.4.3,  $R^*$  is a smooth hyperquadric. With the notation in Claim 4.4.3,  $Y \simeq \mathcal{Q}^n$ , whence we have a birational morphism  $g$  of  $X$  onto  $\mathcal{Q}^n$ . Since  $K_X \simeq -nL \simeq -ng^*\pi^*O_W(1) \simeq g^*(K_{\mathcal{Q}^n})$ ,  $g$  is an isomorphism. Therefore  $h^0(X, L) = n+2$ , which contradicts our assumption  $h^0(X, L) = n+1$ . Thus we complete the proof of Lemma 4.4, hence of Proposition 3.1.  
q.e.d.

## 5. Moishezon fourfolds homeomorphic to $\mathcal{Q}_c^4$

The purpose of this section is to prove:

**Theorem 5.1.** *Let  $X$  be a Moishezon 4-fold homeomorphic to  $\mathcal{Q}^4$ ,*



and  $L$  a line bundle on  $X$  with  $L^4 = 2$ . Assume that  $h^0(X, L) \geq 5$ . Then  $X \simeq \mathcal{Q}^4$ .

Our proof of Theorem 5.1 is completed in (5.7).

**Lemma 5.2.** *Under the assumptions in Theorem 5.1, let  $D$  and  $D'$  be distinct members of  $|L|$ ,  $\tau$  the scheme-theoretic complete intersection  $D \cap D'$ . Then we have*

$$(5.2.1) \quad \text{Pic } X \simeq \mathbb{Z}L, \quad K_X \simeq -4L,$$

$$(5.2.2) \quad H^p(X, -qL) = 0 \quad (p=0, q \geq 1, \text{ or } 1 \leq p \leq 3, 0 \leq q \leq 4, \text{ or } p=4, q \leq 3)$$

$$(5.2.3) \quad H^p(D, -qL_D) = 0 \quad (p=0, q \geq 1, \text{ or } p=1, 2, 0 \leq q \leq 3, \text{ or } p=3, q \leq 2)$$

$$(5.2.4) \quad H^p(\tau, -qL_\tau) = 0 \quad (p=0, q=1, 2, \text{ or } p=1, 0 \leq q \leq 2, \text{ or } p=2, q=0, 1)$$

$$(5.2.5) \quad H^0(X, \mathcal{O}_X) \simeq H^0(D, \mathcal{O}_D) \simeq H^0(\tau, \mathcal{O}_\tau) \simeq \mathbb{C},$$

$$(5.2.6) \quad |L|_D = |L_D| \text{ and } |L|_\tau = |L_\tau|.$$

**Proof.** The proof of (5.2.1) is similar to [15]. The vanishing of  $H^p(X, -qL)$  for  $p \neq 2$  is proved in the same way as in [15]. Since  $X$  is homeomorphic to  $\mathcal{Q}^4$ , we have

$$\chi(X, -qL) = \chi(\mathcal{Q}^4, \mathcal{O}_{\mathcal{Q}^4}(-q)) = (q-1)(q-2)^2(q-3)/12$$

for any  $q$  in view of (5.2.1). This proves the vanishing of  $H^2(X, -qL)$  for  $0 \leq q \leq 4$ . The remaining assertions are easy. q.e.d.

**Lemma 5.3.** *Under the assumptions in Theorem 5.1, let  $B := \text{Bs } |L|$  be the scheme-theoretic base locus of  $|L|$ ,  $h: X \rightarrow \mathbb{P}^m$  a rational map associated with  $|L|$ , and  $W$  the closure of  $h(X \setminus B)$ , where  $m = h^0(X, L) - 1$ . Then  $\dim W \geq 3$ .*

**Proof.** Let  $d = \deg W$ . Then  $d \geq m + 1 - \dim W$ . If  $\dim W = 1$ , then  $d = 1$ ,  $m = 1$  by  $\text{Pic } X \simeq \mathbb{Z}L$ , which contradicts  $m \geq 2$ . Therefore we assume  $\dim W = 2$  to derive a contradiction. So  $d = \deg W \geq m - 1 \geq 2$ .

Then by choosing general  $D$  and  $D' \in |L|$ , we have reduced irreducible components  $Z_i$  ( $1 \leq i \leq dr$ ) of  $\tau := D \cap D'$  outside  $B$  for some positive integer  $r$ , where  $r$  is the number of irreducible components of a general fiber of  $h$ . Each  $Z_i$  is nonsingular outside  $B$  by Bertini's theorem. Let  $Z = Z_1$ , and let  $v: Y \rightarrow Z$  be the normalization of  $Z$ ,  $f: S \rightarrow Y$  the minimal resolution of  $Y$ ,  $g = v \cdot f$ . Then as in the proof of Lemma 3.2, there exist effective Cartier divisors  $E$  and  $G$  on  $S$  with no components in common

such that the canonical sheaf of  $S$  is given by  $K_S = \mathcal{O}_S(-2g^*L - E - G)$ , where the base locus  $\text{Bs } g^*|L|$  contains  $\text{supp}(E + G)$  if  $D$  and  $D'$  are sufficiently general. Since  $h^0(X, L) \geq 3$  and  $Z \not\subset B$ ,  $g^*L$  is effective. By  $P_m(S) = 0$ ,  $S \simeq \mathbf{P}^2$  or  $S$  is ruled. The proof of Lemma 5.3 is now divided into CASES 1-1, 1-2 and CASE 2.

CASE 1. Assume  $S \simeq \mathbf{P}^2$ . Then we have  $G = 0$  and  $S \simeq Y$ . Let  $H \in g^*|L|$ . Then  $K_S = -2H - E$ . Since  $K_{\mathbf{P}^2}$  is indivisible by 2, we have  $E \neq 0$  and  $H = E \in |\mathcal{O}_{\mathbf{P}^2}(1)|$  in view of  $E_{\text{red}} \subset H_{\text{red}}$ . Hence  $g^*(D'') (= H)$  is independent of the choice of  $D'' \in |L|$ . Moreover  $g|_E$ , the restriction of  $g$  to  $E$ , is generically one to one because  $(Lg_*(E)) = (g^*(L)E) = 1$ .

Since the coefficient of  $E$  in  $-K_S$  relevant to  $\text{Sing } \tau$  is equal to 1, there exists by Theorem 1.14 a Zariski open subset  $V$  of  $Z$  with  $E \cap v^{-1}(V) \neq \emptyset$  such that

$$e(Q_V, E_U) + e(Q''_V, E_U) - e(Q'_V, E_U) = 1.$$

where  $U := v^{-1}(V)$  and  $E_U = E \cap U$ . By Theorem 1.11, (1.12) and (1.13), we have  $e(Q_V, E_U) = 1$ ,  $e(Q''_V, E_U) = e(Q'_V, E_U) = 0$ . See Appendix to section one for the detail. Let  $p$  be any point of  $E \cap U$ . By Lemma 1.6, there exists a local parameter system  $x, y, z$  and  $w$  at  $p$  and another irreducible component  $Z^*$  of  $\tau$  at  $p$  such that

$$I_{\tau, p} = (xy, z), \quad I_{Z, p} = (x, z), \quad I_{Z^*, p} = (y, z).$$

Hence  $Z$  is smooth along  $E \cap U$ . It follows that  $\text{Sing } Z$  is finite for general  $D$  and  $D'$ .

There are two subcases  $Z^* \subset B$  or  $Z^* \not\subset B$ . Let  $\bar{E} := g(E)_{\text{red}}$ .

CASE 1-1. Assume  $Z^* \subset B$ . Let  $l$  be a line ( $\neq E$ ) on  $S$ ,  $C := g(l)_{\text{red}}$ . Hence  $g_*(l) = aC$  for an integer  $a \geq 1$ . Then  $a(LC)_X = (g^*(L)l)_S = 1$ , whence  $a = 1$  and  $l$  is mapped generically isomorphically onto  $C$  by  $g$ . Take a general  $D'' \in |L|$ . For any point  $q \in Z \cap D''$ , there exists a line  $l$  on  $S$  such that  $q \in g(l)$ . Therefore any general  $D'' \in |L|$  is smooth along  $Z \cap D''$  by  $(D''g_*(l))_X = 1$ . Hence by choosing  $D$  sufficiently general,  $D$  is also smooth along  $Z$ . Therefore  $Z$  is a Cartier divisor of  $D$ , so that  $Z$  is Gorenstein everywhere. Since  $\text{Sing } Z$  is finite, this implies that  $Z$  is normal. Hence  $S \simeq Y \simeq Z \simeq \mathbf{P}^2$ ,  $Z \cap D'' \simeq \bar{E} \simeq E \simeq \mathbf{P}^1$  and  $\mathcal{O}_Z(L) \simeq \mathcal{O}_Z(\bar{E})$ .

Since  $\bar{E} \subset B$ , we have  $\bar{E} \simeq B \cap Z$ . Hence  $B$  is a smooth Cartier divisor of  $D$  along  $\bar{E}$ . The surfaces  $B$  and  $Z$  intersect transversally along  $\bar{E}$ .

**Claim 5.3.1.**  $I_B \mathcal{O}_\tau \simeq \mathcal{O}_Z(-\bar{E})$  along  $\bar{E}$ .

Proof. Let  $I_{\bar{E}/Z}$  be the ideal of  $O_Z$  defining  $\bar{E}$ . We may assume  $O_X \simeq \mathcal{C}\{x, y, z, w\}$ ,  $I_\tau = (xy, z)$ ,  $I_Z = (x, z)$ ,  $I_B = (y, z)$ . Then  $I_{\bar{E}/Z} = I_B O_Z \simeq y\mathcal{C}\{y, w\}$  by  $\bar{E} \simeq B \cap Z$ . Thus we see

$$I_B O_\tau \simeq y\mathcal{C}\{x, y, w\}/(xy) \simeq y\mathcal{C}\{y, w\} \simeq I_{\bar{E}/Z} \simeq O_Z(-\bar{E}). \quad \text{q.e.d.}$$

CASE 1-1 (Continued). We also see that

$$1 = (HE)_S = (L_Z B)_Z = (\bar{E}B)_Z = ((Z+B)BZ)_X = (\bar{E}^2)_B + (\bar{E}^2)_Z = (\bar{E}^2)_B + 1,$$

whence  $(\bar{E}^2)_B = 0$ . Hence the unique irreducible component  $B_0$  of  $B$  intersecting  $Z$  is a (possibly singular) ruled surface with  $\bar{E}$  a general fiber. Moreover  $Z$  intersects the irreducible components of  $\tau$  other than  $Z$  and  $B_0$  in (at most) finitely many points outside  $\bar{E}$ . This is true for any  $Z_i$ . Since  $\tau$  is Gorenstein, this implies that  $Z$  meets no irreducible components of  $\tau$  other than  $B_0$ . Therefore for general  $D$  and  $D'$ , we have  $Z_i \simeq \mathbf{P}^2$ ,  $Z_i \cap Z_j = \emptyset$  ( $i \neq j$ ) and  $\bar{E}_i := Z_i \cap B \simeq \mathbf{P}^1$  for  $1 \leq i \leq dr$ . Hence we have,

$$H^0(I_B L_\tau) \supset \bigoplus_{i=1}^{dr} H^0(Z_i, I_{\bar{E}_i/Z_i} O_{Z_i}(L)) \simeq \bigoplus_{i=1}^{dr} H^0(Z_i, O_{Z_i}),$$

which shows  $m-1 = h^0(L_\tau) = h^0(I_B L_\tau) \geq dr \geq (m-1)r$ . Hence  $r=1$ ,  $d=m-1 \geq 3$ . Then by [5, Theorem 1],  $W$  is a cone over a smooth variety of minimal degree. In this case,  $W$  is either the Veronese surface in  $\mathbf{P}^5$  with  $d=4$  or the cone over a normal rational curve of degree  $m-1$  in  $\mathbf{P}^{m-1}$  with  $d=m-1$ . In either case, there is a reducible or a nonreduced hyperplane section of  $W$ , which contradicts  $\text{Pic } X \simeq \mathbf{Z}L$ . See also [7, 5.3.11].

CASE 1-2. Assume  $Z^* \not\subset B$ . We may assume that  $Z^* \simeq \mathbf{P}^2$  by choosing general  $D$  and  $D'$ . By the same argument as in CASE 1-1,  $Z$  and  $Z^*$  intersect transversally along  $\bar{E}$ , where  $Z \cap Z^* \simeq \bar{E} \simeq \mathbf{P}^1$ .

Let  $\sigma$  be the sum of all the primary components of  $\tau$  other than  $Z$  and  $Z^*$ . Then  $\sigma \cap (Z \cup Z^*)$  is finite. This implies that  $\sigma = \emptyset$  and  $\tau = Z \cup Z^*$  because  $\tau$  is Gorenstein and connected by (5.2.5). Thus we have an exact sequence of  $O_\tau$ -modules,

$$0 \rightarrow O_\tau \rightarrow O_Z \oplus O_{Z^*} \rightarrow O_{\bar{E}} \rightarrow 0.$$

It follows from the exact sequence that  $h^0(X, L) = h^0(\tau, L_\tau) + 2 = 6$ ,  $B = \text{Bs } |L_\tau| = \emptyset$ . This contradicts  $\bar{E} \subset B$  by the general choice of  $D$  and  $D'$ .

CASE 2. Assume that  $S$  is a ruled surface. We let  $\pi: S \rightarrow T$  be the ruling,  $F (\simeq \mathbf{P}^1)$  a general fiber of  $\pi$ . Let  $H \in g^*[L]$ , and let  $M$  (resp.  $N$ ) be the movable part (resp. the fixed part) of  $H$  in  $g^*[L]$ . Since  $F \simeq \mathbf{P}^1$ , we have

$$-2 = K_S F + F^2 = K_S F = -2HF - (E + G)F.$$

Since  $E_{\text{red}} + G_{\text{red}} \subset H_{\text{red}}$ , we have  $HF = 1$ ,  $EF = GF = 0$ . Therefore there exists a unique irreducible component  $\Gamma$  of  $H$  with  $\Gamma F = 1$  and  $\Gamma \not\subset E + G$ . Since  $\dim W = 2$ , we have  $M = 0$  and  $\Gamma \subset N$ . Any general  $D' \in |L|$  is smooth generically along  $g(\Gamma)$  (or at  $g(\Gamma)$  if  $g(\Gamma)$  is a point) by  $(D''g_*(F))_X = (HF)_S = 1$ . Assume that  $g(\Gamma)$  is a curve on  $X$ . Then any irreducible component  $Z_i$  of  $\tau$  contains  $g(\Gamma)$  because  $g(\Gamma)$  is a curve component of  $B$  by  $\Gamma \subset N$ . However since  $\Gamma \not\subset E$ ,  $\tau$  is smooth at a generic point of  $g(\Gamma)$ . This shows  $d = 1$ , which contradicts  $d \geq m - 1 \geq 2$ . Therefore  $g(\Gamma)$  is a point. We note that this can happen if the connected component of the  $g$ -exceptional set containing  $\Gamma$  corresponds to one of Du Val singularities.

Let  $p := g(\Gamma)$  and  $q := f(\Gamma)$ .

**Claim 5.3.2.**  $q \notin f(E)$ .

Proof. Since  $g(\Gamma)$  is a point,  $H\Gamma = (g^*L)\Gamma = 0$  and  $\Gamma^2 < 0$ . It follows that  $K_S\Gamma + \Gamma^2 = \Gamma^2 - (E + G)\Gamma \leq -1$ , whence  $K_S\Gamma = E\Gamma = G\Gamma = 0$ ,  $\Gamma^2 = -2$ , and  $\Gamma \simeq \mathbf{P}^1$  in view of the minimality of the resolution  $f$ . Assume  $q \in f(E)$ . Then there is a sequence  $N_i$  ( $1 \leq i \leq s$ ) of irreducible components of  $N$  with  $q = f(N_i)$  such that  $N_1\Gamma > 0$ ,  $N_{i-1}N_i > 0$  and  $N_sE > 0$ . Then since  $E\Gamma = G\Gamma = 0$ ,  $N_1$  is not contained in  $E + G$ . Hence  $K_S N_1 = HN_1 = EN_1 = GN_1 = 0$ ,  $N_1^2 = -2$  in the same manner as for  $\Gamma$ . Therefore  $s \geq 2$  and  $N_2 \not\subset E + G$ . Repeating the same arguments as above, we see that  $K_S N_i = HN_i = EN_i = GN_i = 0$ ,  $N_i^2 = -2$  for any  $i$ . This contradicts  $N_sE > 0$ . Therefore  $q \notin f(E)$ . q.e.d.

**Claim 5.3.3.**  $p \notin g(E)$ .

Proof. Assume the contrary. Let  $V$  be a sufficiently small open neighborhood of  $f^{-1}(q)$  in  $S$ . Note that  $\Gamma \subset V$ . Since  $f(V) \setminus \{q\}$  is disjoint from  $f(E)$ ,  $g(V) \setminus \{p\}$  is disjoint from  $g(E)$ . Therefore the germ  $(g(V), p)$  is a locally irreducible component of  $(\tau, p)$  which intersects the other locally irreducible components at the point  $p$  only. This shows that  $\tau \setminus \{p\}$  has at least two local connected components, which contradicts that  $\tau$  is Gorenstein. Hence  $p \notin g(E)$ . q.e.d.

(5.3.4) COMPLETION OF THE PROOF OF LEMMA 5.3. Claim 5.3.3 shows that the point  $p \in g(N) \subset B$  is isolated in  $B$ , whence any irreducible component  $Z_i$  of  $\tau$  passes through  $p$ . Since  $d \geq m-1 \geq 2$ , there is another component  $Z_2$  of  $\tau$  with  $p \in Z_2$ . Since  $p \notin g(E)$ ,  $\text{Sing } \tau_{\text{red}}(\subset B)$  is isolated at  $p$ , whence  $Z_1 \cap Z_2$  is isolated at  $p$ . However since  $\tau$  is Gorenstein,  $Z_1 \cap Z_2$  has a curve component at  $p$ , a contradiction. This completes the proof of Lemma 5.3. q.e.d.

**Lemma 5.4.** *If  $\dim W = 3$ , then  $W \simeq Q^3$ , a smooth hyperquadric in  $P^4$ .*

Proof. Since  $\dim W = 3$ ,  $\tau := D \cap D'$  is irreducible nonsingular outside  $B$  for general  $D, D' \in |L|$  by Bertini's theorem. Let  $Z$  be the unique irreducible component of  $\tau$  outside  $B$ ,  $g: S \rightarrow Z$  the minimal resolution of the normalization of  $Z$ . We see that  $S \not\simeq P^2$ . In fact, if  $S \simeq P^2$ , then  $g^*[L]$  has no movable components by the same arguments as in the proof of Lemma 3.2 CASE 1, whence  $\dim W \leq 2$ , a contradiction. Hence  $S$  is a ruled surface with  $F \simeq P^1$  a general fiber. Under the same notation as before,  $H := g^*(D'') \in g^*[L]$  has an irreducible component  $\Gamma$  with  $F\Gamma = 1$ . We see  $EF = GF = 0$  and  $\Gamma \not\subset E + G$ . If the movable part  $M$  of  $H$  contains  $\Gamma$ , then  $M = \Gamma$  by  $HF = \Gamma F = 1$ , which shows that  $d = \deg W = 1 \geq m-2 \geq 2$ , a contradiction. Therefore the fixed part  $N$  of  $H$  contains  $\Gamma$ .

Since  $HF = NF = \Gamma F = 1$ , the movable part  $M$  satisfies  $MF = 0$  so that  $M^2 = 0$  and that  $M$  is a union of general fibers  $F_i$  of the ruling,  $M = F_1 + \cdots + F_d$ . Let  $C_i = g(F_i)_{\text{red}}$ , and  $\bar{M} = C_1 + \cdots + C_d$ . Then  $g_*(F_i) = C_i$  and  $(LC_i) = 1$  for any  $i$  by  $HF_i = 1$ . We note that  $\bar{M}$  is the movable part of the intersection  $l := D \cap D' \cap D''$ . The image  $C_i = g(F_i)$  is a rational curve intersecting  $g(\Gamma)$  (passing through  $g(\Gamma)$  if  $g(\Gamma)$  is a point) with  $(LC_i)_X = (HF_i)_S = 1$ , whence both  $C_i$  and  $D''$  are smooth at  $C_i \cap g(\Gamma)$ . Since  $g|_{S \setminus N}$  is an isomorphism,  $C_i \setminus g(\Gamma)$  is smooth. Hence  $C_i$  is a smooth rational curve.

Assume that  $g(\Gamma)$  is a point. Then in the same manner as in the proofs of Claims 5.3.2 and 5.3.3 we see that  $(\Gamma^2)_S = -2$  and  $p := g(\Gamma) \notin g(E)$ . Hence  $p$  is isolated in  $B$ . Therefore (2.1.3) is possible and  $d = 2$ . Moreover  $p$  is the point where  $C_1$  and  $C_2$  intersect. By a suitable coordinate system at  $p$ , we have

$$I_{l,p} = (x, y, zw), \quad I_{C,p} = (x, y, z), \quad I_{B,p} = (x, y, z, w).$$

Therefore if we choose general  $\tau := D \cap D'$ , then we may assume  $I_{\tau,p} = (x + \alpha zw, y + \beta zw)$  for some  $\alpha, \beta \in O_{X,p}$ , whence by rechoosing  $x, y$  modulo  $zw$ , we may assume  $I_{\tau,p} = (x, y)$ . Therefore  $\tau$  is smooth at  $p$ ,

whence  $\tau \simeq Z$  at  $p$  and  $Z$  is smooth at  $p$ . This contradicts that  $\Gamma$  is contracted into a singular point of  $Y$  by  $f$ . Thus  $g(\Gamma)$  is a curve.

Then since  $(LC_i)=1$ , a general  $D''' \in |L|$  is smooth along a sufficiently small Zariski open subset  $V$  of  $g(\Gamma)$ , and  $C_i$  intersects general  $D''' \in |L|$  transversally at a point of  $V$ . We also see that  $Z$  is smooth along  $V$  because  $\Gamma \not\subset E$ . Moreover  $\tau \simeq Z$  along  $V$ , whence  $\tau \simeq Z$  over a smooth Zariski open subset  $U$  of  $Z$  containing both  $V$  and  $\bar{M}$ . Then we have a natural exact sequence

$$0 \rightarrow O_\tau \rightarrow O_\tau(\bar{M}) \rightarrow \bigoplus_{i=1}^d O_{C_i}(\bar{M}) (\simeq \bigoplus_{i=1}^d O_{C_i}) \rightarrow 0.$$

Since  $h^1(\tau, O_\tau)=0$  in (5.2.4), we infer from the above sequence that  $h^0(\tau, O_\tau(\bar{M}))=d+1$ . Hence we have

$$m-1=h^0(\tau, L_\tau) \geq h^0(\tau, \bar{M})=d+1 \geq m-2+1.$$

Hence  $d=m-2 \geq 2$ . If  $W$  is not a smooth hyperquadric  $\mathcal{Q}^3$ ,  $W$  is a cone over a smooth variety of minimal degree  $d$  with a reducible or nonreduced hyperplane section [5, Theorem 1]. This contradicts  $\text{Pic } X \simeq \mathbb{Z}L$ . Hence  $W \simeq \mathcal{Q}^3$ . q.e.d.

**Lemma 5.5.** *Any line on  $\mathcal{Q}^3$  is algebraically equivalent to each other.*

*Proof.* I learned this proof from I. Shimada. Let  $p$  be a point of  $\mathcal{Q}^3$ . Then those lines on  $\mathcal{Q}^3$  passing through  $p$  are parametrized by a smooth conic in  $\mathbf{P}(T_{\mathcal{Q}^3, p}^*)$ . Therefore the Hilbert scheme of lines on  $\mathcal{Q}^3$  is dominated by a smooth conic bundle over  $\mathcal{Q}^3$  in  $\mathbf{P}(T_{\mathcal{Q}^3}^*)$ . Hence it is irreducible. See also [4]. q.e.d.

**Lemma 5.6.**  $\dim W=4$ .

*Proof.* STEP 1. Assume  $\dim W=3$ . Then  $W \simeq \mathcal{Q}^3$  and  $h^0(X, L)=h^0(\tau, L_\tau)+2=5$  by Lemma 5.4. Let  $H$  be a general member of  $|O_W(1)|$ ,  $\hat{D}:=\hat{h}^*H$  and  $D:=\varphi_*(\hat{D}) \in |L|$ . Then  $\text{Sing } D \subset B:=\text{Bs } |L|$  and  $\text{Sing } \hat{D} \subset \hat{B}:=\varphi^{-1}(B)$  by Bertini's theorem.

Let  $Q, Q'$  and  $Q''$  be general members of  $|O_W(1)|$ . Then the complete intersection  $Q \cap Q' \cap Q''$  consists of two distinct points  $p_1$  and  $p_2$ . Since  $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ , we have two lines  $l_i$  on  $Q$  with  $(l_1 l_2)_Q=1$ ,  $(l_i)_Q^2=0$ . There exist (general) members  $f_i \in |O_Q(l_i)|$  such that  $p_i \in f_i$  and  $p_i \notin f_j$  ( $i \neq j$ ). By choosing new  $Q'$  and  $Q''$  if necessary, we may assume that  $Q \cap Q' \cap Q'' = \{p_1, p_2\}$ ,  $Q \cap Q'$  is an irreducible rational curve ( $\in |O_Q(l_1 + l_2)|$ ) on  $Q$ , while

$Q \cap Q'' = f_1 + f_2$ . Let  $D := \varphi(\hat{h}^*(Q))$ ,  $D' := \varphi(\hat{h}^*(Q'))$  and  $D'' := \varphi(\hat{h}^*(Q''))$ . Let  $\tau := D \cap D'$  and  $\sigma := D \cap D''$ .

STEP 2. By Bertini's theorem we have a unique irreducible component  $Z$  of  $\tau$  outside  $B$ . Let  $g: S \rightarrow Z$  be the minimal resolution of the normalization of  $Z$ . By the proof of Lemma 5.4  $S$  is a ruled surface. Under the same notation as in the proof of Lemma 5.4 there is an irreducible component  $\Gamma$  of the fixed part  $N$  of  $g^*[L]$  with  $(\Gamma F)_S = 1$  where  $F$  is a general fiber of the ruling of  $S$ . By the proof of Lemma 5.4,  $g(\Gamma)$  is a curve, along which  $\tau \simeq Z$  and  $Z$  is smooth generically. Moreover the movable part  $M$  of  $g^*[L]$  consists of a pair of smooth rational curves  $F_1$  and  $F_2$ . The complete intersection  $l := D \cap D' \cap D''$  is therefore a union of smooth rational curves  $C_i := g(F_i)$  ( $i=1, 2$ ) outside  $B$ , where  $C_i$  intersect  $g(\Gamma)$  transversally at distinct points as was given in (2.1.2). Moreover the proof of Lemma 5.4 shows that the linear system  $|M|$  on  $\tau$ , hence  $|L_\tau|$  separates  $C_1$  and  $C_2$ , that is,  $h(C_1 \setminus B) \neq h(C_2 \setminus B)$ . It follows that  $\hat{h}^{-1}(w) \simeq \mathbf{P}^1$  for any general  $w \in W$ .

STEP 3. Let  $\hat{Z}_i := \hat{h}^{-1}(f_i)$ ,  $Z_i := \varphi(\hat{Z}_i)_{\text{red}}$ . Then since  $\hat{h}^{-1}(w)$  is irreducible,  $Z_i$  is an irreducible component of  $\sigma$  outside  $B$ . Since  $\text{Bs}|O_Q(l_i)| = \emptyset$ ,  $\hat{Z}_i$  is smooth outside  $\text{Sing } \hat{D}$ , whence smooth outside  $\hat{B}$ . As  $X \setminus B \simeq \hat{X} \setminus \hat{B}$ , we have  $Z_i \setminus B \simeq \hat{Z}_i \setminus \hat{B}$ , whence  $Z_i$  is smooth outside  $B$ . Moreover  $Z_1 \cap Z_2 \setminus B \simeq \hat{Z}_1 \cap \hat{Z}_2 \setminus \hat{B} \simeq \hat{h}^{-1}(f_1 \cap f_2) \setminus \hat{B}$ . Hence  $Z_1 \cap Z_2$  is a smooth rational curve  $C := \varphi(\hat{h}^{-1}(f_1 \cap f_2))$  outside  $B$ , along which  $Z_1$  and  $Z_2$  intersect transversally.

Let  $v_i: Y_i \rightarrow Z_i$  be the normalization of  $Z_i$ ,  $f_i: S_i \rightarrow Y_i$  the minimal resolution of  $Y_i$ ,  $g_i = v_i \cdot f_i$ . Then as in the proof of Lemma 3.2, there exist effective Cartier divisors  $E_i$  and  $G_i$  on  $S_i$  with no components in common such that the canonical sheaf of  $S_i$  is given by  $K_{S_i} = O_{S_i}(-2g_i^*L - E_i - G_i)$ . Let  $A_i$  be a unique smooth rational curve on  $S_i$  such that  $g_i(A_i) = C$ . Then since  $Z_1 \cap Z_2 \simeq C$  generically along  $C$ , we have by Theorem 1.14

$$A_i \subset E_i, \quad A_i \not\subset E'_i := E_i - A_i, \quad E'_{i,\text{red}} + G_{i,\text{red}} \subset g_i^{-1}(B)_{\text{red}}$$

if  $f_1$  and  $f_2$  are sufficiently general. See also the proof of Lemma 5.3.

Let  $M_i$  (resp.  $N_i$ ) be the movable part (resp. the fixed part) of  $g_i^*D'$ . Then since  $\varphi^*L \simeq \hat{h}^*O_W(1)$  on  $\hat{X} \setminus \hat{B}$ ,  $L_{Z_i} \simeq ((\varphi|_{\hat{Z}_i \setminus \hat{B}})^{-1})^* \hat{h}^*O_Q(f_1 + f_2) \simeq O_{Z_i}(A_i)$  on  $Z_i \setminus B$ . Hence we have  $A_i \in |M_i|$ . It follows that  $-K_{S_i} = 3A_i + 2N_i + E'_i + G_i$ . Hence  $S_i$  is either  $\mathbf{P}^2$  or ruled.

CASE 1. Assume  $S_1 \simeq \mathbf{P}^2$ . Since  $Z_i$  is algebraically equivalent to

each other by Lemma 5.5, we may assume that  $S_2 \simeq \mathbf{P}^2$ . Then we have  $A_i \in |O_{S_i}(1)|$  and  $N_i = E'_i = G_i = 0$ . Hence  $S_i \simeq Y_i$ . By the argument in the proof of Lemma 5.3 CASES 1-1 and 1-2, we see  $D \cap D' = Z_1 \cup Z_2$ ,  $h^0(X, L) = 6$ , which contradicts  $h^0(X, L) = 5$ .

CASE 2. Assume that  $S_1$  is ruled. By Lemma 5.5, we may assume that  $S_2$  is also ruled. Let  $\pi_i: S_i \rightarrow \mathbf{P}^1$  be a ruling of  $S_i$  with  $\tilde{F}_i$  a general fiber. Then we have

$$2 = -K_{S_i} \tilde{F}_i = (3A_i + 2N_i + E'_i + G_i) \tilde{F}_i,$$

whence  $A_i \tilde{F}_i = 0$ . Hence  $A_i \in |\tilde{F}_i|$ . There exists a unique irreducible component  $\Gamma_i$  of  $N_i$  such that  $\Gamma_i \tilde{F}_i = 1$  because  $E'_{i,\text{red}} + G_{i,\text{red}} \subset N_i$ . Hence we have  $E'_i \tilde{F}_i = G_i \tilde{F}_i = 0$ .

Let  $C'_i := g_i(M_i)$ . Then since  $l \simeq \sigma \cap D' \simeq \tau \cap D''$ ,  $l$  has  $C'_1$  and  $C'_2$  as irreducible components outside  $B$ . Hence we may assume by STEP 2 that  $C_i = C'_i$  ( $i = 1, 2$ ).

STEP 4. Next we prove that  $g_i(\Gamma_i)$  is a curve ( $i = 1, 2$ ). Assume the contrary. Hence  $g_1(\Gamma_1)$  is a point, say  $p \in B$ . By STEP 2  $(LC)_X = 1$  where  $C = \varphi(\tilde{h}^{-1}(f_1 \cap f_2))$ . Hence  $C$  passes through a unique point of  $B$ , hence through the point  $p$  by  $(\Gamma_1 A_1)_{S_1} = (\Gamma_1 \tilde{F}_1)_{S_1} = 1$ . As  $Z_1$  and  $Z_2$  are algebraically equivalent,  $g_2(\Gamma_2)$  is also a point of  $B$ , which  $C$  passes through too. Hence  $g_2(\Gamma_2) = p$ . It follows that  $C_1$  and  $C_2$  intersect at  $p$ , which contradicts that  $C_1$  and  $C_2$  intersect  $g(\Gamma)$  transversally at distinct points. Hence  $g_i(\Gamma_i)$  is a curve component of  $B$  by  $\Gamma_i \subset N_i \subset g_i^{-1}(B)$ . Since  $Z_1$  and  $Z_2$  are algebraically equivalent,  $g_2(\Gamma_2) \subset Z_2$  implies  $g_2(\Gamma_2) \subset Z_1$ , whence  $g_2(\Gamma_2) \subset Z_1 \cap Z_2$ . By (2.1.2) and by STEP 2,  $C_i$  intersects a unique curve component  $g(\Gamma)$  of  $B$ , while  $C_i$  intersects  $g_i(\Gamma_i)$  by  $(M_i \Gamma_i)_{S_i} = 1$ . Hence  $g(\Gamma) = g_1(\Gamma_1) = g_2(\Gamma_2)$ . However since  $g_i(\Gamma_i) \subset Z_1 \cap Z_2$ , we have  $\Gamma_1 \subset E'_1$  and  $\Gamma_2 \subset E'_2$  by Theorem 1.14, which contradicts  $\Gamma_i \tilde{F}_i = 1$ ,  $E'_i \tilde{F}_i = 0$ . q.e.d.

(5.7) COMPLETION OF THE PROOF OF THEOREM 5.1. Since  $\dim W = 4$ ,  $D_1 \cap D_2 \cap D_3$  is irreducible nonsingular outside  $B$  for general  $D_i$  by Bertini's theorem. Let  $C$  be the unique irreducible component of  $D_1 \cap D_2 \cap D_3$  outside  $B$ . If  $LC = 0$ , then  $C$  is mapped to a point by the rational map associated with  $|pL|$  for any  $p$ . Since  $C$  sweeps out an open subset of  $X$ , this contradicts that  $X$  is Moishezon with  $b_2(X) = 1$ . Hence  $LC \geq 1$ . In view of Lemma 2.1,  $LC = 1$  or  $2$ . If  $LC = 1$ , then (2.1.2) is possible for  $D_i$  general. In this case,  $C \cap B \neq \emptyset$ , whence  $C$  is mapped to a point by the rational map  $h$ . This shows that  $\dim W \leq 3$ , which



contradicts Lemma 5.6. Thus (2.1.1) only is possible, so that  $LC=2$ . Theorem 5.1 now follows from Proposition 3.1. q.e.d.

(5.8) REMARK. It is plausible that Theorem 5.1 is true by assuming only  $h^0(X, L) \geq 3$  instead of  $h^0(X, L) \geq 5$ . We were however unable to prove even that  $h^0(X, L) = 4$  is impossible. Here we make some comments.

Assume that  $X$  is a Moishezon 4-fold homeomorphic to  $\mathcal{Q}^4$  with  $h^0(X, L) = 4$ . Then the proof in Lemmas 5.2–5.7 fails only at two points in the proofs of Lemmas 5.3 and 5.4. The first point is corrected by a slight modification of the previous proof, while it is difficult to do so for the second.

In what follows we keep the previous notation. First in the proof of Lemma 5.3 CASE 1–1,  $m=3$ ,  $d=m-1=2$  and  $W$  is a (possibly singular) quadric surface in  $\mathbf{P}^3$ . Then  $W$  has a reducible hyperplane section, which contradicts  $\text{Pic } X \simeq \mathbf{Z}L$ .

The second point is in the proof of Lemma 5.4, where  $W \simeq \mathbf{P}^3$ . In view of the proof of Lemma 5.4,  $S$  is ruled,  $(M+N)F=1$  and  $FE=FG=0$ . Moreover we see that there are two cases.

CASE 1.  $M=\Gamma$ ,  $\Gamma F=1$ .

CASE 2.  $M=F$ ,  $\Gamma F=1$ ,  $\Gamma \subset N$ .

**Claim 5.8.1.** CASE 1 is impossible.

NOTE. We do not know whether CASE 2 is impossible.

Proof. By the proof of Lemma 5.4, we have  $K_S = -(2g^*(L) + E + G) = -(2\Gamma + 2N + E + G)$ . By Lemma 2.1, the movable component  $g(\Gamma)$  of the scheme-theoretic complete intersection  $l$  of 3 general members of  $|L|$  is a smooth rational curve. Therefore  $\Gamma \simeq g(\Gamma) \simeq \mathbf{P}^1$  and  $S$  is a rational ruled surface. We also see that any general member of  $|L|$  as well as a complete intersection  $\tau$  of two general members of  $|L|$  is smooth along  $g(\Gamma)$  by Lemma 2.1. We have

$$2 = -(K_S + \Gamma)\Gamma = (\Gamma + 2N + E + G)\Gamma.$$

Since  $\Gamma$  is movable, we have  $\Gamma^2 \geq 0$ , whence  $0 \leq \Gamma^2 \leq 2$ .

Now we prove  $\Gamma^2 = 0$ . If  $\Gamma^2 = 2$ , then  $N\Gamma = E\Gamma = G\Gamma = 0$ . Therefore  $B \cap g(\Gamma) = \emptyset$ . On the other hand,  $h(g(\Gamma))$  is one point. However we have

$$2 = ((\Gamma + N)\Gamma)_S = (g^*(L)\Gamma)_S = (Lg_*(\Gamma))_X = \deg \text{Bs } g^*|L|_\Gamma = 0,$$

which is a contradiction. Next we assume  $\Gamma^2=1$ . Then  $N\Gamma=0$ ,  $(E+G)\Gamma=1$ , which contradicts  $E_{\text{red}}+G_{\text{red}}\leq N_{\text{red}}$ . Therefore  $\Gamma^2=0$ ,  $N\Gamma=1$  and  $E\Gamma=G\Gamma=0$ .

Since  $\Gamma^2=0$ ,  $F^2=0$  and  $\Gamma F=1$ , we have a birational morphism  $\eta: S \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$  ( $\subset \mathbf{P}^3$ ) associated with the linear system  $|\Gamma+F|$ . The surface  $S$  is obtained from  $\mathbf{P}^1 \times \mathbf{P}^1$  by repeating blowing-ups. Since  $K_S = -(2\Gamma+2N+E+G)$  and since  $E_{\text{red}}+G_{\text{red}}\leq N_{\text{red}}$ , any irreducible component of  $2N+E+G$  has multiplicity at least two in  $-K_S$ . Therefore the anticanonical divisor  $-K_{\mathbf{P}^1 \times \mathbf{P}^1}$  is an effective divisor with multiple components only. That is,  $-K_{\mathbf{P}^1 \times \mathbf{P}^1} = -2(\eta_*(\Gamma) + \eta_*(F))$  for some fiber  $F$  with  $2\eta_*(F) = \eta_*(2N+E+G)$ . Since  $\Gamma^2=0$ , the centers of blowing-ups are chosen from the outside of  $\eta(\Gamma)$  (or its proper transform). Hence the proper transform of the  $(-1)$ -curve arising from (any of) the first blowing-up appears in  $2N+E+G$  as a component with multiplicity exactly one. This is a contradiction if  $S$  is not isomorphic to  $\mathbf{P}^1 \times \mathbf{P}^1$ . It follows that  $S \simeq \mathbf{P}^1 \times \mathbf{P}^1$ ,  $N=F$  and  $E=G=0$ , whence  $S \simeq Y \simeq \mathbf{P}^1 \times \mathbf{P}^1$ . Since  $E=0$ ,  $S \simeq Z$  outside a finite set  $\Sigma$  of  $Z$ .

Moreover  $Z$  intersects the other irreducible components of  $\tau$  in a subset of  $\Sigma$  only. Since  $\tau$  is Gorenstein and connected by (5.2.5), this implies that  $\tau_{\text{red}} \simeq Z$ . As  $\tau$  is Gorenstein and generically reduced along  $Z$ ,  $\tau$  is reduced everywhere, so that  $\tau \simeq Z$ . Therefore  $Z$  is Gorenstein and has isolated singularities only, whence  $Z$  is normal. Consequently  $\tau \simeq Z \simeq S \simeq \mathbf{P}^1 \times \mathbf{P}^1$ . Hence  $h^0(X, L) = h^0(\tau, L_\tau) + 2 = 6$  by (5.2.6), contradicting the assumption  $h^0(X, L) = 4$ . q.e.d.

## NOTATION.

$A^n$	$\text{Spec } C[x_1, \dots, x_n]$
$B, B_S   L $	the scheme-theoretic base locus of $ L $
$B^*, \hat{B}, \hat{B}^*$	(3.5), (4.1)
$c(E)$	the total Chern class $\sum_{i \in \mathbf{Z}} c_i(E)$ of a vector bundle $E$
$c_i(E)$	the $i$ -th Chern class of a vector bundle $E$
$c_i(X)$	the $i$ -th Chern class of $X$
$e(Q_C^{(v)}, p)$	(1.10), (1.11), [15, (2.6)]
$e(Q_V^{(v)}, B_V)$	(1.12), (1.13), [15, (2.A)]
$F_b$	$\text{Proj}(O_{P^1}(b) \oplus O_{P^1})$
$\bar{F}_2$	$F_2$ with the unique $(-2)$ -curve contracted, isomorphic to a singular quadric surface in $P^3$
$g^*[L $	$\{g^*D ; D \in  L \}$
$h^q(X, F)$	$\dim H^q(X, F)$ for a coherent sheaf $F$
$\hat{l}(l'), l(l')$	(4.1)
$l(Q_C^{(v)}, p)$	(1.2), (1.3)
$l_q, \hat{l}_q, l'_q$	(4.3.2)
$N_{C/X}$	the normal bundle of $C$ in $X$
$O_X, O_S, O_Z$	the structure sheaf of $X, S, Z$ respectively
$\hat{O}_X$	the formal completion of $O_X$
$\text{Pic } X$	$H^1(X, O_X^*)$
$P(l')$	(4.3.5)
$R^*$	(3.5), (4.1)
$S \xrightarrow{v} Y \xrightarrow{f} Z$	(3.2)
$\sigma_q$	(4.3.2)
$W, X \xrightarrow{\hat{h}} W$	(3.2), (4.1), (5.3)
$\hat{X} \xrightarrow{\hat{h}} W$	(3.5), (4.1)
$\hat{X} \xrightarrow{\phi} X$	(3.5), (4.1)
$\chi(X, F)$	$\sum_{q \in \mathbf{Z}} (-1)^q h^q(X, F)$
$( )_S, ( )_X$	the intersection numbers on $S, X$

---

## Bibliography

- [1] M. Artin: *On the solutions of analytic equations*, Invent. Math. **5** (1968), 277–291.
- [2] A. Altman-S. Kleiman: Introduction to Grothendieck duality theory, Lecture Notes in Math. **146**, Springer Verlag (1970).
- [3] E. Brieskorn: *Ein Satz über die komplexen Quadriken*, Math. Ann. **155** (1964), 184–193.
- [4] W. Barth-A. Van de Ven: *Fano varieties of lines on hypersurfaces*, Arch. Math. **31** (1978), 94–104.
- [5] D. Eisenbud-J. Harris: *On varieties of minimal degree*, Algebraic Geometry in Bowdoin 1985, Proc. Symp. Pure Math. **46** (1987), 3–14.
- [6] H. Hironaka: *An example of non-Kählerian complex-analytic deformation of Kählerian complex structures*, Ann. of Math. **75** (1962), 190–208.
- [7] J. Kollar: *Flips, flops, minimal models etc.*, Preprint (1990).
- [8] S. Kobayashi-T. Ochiai: *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. **13** (1973), 31–47.
- [9] S. Mori: *Projective manifolds with ample tangent bundles*, Ann. of Math. **110** (1979), 593–606.
- [10] S. Mori: *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. **116** (1982), 133–176.
- [11] J. Morrow: *A survey of some results on complex Kähler manifolds*, Global Analysis, Univ. Tokyo Press and Princeton University Press (1969), 315–324.
- [12] I. Nakamura: *Moishezon threefolds homeomorphic to  $\mathbb{P}^3$* , J. Math. Soc. Japan **39** (1987), 521–535.
- [13] I. Nakamura: *Threefolds homeomorphic to a hyperquadric in  $\mathbb{P}^4$* , Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Kinokuniya, Tokyo Japan (1987), 379–404.
- [14] I. Nakamura: *A subadjunction formula and Moishezon fourfolds homeomorphic to  $\mathcal{P}_c^4$* , Proc. Japan Acad. **67A** (1991), 65–67.
- [15] I. Nakamura: *On Moishezon manifolds homeomorphic to  $\mathbb{P}_c^n$* , Jour. Math. Soc. Japan **44** (1992), 667–692.
- [16] T. Peternell: *A rigidity theorem for  $\mathbb{P}_3(\mathbb{C})$* , Manuscripta Math. **50** (1985), 397–428.
- [17] T. Peternell: *Algebraic structures on certain 3-folds*, Math. Ann. **274** (1986), 133–156.
- [18] K. Ueno: *Classification theory of algebraic varieties and compact complex spaces*, Lectures Notes in Math. **439**, Springer-Verlag, Berlin, Heidelberg, New York (1975).

Department of Mathematics  
Hokkaido University  
Sapporo, 060, Japan

