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# ÉTALE ENDOMORPHISMS OF 3-FOLDS. I

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## Abstract

This paper is the first part of our project towards classifications of smooth projective 3-folds  $X$  with  $\kappa(X) = -\infty$  admitting a non-isomorphic étale endomorphism. We can prove that for any extremal ray  $R$  of divisorial type, the contraction morphism  $\pi_R: X \rightarrow X'$  associated to  $R$  is the blowing-up of a smooth 3-fold  $X'$  along an elliptic curve. The difficulty is that there may exist infinitely many extremal rays on  $X$ . Thus we introduce the notion of an ‘ESP’ which is an infinite sequence of non-isomorphic finite étale coverings of 3-folds with constant Picard number. We can run the minimal model program (‘MMP’) with respect to an ESP and obtain the ‘FESP’  $Y_\bullet$  of  $(X, f)$  which is a distinguished ESP with *extremal rays of fiber type* (cf. Definition 3.6). We first classify  $Y_\bullet$  and then blow-up  $Y_\bullet$  along elliptic curves to recover the original  $X$ . The finiteness of extremal rays of  $\overline{\text{NE}}(X)$  is verified in certain cases (cf. Theorem 1.4). We encounter a new phenomenon showing that our étaleness assumption is related with torsion line bundles on an elliptic curve (cf. Theorem 1.5).

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## 1. Introduction

**1.1. Main results.** We work over the field  $\mathbb{C}$  of complex numbers. An endomorphism of a projective variety  $X$  means a morphism (holomorphic map) from  $X$  to itself. The study of surjective endomorphisms of a given variety  $X$ , such as the complex projective space  $\mathbb{P}^n$  or a K3 surface, is a chief concern of study in complex dynamical systems. On the other hand, it is also related to the classification of projective manifolds  $X$  admitting a non-

isomorphic surjective endomorphism, which is our deep concern. This is the first part of a series of articles which study structures of smooth projective 3-folds  $X$  with negative Kodaira dimension *admitting a non-isomorphic étale endomorphism*. The main purpose of this paper is to develop a strategy for classifications of such 3-folds  $X$ ; i.e., the application of the MMP to endomorphisms (cf. Proposition 1.1, Corollary 1.2), finiteness of extremal rays (cf. Theorem 1.4), and clarifications of new phenomena in the case of  $\kappa(X) = -\infty$  (cf. Theorem 1.5, Remark 1.3).

Our slogan towards classifications is to apply the *minimal model program* (‘MMP’ for short) to the ‘ESP’ and construct its ‘FESP’ which is the counterpart of the minimal model in the case of  $\kappa(X) \geq 0$ . Compared with the case of  $\kappa(X) \geq 0$ , an extremal ray  $R$  of  $\overline{\text{NE}}(X)$  may not necessarily be preserved by a suitable power  $f^k$  ( $k > 0$ ), which is one of the serious troubles. We first explain the notions of an ESP and an FESP (cf. Definitions 2.3, 2.4, and 3.6). Let us consider the following infinite sequence;  $Z_\bullet = (v_n : Z_n \rightarrow Z_{n+1})_{n \in \mathbb{Z}}$  such that

- any  $Z_n$  is a smooth projective variety,
- any  $v_n$  is a non-isomorphic finite étale covering, and
- the Picard number  $\rho(Z_n)$  is constant.

Then we say that the above sequence  $Z_\bullet$  is an *étale sequence of constant Picard number* (ESP, for short). A variety  $Z_n$  appearing in an ESP  $Z_\bullet$  has strong conditions. For example:

- $\chi(\mathcal{O}_{Z_n}) = \chi_{\text{top}}(Z_n) = K_{Z_n}^{\dim Z_n} = 0$  for any  $n$  (cf. Lemma 2.1).
- If  $Z_n \simeq \mathbb{P}_{C_n}(\mathcal{E}_n)$  for a vector bundle  $\mathcal{E}_n$  on an elliptic curve  $C_n$  for any  $n$ , then  $\mathcal{E}_n$  is *semi-stable* (cf. Proposition 4.1).

Furthermore, we say that an ESP  $Y_\bullet = (g_n : Y_n \rightarrow Y_{n+1})_n$  is an *ESP of fiber type* (FESP, for short) if any  $Y_n$  ( $n \in \mathbb{Z}$ ) is a smooth projective 3-fold with an extremal ray of *fiber type*. Even if we consider a non-isomorphic étale endomorphism, it is more natural to consider an ESP, on which the MMP works well. Thus we introduce a new notion. We say that an ESP (resp. an FESP)  $Y_\bullet = (g_n : Y_n \rightarrow Y_{n+1})_n$  is a ‘*constant ESP*’ (resp. ‘*constant FESP*’) if there exists a non-isomorphic étale endomorphism  $g : Y \rightarrow Y$  such that  $Y_n = Y$  and  $g_n = g$  for any  $n$  (cf. Definitions 2.4, 3.6). We denote it by  $Y_\bullet = (Y, g)$ .

Now we state one of our main results.

**Proposition 1.1.** *Let  $Y_\bullet = (f_n : Y_n \rightarrow Y_{n+1})_n$  be an ESP of smooth projective 3-folds  $Y_n$ . Suppose that the canonical bundle  $K_{Y_0}$  of  $Y_0$  is not nef. Let  $R_0$  be an extremal ray of  $\overline{\text{NE}}(Y_0)$  such that the contraction morphism  $\pi_0 := \text{Cont}_{R_0} : Y_0 \rightarrow Z_0$  associated to  $R_0$  is a birational morphism. If we put  $R_i := (f_{i-1} \circ \cdots \circ f_0)_*(R_0)$  and  $R_{-i} := (f_{-1} \circ \cdots \circ f_{-i})^*R_0$  for each  $i > 0$ , then the following assertions hold for any integer  $n$ :*

- (1)  $R_n$  is an extremal ray of  $\overline{\text{NE}}(Y_n)$  and the contraction morphism  $\pi_n := \text{Cont}_{R_n} : Y_n \rightarrow Z_n$  associated to  $R_n$  is a divisorial contraction, and is (the inverse of) the blowing-up of a smooth projective 3-fold  $Z_n$  along an elliptic curve  $C_n$  on  $Z_n$ .
- (2) There exists an ESP  $Z_\bullet = (g_n : Z_n \rightarrow Z_{n+1})_n$  of smooth projective 3-folds  $Z_n$  and a Cartesian morphism  $\pi_\bullet := (\pi_n)_n : Y_\bullet \rightarrow Z_\bullet$ , i.e.,  $\pi_{n+1} \circ f_n = g_n \circ \pi_n$ .
- (3)  $g_n^{-1}(C_{n+1}) = C_n$  and  $C_\bullet = (g_n|_{C_n} : C_n \rightarrow C_{n+1})_n$  is an ESP of elliptic curves.
- (4) The normal bundle  $\mathcal{N}_n$  of  $C_n$  in  $Z_n$  is a semi-stable vector bundle of rank 2 and degree 0. Moreover,  $\mathcal{N}_n \simeq (g_n|_{C_n})^*\mathcal{N}_{n+1}$ .

$$\begin{array}{ccccccc}
X_{\bullet} : \cdots & \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \longrightarrow & \cdots \\
& & \pi_n^{(0)} \downarrow & & \downarrow \pi_{n+1}^{(0)} & & \\
& & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \\
X_{\bullet}^{(i)} : \cdots & \longrightarrow & X_n^{(i)} & \xrightarrow{f_n^{(i)}} & X_{n+1}^{(i)} & \longrightarrow & \cdots \\
& & \pi_n^{(i)} \downarrow & & \downarrow \pi_{n+1}^{(i)} & & \\
X_{\bullet}^{(i+1)} : \cdots & \longrightarrow & X_n^{(i+1)} & \xrightarrow{f_n^{(i+1)}} & X_{n+1}^{(i+1)} & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \\
X_{\bullet}^{(k)} : \cdots & \longrightarrow & X_n^{(k)} & \xrightarrow{f_n^{(k)}} & X_{n+1}^{(k)} & \longrightarrow & \cdots
\end{array}$$

Fig. 1. FESP

**Corollary 1.2** (Construction of an FESP). *Let  $X_{\bullet} = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Then there exists a Cartesian diagram as in Figure 1 which satisfies the following conditions:*

- (1) *For all  $0 \leq i \leq k$ , the  $i$ -th row  $X_{\bullet}^{(i)} = (f_n^{(i)}: X_n^{(i)} \rightarrow X_{n+1}^{(i)})_n$  is an ESP.*
- (2) *For all  $n$  and  $1 \leq i < k$ ,  $\pi_n^{(i-1)}: X_n^{(i-1)} \rightarrow X_n^{(i)}$  is (the inverse of) the blowing-up along an elliptic curve  $C_n^{(i)}$  on  $X_n^{(i)}$ .*
- (3)  *$(f_n^{(i)})^{-1}(C_{n+1}^{(i)}) = C_n^{(i)}$  for all  $n$  and  $1 \leq i \leq k$ . That is,  $C_{\bullet}^{(i)} := (f_n^{(i)}: C_n^{(i)} \rightarrow C_{n+1}^{(i)})_n$  is an ESP of elliptic curves and the inclusion  $C_{\bullet}^{(i)} \hookrightarrow X_{\bullet}^{(i)}$  is a Cartesian morphism.*
- (4) *The normal bundle  $\mathcal{N}_n^{(i)}$  of  $C_n^{(i)}$  in  $X_n^{(i)}$  is a semi-stable vector bundle of rank 2 and degree 0. Moreover,  $(f_n^{(i)}|_{C_n^{(i)}})^*(\mathcal{N}_{n+1}^{(i)}) \simeq \mathcal{N}_n^{(i)}$ .*
- (5) *On the bottom  $k$ -th row  $X_{\bullet}^{(k)} = (f_n^{(k)}: X_n^{(k)} \rightarrow X_{n+1}^{(k)})_n$ , all the extremal rays of  $\overline{\text{NE}}(X_n^{(k)})$  are of fiber type. In particular,  $X_{\bullet}^{(k)}$  is an FESP (cf. Definition 3.6).*

In Corollary 1.2, we call the vertical sequence

$$X_n \xrightarrow{\pi_n^{(0)}} \cdots \longrightarrow X_n^{(i)} \xrightarrow{\pi_n^{(i)}} X_n^{(i+1)} \longrightarrow \cdots \longrightarrow X_n^{(k)}$$

a sequence of blowing-downs of an ESP. Furthermore,  $X_{\bullet}^{(k)}$  is called an FESP constructed from  $X_{\bullet}$  by a sequence of blowing-downs of an ESP (cf. Definition 3.7). In this paper, by an extremal ray  $R$  of a smooth projective variety  $X$ , we always mean a  $K_X$ -negative extremal ray of  $\overline{\text{NE}}(X)$ . Furthermore,  $\mathcal{F}_r$  denotes a unique indecomposable vector bundle of rank  $r$  and degree 0 on an elliptic curve  $E$  with  $\Gamma(E, \mathcal{F}_r) \neq 0$  (cf. Theorem 4.11). We are particularly interested in the case of  $r = 2$ . We call the  $\mathbb{P}^1$ -bundle  $\mathbb{P}_E(\mathcal{F}_2)$  over  $E$  associated with  $\mathcal{F}_2$  the ‘Atiyah surface’ and denote it by  $\mathbb{S}$  (cf. Definition 5.1). It is worthwhile to note that we

can construct an example of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$  such that the following hold (cf. Remark 8.3);

- There exists an extremal ray  $R$  of  $\overline{\text{NE}}(X)$  such that the exceptional divisor of the contraction morphism  $\text{Cont}_R: X \rightarrow X'$  is isomorphic to  $\mathbb{S}$ .
- No finite étale covering  $\widetilde{X}$  of  $X$  is isomorphic to the product of a uniruled surface and an elliptic curve.

Thus in the case of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  with  $\kappa(X) = -\infty$ , the same conclusion as in the case of  $\kappa(X) \geq 0$  (cf. [9], [13]) *does not hold*. This is one of the new phenomena appearing for an étale endomorphism in the case of  $\kappa(X) = -\infty$ .

Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Then we can apply Corollary 1.2 to a constant ESP  $(X, f)$  and construct an FESP  $Y_\bullet$ . Let  $\mathcal{R}$  be the set of extremal rays of  $X$ . If  $\mathcal{R}$  is a finite set, then we can take  $Y_\bullet$  as a constant FESP  $(Y, g)$  induced from a non-isomorphic étale endomorphism  $g: Y \rightarrow Y$  (cf. Proposition 3.8). The difficulty is that  $\mathcal{R}$  may be an infinite set and it is not clear if one of them is  $f$ -invariant. Thus the MMP does not necessarily work compatibly with étale endomorphisms (cf. Remark 3.9). Thus we shall introduce the notion of an ‘FESP’ and study the structure of an endomorphism  $f: X \rightarrow X$  through its FESP  $Y_\bullet$ . The notions of an ESP and an FESP first appeared in [9], [10]. In particular, the author [10] studied the structure of an ESP  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = 2$ , in which a counterpart of an FESP was called a ‘*modified minimal reduction*’. M. Aprodu, S. Kebekus and T. Peternell [1] also tried to study a constant ESP  $(X, f)$  induced from a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a 3-fold  $X$  with  $\kappa(X) = -\infty$ . In our Corollary 1.2, the condition (4) is new (cf. Proposition 8.4 and Remark 8.3). Notably, if  $X_\bullet = (X, f)$  is a constant ESP induced from a non-isomorphic étale endomorphism  $f: X \rightarrow X$ , then we can say more. It is one of the characteristic features of étale endomorphisms.

**REMARK 1.3.** (1) In the assertion (4) of Corollary 1.2, let  $E_n^{(i)}$  be the  $\pi_n^{(i-1)}$ -exceptional divisor. Then each  $E_n^{(i)}$  is isomorphic to either the Atiyah surface  $\mathbb{S}$  or  $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}_n^{(i)})$ , where  $\mathcal{L}_n^{(i)}$  is a line bundle of degree zero on  $C_n^{(i)}$ .

(2) Furthermore, suppose that  $X_\bullet$  is a constant ESP  $(X, f)$  induced from a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Then, each  $\mathcal{L}_n^{(i)}$  is a *torsion* line bundle on  $C_n^{(i)}$ . In particular, let  $R$  be an arbitrary extremal ray of divisorial type on  $X$  and  $E_R$  the exceptional divisor of the contraction morphism  $\pi_R := \text{Cont}_R: X \rightarrow Z$  associated to  $R$ . Then  $E_R$  is isomorphic to either the Atiyah surface  $\mathbb{S}$ , or  $\mathbb{P}_C(\mathcal{O} \oplus \mathcal{L})$  for a *torsion* line bundle  $\mathcal{L}$  on an elliptic curve  $C$ .

It is remarkable that Remark 1.3 (2) holds even if the set  $\mathcal{R}$  of extremal rays of  $\overline{\text{NE}}(X)$  is an infinite set. It will be proved in our subsequent Part III article as an application of Theorem 1.5 below.

The following theorem describes the finiteness of  $\mathcal{R}$  in certain cases:

**Theorem 1.4.** *Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$  and  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  an FESP constructed from  $X_\bullet$  by a sequence of blowing-downs of an ESP. Let  $R_n$  be an extremal ray of fiber type on  $\overline{\text{NE}}(Y_n)$  such that  $(g_n)_*R_n = R_{n+1}$  for any  $n$  and set  $R_\bullet := (R_n)_n$ . Suppose that  $(Y_\bullet, R_\bullet)$  is of type  $(C_1)$  or  $(C_0)$ . Then for any  $n$ , there exist at most finitely many extremal rays of divisorial type on  $\overline{\text{NE}}(X_n)$ .*

Now, we shall explain briefly our terminology. Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Then there exists an FESP  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  constructed from  $X_\bullet$  by a sequence of blowing-downs of an ESP (cf. Corollary 1.2). Take an extremal ray  $R_0$  of fiber type on  $\overline{\text{NE}}(Y_0)$  arbitrarily. For each  $n > 0$ , we set  $R_n := (g_{n-1} \circ \cdots \circ g_0)_*(R_0)$  and  $R_{-n} := (g_{-1} \circ \cdots \circ g_{-n})^*(R_0)$ . Then, any  $R_n$  is an extremal ray of fiber type on  $Y_n$ . Furthermore, with the aid of classifications of extremal rays of 3-folds due to Mori [29], we see that any  $R_n$  is of the same type C or of the same type D in the sense of [29].

Note that an FESP  $Y_\bullet$  is *not uniquely determined* for a given étale endomorphism  $f: X \rightarrow X$  (cf. Remark 8.3). Let  $R_\bullet = (R_n)_n$  be the set of extremal rays of fiber type on  $\overline{\text{NE}}(Y_\bullet)$ . Let  $(Y_\bullet, R_\bullet)$  be the pair of  $Y_\bullet$  and  $R_\bullet$ . Then we say that the FESP  $(Y_\bullet, R_\bullet)$  is of type (C) if the extremal contraction  $\varphi_n := \text{Cont}_{R_n}: Y_n \rightarrow S_n$  associated to  $R_n$  is a conic bundle over a smooth algebraic surface  $S_n$  (cf. Definition 3.6) for any  $n$ . Then by Propositions 3.1, 6.4 and Lemma 7.1, we see that a suitable finite étale covering  $\widetilde{S}_n$  of  $S_n$  is isomorphic to one of the following:

- the direct product  $B \times E$  of a smooth curve  $B$  of  $g(B) \geq 2$  and an elliptic curve  $E$  (i.e., the FESP  $(Y_\bullet, R_\bullet)$  is called of type  $(C_1)$ ), or
- an abelian surface (i.e., the FESP  $(Y_\bullet, R_\bullet)$  is called of type  $(C_0)$ ), or
- a  $\mathbb{P}^1$ -bundle over an elliptic curve  $C$  associated to a semi-stable vector bundle of rank two on  $C$  (i.e., the FESP  $(Y_\bullet, R_\bullet)$  is called of type  $(C_{-\infty})$ ).

Next, we say that the FESP  $(Y_\bullet, R_\bullet)$  is of type (D) if for any  $n$ , the contraction morphism  $\varphi_n := \text{Cont}_{R_n}: Y_n \rightarrow C_n$  associated to  $R_n$  is a del Pezzo fiber space over a curve  $C_n$  (cf. Definition 3.6). Then in this case,  $\varphi_n$  is smooth and  $C_n$  is an elliptic curve for any  $n$  (cf. Proposition 7.3).

Now we shall state another main result concerning *torsion line bundles* on an elliptic curve.

**Theorem 1.5** (Torsion theorem. I). *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})$  be an FESP constructed from the constant ESP  $(X, f)$  by a sequence of blowing-downs of an ESP. Let  $R_n$  be an extremal ray of fiber type on  $\overline{\text{NE}}(Y_n)$  such that  $(g_n)_*R_n = R_{n+1}$  for any  $n$  and set  $R_\bullet = (R_n)_n$ . Suppose that the FESP  $(Y_\bullet, R_\bullet)$  is of type  $(C_{-\infty})$ , i.e., each  $Y_n$  is a conic bundle over an elliptic ruled surface  $S_n$ . Furthermore, suppose that any  $S_n$  is isomorphic to a  $\mathbb{P}^1$ -bundle  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$  for a line bundle  $\ell_n$  of degree 0 on the Albanese elliptic curve  $C$  of  $X$ . Then  $\ell_n \in \text{Pic}^0(C)$  is of finite order for any  $n$ .*

Theorem 1.5 shows that the base surface  $S_n$  of the conic bundle  $Y_n$  is of very limited type; A suitable finite étale covering  $\widetilde{S}_n$  of  $S_n$  is isomorphic to either the Atiyah surface  $\mathbb{S}$  over  $C$  or the direct product  $C \times \mathbb{P}^1$ , where  $C$  is the Albanese elliptic curve of  $X$  (cf. Corollary 10.6). Note that the above fact does not hold without the assumption that the non-isomorphic endomorphism  $f: X \rightarrow X$  is *étale* (cf. Remark 10.5). Furthermore, it does not hold if we replace the étale endomorphism  $f: X \rightarrow X$  by the general ESP  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  (cf. Remark 3.2). Though Theorem 1.5 seems to be very technical at first glance, it distinguishes the structure of an FESP obtained from a non-isomorphic étale endomorphism  $f: X \rightarrow X$  from an FESP obtained from a general ESP  $X_\bullet$ . Using Theorem 1.5, we can study the structure of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  and especially the set  $\mathcal{R}$  of

extremal rays of  $\overline{\text{NE}}(X)$  through its FESP  $Y_\bullet$  in great detail.

Now, we go back to the construction of an FESP. Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . If there exist at most finitely many extremal rays, then we can replace  $f$  by its suitable power  $f^k$  ( $k > 0$ ) so that  $f_*R = R$  for any extremal ray  $R$  of  $\overline{\text{NE}}(X)$ . Thus the MMP works compatibly with étale endomorphisms and by a succession of equivariant blowing-downs  $\pi_\bullet: X_\bullet = (X, f) \rightarrow Y_\bullet = (Y, g)$ , we obtain a constant FESP  $Y_\bullet$  induced from a non-isomorphic étale endomorphism  $g: Y \rightarrow Y$ . For example, this holds true in the case where there exists an FESP of type  $(C_1)$  or of type  $(C_0)$  constructed from  $f: X \rightarrow X$  by a sequence of blowing-downs of an ESP (cf. Theorem 1.4 and Corollary 8.1). Furthermore, even if there exist infinitely many extremal rays of  $\overline{\text{NE}}(X)$ , it is sufficient to find an extremal ray  $R$  such that  $(f^k)_*R = R$  for some  $k > 0$ . Running the MMP again by the same method as above, we obtain a constant FESP  $Y_\bullet = (Y, g)$ . Fortunately, *in most cases, these nice conditions concerning extremal rays are satisfied*; Actually applying Theorem 1.5 to the case where there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_{-\infty})$  or of type  $(D)$ , we can show the following:

**Fact:** Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Then we can find a  $f^k$ -invariant extremal ray  $R$  of  $\overline{\text{NE}}(X)$  for some  $k > 0$  and run the MMP compatibly with étale endomorphisms except the following case: There exists a finite étale covering  $\tilde{X} \rightarrow X$  of  $X$  such that  $\tilde{X}$  is isomorphic to the product  $S \times E$  of a rational surface  $S$  and an elliptic curve  $E$ .

We shall show this in our subsequent articles. In summary, our **strategy** is as follows:

- (1) (Construction of an FESP): Using Corollary 1.2, we shall first construct an FESP  $Y_\bullet$  of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  by a sequence of blowing-downs of an ESP.
- (2) (Classifications of an FESP): Using results in Section 7, Theorems 1.5, 10.1 and Corollary 10.6, we shall study the structure of the FESP  $Y_\bullet$  roughly according as the type of the pair  $(Y_\bullet, R_\bullet)$ .
- (3) (Finiteness of extremal rays of  $\overline{\text{NE}}(X)$ ): Applied Theorems 1.4, 1.5 to the FESP  $(Y_\bullet, R_\bullet)$ , we shall find a  $f^k$ -invariant extremal ray  $R$  of  $\overline{\text{NE}}(X)$  for some  $k > 0$ . This is possible in most cases as stated in the above **Fact**. For example, in the case where  $(F_\bullet, R_\bullet)$  is of type  $(C_1)$  or  $(C_0)$  (resp. type  $(C_{-\infty})$ ), we can apply Theorem 1.4 (resp. Theorem 1.5).
- (4) (Construction of a constant FESP): If (3) holds true, then we shall again run the MMP compatibly with étale endomorphisms and obtain another constant FESP  $(Z, v)$  induced from a non-isomorphic étale endomorphism  $v: Z \rightarrow Z$ .
- (5) (Study the structure of a constant FESP): We shall again study the structure of the constant FESP  $(Z, v)$  according as its type.
- (6) (Blowing-ups of an FESP): Using Lemma 3.3, we shall find a  $v$ -invariant elliptic curve  $E$  on  $Z$  and perform an equivariant blowing-ups along  $E$  to recover the original endomorphism  $f: X \rightarrow X$ . The structure of  $X$  can be studied in great detail.

In this Part I article, we shall mainly focus our attention to the topics (1), (2). The other topics (3), (4), (5), (6) will be studied in our subsequent articles.

**1.2. Announcement of results in our subsequent articles.** Now we shall announce the main results of our subsequent articles.

**Classification results:**

Let  $X$  be a smooth projective 3-fold with  $\kappa(X) = -\infty$  admitting a non-isomorphic étale endomorphism. Then up to finite étale covering,  $X$  satisfies one of the following 6 conditions:

- (1)  $X \simeq S \times E$  for an elliptic curve  $E$  and a uniruled algebraic surface  $S$ .
- (2)  $X$  is a  $\mathbb{P}^1$ -bundle over an abelian surface.
- (3)  $X$  is obtained by a succession of blowing-ups along elliptic curves from a  $\mathbb{P}^1$ -bundle  $Y$  over the product  $S := C \times E$ , where  $C$  is a curve of genus  $g(C) \geq 1$  and  $E$  is an elliptic curve.
- (4)  $X$  is a  $\mathbb{P}^2$ -bundle over an elliptic curve  $C$ .
- (5)  $X$  is a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle over an elliptic curve  $C$ .
- (6) The Albanese map  $\alpha: X \rightarrow C$  is a *fiber bundle* over the Albanese elliptic curve  $C$  of  $X$  whose fiber is a blown-up of a Hirzebruch surface.

To be more precise, ‘up to finite étale covering’ means the following; There exist a finite étale covering  $\tilde{X} \rightarrow X$  of  $X$  and a non-isomorphic étale endomorphism  $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$  which is a lift of some power  $f^k (k > 0)$  of the original non-isomorphic étale endomorphism  $f: X \rightarrow X$  such that  $\tilde{X}$  satisfies one of the 6 conditions listed as above.

We note that these 6 classes are not necessarily mutually disjoint from each other. For example, the product of the Hirzebruch surface  $\mathbb{F}_1$  and an elliptic curve satisfies both of the conditions (1) and (6). Comparing with classification of 3-folds with non-negative Kodaira dimension admitting non-isomorphic étale endomorphisms (cf. [9], [13]), our classification is not so simple and not strong enough. We mainly give a necessary condition for the existence of such varieties and state a sufficient condition in some special cases. As an application, combining Theorem 1.5 with our classification result, we shall prove the fact stated in Remark 1.3.

**Contents of Part II:** Applied Corollary 1.2 and Theorem 1.4, we can run the MMP compatibly with étale endomorphisms. We shall study the structure of  $X$  satisfying the condition (2) and (3) as above:

**Classification in the case (3):** In this case, there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_1)$  or  $(C_0)$  constructed from  $X$  by blowing-downs. Theorem 1.4 shows that there exists at most finitely many extremal rays of  $\overline{NE}(X)$ . Thus if we replace  $f$  by its suitable power  $f^k (k > 0)$ , we can choose  $Y_\bullet$  as a *constant* FESP  $(Y, g)$  induced from a non-isomorphic étale endomorphism  $g: Y \rightarrow Y$  of a smooth projective 3-fold  $Y$  with  $\kappa(Y) = -\infty$ . Applied the results in Sections 4, 7, 9, we shall first study the structure of  $Y$ . Next, performing a succession of equivariant blowing-ups of  $Y$  along elliptic curves (cf. Lemma 3.3 and Corollary 7.9), we shall recover the original endomorphism  $f: X \rightarrow X$  from its FESP  $Y$ . Thus, up to finite étale covering,  $X$  satisfies one of the following conditions:

- (i)  $X$  is isomorphic to the direct product of a uniruled surface and an elliptic curve.
- (ii)  $X \simeq Y$  and  $X$  is a  $\mathbb{P}^1$ -bundle over an abelian surface.
- (iii)  $Y$  is a fiber bundle over a smooth curve  $B$  of genus  $g(B) \geq 1$  whose fiber is the Atiyah surface  $\mathbb{S}$  over an elliptic curve  $E$ .
- (iv) There exists a smooth morphism  $\psi: Y \rightarrow B$  such that
  - the general fiber of  $\psi$  is isomorphic to the Atiyah surface  $\mathbb{S}$  over an elliptic curve  $E$  and,

- the special fiber of  $\psi$  is isomorphic to the direct product  $E \times \mathbb{P}^1$ .

Furthermore, in both cases (iii) and (iv),  $X$  is a fiber bundle over  $E$ .

### Contents of Part III:

Combining Theorem 1.5 with Atiyah's theory of vector bundles on an elliptic curve (cf. [2]), we shall study the structure of  $X$  in the case where there exist an FESP  $(Y_\bullet, R_\bullet)$  of type (D) constructed from  $X$  by a sequence of blowing-downs of an ESP. Then up to finite étale covering, we can choose the FESP  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  as a  $\mathbb{P}^2$ -bundle, or a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle over the Albanese elliptic curve  $C$  of  $X$ . Applying Theorems 1.5 and 10.1, we can show:

(i) If  $Y_n$  is a  $\mathbb{P}^2$ -bundle over  $C$ , then  $Y_n \simeq \mathbb{P}_C(\mathcal{E}_n)$  for a semi-stable vector bundle  $\mathcal{E}_n$  of rank 3 on  $C$ . Furthermore,  $\mathcal{E}_n$  is isomorphic to one of the following; a stable bundle,  $\mathcal{F}_2 \oplus \mathcal{P}_n$ , or  $\mathcal{O}_C \oplus \mathcal{M}_n \oplus \mathcal{N}_n$ , where  $\mathcal{P}_n, \mathcal{M}_n$  and  $\mathcal{N}_n$  are all *torsion* line bundles on  $C$ . We show that up to finite étale covering,  $X$  satisfies the condition (4), (6) or (1) respectively.

(ii) If  $Y_n$  is a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle over  $C$ , then up to finite étale covering,  $Y_n$  is isomorphic to one of the following;  $C \times \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathbb{S} \times \mathbb{P}^1$ , or  $\mathbb{S} \times_C \mathbb{S}$  for the Atiyah surface  $\mathbb{S}$ .

From these data, we can always find a  $f^k$ -invariant extremal ray  $R$  of  $\overline{\text{NE}}(X)$  for some  $k > 0$  and run again the MMP compatibly with étale endomorphisms except only two cases where  $Y_n \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{M}_n \oplus \mathcal{N}_n)$  or  $\tilde{Y}_n \simeq C \times \mathbb{P}^1 \times \mathbb{P}^1$  for a suitable finite étale covering  $\tilde{Y}_n$  of  $Y_n$ . Then after a succession of equivariant blowing-ups along elliptic curves (cf. Lemma 3.3, Corollary 7.9 and Proposition 7.10), we can recover the structure of the original endomorphism  $f: X \rightarrow X$ . We show that  $X$  satisfies the condition (1), (4), (5) or (6) respectively.

**What is new in this article:** Although our techniques of studying ESPs are new, many arguments appearing in the proofs of the various classification results have already appeared in our older papers on endomorphisms of threefolds and surfaces (cf. [9], [10], [12], [13], [14]). Since our new results seem to be hidden behind these classical arguments, we shall briefly describe ‘*what is new in this article*’.

- (Construction of an FESP): Proposition 1.1 and Corollary 1.2 have been already stated implicitly (cf. [9], [13]) in the special case where  $X_\bullet = (X, f)$  is a constant ESP introduced from a non-isomorphic étale endomorphism  $f: X \rightarrow X$ . The assertions in Proposition 1.1 (4) and in Corollary 1.2 (4) are new which relate with Propositions 4.1 and 6.8.
- (Classifications of an FESP): In Propositions 7.3, 7.5, and 7.8, classifications of an FESP of 3-folds with negative Kodaira dimension are treated. Its proofs are built upon classifications of an ESP of smooth algebraic surfaces in Section 6 which essentially appeared in [9], [10]. On the other hand, some of its proofs are simplified.
- (Finiteness of extremal rays): Theorems 1.4 and 8.6 show finiteness of extremal rays in certain cases.
- (New phenomena): Theorems 1.5, 10.1, Corollary 10.6, Remark 1.3 and Proposition 3.4 are related with *torsion* line bundles on an elliptic curve and distinguish an FESP constructed by an *étale endomorphism* from that of a general ESP. Propositions 6.2, 8.4 and Remarks 8.3, 8.5 are related with the Atiyah surface  $\mathbb{S}$  in the ESPs.

**Organization of this article.** In Section 2, we recall fundamental facts and techniques concerning endomorphisms. Moreover, we introduce the notions of an ESP (cf. Definition 2.3) and a *constant* ESP (cf. Definition 2.4). We show that very strong conditions are

imposed on such varieties appearing in an ESP (cf. Lemma 2.1 and Proposition 2.2).

In Section 3, we shall prove Proposition 1.1 which summarizes fundamental results of extremal rays on smooth projective 3-folds  $X$  appearing in an ESP. In Definition 3.6, we shall introduce a notion called an FESP. Applying Proposition 1.1, we shall give a proof of Corollary 1.2 which shows the existence of an FESP  $Y_\bullet$  constructed from the given étale endomorphism  $f: X \rightarrow X$  by a sequence of blowing-downs of an ESP (cf. Definition 3.7). Furthermore, we show that if there exist finitely many extremal rays of  $\overline{\text{NE}}(X)$ , then we can take  $Y_\bullet$  as a constant FESP (cf. Proposition 3.8). Remark 3.9 (3) shows that the FESP  $Y_\bullet$  is not uniquely determined.

In Section 4, we study the structure of an elliptic ruled surface admitting a non-isomorphic étale endomorphism. Proposition 4.1 shows that for a vector bundle  $\mathcal{E}$  on an elliptic curve  $C$ , if  $\mathbb{P}_C(\mathcal{E})$  appears in an ESP of projective bundles, then  $\mathcal{E}$  is semi-stable. Furthermore, we recall Atiyah's vector bundle  $\mathcal{P}_r$  on an elliptic curve (cf. [2]) and study the structure of its associated projective bundles from the viewpoint of an ESP (cf. Proposition 4.13).

In Section 5, we consider the structure of surjective morphisms between elliptic ruled surfaces which are not necessarily endomorphisms. Proposition 5.10 gives the structure of elliptic ruled surfaces appearing in an ESP. Proposition 5.5 describes endomorphisms of the Atiyah surface  $\mathbb{S}$ . Proposition 5.16 is applied to the proof of Theorem 10.1.

In Section 6, we first recall basic facts about elliptic fibrations and abelian fibrations. In Proposition 6.4, we shall study the structure of an ESP of algebraic surfaces. Next, we consider a pair of ESPs  $(C_\bullet, Z_\bullet)$  such that  $C_\bullet$  is an ESP of elliptic curves and the inclusion map  $i_\bullet: C_\bullet \rightarrow Z_\bullet$  is a Cartesian morphism of ESPs. Proposition 6.8 shows that the normal bundle of each elliptic curve appearing in the ESP  $C_\bullet$  is a *semi-stable* vector bundle of degree zero. Proposition 6.9 classifies such a pair  $(C_\bullet, S_\bullet)$  in the case where  $S_\bullet$  is an ESP of algebraic surfaces. Proposition 6.2 (cf. Remark 8.5) treats some new phenomenon of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  with  $\kappa(X) = -\infty$  such that  $X$  has no Seifert elliptic fiber space structure.

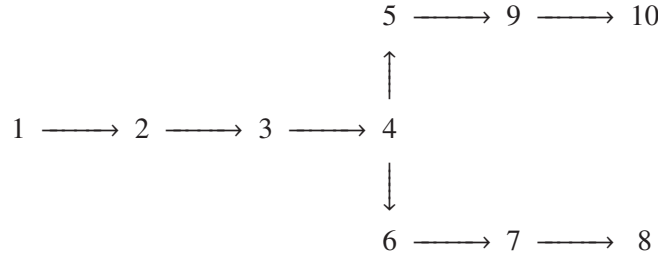
In Section 7, we shall classify the FESP  $(Y_\bullet, R_\bullet)$  constructed from an ESP  $X_\bullet$  of smooth projective 3-folds with negative Kodaira dimension. They are classified into two types according as the extremal contraction of  $R_\bullet$  gives a conic bundle structure (type (C)) or a del Pezzo fibration (type (D)) (cf. Proposition 7.3). Corollary 7.9 and Proposition 7.10 describe the blowing-up centers when we recover the original endomorphism  $f: X \rightarrow X$  from its FESP  $Y_\bullet$  by a sequence of blowing-ups of an ESP.

In Section 8, we shall prove Theorems 1.4 and 8.6 concerning finiteness of extremal rays on an ESP of 3-folds in certain cases. In Remark 8.3, we shall give an example of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  such that an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_1)$  or  $(C_0)$  is not uniquely determined. Proposition 8.4 and Remark 8.5 describe a new phenomenon concerning extremal rays.

In Section 9, we shall prove some technical lemma (cf. Theorems 9.3, 9.5 and 9.6) in the case where there exists an FESP  $(Y_\bullet, R_\bullet)$  of type (C) constructed from an endomorphism  $f: X \rightarrow X$ . We introduce a method for reducing the FESP  $Y_\bullet$  of conic bundles to an FESP of  $\mathbb{P}^1$ -bundles over smooth algebraic surfaces.

In Section 10, based on the results of Sections 5 and 9, we shall prove Theorems 10.1 and 1.5 which are one of our main results.

**Section Dependency.** The contents of each section are related in the following diagram;



## 2. Preliminaries

**2.1. Notation.** In this paper, we work over the complex number field  $\mathbb{C}$ .

A variety means a reduced and irreducible complex algebraic scheme. A projective variety is a complex variety embedded in a projective space, and a quasi-projective variety is a Zariski open subset of a projective variety. By a smooth projective  $n$ -fold, we mean a nonsingular projective variety of dimension  $n$ . By an endomorphism  $f: X \rightarrow X$ , we mean a morphism from a projective variety  $X$  to itself.

The following symbols are used for a variety  $X$ .

$K_X$ : the canonical divisor of  $X$ .

$T_X$ : the tangent bundle of  $X$ .

$\kappa(X)$ : the Kodaira dimension of  $X$ .

$c_i(X)$ : the  $i$ -th Chern class of  $X$ .

$b_i(X)$ : the  $i$ -th Betti number of  $X$ .

$\chi(\mathcal{O}_X)$ : the Euler–Poincaré characteristic of the structure sheaf  $\mathcal{O}_X$ .

$\chi_{\text{top}}(X)$ : the topological Euler characteristic of  $X$ .

$\text{Sing}(X)$ : the singular locus of  $X$ .

$\text{Aut}(X)$ : the algebraic group of automorphisms of  $X$ .

$\text{Aut}^0(X)$ : the identity component of  $\text{Aut}(X)$ .

$N_1(X) := (\{1\text{-cycles on } X\} / \equiv) \otimes_{\mathbb{Z}} \mathbb{R}$ , where  $\equiv$  means a numerical equivalence.

$N^1(X) := (\{\text{Cartier divisors on } X\} / \equiv) \otimes_{\mathbb{Z}} \mathbb{R}$ , where  $\equiv$  means a numerical equivalence.

$\text{NE}(X)$ : the smallest convex cone in  $N_1(X)$  containing all effective 1-cycles.

$\overline{\text{NE}}(X)$ : the Kleiman–Mori cone of  $X$ , i.e., the closure of  $\text{NE}(X)$  in  $N_1(X)$  for the metric topology.

$\text{Nef}(X)$ : the smallest closed convex cone in  $N^1(X)$  containing all nef divisors.

$\rho(X) := \dim_{\mathbb{R}} N_1(X)$ , the Picard number of  $X$ .

$[C]$ : the numerical equivalence of a 1-cycle  $C$ .

$\text{cl}(D)$ : the numerical equivalence class of a Cartier divisor  $D$ .

$\sim_{\mathbb{Q}}$ : the  $\mathbb{Q}$ -linear equivalence of  $\mathbb{Q}$ -divisors of  $X$ .

$g(C)$ : the genus of a smooth curve  $C$ .

$\text{Hom}_{\text{group}}(C, C')$ : the set of group homomorphisms from an elliptic curve  $C$  to  $C'$ .

For an endomorphism  $f: X \rightarrow X$  and an integer  $k > 0$ ,  $f^k$  stands for the  $k$ -times composite  $k \circ \cdots \circ f$  of  $f$ .

**Extremal rays:** For a smooth projective variety  $X$ , an extremal ray  $R$  means a  $K_X$ -negative extremal ray of  $\overline{\text{NE}}(X)$ , i.e., a 1-dimensional face of  $\overline{\text{NE}}(X)$  with  $K_X R < 0$ . A

extremal ray  $R$  defines a proper surjective morphism with connected fibers  $\text{Cont}_R: X \rightarrow Y$  such that, for an irreducible curve  $C \subset X$ ,  $\text{Cont}_R(C)$  is a point if and only if  $[C] \in R$ . This is called the contraction morphism associated to  $R$ .

**Fibrations:** A proper surjective morphism  $\pi: X \rightarrow S$  is called a fibration or a fiber space if  $X$  and  $S$  are normal projective varieties and  $\pi$  has a connected fiber. The closed subset  $\Delta_\pi := \{s \in S \mid \pi \text{ is not smooth at some point of } \pi^{-1}(s)\}$  is called the discriminant locus of  $\pi$ .

**Elliptic fibrations:** A fibration  $f: X \rightarrow S$  is called an elliptic fibration or an elliptic fiber space if the general fibers of  $f$  are elliptic curves.

**ESP:** An étale sequence  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_{n \in \mathbb{Z}}$  of constant Picard number ('ESP', for short) is an infinite sequence

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

of smooth projective varieties  $X_n$  such that each  $f_n$  is a non-isomorphic finite étale covering and each Picard number  $\rho(X_n)$  is constant. For simplicity, we often omit the subscript  $n \in \mathbb{Z}$  and denote it by  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$ .

**$\mathbb{P}^1$ -fiber space:** A fibration  $g: Y \rightarrow T$  is called a  $\mathbb{P}^1$ -fiber space if the general fibers of  $g$  are isomorphic to the complex projective line  $\mathbb{P}^1$ .

**2.2. Basic facts and techniques.** In this section, we invoke some fundamental facts and techniques related with finite étale coverings and étale endomorphisms which will be frequently used in later Sections. For the proofs, the reader can consult [9].

**Lemma 2.1.** *Suppose that there exists an infinite sequence*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

*of smooth projective  $k$ -folds such that each  $f_n$  is a non-isomorphic finite étale covering. Then  $\chi(\mathcal{O}_{X_n}) = \chi_{\text{top}}(X_n) = (K_{X_n})^k = 0$  for any  $n$ .*

The existence of a non-isomorphic étale endomorphism imposes on the variety  $X$  strong conditions as the following proposition shows.

**Proposition 2.2.** (cf. [9, Lemma 2.3, Proposition 2.6]) *Let  $f: X \rightarrow X$  be a surjective endomorphism of a smooth projective  $m$ -fold  $X$ . Then,*

- (1)  *$f$  is a finite morphism.*
- (2) *If  $\kappa(X) \geq 0$ , then  $f$  is an étale endomorphism.*
- (3) *If  $\kappa(X) = \dim X$ , then  $f$  is an automorphism.*
- (4) *If  $f$  is a non-isomorphic finite étale endomorphism, then*
  - $\chi(\mathcal{O}_X) = \chi_{\text{top}}(X) = K_X^m = 0$  and,
  - *the fundamental group  $\pi_1(X)$  of  $X$  is an infinite group.*

Now we introduce an important notion called an 'ESP' which includes as a special case the non-isomorphic étale endomorphisms.

**DEFINITION 2.3.** Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an infinite sequence of finite coverings between smooth projective  $k$ -folds  $X_n$ . Then we say that the sequence  $X_\bullet$  is an étale sequence of constant Picard number ('ESP' for short) if the following conditions are satisfied:

- (1) The Picard number  $\rho(X_n)$  is independent of  $n$ .

(2) Any  $f_n$  is a non-isomorphic finite étale covering.

DEFINITION 2.4. Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective varieties.

(1) If there exists a smooth projective variety  $X$  such that  $X_n = X$  for any  $n$ , then we say that  $X_\bullet$  is a ‘stable ESP’.

(2) Furthermore, if  $X_n = X$  and  $f_n = f$  for any  $n$  for a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a smooth projective variety  $X$ , then we say that  $X_\bullet$  is a ‘constant ESP’ induced by  $f: X \rightarrow X$  and denote it by  $X_\bullet = (X, f)$ .

DEFINITION 2.5. Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  and  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be ESP’s. Then by a Cartesian morphism  $\varphi_\bullet = (\varphi_n)_n: X_\bullet \rightarrow Y_\bullet$ , we mean that the following Cartesian diagram holds for any  $n$ :

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & X_{n+1} \\ \varphi_n \downarrow & & \downarrow \varphi_{n+1} \\ Y_n & \xrightarrow{g_n} & Y_{n+1}. \end{array}$$

The next lemma, which is stated implicitly in the papers [9] and [13], will provide the key to many of the results in later Sections.

**Lemma 2.6.** *Let  $\lambda: V \rightarrow S$  and  $\nu: W \rightarrow T$  be fiber spaces between normal varieties, i.e.,  $\lambda$  and  $\nu$  are proper surjective morphisms with connected fibers. Furthermore, suppose that  $g: V \rightarrow W$  and  $u: S \rightarrow T$  are finite surjective morphisms with the following commutative diagram:*

$$\begin{array}{ccc} V & \xrightarrow{g} & W \\ \lambda \downarrow & & \downarrow \nu \\ S & \xrightarrow[u]{} & T. \end{array}$$

*Suppose that  $g^{-1}(B) = A$  for irreducible subvarieties  $A(\subset V)$  and  $B(\subset W)$ . If the above commutative diagram is Cartesian, then  $\lambda(A) = u^{-1}(\nu(B))$ .*

Proof. By definition, we have  $u(\lambda(A)) = \nu(g(A)) = \nu(B)$ . For the inverse image  $\Gamma$  of  $\nu(B)$  by  $u$ , let us consider the fiber product  $\widetilde{\Gamma} := \Gamma \times_T B$ . Since  $u(\Gamma) = \nu(B)$ , by the universality, there exists a morphism  $\sigma: \widetilde{\Gamma} \rightarrow V$  such that

- $g \circ \sigma = p_2$  for the second projection  $p_2: \widetilde{\Gamma} \rightarrow B$ , and
- $\lambda \circ \sigma = p_1$  for the first projection  $p_1: \widetilde{\Gamma} \rightarrow \Gamma$ .

Again, by the universality, there exists a morphism  $\rho: \widetilde{\Gamma} \rightarrow V \times_W B$  such that the following commutative diagram holds:

$$\begin{array}{ccccc}
V \times_W B & \xlongequal{\quad} & V \times_W B & \longrightarrow & B \\
\rho \uparrow & & \downarrow & & \downarrow \\
\widetilde{\Gamma} & \xrightarrow{\sigma} & V & \xrightarrow{g} & W \\
p_1 \downarrow & & \downarrow \lambda & & \downarrow \nu \\
\Gamma & \longrightarrow & S & \xrightarrow{u} & T.
\end{array}$$

Hence,  $\Gamma$  is contained in the image of the composite map  $V \times_W B \rightarrow V \xrightarrow{\lambda} S$ . Note that by construction, the image of the natural projection  $V \times_W B \rightarrow V$  is  $g^{-1}(B) = A$ . Hence we infer that  $\Gamma \subset \lambda(A)$ .  $\square$

### 3. Extremal contractions and a construction of an FESP

In this section, we shall apply the minimal model program (MMP) to a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . One of the serious troubles is that there may exist infinitely many extremal rays of divisorial type on  $X$ . Compared with our previous results [9], [13], it is unknown whether there exists an extremal ray  $R$  of  $\overline{\text{NE}}(X)$  which is preserved by an endomorphism  $f$  of  $X$ . Hence, in general, we cannot run the MMP compatibly with étale endomorphisms. Thus, even if our concern is the study of endomorphisms, it is necessary to consider an ESP of 3-folds all of which are birationally equivalent to  $X$ . Using Proposition 1.1 and Theorem 3.10, we shall apply the MMP to a non-isomorphic étale endomorphism  $f: X \rightarrow X$  and reduce the study of  $f: X \rightarrow X$  to that of an *ESP of fiber type* (called an ‘FESP’ for short) which is not necessarily an endomorphism. We shall explain this reduction. Actually, we will consider a slightly more general set-up and give a proof of Corollary 1.2.

First, we recall some basic properties of non-isomorphic surjective endomorphisms from [9]. We shall use the standard terminology and results of the MMP as in [29] and [21]. We have proved the following results related to the extremal rays in [9].

**Proposition 3.1.** (cf. [9, Proposition 4.2 and 4.12]) *Let  $f: Y \rightarrow X$  be a finite surjective morphism between smooth projective  $n$ -folds with  $\rho(X) = \rho(Y)$ . Then the following assertions hold:*

- (1) *The push-forward map  $f_*: N_1(Y) \rightarrow N_1(X)$  is an isomorphism and  $f_*\overline{\text{NE}}(Y) = \overline{\text{NE}}(X)$ .*
- (2) *Let  $f_*: N^1(Y) \rightarrow N^1(X)$  be the map induced from the push-forward map  $D \mapsto f_*D$  of divisors  $D$ . Then the dual  $f^*: N_1(X) \rightarrow N_1(Y)$  (called the pullback map) is an isomorphism and  $f^*\overline{\text{NE}}(X) = \overline{\text{NE}}(Y)$ .*
- (3) *If  $f$  is étale and the canonical divisor  $K_X$  is not nef, then there is a one-to-one correspondence between the set of extremal rays of  $X$  and the set of extremal rays of  $Y$  under the isomorphisms  $f_*$  and  $f^*$ .*
- (4) *Under the same assumption as in (3), let  $\phi: X \rightarrow X'$  be the contraction morphism  $\text{Contr}_R$  associated to an extremal ray  $R \subset \overline{\text{NE}}(X)$  and let  $\psi: Y \rightarrow Y'$  be the contraction morphism associated to the extremal ray  $f^*R$ . Then there exists a finite surjective morphism  $f': Y' \rightarrow X'$  and the Cartesian diagram below such that  $f^{-1}(\text{Exc}(\phi)) = \text{Exc}(\psi)$  and  $f'^{-1}(\phi(\text{Exc}(\phi))) = \psi(\text{Exc}(\psi))$ :*

$$\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\psi \downarrow & & \downarrow \phi \\
Y' & \xrightarrow{f'} & X' .
\end{array}$$

Moreover,  $\phi$  is a birational (resp. divisorial) contraction if and only if  $\psi$  is a birational (resp. divisorial) contraction.

Proposition 3.1 is applied to the following fundamental result on extremal rays of a smooth projective 3-fold, which is proved in [9] except the assertion (4).

Proof of Proposition 1.1. By Proposition 3.1, for any  $n$ ,

- $R_n$  is an extremal ray,
- there exists a finite morphism  $g_n: Z_n \rightarrow Z_{n+1}$  such that  $\pi_{n+1} \circ f_n = g_n \circ \pi_n$  and,
- $f_n^{-1}(E_n) = E_{n+1}$  for the exceptional divisor  $E_n$  of  $\pi_n$ .

In [29], extremal divisorial contractions of smooth projective 3-folds are classified into 5 types. In our situation, we shall exclude 4 cases where a prime divisor is contracted to a point. Suppose that  $E_n$  is contracted to a point by  $\pi_n := \text{Contr}_{R_n}$  for some  $n > 0$ . Then by [29],  $E_n$  is isomorphic to  $\mathbb{P}^2$ ,  $\mathbb{P}^1 \times \mathbb{P}^1$ , or a singular quadric surface and is simply-connected. Since  $(f_{n-1})^{-1}(E_n) = E_{n-1}$  and  $f_{n-1}$  is a non-isomorphic étale covering,  $E_{n-1}$  is not connected. Thus a contradiction is derived. Hence  $\pi_n(E_n)$  is not a point for any  $n$ . Thus by [29],  $\pi: Y_n \rightarrow Z_n$  is the (inverse of the) blowing-up of a smooth projective 3-fold  $Z_n$  along a smooth curve  $C_n$  for any  $n$ . By the purity of branch loci,  $g_n$  is also a finite étale covering and hence  $Y_n \cong Y_{n+1} \times_{Z_{n+1}} Z_n$  and  $\deg g_n = \deg f_n > 1$  for each  $n$ . Thus, by Proposition 3.1,  $g_n^{-1}(C_{n+1}) = C_n$  for any  $n$ . Thus there is induced an ESP  $C_\bullet = (w_n: C_n \rightarrow C_{n+1})_n$  of smooth curves  $C_n$ , where we put  $w_n := g_n|_{C_n}$ . Hence  $C_n$  is an elliptic curve for any  $n$ .

Finally, we shall prove the assertion (4). It is sufficient to show the claim for  $n = 0$ . First, we show that  $\deg \mathcal{N}_0 = 0$ . For the defining ideal  $\mathcal{I}_n$  of  $C_n$ , we have  $g_n^*(\mathcal{I}_{n+1}^j) = \mathcal{I}_n^j$  for any  $j$  and  $n$ , since  $g_n$  is étale. Hence there exists an isomorphism  $(w_n)^* \mathcal{N}_{n+1} \cong \mathcal{N}_n$  for the normal bundle  $\mathcal{N}_n$  of  $C_n$  in  $Z_n$ . Since  $K_{Z_n} \sim g_n^* K_{Z_{n+1}}$ , by the projection formula we have

$$(K_{Z_n}, C_n) = (K_{Z_{n+1}}, (g_n)_*(C_n)) = (\deg g_n) (K_{Z_{n+1}}, C_{n+1})$$

for any  $n$ . Hence for any  $n > 0$ ,

$$(K_{Z_0}, C_0) = (K_{Z_n}, C_n) \prod_{i=0}^{n-1} \deg g_i .$$

Then  $(K_{Z_0}, C_0) = 0$ , since it is divisible by infinitely many positive integers. Hence,  $\deg \mathcal{N}_0 = -(K_{Z_0}, C_0) = 0$ .

The semi-stability of  $\mathcal{N}_0$  will be proved by contradiction. Suppose that  $\mathcal{N}_0$  is unstable. Since  $(w_n)^* \mathcal{N}_{n+1} \cong \mathcal{N}_n$ , any  $\mathcal{N}_n$  is unstable. Since  $\mathcal{N}_n$  is an unstable vector bundle of degree zero on an elliptic curve  $C_n$ ,  $\mathcal{N}_n \simeq \mathcal{L}_n \oplus \mathcal{V}_n$  for line bundles  $\mathcal{L}_n$  of degree  $d_n > 0$  and  $\mathcal{V}_n$  of degree  $-d_n < 0$  on  $C_n$ . Since  $\mathcal{L}_0 \simeq (w_{n-1} \circ \cdots \circ w_1 \circ w_0)^* \mathcal{L}_n$ , we have the following equality for  $n > 0$ :

$$d_0 = d_n \prod_{i=0}^{n-1} \deg w_i .$$

Since  $\deg w_i = \deg g_i = \deg f_i \geq 2$  for all  $i$ , we see that  $0 < d_n < 1$  for a sufficiently positive integer  $n$ . This contradicts the fact that  $d_n \in \mathbb{Z}$ . Hence  $\mathcal{N}_0$  is semi-stable.  $\square$

**REMARK 3.2.** Let  $E_n$  be the exceptional divisor of  $\pi_n$ . Then, by Proposition 1.1, (4) and Atiyah's classification (cf. [2]),  $E_n$  is isomorphic to either the Atiyah surface  $\mathbb{S}$  or  $\mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \ell_n)$  for some line bundle  $\ell_n$  of degree zero on  $C_n$ . In fact, all these types can occur. However, in the case where  $Y_\bullet$  is a constant ESP  $(Y, f)$  induced from a non-isomorphic étale endomorphism  $f: Y \rightarrow Y$ , the case where  $\ell_n \in \text{Pic}^0(C_n)$  is of infinite order cannot occur (cf. Theorems 10.1 and 1.3). We shall prove these facts in our subsequent article; Part III.

The following asserts the existence of 'Cartesian blowing-ups'.

**Lemma 3.3.** *Let  $V_\bullet = (g_n: V_n \rightarrow V_{n+1})_n$  be an ESP of 3-folds  $V_n$ . Suppose that there exists an ESP  $C_\bullet = (g_n|_{C_n}: C_n \rightarrow C_{n+1})_n$  of elliptic curves on  $V_\bullet$  such that the inclusion morphism  $i_\bullet: C_\bullet \rightarrow V_\bullet$  is Cartesian. Let  $\pi_n: W_n := \text{Bl}_{C_n}(V_n) \rightarrow V_n$  be the blowing-up of  $V_n$  along  $C_n$  for any  $n$ . Then there is induced an ESP  $W_\bullet = (f_n: W_n \rightarrow W_{n+1})_n$  of  $W_n$  such that  $\pi_\bullet = (\pi_n)_n: W_\bullet \rightarrow V_\bullet$  is a Cartesian morphism.*

*Proof.* By the universality of the blowing-up, there exists a unique morphism  $f_n: W_n \rightarrow W_{n+1}$  such that  $\pi_{n+1} \circ f_n = g_n \circ \pi_n$  for any  $n$ . Since  $g_n^{-1}(C_{n+1}) = C_n$ , we see that  $f_n^* E_{n+1} = E_n$  for the  $\pi_n$ -exceptional divisors  $E_n$  for any  $n$ . Hence  $K_{W_n} \sim f_n^* K_{W_{n+1}}$  and  $f_n$  is a finite étale covering. Since  $\rho(W_n) = \rho(V_n) + 1$  is constant,  $W_\bullet := (f_n: W_n \rightarrow W_{n+1})_n$  is also an ESP of  $W_n$ . By construction,  $\pi_\bullet$  is a Cartesian morphism.  $\square$

The following is concerned with some new phenomenon related with Remark 1.3.

**Proposition 3.4.** *There exists an example of an ESP  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  of 3-folds  $X_n$  with extremal rays  $R_\bullet = (R_n)_n$  of  $\overline{\text{NE}}(X_\bullet)$  such that the following hold:*

- (1)  $R_\bullet$  is of divisorial type and  $(f_n)_* R_n = R_{n+1}$  for any  $n$ .
- (2) Let  $D_n$  be the exceptional divisor of the contraction morphism  $\text{Cont}_{R_n}: X_n \rightarrow Y_n$  associated to  $R_n$  and  $D_\bullet := (f_n|_{D_n}: D_n \rightarrow D_{n+1})_n$  an induced ESP of  $D_n$ . Then any  $D_n$  is isomorphic to the elliptic ruled surface  $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{M}_n^{\otimes -1})$  associated to a line bundle  $\mathcal{M}_n \in \text{Pic}^0(E)$  of infinite order on a fixed elliptic curve  $E$ .
- (3)  $X_n \not\cong X_m$  for any  $n \neq m$ . In particular,  $X_\bullet$  is not a constant ESP induced from a non-isomorphic étale endomorphism.

*Proof. Step 1:* Let  $E$  be an elliptic curve and we fix two line bundles  $\mathcal{L}, \mathcal{M}$  on  $E$ . Suppose that both  $\mathcal{L}$  and  $\mathcal{M}$  are of infinite order in  $\text{Pic}^0 E \cong E$  and there exist no pair of integers  $(m, n) \neq (0, 0)$  such that  $\mathcal{L}^{\otimes m} \cong \mathcal{M}^{\otimes n}$ . We fix a positive integer  $k > 1$ . Note that  $\text{Pic}^0 E$  is divisible by any positive integer. Thus there exists a sequence  $\{\mathcal{L}_n\}_n$  (resp.  $\{\mathcal{M}_n\}_n$ ) of line bundles of degree zero on  $E$  such that  $\mathcal{L}_0 \cong \mathcal{L}$  and  $\mathcal{L}_{n+1}^{\otimes k} \cong \mathcal{L}_n$  (resp.  $\mathcal{M}_0 \cong \mathcal{M}$  and  $\mathcal{M}_{n+1}^{\otimes k} \cong \mathcal{M}_n$ ) for all  $n$ . For any  $n$ , let  $\mathcal{E}_n := \mathcal{O}_E \oplus \mathcal{L}_n \oplus \mathcal{M}_n$  be the decomposable vector bundle of rank 3 and degree 0 on  $E$ . Let  $\alpha_n: Y_n := \mathbb{P}_E(\mathcal{E}_n) \rightarrow E$  be the associated  $\mathbb{P}^2$ -bundle over  $E$ . For the multiplication map  $\mu_k: E \rightarrow E$  by  $k$ , there exist isomorphisms  $\mu_k^* \mathcal{L}_{n+1} \cong \mathcal{L}_{n+1}^{\otimes k} \cong \mathcal{L}_n$  and  $\mu_k^* \mathcal{M}_{n+1} \cong \mathcal{M}_{n+1}^{\otimes k} \cong \mathcal{M}_n$ . Hence there exists an isomorphism

$$\psi_n: Y_n \cong \widetilde{Y_{n+1}} := Y_{n+1} \times_{E, \mu_k} E.$$

Let  $p_n: \widetilde{Y}_n \rightarrow Y_n$  be the first projection. Then the composite map  $g_n := p_{n+1} \circ \psi_n: Y_n \rightarrow Y_{n+1}$  is a non-isomorphic finite étale covering of degree  $k^2 (\geq 4)$  for all  $n$ . Thus we have constructed an ESP  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  of  $\mathbb{P}^2$ -bundles over  $E$ . Note that  $Y_q \not\cong Y_p$  for any  $q > p$  as an abstract variety.

(The reason is as follows: Suppose the contrary that there exists an isomorphism  $v_{q,p}: Y_q \simeq Y_p$ . Then the composite map  $v_{q,p} \circ g_{q-1} \circ \cdots \circ g_p: Y_p \rightarrow Y_p$  gives a non-isomorphic étale endomorphism of  $Y_p$ . Then applying Proposition 4.7 and Remark 1.3, we see that both  $\mathcal{L}_p, \mathcal{M}_p \in \text{Pic}^0(E)$  are of finite order. Thus a contradiction is derived. The details will be given in our subsequent part III article.)

**Step 2:** Let  $s_n: E \rightarrow Y_n$  be a holomorphic section of  $\alpha_n$  corresponding to a surjection  $\mathcal{E}_n \twoheadrightarrow \mathcal{L}_n$  and put  $C_n := s_n(E)$ . Let  $\pi_n: X_n := \text{Bl}_{C_n}(Y_n) \rightarrow Y_n$  be the blowing-up of  $Y_n$  along  $C_n$ . Since  $g_n^{-1}(C_{n+1}) = C_n$  by construction,  $C_\bullet = (g_n|_{C_n}: C_n \rightarrow C_{n+1})_n$  is an ESP of elliptic curves such that the inclusion morphism  $i_\bullet: C_\bullet \rightarrow Y_\bullet$  is Cartesian. Hence by Lemma 3.3, there exists an ESP  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  such that  $\pi_\bullet = (\pi_n)_n: X_\bullet \rightarrow Y_\bullet$  is the Cartesian blowing-up along  $C_\bullet$ . Then the normal bundle  $\mathcal{N}_{C_n/Y_n} \cong \mathcal{O}_E \oplus \mathcal{M}_n$  is a decomposable, semi-stable vector bundle of rank 2 and degree 0 on  $E$ . By construction, we see that the  $\pi_n$ -exceptional divisor  $D_n$  is isomorphic to the  $\mathbb{P}^1$ -bundle  $\mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{M}_n^{\otimes -1})$  over  $E$ , where  $\mathcal{M}_n \in \text{Pic}^0(E)$  is of infinite order for all  $n$ . By applying the same argument as in the proof of Theorems 1.5, 10.1 and Remark 1.3, we see that  $X_n \not\cong X_m$  for any  $n \neq m$ . (The details will be proved in our next article, Part III).  $\square$

We will now apply Propositions 3.1 and 1.1 to a non-isomorphic, finite étale endomorphism  $f: X \rightarrow X$  of a 3-fold  $X$  with  $\kappa(X) = -\infty$ .

**Proposition 3.5.** *Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an ESP of smooth projective 3-folds  $Y_n$  with negative Kodaira dimension. Then the following are equivalent:*

- (1) *For any integer  $n$ , there exist no extremal rays whose contraction morphisms are birational.*
- (2) *For some integer  $n$ , there exist no extremal rays whose contraction morphisms are birational.*
- (3) *For any integer  $n$ , there exist no extremal rays whose contraction morphisms are birational and contract an irreducible divisor to an elliptic curve.*
- (4) *For any extremal ray  $R_n$  of  $\overline{\text{NE}}(Y_n)$  such that  $(g_n)_*R_n = R_{n+1}$ , the contraction morphism  $\varphi_n := \text{Cont}_{R_n}: Y_n \rightarrow W_n$  induces a Cartesian morphism  $\varphi_\bullet = (\varphi_n)_n: Y_\bullet \rightarrow W_\bullet$  such that either of the following hold.*
  - $\varphi_n$  is a conic bundle over a smooth algebraic surface  $W_n$  for any  $n$ .
  - $\varphi_n$  is a del Pezzo fibration over a smooth curve  $W_n$  for any  $n$ .

Proof. (1)  $\Rightarrow$  (2): Trivial.

(2)  $\Rightarrow$  (1) follows immediately from Proposition 3.1.

(1)  $\Rightarrow$  (3): Trivial.

(3)  $\Rightarrow$  (1) follows immediately from Proposition 1.1.

(4)  $\Rightarrow$  (1): Trivial.

(1)  $\Rightarrow$  (4): Any  $Y_n$  is not a Fano manifold, since Fano manifolds are simply connected and  $g_n$  is a non-isomorphic finite étale covering. Hence the assertion follows immediately from Proposition 3.1 (4) and the classification of extremal rays of smooth projective 3-folds

by Mori [29]. □

Accordingly, we make the following definition:

- DEFINITION 3.6. (1) Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an ESP of smooth projective 3-folds with negative Kodaira dimension. Then  $Y_\bullet$  is called an *ESP of fiber type* ('FESP' for short), if there exists an extremal ray  $R_n$  of fiber type on any  $\overline{\text{NE}}(Y_n)$ , that is, the contraction morphism  $\text{Cont}_{R_n}: Y_n \rightarrow W_n$  associated to  $R_n$  is a Mori fiber space, i.e.,  $\dim W_n < \dim Y_n$  for any  $n$ .
- (2) Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an FESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Let  $R_\bullet = (R_n)_n$  be an extremal ray of fiber type on  $\overline{\text{NE}}(X_n)$  such that  $(f_n)_* R_n = R_{n+1}$  for any  $n$ . Then
- the pair  $(Y_\bullet, R_\bullet)$  is called of type (C) if  $\varphi_n := \text{Cont}_{R_n}: X_n \rightarrow S_n$  is a conic bundle over a smooth algebraic surface  $S_n$  and,
  - the pair  $(Y_\bullet, R_\bullet)$  is called of type (D) if  $\varphi_n := \text{Cont}_{R_n}: X_n \rightarrow C_n$  is a del Pezzo fibration over a curve  $C_n$ .
- (3) Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an FESP. Suppose that there exists a non-isomorphic étale endomorphism  $g: Y \rightarrow Y$  of a smooth projective 3-fold  $Y$  such that  $Y_n = Y$  and  $g_n = g$  for any  $n$ . Then we say that  $Y_\bullet$  is a '*constant FESP*' induced from  $g: Y \rightarrow Y$ . We denote it by  $Y_\bullet = (Y, g)$ .

Using Proposition 1.1, we are ready to prove Corollary 1.2.

Proof of Corollary 1.2. Since  $\kappa(X_0) = -\infty$ ,  $X_0$  is uniruled and  $K_{X_0}$  is not nef by the 3-dimensional abundance theorem (cf. [19], [26], [27]). Hence there exists an extremal ray. Suppose that there exists *no* extremal ray of divisorial type on  $\overline{\text{NE}}(X_0)$ . Then  $X_\bullet$  itself is already an FESP and there is nothing to prove. Hereafter, we may assume that there exists some extremal ray  $R_0$  of *divisorial* type on  $\overline{\text{NE}}(X_0)$ . Let us choose such a ray  $R_0$  arbitrarily and put  $R_n := (f_{n-1} \circ \cdots \circ f_0)_* R_0$  and  $R_{-n} := (f_{-1} \circ \cdots \circ f_{-n})^* R_0$  for any  $n > 0$ . Then, by Proposition 1.1, any  $R_n$  is also an extremal ray of divisorial type on  $\overline{\text{NE}}(X_n)$  and the contraction morphism  $\pi_n^{(0)} := \text{Cont}_{R_n}: X_n \rightarrow X_n^{(1)}$  is the blowing-up of a smooth projective 3-fold  $X_n^{(1)}$  along an elliptic curve  $C_n^{(1)} (\subset X_n^{(1)})$ . Moreover, it induces the ESP  $X_\bullet^{(1)} := (f_n^{(1)}: X_n^{(1)} \rightarrow X_{n+1}^{(1)})_n$  of smooth projective 3-folds  $X_n^{(1)}$  such that

- $\pi_{n+1}^{(0)} \circ f = f_n^{(1)} \circ \pi_n^{(0)}$  and,
- $(f_n^{(1)})^{-1}(C_{n+1}^{(1)}) = C_n^{(1)}$ .

Namely, there exists a Cartesian blowing-up  $\pi_\bullet^{(0)}: X_\bullet^{(0)} \rightarrow X_\bullet^{(1)}$  along an ESP  $C_\bullet^{(1)} := (f_n^{(1)}|_{C_n^{(1)}}: C_n^{(1)} \rightarrow C_{n+1}^{(1)})_n$  of elliptic curves.

If there exists no extremal ray of divisorial type on  $X_0^{(1)}$ , then we stop here. Otherwise, we repeat the same procedure as above. We again apply Proposition 1.1 to the new ESP  $X_\bullet^{(1)}$  and consider the contraction morphism  $\pi_n^{(1)}: X_n^{(1)} \rightarrow X_n^{(2)}$ . In this way, for each positive integer  $n$ , we have successive contractions of extremal rays  $X := X_n^{(0)} \rightarrow X_n^{(1)} \rightarrow X_n^{(2)} \rightarrow \cdots$  with a strictly decreasing sequence  $\rho(X) > \rho(X_n^{(1)}) > \cdots$  of Picard numbers. Note that no flipping contractions can occur in our situation and  $X_n^{(i)}$  is nonsingular for all  $n$  and  $0 \leq i \leq k$ . Thus, there exists some positive integer  $k$  such that  $X_n^{(k)}$  has no extremal rays of divisorial type for any  $n$ . Furthermore, they form an FESP  $X_\bullet^{(k)} := (f_n^{(k)}: X_n^{(k)} \rightarrow X_{n+1}^{(k)})_n$ .

To sum up, we have the Cartesian diagram in Figure 1 such that the following hold;

- For each  $n$  and  $1 \leq i < k$ ,  $\pi_n^{(i-1)}: X_n^{(i-1)} \rightarrow X_n^{(i)}$  is (the inverse of) the blowing-up along an elliptic curve  $C_n^{(i)}$  on  $X_n^{(i)}$ .
- For  $0 \leq i \leq k$ ,  $X_\bullet^{(i)} := (f_n^{(i)}: X_n^{(i)} \rightarrow X_{n+1}^{(i)})_n$  is an ESP.
- $(f_n^{(i)})^{-1}(C_{n+1}^{(i)}) = C_n^{(i)}$  for all  $1 \leq i \leq k$ . That is,  $C_\bullet^{(i)} := (f_n^{(i)}|_{C_n^{(i)}}: C_n^{(i)} \rightarrow C_{n+1}^{(i)})_n$  is an ESP of elliptic curves and the inclusion  $C_\bullet^{(i)} \hookrightarrow X_\bullet^{(i)}$  is a Cartesian morphism.
- The bottom row  $X_\bullet^{(k)} = (f_n^{(k)}: X_n^{(k)} \rightarrow X_{n+1}^{(k)})_n$  is an FESP.

Thus, after finitely many divisorial contractions, we have obtained an FESP  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$ , where we put  $Y_n := X_n^{(k)}$  and  $g_n := f_n^{(k)}$  for any  $n$ . All the assertions except (4) follow by construction. The assertion (4) follows from Proposition 1.1 (4).  $\square$

DEFINITION 3.7. In Corollary 1.2, the vertical sequence

$$X_n \xrightarrow{\pi_n^{(0)}} \cdots \rightarrow X_n^{(i)} \xrightarrow{\pi_n^{(i)}} X_n^{(i+1)} \rightarrow \cdots \rightarrow X_n^{(k)}$$

is called a sequence of blowing-downs of an ESP. Furthermore, we call  $X_\bullet^{(k)}$  the FESP constructed from  $X_\bullet$  by a sequence of blowing-downs of an ESP. In other words,  $X_\bullet$  is reconstructed from the FESP  $X_\bullet^{(k)}$  by a sequence of blowing-ups of an ESP.

In particular, we consider the special case where  $f: X \rightarrow X$  is a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . First, suppose that the constant ESP  $X_\bullet := (X, f)$  is already an FESP, i.e., for *some* extremal ray  $R$  of  $X$ , the contraction morphism  $\pi := \text{Contr}_R: X \rightarrow X'$  is *not birational*. Then we do not need to consider the reduction any more. By Proposition 3.1 and [29], the FESP  $(X_\bullet, R_\bullet)$  is of type either (C) or (D). By Theorem 3.10,  $(f^k)_*R = R$  for some positive integer  $k$ . Thus, if we replace  $f$  by the power  $f^k$ , we may assume that  $f_*R = R$ . Hence by Proposition 3.5, if  $(X_\bullet, R_\bullet)$  is of type (C) (resp. (D)), then the conic bundle  $\pi: (X, f) \rightarrow (S, g)$  (resp. the del Pezzo fibration  $\pi: (X, f) \rightarrow (C, h)$ ) gives the Cartesian morphism of constant ESPs. Thus in this case, the MMP works compatibly with étale endomorphisms.

Next, suppose that  $X_\bullet = (X, f)$  is not an FESP, i.e., for *any*  $K_X$ -neative extremal ray  $R$  of  $X$ , the contraction morphism  $\text{Contr}_R: X \rightarrow X'$  is *birational*. Then we can apply the above procedure to the constant ESP  $X_\bullet := (X, f)$ . Thus, by Corollary 1.2, after finitely many blowing-downs of an ESP, we can obtain an FESP  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  as above. The following proposition gives a sufficient condition for the existence of a constant FESP (cf. Definition 3.6).

**Proposition 3.8.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a 3-fold  $X$  and  $X_\bullet := (X, f)$  a constant ESP. Suppose that there exist finitely many extremal rays of divisorial type on  $X$ . Then by replacing  $f$  by a suitable power  $f^k$  ( $k > 0$ ), there exist a constant ESP  $Y_\bullet = (Y, g)$  and a Cartesian morphism  $\pi_\bullet: X_\bullet \rightarrow Y_\bullet$  induced from a divisorial contraction  $\pi: X \rightarrow Y$ .*

*Proof.* By Proposition 3.1 (3),  $f$  induces a permutation of the finite set  $\mathcal{R}$  consisting of extremal rays of divisorial type. Hence replacing  $f$  by a suitable power  $f^k$  ( $k > 0$ ),  $f$  induces an identity permutation of  $\mathcal{R}$ . Thus applying Proposition 1.1, the proof is finished.  $\square$

Suppose that there exist finitely many extremal rays of divisorial type on  $Y$ . Then applying Proposition 3.8 again to the constant ESP  $Y_\bullet$ , we can obtain a constant ESP  $Z_\bullet = (Z, h)$  and a Cartesian morphism  $\pi'_\bullet: Y_\bullet \rightarrow Z_\bullet$  induced from a divisorial contraction  $\pi': Y \rightarrow Z$ . If there exist only finitely many extremal rays of divisorial type in each ESP, we can continue this procedure and eventually obtain a constant FESP  $W_\bullet = (W, v)$  induced by a non-isomorphic étale endomorphism  $v: W \rightarrow W$ .

REMARK 3.9. (1) Suppose that  $f: X \rightarrow X$  is a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) \geq 0$ . Then there exist only finitely many extremal rays of  $\overline{NE}(X)$ . Hence by Proposition 3.8, there exists a constant FESP  $Y_\bullet = (Y, g)$ . Furthermore,  $Y$  is a smooth and a unique minimal model of  $X$ .  $Y_\bullet$  is called a *minimal reduction* in [9], [13].

(2) On the contrary, in the case where  $\kappa(X) = -\infty$ , the trouble is that there exists a smooth projective 3-fold  $X$  with infinitely many extremal rays of divisorial type and admits a non-isomorphic étale endomorphism. For example (cf. [15], Remark A.9), let  $S$  be a rational elliptic surface with global sections whose Mordell-Weil rank is positive. It is obtained as 9-points blowing-up of  $\mathbb{P}^2$ . We regard  $S$  as an elliptic curve  $C_K$  defined over the function field  $K$  of the base curve. Since  $S$  is relatively minimal, the translation mapping  $C_K \rightarrow C_K$  given by the non-torsion section  $\gamma$  induces a relative automorphism  $g: S \simeq S$  over  $C$ , which is of infinite order. Let  $X := S \times E$  be the product variety of  $S$  and an elliptic curve  $E$ . Since  $\gamma$  is a  $(-1)$ -curve of  $S$ , the curve  $\gamma \times \{o\}$  for a point  $o \in E$  spans the extremal ray  $R$  of divisorial type. If we denote by  $\mu_k: E \rightarrow E$  multiplication by  $k > 1$ , then the product mapping  $f := g \times \mu_k: X \rightarrow X$  gives a non-isomorphic étale endomorphism of  $X$ . By construction,  $(f^k)_*R \neq R$  and  $(f^k)^*R \neq R$  for any positive integer  $k$ . Furthermore, we can take an FESP of the constant ESP  $X_\bullet = (X, f)$  as  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$ , where  $Y_n \simeq \mathbb{P}^2 \times E$  for any  $n$ . However, the étale endomorphism  $g_n$  depends on  $n$ , since the contraction morphism  $\pi_n: X \rightarrow Y_n$  depends on  $n$ . Thus  $Y_\bullet$  is *not* a constant FESP, but is a stable FESP (cf. Definition 2.4).

(3) If  $f: X \rightarrow X$  is a non-isomorphic étale endomorphism with  $\kappa(X) = -\infty$ , then an FESP  $Y_\bullet$  of the constant ESP  $(X, f)$  is not uniquely determined (cf. Remark 7.14).

The following theorem is useful for studying the structure of an étale endomorphism of a smooth projective variety which has a structure of a Mori fiber space.

**Theorem 3.10** (Fujimoto, Nakayama [15]). *Let  $X$  be a normal projective variety defined over an algebraically closed field of characteristic zero such that  $X$  has only log-terminal singularities. Let  $R \subset \overline{NE}(X)$  be an extremal ray such that  $K_X R < 0$  and the associated contraction morphism  $\text{Contr}_R$  is a fibration to a lower dimensional variety. Then, for any surjective endomorphism  $f: X \rightarrow X$ , there exists a positive integer  $k$  such that  $(f^k)_*(R) = R$  for the automorphism  $(f^k)_*: N_1(X) \cong N_1(X)$  induced from the power  $f^k = f \circ \cdots \circ f$ .*

REMARK 3.11. Theorem 3.10 holds for any positive dimensional log terminal variety  $X$  and for any surjective endomorphisms  $f$  including automorphisms and finite ramified coverings of  $X$ . However, we cannot allow the case where  $\text{Contr}_R$  is a birational morphism. An easy counterexample is given in [15], Remark A.9.

In proving theorems concerning finiteness of extremal rays, the following fact is often useful.

**Lemma 3.12.** *Let  $\varphi: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$ . Suppose that there exists a finite étale covering  $p: X' \rightarrow X$ . Then*

(1) *for any extremal ray  $R$  of divisorial type on  $X$ , there exists some extremal ray  $R'$  of divisorial type on  $X'$  such that  $p_*R' = R$ .*

(2) *Furthermore, suppose that there exists a non-isomorphic étale endomorphism  $\varphi': X' \rightarrow X'$  such that  $p \circ \varphi' = \varphi \circ p$ . Then the contraction morphism  $\text{Contr}_R: X' \rightarrow W'$  is (the inverse of) the blowing-up of a smooth projective 3-fold  $W'$  along an elliptic curve  $C'$  on  $W'$ .*

*Proof.* Let  $R(\subset \overline{\text{NE}}(X))$  be any extremal ray of divisorial type and  $\pi := \text{Contr}_R: X \rightarrow Y$  the contraction morphism associated to  $R$ . Then by Proposition 1.1,  $\pi$  is the inverse of the blowing-up of  $Y$  along an elliptic curve. Let  $e$  be any extremal rational curve on  $X$  whose numerical class  $[e]$  spans  $R$ . Let  $e'$  be one of the connected components of  $p^{-1}(e)$ . Since  $K_{X'} \sim p^*K_X$ , we infer by the projection formula that  $(K_{X'}, e') = (K_X, e) = -1 < 0$ . For an ample divisor  $H'$  of  $X'$ , we take a sufficiently small positive real number  $\epsilon > 0$  such that  $(K_{X'} + \epsilon H, e') < 0$ . Then, by the cone theorem [29], we have:  $\overline{\text{NE}}(X') = \overline{\text{NE}}_{K_{X'} + \epsilon H \geq 0}(X') + \sum_{j=1}^r R_j$ , where  $R_j$  is an extremal ray spanned by a numerical equivalence class  $[e_j]$  of an extremal rational curve  $e_j$  on  $X'$ . Hence  $e' \equiv c' + \sum_{j=1}^r a_j e_j$ , where  $[c'] \in \overline{\text{NE}}_{K_{X'} + \epsilon H \geq 0}(X')$ , and  $a_j \geq 0$  for any  $j$ . By construction, we have  $a_i > 0$  for some  $i$ . Let  $\psi := \pi \circ p: X' \rightarrow Y$  be the composite map. Then  $0 = \psi_* e' \equiv \psi_* c' + \sum_{j=1}^r a_j \psi_* e_j$ . Hence  $\psi_*(e_i) \equiv 0$ . Thus the curve  $p(e_i)$  on  $X$  is contracted to a point by  $\pi: X \rightarrow Y$ , hence its numerical equivalence class  $[p(e_i)]$  spans the extremal ray  $R$  on  $X$ . Thus  $p_*R_i = R$ . We set  $R' := R_i$ .

Next, we show that the extremal ray  $R'$  is of divisorial type. Assume the contrary. Let  $\pi': X' \rightarrow Y'$  be the contraction morphism. Then there exists a finite morphism  $p': Y' \rightarrow Y$  such that  $p' \circ \pi' = \pi \circ p$ . Since  $\dim Y' \leq 2$ , this contradicts the finiteness of  $p'$ . Hence  $R'$  is of divisorial type and the assertion (1) has been proved. The assertion (2) follows immediately from the assertion (1) and Proposition 1.1.  $\square$

#### 4. Projective bundles over an elliptic curve

In this section, we shall study projective bundles  $X$  over an elliptic curve  $C$  which admit non-isomorphic étale endomorphisms  $f: X \rightarrow X$ . We can show that such  $X$  is associated to a semi-stable vector bundle on  $C$ . A key ingredient is the following Proposition 4.1 which relates the semi-stability of vector bundles on an elliptic curve with the existence of non-isomorphic étale endomorphisms of associated projective bundles. As an immediate application, we shall relate torsion line bundles on an elliptic curve with an elliptic ruled surface admitting a non-isomorphic étale endomorphism (cf. Proposition 4.8). Furthermore, using Atiyah's vector bundle  $\mathcal{F}_r$  ([2]) on an elliptic curve, we shall construct concrete examples of such projective bundles (cf. Proposition 4.13). We also remark that the étaleness assumption is essentially used in our paper (cf. Remark 4.3).

**Proposition 4.1.** *Let  $r > 1$  be a fixed integer. For each integer  $n$ , let  $\mathcal{E}_n$  be a vector bundle of rank  $r$  on an elliptic curve  $C_n$  and  $\varphi_n: X_n := \mathbb{P}_{C_n}(\mathcal{E}_n) \rightarrow C_n$  the associated  $\mathbb{P}^{r-1}$ -bundle. Suppose that there exists an ESP  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  of  $X_n$ . Then  $\mathcal{E}_n$  is semi-stable for all  $n$ .*

Proof. By considering the following truncated sequence

$$X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$$

starting from  $X_n$  for each  $n$ , it is sufficient to show that  $\mathcal{E}_0$  is semi-stable without loss of generality. Assuming that  $\mathcal{E}_0$  is unstable, we derive a contradiction. Since  $\varphi_n$  gives the Albanese map of  $X_n$ , there exists a unique finite étale covering  $h_n: C_n \rightarrow C_{n+1}$  with  $h_n \circ \varphi_n = \varphi_{n+1} \circ f_n$  for each  $n$ . Since  $f_n$  is étale and  $\mathbb{P}^{r-1}$  is simply connected,  $f_n$  is of degree one on each fiber of  $\varphi_n$  and  $\deg h_n = \deg f_n > 1$ . Thus, there exists the following Cartesian morphism of ESPs;

$$\varphi_\bullet = (\varphi_n)_n: X_\bullet \rightarrow C_\bullet := (h_n: C_n \rightarrow C_{n+1})_n.$$

In particular, there exists an isomorphism  $h_n^* \mathcal{E}_{n+1} \cong \mathcal{E}_n \otimes \ell_n$  for some line bundle  $\ell_n$  on  $C_n$ . Since the semi-stability of a vector bundle is preserved after taking a tensor product with a line bundle or pulling-back under a finite étale base change,  $\mathcal{E}_n$  is unstable for all  $n \geq 0$ . For a sufficiently large positive integer  $n$ , take a maximal destabilizing subbundle  $\mathcal{G}_n$  of  $\mathcal{E}_n$  with  $r' := \text{rank } \mathcal{G}_n < r$ . Then there exists an isomorphism  $\mathcal{E}_0 \cong (h_{n-1} \circ \cdots \circ h_0)^* \mathcal{E}_n \otimes \mathcal{L}_0^{-1}$  for some line bundle  $\mathcal{L}_0$  on  $C$ . Now we take a coherent subsheaf  $\mathcal{G}_0 := (h_{n-1} \circ \cdots \circ h_0)^* \mathcal{G}_n \otimes \mathcal{L}_0^{-1}$  of  $\mathcal{E}_0$ . Then

$$\mu(\mathcal{G}_0) + \deg \mathcal{L}_0 = \mu(\mathcal{G}_n) \cdot \prod_{i=0}^{n-1} \deg h_i,$$

for the slope  $\mu(\mathcal{G}_0) := \deg \mathcal{G}_0 / r'$ . Similarly,

$$\mu(\mathcal{E}_0) + \deg \mathcal{L}_0 = \mu(\mathcal{E}_n) \cdot \prod_{i=0}^{n-1} \deg h_i,$$

for the slope  $\mu(\mathcal{E}_0)$  of  $\mathcal{E}_0$ . Hence, we have an equality:

$$\mu(\mathcal{G}_0) - \mu(\mathcal{E}_0) = (\mu(\mathcal{G}_n) - \mu(\mathcal{E}_n)) \cdot \prod_{i=0}^{n-1} \deg h_i.$$

Since  $\mu(\mathcal{G}_n) > \mu(\mathcal{E}_n)$ ,  $\mathcal{G}_0$  is unstable. Furthermore, due to the uniqueness of the Harder-Narasimhan filtration,  $\mathcal{G}_0$  is also a maximal destabilizing subsheaf of  $\mathcal{E}_0$  and does not depend on the choice of  $n$ . By definition, the positive rational number  $\mu(\mathcal{G}_n) - \mu(\mathcal{E}_n)$  is uniformly bounded below by the constant  $(rr')^{-1} > r^{-2}$ , which is independent of  $n$ . Since  $\deg h_i \geq 2$  for each  $i$ , we have the following inequality for  $n \gg 0$ ;

$$\mu(\mathcal{G}_0) - \mu(\mathcal{E}_0) \geq 2^n / r^2.$$

Thus a contradiction is derived.  $\square$

The following corollary is obtained immediately.

**Corollary 4.2.** *Let  $S_\bullet = (f_n: S_n \rightarrow S_{n+1})_n$  be an ESP of relatively minimal elliptic ruled surfaces  $S_n$ . Suppose that  $S_n = \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$  for an invertible sheaf  $\mathcal{L}_n$  on an elliptic curve  $C_n$  for each  $n$ . Then  $\deg(\mathcal{L}_n) = 0$  for any  $n$ .*

REMARK 4.3. (1) Proposition 4.1 does not hold if we consider an infinite sequence of bounded above

$$\cdots \longrightarrow X_n \xrightarrow{g_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{g_1} X_0$$

of non-isomorphic finite étale coverings of projective bundles over elliptic curves. We shall give such an example. Let  $C$  be an elliptic curve with  $o \in C$  as the identity element and  $\mu_N: C \rightarrow C$  the multiplication map by an integer  $N(> 1)$ . Let  $\mathcal{O}_C([o])$  be the invertible sheaf associated to the prime divisor  $[o]$  consisting of  $o$ . Then  $\mu_N^* \mathcal{O}_C([o]) \cong \mathcal{O}_C([o])^{\otimes N^2}$ . If we put  $\mathcal{L}_n := (\mu_{nN})^* \mathcal{O}_C([o])$ , then the direct sum  $\mathcal{E}_n := \mathcal{O}_C \oplus \mathcal{L}_n$  is an unstable vector bundle on  $C$  and  $\mu_N^* \mathcal{E}_n \cong \mathcal{E}_{n+1}$ . Let  $S_n := \mathbb{P}(\mathcal{E}_n)$  be a  $\mathbb{P}^1$ -bundle over  $C_n := C$  associated to  $\mathcal{E}_n$ . Then for each  $n$ , there exists an isomorphism  $S_{n+1} \cong S_n \times_{C, \mu_N} C$  and the natural projection  $g_n: S_{n+1} \rightarrow S_n$  gives a finite étale covering of degree  $N^2(> 1)$ .

- (2) Proposition 4.1 does not hold without the assumption that  $f_n$  is finite étale. In fact, by [38, Proposition 5], any  $\mathbb{P}^1$ -bundle over an elliptic curve admits non-isomorphic surjective endomorphisms. In particular, let  $S := \mathbb{P}_C(\mathcal{E})$  be a  $\mathbb{P}^1$ -bundle associated with an unstable vector bundle  $\mathcal{E}$  on an elliptic curve  $C$ . Then  $S$  admits a surjective endomorphism  $f: S \rightarrow S$  which is a finite ramified covering. Thus, there exists an infinite sequence of non-isomorphic finite ramified coverings between elliptic ruled surfaces:

$$\cdots \xrightarrow{f} S \xrightarrow{f} S \xrightarrow{f} \cdots$$

The following result will be used later.

**Proposition 4.4.** (cf. [39], [44], [46]) *Let  $\pi: X := \mathbb{P}_C(\mathcal{E}) \rightarrow C$  be a  $\mathbb{P}^1$ -bundle over an elliptic curve  $C$  associated to a stable vector bundle  $\mathcal{E}$  of rank two. Then the following hold:*

(1) *The anti-pluricanonical system  $| -2K_X |$  is base point free and defines a Seifert elliptic fibration (cf. Definition 6.1)  $\Phi: X \rightarrow \mathbb{P}^1$  such that  $\mathcal{O}_X(-2K_X) \simeq \Phi^* \mathcal{O}(1)$ . Furthermore,  $\Phi$  has three multiple singular fibers of type  $2I_0$ .*

(2) *Let  $\mu_2: C \rightarrow C$  be a multiplication mapping by two. Then the pull-back  $X \times_{C, \mu_2} C$  is isomorphic over  $C$  to the trivial  $\mathbb{P}^1$ -bundle  $\mathbb{P}^1 \times C$ .*

We recall the following theorem due to Miyaoka concerning a numerical characterization for a semistable vector bundle on curves.

**Theorem 4.5** (Miyaoka [25]). *Let  $\mathcal{E}$  be a vector bundle of rank two on a smooth projective curve  $C$  and  $\pi: X := \mathbb{P}_C(\mathcal{E}) \rightarrow C$  the associated  $\mathbb{P}^1$ -bundle. Then the following conditions are equivalent:*

- (1)  $\mathcal{E}$  is semistable.
- (2) The relative anti-canonical divisor  $-K_{X/C} := -K_X + \pi^* K_C$  is nef.
- (3)  $\text{Nef}(X) = \mathbb{R}_{\geq 0}[-K_{X/C}] + \mathbb{R}_{\geq 0}[F]$ , where  $F$  is a fiber of  $\pi$ .
- (4)  $\overline{\text{NE}}(X) = \mathbb{R}_{\geq 0}[-K_{X/C}] + \mathbb{R}_{\geq 0}[F]$ .
- (5) Every effective divisor on  $X$  is nef.

Now, we need the following preliminary result.

**Lemma 4.6.** *Let  $\varphi: Y \rightarrow X$  be a surjective morphism of normal projective varieties  $X$  and  $Y$ . Suppose that  $\varphi^*\mathcal{L} \cong \mathcal{O}_Y$  for a line bundle  $\mathcal{L}$  on  $X$ . Then  $\mathcal{L}$  is of finite order in  $\text{Pic}(X)$ . Furthermore, if  $\varphi$  is generically finite, then  $\text{ord}(\mathcal{L})|\deg \varphi$ .*

*Proof.* With the aid of the Stein factorization, the claim can be reduced to the following two cases:

- The case where  $\varphi: Y \rightarrow X$  is a fiber space.
- The case where  $\varphi$  is a finite morphism.

(1) In the former case, since  $\varphi_*\mathcal{O}_Y \simeq \mathcal{O}_X$ , it follows immediately by the projection formula that  $\mathcal{L} \simeq \mathcal{O}_X$ .

(2) In the latter case, there exists some Cartier divisor  $D$  of  $X$  such that  $\mathcal{L} \simeq \mathcal{O}_X(D)$ . Then by assumption, we infer that  $\varphi^*D \sim 0$ . We regard  $\mathcal{L}$  as a Cartier divisor and consider the push-forward map  $\varphi_*: N^1(Y) \rightarrow N^1(X)$  of Cartier divisors. Since  $\varphi_*\varphi^*(D) \sim (\deg \varphi)D$ , it follows that  $(\deg \varphi)D \sim 0$  and  $\mathcal{L}$  is of finite order in  $\text{Pic}(X)$ . In particular, we see that  $\text{ord}(\mathcal{L})|\deg \varphi$ .  $\square$

As an application of this technique, we can now state the following result:

**Proposition 4.7.** *Let  $A$  be an  $m$ -dimensional abelian variety,  $X$  an  $m$ -dimensional projective variety and  $f, g: X \rightarrow A$  surjective morphisms. Suppose that*

- $f - g: X \rightarrow A$  is a surjective morphism and;
- $f^*\mathcal{L} \simeq g^*\mathcal{L}$  for a line bundle  $\mathcal{L} \in \text{Pic}^0(A)$ .

*Then  $\mathcal{L}$  is of finite order in  $\text{Pic}^0(A)$ .*

*Proof.* By [32, p75, (ii)], we have  $(f - g)^*\mathcal{L} \simeq \mathcal{O}_X$ . Then applying Lemma 4.6, we infer that  $\mathcal{L}$  is of finite order in  $\text{Pic}(A)$ .  $\square$

We are ready to prove the following fundamental result.

**Proposition 4.8.** *Let  $g: S \rightarrow S$  be a non-isomorphic étale endomorphism of a relatively minimal elliptic ruled surface  $S$ . Suppose that  $S = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$  for an invertible sheaf  $\mathcal{L}$  on an elliptic curve  $C$ . Then  $\mathcal{L} \in \text{Pic}(C)$  is of finite order.*

*Proof.* Let  $\varphi: S \rightarrow C$  be the Albanese map of  $S$ , which gives  $S$  a  $\mathbb{P}^1$ -bundle structure over the elliptic curve  $C$ . Then there exists an étale endomorphism  $h: C \rightarrow C$  of  $C$  with  $\varphi \circ g = h \circ \varphi$ . Since  $g$  is étale and  $\mathbb{P}^1$  is simply connected, there exists an isomorphism

$$S := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \cong S \times_{C, h} C \simeq \mathbb{P}(\mathcal{O}_C \oplus h^*\mathcal{L})$$

over  $C$  and  $\deg h = \deg g$ . Thus there exists an isomorphism

$$\mathcal{O}_C \oplus \mathcal{L} \cong (\mathcal{O}_C \oplus h^*\mathcal{L}) \otimes \ell$$

for some invertible sheaf  $\ell$  on  $C$ . If  $h^*\mathcal{L} \simeq \mathcal{O}_C$ , then  $\text{ord}(\mathcal{L})|\deg h$  by Lemma 4.6. Suppose that  $h^*\mathcal{L} \not\cong \mathcal{O}_C$ . Then by [3, Theorem 1], this is possible if and only if

- $\ell \cong \mathcal{O}_C$  and  $h^*\mathcal{L} \cong \mathcal{L}$ , or
- $\ell \cong \mathcal{L}$  and  $h^*\mathcal{L} \cong \mathcal{L}^{-1}$ .

Then since  $\deg h > 1$ , we see that  $\deg \mathcal{L} = 0$ . Since both  $h - \text{id}_C$  and  $h + \text{id}_C$  are surjective endomorphisms of  $C$ , Proposition 4.7 implies that  $\mathcal{L} \in \text{Pic}^0(C)$  is of finite order.  $\square$

The following follows immediately from Lemma 4.6 and the proof of Proposition 4.8.

**Corollary 4.9.** *Under the same notation as in the proof of Proposition 4.8,  $\text{ord}(\mathcal{L})$  divides  $\deg(h + \text{id}_C)$ ,  $\deg(h - \text{id}_C)$ , or  $\deg(h)$ .*

**REMARK 4.10.** Proposition 4.8 does not hold without the assumption that an endomorphism  $f: X \rightarrow X$  is étale. For example, let  $\mathcal{L}$  be a line bundle of degree zero on an elliptic curve  $C$  which is of infinite order in  $\text{Pic}^0(C)$ . Let  $S := \mathbb{P}_C(\mathcal{E})$  be a  $\mathbb{P}^1$ -bundle over  $C$  associated to a decomposable vector bundle  $\mathcal{E} := \mathcal{O}_C \oplus \mathcal{L}$  of rank two and degree zero on  $C$ . Then, by [38],  $S$  admits a non-isomorphic surjective endomorphism  $f: S \rightarrow S$ , which gives a finite ramified covering of  $S$ .

The following is a well-known fact (cf. [42]).

**Fact:**

- (1) Let  $\mathcal{F}$  and  $\mathcal{F}'$  be two semi-stable vector bundles over an elliptic curve  $C$  with slope  $\mu$  and  $\mu'$  respectively. If  $\mu < \mu'$ , then  $\text{Ext}_{\mathcal{O}_C}^1(\mathcal{F}, \mathcal{F}') = 0$ .
- (2) An arbitrary unstable vector bundle of finite rank on an elliptic curve is a direct sum of semi-stable bundles.
- (3) Any indecomposable vector bundle on an elliptic curve is semi-stable.

Now we quote some deep results due to Atiyah concerning vector bundles on an elliptic curve which we need throughout our articles. For a proof, the reader may consult [2].

**Theorem 4.11** (Atiyah [2]). *Let  $C$  be an elliptic curve. Then the following hold:*

- (1) *There exists on  $C$  an indecomposable vector bundle  $\mathcal{F}_r$  of rank  $r$  and degree 0 with  $H^0(C, \mathcal{F}_r) \neq 0$ , unique up to isomorphism. Moreover,  $\mathcal{F}_r$  is defined inductively as follows;  $\mathcal{F}_1 = \mathcal{O}_C$  and  $\mathcal{F}_r$  is the unique non-trivial extension of  $\mathcal{F}_{r-1}$  by  $\mathcal{O}_C$ .*
- (2) *Let  $F$  be an indecomposable vector bundle on  $C$  of rank  $r$  and degree 0. Then  $F \cong \mathcal{F}_r \otimes \mathcal{L}$  for a unique line bundle  $\mathcal{L}$  of degree 0 on  $C$  such that  $\mathcal{L}^{\otimes r} \cong \det F$ .*
- (3) *The bundle  $\mathcal{F}_r$  is self-dual, that is,  $\mathcal{F}_r^\vee \simeq \mathcal{F}_r$ , where  $\mathcal{F}_r^\vee := \text{Hom}_{\mathcal{O}_C}(\mathcal{F}_r, \mathcal{O}_C)$ .*

The following is an immediate consequence of Theorem 4.11.

**Lemma 4.12.** *Let  $h: C \rightarrow C$  be a non-isomorphic étale endomorphism of an elliptic curve  $C$ . Then  $h^*\mathcal{F}_r \simeq \mathcal{F}_r$  for any  $r > 0$ .*

**Proof.** By construction, the vector bundle  $\mathcal{F}_r$  is constructed as a successive extension of trivial line bundles by the following non-split exact sequence;

$$(E_r): 0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow 0.$$

We use induction on  $r$ . Since  $\mathcal{F}_1 \simeq \mathcal{O}_C$ , the claim is trivial for  $r = 1$ . Assume that the claim holds true for  $r - 1$ . Let  $\delta_r (\neq 0)$  be the non-trivial extension class in  $\text{Ext}^1(\mathcal{F}_{r-1}, \mathcal{O}_C) \simeq H^1(C, \mathcal{F}_{r-1}^\vee) \simeq \mathbb{C}$  corresponding to the non-split exact sequence  $(E_r)$ . Then pulling-back  $(E_r)$  by  $h: C \rightarrow C$ , we obtain the following exact sequence:

$$(1) \quad 0 \longrightarrow \mathcal{O}_C \longrightarrow h^*\mathcal{F}_r \longrightarrow h^*\mathcal{F}_{r-1} \longrightarrow 0.$$

Thanks to the projection formula and the finiteness of  $h$ , we infer that

$$\mathrm{Ext}^1(h^*\mathcal{F}_{r-1}, \mathcal{O}_C) \simeq H^1(C, h^*\mathcal{F}_{r-1}^\vee) \simeq H^1(C, h_*h^*\mathcal{F}_{r-1}^\vee) \simeq H^1(C, \mathcal{F}_{r-1}^\vee \otimes h_*\mathcal{O}_C).$$

On the other hand, the trace map  $\mathrm{Tr}_{C/C}: h_*\mathcal{O}_C \rightarrow \mathcal{O}_C$  gives rise to a splitting of the natural inclusion  $\mathcal{O}_C \hookrightarrow h_*\mathcal{O}_C$ . Hence  $\mathcal{F}_{r-1}^\vee \rightarrow \mathcal{F}_{r-1}^\vee \otimes h_*\mathcal{O}_C$  likewise splits. Thus  $\mathrm{Ext}^1(\mathcal{F}_{r-1}, \mathcal{O}_C) \simeq H^1(C, \mathcal{F}_{r-1}^\vee)$  embeds as a direct summand of  $\mathrm{Ext}^1(h^*\mathcal{F}_{r-1}, \mathcal{O}_C)$ . Hence the natural homomorphism  $h^*: \mathrm{Ext}^1(\mathcal{F}_r, \mathcal{O}_C) \rightarrow \mathrm{Ext}^1(h^*\mathcal{F}_r, \mathcal{O}_C)$  is injective. Thus  $h^*\delta_r \neq 0$  and the exact sequence (1) does not split. By assumption, we have  $h^*\mathcal{F}_{r-1} \simeq \mathcal{F}_{r-1}$ . Since  $H^0(C, \mathcal{F}_{r-1}) \neq 0$  and  $H^0(C, h^*\mathcal{F}_{r-1}) \neq 0$ , we infer that  $h^*\mathcal{F}_r \simeq \mathcal{F}_r$  by Theorem 4.11.  $\square$

The following proposition gives a sufficient condition for a projective bundle over an elliptic curve to have a non-isomorphic étale endomorphism.

**Proposition 4.13.** *For a positive integer  $r > 1$ , let  $\varphi: Y := \mathbb{P}_C(\mathcal{F}_r) \rightarrow C$  be a  $\mathbb{P}^{r-1}$ -bundle associated to the vector bundle  $\mathcal{F}_r$  on an elliptic curve  $C$ . Then  $Y$  admits a non-isomorphic étale endomorphism.*

*Proof.* For any positive integer  $k (> 1)$ , let  $\mu_k: C \rightarrow C$  be multiplication by  $k$ . Then by Lemma 4.12, there exists an isomorphism

$$\widetilde{Y} := Y \times_{C, \mu_k} C \simeq \mathbb{P}_C(\mu_k^*\mathcal{F}_m) \simeq \mathbb{P}_C(\mathcal{F}_m) =: Y.$$

Hence the natural morphism  $Y \simeq \widetilde{Y} \rightarrow Y$  gives an étale endomorphism of  $Y$  of degree  $k^2 (> 1)$ .  $\square$

## 5. Surjective morphisms between elliptic ruled surfaces

In this section, we shall study the structure of surjective morphisms between  $\mathbb{P}^1$ -bundles over elliptic curves which are not necessarily endomorphisms. Propositions 5.6, 5.15 and 5.16 are our main results, which may be new and are necessary for use in the proof of Theorem 10.1. Our methods of proof are quite elementary and based on the one-to-one correspondence between the Kleiman-Mori cones of the source variety and target variety. Proposition 5.5 is devoted to the study of endomorphisms of the ‘Atiyah surface’. Combining all the results stated as above, we shall study ESPs of elliptic ruled surfaces in Proposition 5.10. First we recall some remarkable properties of Atiyah’s vector bundle  $\mathcal{F}_2$ . Hereafter, we shall use the following terminology.

**DEFINITION 5.1.** Let  $\mathcal{F}_2$  be an indecomposable semi-stable locally free sheaf of rank 2 and degree 0 on an elliptic curve  $C$  (cf. Theorem 4.11). Let  $\pi: \mathbb{S} := \mathbb{P}_C(\mathcal{F}_2) \rightarrow C$  be the  $\mathbb{P}^1$ -bundle associated with  $\mathcal{F}_2$  and  $s_\infty$  the section of  $\pi$  corresponding to a surjection  $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_C$ . Then we call  $\mathbb{S}$  an ‘Atiyah surface’ (over  $C$ ) and  $s_\infty$  the ‘canonical section’ of  $\pi$ .

The following seems to be well-known.

**Proposition 5.2.** *Let  $\pi: \mathbb{S} \rightarrow C$  be the Atiyah surface. Then the following hold:*

- (1) *The canonical section  $s_\infty$  is a unique irreducible curve on  $\mathbb{S}$  such that its self-intersection number is zero and is not contained in any fiber of  $\pi$ .*
- (2)  *$h^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a s_\infty)) = 1$  for any positive integer  $a$ .*

**Corollary 5.3.** *There exists no elliptic fiber space structure on the Atiyah surface  $\mathbb{S}$ .*

*Proof.* Suppose that there exists on  $\mathbb{S}$  an elliptic surface structure  $\rho: \mathbb{S} \rightarrow \Delta$ . Since  $K_{\mathbb{S}} \sim -2s_{\infty}$  and  $s_{\infty}^2 = 0$ , the anti-canonical divisor  $-K_{\mathbb{S}}$  of  $\mathbb{S}$  is nef and the intersection number  $(K_{\mathbb{S}}, \gamma)$  is an even integer for every irreducible curve  $\gamma$  on  $\mathbb{S}$ . Hence  $\rho$  is a relatively minimal elliptic fibration. Clearly, we have  $(K_{\mathbb{S}}, F) = F^2 = 0$  for a general fiber  $F$  of  $\rho$ . Then either  $s_{\infty} \cap F = \emptyset$  or  $s_{\infty} = F$ , since  $s_{\infty}^2 = 0$ . Then Proposition 5.2 implies that  $F = s_{\infty}$ . Thus  $\mathbb{S} = s_{\infty}$ , which derives a contradiction.  $\square$

**REMARK 5.4.** By Proposition 4.13, the Atiyah surface  $\mathbb{S}$  admits a non-isomorphic étale endomorphism. (cf. [38])

One basic fact worth mentioning is the following.

**Proposition 5.5.** *Let  $\pi: \mathbb{S} \rightarrow C$  be the Atiyah surface. Then every surjective endomorphism  $\varphi: \mathbb{S} \rightarrow \mathbb{S}$  is a finite étale covering satisfying  $\varphi^*(s_{\infty}) = s_{\infty}$ .*

*Proof.* Since the pull-back map  $\varphi^*: N_1(\mathbb{S}) \rightarrow N_1(\mathbb{S})$  induces an isomorphism of the Kleiman-Mori cone  $\overline{NE}(\mathbb{S})$ , which is a 2-dimensional closed convex polyhedral cone. Hence  $\varphi^*$  induces a permutation of the two extremal rays of  $\overline{NE}(\mathbb{S})$ . Since  $\pi$  is the Albanese map of  $\mathbb{S}$ , there exists an étale endomorphism  $h: C \rightarrow C$  with  $\pi \circ \varphi = h \circ \pi$  by the universality of the Albanese map. Hence both extremal rays  $\mathbb{R}_{\geq 0}[s_{\infty}]$  and  $\mathbb{R}_{\geq 0}[f]$  are preserved by  $\varphi^*$ . We have  $K_{\mathbb{S}} \sim \varphi^* K_{\mathbb{S}} + R_{\varphi}$ , where  $R_{\varphi}$  is an effective divisor called the ramification divisor of  $\varphi$ . Then  $K_{\mathbb{S}} \equiv -2s_{\infty}$ , since  $K_{\mathbb{S}} \sim -2L + \pi^*(K_C + \det \mathcal{F}_2)$  for the tautological divisor  $L := \mathcal{O}_{\mathbb{P}(\mathcal{F}_2)}(1)$ . Hence  $[R_{\varphi}]$  belongs to the ray  $\mathbb{R}_{\geq 0}[s_{\infty}]$ .

The proof is by contradiction. Suppose the contrary that  $R_{\varphi} \neq 0$  and let  $R_{\varphi} = \sum_i a_i D_i$  be an irreducible decomposition. Since  $[s_{\infty}]$  spans the extremal ray of  $\overline{NE}(\mathbb{S})$ , we have  $[D_i] \in \mathbb{R}_{\geq 0}[s_{\infty}]$  for all  $i$ . Then,  $D_i = s_{\infty}$  for all  $i$  by Proposition 5.2. Hence we have  $R_{\varphi} = b s_{\infty}$  for some positive integer  $b$  and  $\varphi$  is unramified over  $\mathbb{S} \setminus s_{\infty}$ . Applying Lemma 4.12, there exists an isomorphism  $\mathbb{S} \times_{h,C} C \cong \mathbb{P}_C(h^* \mathcal{F}_2) \cong \mathbb{P}_C(\mathcal{F}_2) =: \mathbb{S}$ . Since  $h$  is étale, the natural morphism  $\psi: \mathbb{S} \rightarrow \mathbb{S} \times_{h,C} C \simeq \mathbb{S}$  is also unramified over  $\mathbb{S} \setminus s_{\infty}$ . For any fiber  $f$  of  $\pi$ , set  $p := \pi(f)$  and  $F := \varphi(f)$ . Then  $F = \pi^{-1}(h(p))$  is a disjoint union of fibers of  $\pi$  and  $f$  is one of its connected components. We have  $K_{\mathbb{S}} \sim \varphi^* K_{\mathbb{S}} + b s_{\infty}$ . The restriction of this to the fiber  $f$  of  $\pi$  gives the following equality:

$$K_f \sim (\varphi|_f)^* K_F + b s_{\infty}|_f.$$

Hence the restriction of the morphism  $\psi$  to  $f$  gives a finite ramified covering from  $\mathbb{P}^1$  to itself branched over only one point  $s_{\infty} \cap f$ . Thus a contradiction is derived. Hence  $\varphi$  is a finite étale covering and there exists an isomorphism  $\psi: \mathbb{S} \cong \mathbb{S} \times_{h,C} C$ . Since  $\varphi^*[s_{\infty}] \in \mathbb{R}_{\geq 0}[s_{\infty}]$ , Proposition 5.2 implies that  $\varphi^{-1}(s_{\infty}) = s_{\infty}$ .  $\square$

We prove some simple propositions that provide the key to the proof of Theorem 1.5. First, we study the structure of a surjective morphism  $\psi$  between  $\mathbb{P}^1$ -bundles over elliptic curves which are associated to semi-stable vector bundles of rank 2. Note that  $\psi$  is not necessarily an endomorphism.

**Proposition 5.6.** *There exist no surjective morphisms from one to another among the following three  $\mathbb{P}^1$ -bundles  $S_i$  over an elliptic curve  $C_i$  ( $1 \leq i \leq 3$ ).*

- (1)  $\pi_1: S_1 = \mathbb{P}_{C_1}(\mathcal{O}_{C_1} \oplus \ell) \rightarrow C_1$  for a line bundle  $\ell$  of degree 0 on  $C_1$ , where  $\ell \in \text{Pic}^0(C_1)$  is torsion.
- (2)  $\pi_2: S_2 = \mathbb{P}_{C_2}(\mathcal{O}_{C_2} \oplus \mathcal{L}) \rightarrow C_2$  for a line bundle  $\mathcal{L}$  of degree 0 on  $C_2$ , where  $\mathcal{L} \in \text{Pic}^0(C_2)$  is of infinite order.
- (3)  $\pi_3: S_3 = \mathbb{S} \rightarrow C_3$  is the Atiyah surface.

Proof. Let  $s_i$  be the section of  $\pi_i$  ( $1 \leq i \leq 3$ ) corresponding to a surjection  $\mathcal{O}_{C_1} \oplus \ell \twoheadrightarrow \ell$  (resp.  $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$  and  $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_{C_3}$ ). It is well-known that  $\rho(S_i) = 2$  and the Kleiman-Mori cone  $\overline{\text{NE}}(S_i)$  is a two-dimensional closed convex polyhedral cone spanned by the numerical equivalence classes  $[f_i]$  and  $[s_i]$  of a fiber  $f_i$  and a section  $s_i$  of  $\pi_i$ , i.e.,  $\overline{\text{NE}}(S_i) = \mathbb{R}_{\geq 0}[f_i] + \mathbb{R}_{\geq 0}[s_i]$ . The proof is by contradiction.

(1) Suppose the contrary that there exists a surjective morphism  $\varphi: S_1 \twoheadrightarrow S_2$ . Since  $\rho(S_i) = 2$  for all  $i$ , by Proposition 3.1, there is induced an isomorphism  $\varphi_*: \overline{\text{NE}}(S_1) \rightarrow \overline{\text{NE}}(S_2)$  of the Kleiman-Mori cone by the push-forward map  $\varphi_*: N_1(S_1) \rightarrow N_1(S_2)$ . Hence there exists a one-to-one correspondence between the extremal rays of  $\overline{\text{NE}}(S_i)$ 's. Then  $\varphi_*(\mathbb{R}_{\geq 0}[f_1]) = \mathbb{R}_{\geq 0}[f_2]$ , since there exists a finite morphism  $h: C_1 \rightarrow C_2$  such that  $\pi_2 \circ \varphi = h \circ \pi_1$  by the universality of the Albanese map. Hence  $\varphi_*(\mathbb{R}_{\geq 0}[s_1]) = \mathbb{R}_{\geq 0}[s_2]$ . Since  $\ell \in \text{Pic}^0(C_1)$  is of finite order, there exists a finite étale covering  $\widetilde{S}_1$  of  $S_1$  which is isomorphic to the trivial  $\mathbb{P}^1$ -bundle  $C_1 \times \mathbb{P}^1 \rightarrow C_1$ . Thus all the irreducible curves  $\Gamma$  on  $S_1$  which are numerically equivalent to  $s_1$  form a positive-dimensional algebraic family. Let  $s'_2$  be another section of  $\pi_2$  corresponding to a surjection  $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$ . Since  $\mathcal{L} \in \text{Pic}(C_2)$  is of infinite order,  $s'_2$  is the unique irreducible curve which is numerically equivalent to  $s_2$ . (The reason is as follows: Let  $D(\neq s_2, s'_2)$  be another irreducible curve such that  $D \equiv s_2 \equiv s'_2$ . Since  $(D, s_2) = (D, s'_2) = 0$ ,  $D, s_2$  and  $s'_2$  are three disjoint sections of  $\pi_2$ . Hence  $\pi_2: S_2 \rightarrow C_2$  is a trivial  $\mathbb{P}^1$ -bundle and a contradiction is derived.) Hence  $\varphi(\Gamma) = s_2$  or  $s'_2$ . Since  $\Gamma$  moves and sweeps out  $S_1$ ,  $\varphi(S_1) \subset s_2 \cup s'_2$  and  $\varphi$  is not surjective. Thus a contradiction.

(2) Suppose the contrary that there exists a surjective morphism  $\psi: S_2 \twoheadrightarrow S_1$ . Then, by Proposition 3.1, the pull-back map  $\psi^*$  gives an isomorphism  $\psi^*: \overline{\text{NE}}(S_1) \cong \overline{\text{NE}}(S_2)$  of the Kleiman-Mori cone. By the similar method as in (1), we can show that  $\psi^{-1}(\Gamma) = s_2 \cup s'_2$  for all the irreducible curve  $\Gamma$  on  $S_1$  which is numerically equivalent to  $s_1$ . Since  $\Gamma$  moves and sweeps out  $S_1$ , we infer that  $\psi^{-1}(S_1) = s_2 \cup s'_2$ . This contradicts the fact that  $\psi$  is a map.

(3) Suppose the contrary that there exists a surjective morphism  $g: S_1 \twoheadrightarrow S_3$ . Then, by Proposition 5.2, the canonical section  $s_\infty$  is the unique irreducible curve on  $S_3$  which is numerically equivalent to  $s_\infty$ . By a similar method as in (1), we can show that  $g(\Gamma) = s_\infty$  for all the irreducible curves  $\Gamma$  on  $S_1$  which are numerically equivalent to  $s_1$ . Since  $\Gamma$  moves and sweeps out  $S_1$ , we infer that  $g(S_1) = s_\infty$ . This contradicts the assumption that  $g$  is surjective.

(4) Suppose the contrary that there exists a surjective morphism  $u: S_3 \twoheadrightarrow S_1$ . Then, by a similar method as in (1), we can show that  $u^{-1}(\Gamma) = s_\infty$  for all the irreducible curves  $\Gamma$  on  $S_1$ , which sweeps out  $S_1$ . Hence  $u^{-1}(S_1) = s_\infty$  and thus a contradiction.

(5) Suppose the contrary that there exists a surjective morphism  $v: S_2 \twoheadrightarrow S_3$ . Then, by the universality of the Albanese map, there exists a finite covering  $h: C_2 \rightarrow C_3$  with  $\pi_3 \circ v = h \circ \pi_2$ . By a similar method as in (1), we can show that  $v_*(\mathbb{R}_{\geq 0}[s_2]) = \mathbb{R}_{\geq 0}([s_3])$ . Then  $K_{S_2} \sim v^*K_{S_3} + R_v$  for the ramification divisor  $R_v$  of  $v$ . Since  $K_{S_2} \equiv -2s_2$ ,  $K_{S_3} \equiv -2s_3$ , we have  $R_v \equiv as_2$  for some non-negative integer  $a \geq 0$  and the branched divisor  $B_v$  of  $v$  is supported on  $s_3$  by Proposition 5.2. If  $a = 0$ , then  $v$  is finite étale and of degree one on each

fiber of  $\pi_2$ . Hence, by Lemma 4.12, there exists an isomorphism  $S_2 \cong S_3 \times_{C_3} C_2 \cong S_3 = \mathbb{S}$ , which derives a contradiction. If  $a > 0$ , then by Proposition 5.2,  $v$  is unramified outside  $v^{-1}(s_3)$ . By the same reason as in the proof of (1), we infer that  $v^{-1}(s_3) = s_2 \sqcup s'_2$ , where  $s_2, s'_2$  are disjoint two sections of  $\pi_2$  corresponding to surjections  $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{O}_{C_2}$  and  $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$ , respectively. For each fiber  $f$  of  $\pi_3$ ,  $f \setminus s_3 \cong \mathbb{C}$  is simply connected. Hence the inverse image  $v^{-1}(f \setminus s_3)$  is a disjoint union of copies of  $\mathbb{C}$ . However, by the above remark,  $v^{-1}(f \setminus s_3) = v^{-1}(f) \setminus (s_2 \sqcup s'_2)$  is a disjoint union of copies of  $\mathbb{C}^*$ . Thus a contradiction.

(6) Suppose the contrary that there exists a surjective morphism  $w: S_3 \twoheadrightarrow S_2$ . Then, by a similar method as in (4), we can show that  $w^{-1}(s_2) = w^{-1}(s'_2) = s_3$ . This contradicts the assumption that  $w$  is a map.  $\square$

We employ the same notation as in Proposition 5.6.

**Corollary 5.7.** *Let  $C'$  be an elliptic curve and*

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{C'}([p]) \longrightarrow 0$$

*a non-split extension for a point  $p \in C'$ . Let  $\pi': V := \mathbb{P}_{C'}(\mathcal{E}) \rightarrow C'$  be the  $\mathbb{P}^1$ -bundle over  $C'$  associated with the stable vector bundle  $\mathcal{E}$ . Then*

- (1) *There exist no surjective morphisms  $V \twoheadrightarrow S_3$  and  $S_3 \twoheadrightarrow V$ .*
- (2) *There exist no surjective morphisms  $V \twoheadrightarrow S_2$  and  $S_2 \twoheadrightarrow V$ .*
- (3) *There exist no finite étale coverings from  $V$  to  $S_1$ .*

*Proof.* (1) Suppose that there exists a surjective morphism  $\varphi: V \twoheadrightarrow S_3$ . By Proposition 4.4, there exists an étale double covering  $p: \widetilde{V} \rightarrow V$  and an isomorphism  $\widetilde{V} \cong \mathbb{P}^1 \times C''$  for some elliptic curve  $C''$ . Then,  $\varphi \circ p$  is a surjective morphism from  $\widetilde{V}$  to  $S_3$ , which contradicts Proposition 5.6. Conversely, suppose that there exists a surjective morphism  $\psi: S_3 \twoheadrightarrow V$ . Let  $\widetilde{S}_3$  be the connected component of the fiber product  $S_3 \times_V \widetilde{V}$  and  $q: \widetilde{S}_3 \rightarrow \widetilde{V}$  the natural projection. Then by Lemma 4.12, the surface  $\widetilde{S}_3$  is isomorphic to the Atiyah surface  $\mathbb{S}$  and this contradicts Proposition 5.6.

(2) Suppose that there exists a surjective morphism  $f: V \twoheadrightarrow S_2$ . Then,  $f \circ p: \widetilde{V} \twoheadrightarrow S_2$  gives a surjective morphism from  $\widetilde{V}$  to  $S_2$ , which contradicts Proposition 5.6. Conversely, suppose that there exists a surjective morphism  $g: S_2 \twoheadrightarrow V$ . Let  $\widetilde{S}_2$  be the connected component of the fiber product  $S_2 \times_V \widetilde{V}$  and  $q: \widetilde{S}_2 \rightarrow \widetilde{V} \simeq \mathbb{P}^1 \times C''$  the natural projection. Then there exist an elliptic curve  $D$  and an isomorphism  $\widetilde{S}_2 \simeq \mathbb{P}_D(\mathcal{O}_D \oplus \ell')$  for some non-torsion line bundle  $\ell' \in \text{Pic}^0(D)$ . Hence a contradiction is derived by Proposition 5.6.

(3) Suppose that there exists a finite étale covering  $u: V \rightarrow S_1$ . Then there exists a finite morphism  $h: C' \rightarrow C_1$  with  $\pi_1 \circ u = h \circ \pi'$ . Since  $u$  is étale,  $u$  is of degree one restricted to each fiber of  $\pi'$  and there exists an isomorphism  $V \simeq S_1 \times_{C_1} C' \simeq \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus h^*\ell)$ . Since  $\mathcal{E}$  is indecomposable, we have a contradiction.  $\square$

**REMARK 5.8.** Corollary 5.7 (3) does not necessarily hold without the assumption that the morphism is étale. In fact, using Proposition 4.4, we can construct an example of a finite ramified covering  $\rho: V \rightarrow S_1$  of degree 4.

There is also an analogous statement for elliptic ruled surfaces which are associated to unstable vector bundles of rank 2 on elliptic curves.

**Lemma 5.9.** *Let  $\pi: \mathbb{S} \rightarrow C$  be the Atiyah surface and  $T := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \ell)$  a  $\mathbb{P}^1$ -bundle, where  $\ell$  is a line bundle on an elliptic curve  $C'$  with  $\deg \ell \neq 0$ . Furthermore, set  $U := \mathbb{P}_{C''}(\mathcal{O}_{C''} \oplus \mathcal{L})$  for a torsion line bundle  $\mathcal{L} \in \text{Pic}^0(C'')$  on an elliptic curve  $C''$ . Then the followings hold:*

- (1) *There exist no surjective morphisms  $\mathbb{S} \twoheadrightarrow T$  and  $T \twoheadrightarrow \mathbb{S}$ .*
- (2) *There exist no surjective morphisms  $\mathbb{S} \twoheadrightarrow U$  and  $U \twoheadrightarrow \mathbb{S}$ .*

*Proof.* Since the proof of (2) is completely analogue to that of (1), we shall only give a proof of (1). Let  $\Delta(\subset T)$  be the section of  $\pi_T: T \rightarrow C$  corresponding to the second projection  $\mathcal{O}_C \oplus \ell \twoheadrightarrow \ell$  and  $F$  a general fiber of  $\pi_T$ . It is well-known that  $\rho(T) = 2$  and the Kleiman-Mori cone  $\overline{\text{NE}}(T)$  is a 2-dimensional, closed convex polyhedral cone spanned by the numerical equivalence classes  $[F]$  and  $[\Delta]$ ; i.e.,  $\overline{\text{NE}}(T) = \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[\Delta]$ .

(1) Suppose that there exists a surjective morphism  $\varphi: \mathbb{S} \twoheadrightarrow T$ . Since  $\rho(\mathbb{S}) = \rho(T) = 2$ , by Proposition 3.1, there is induced an isomorphism  $\varphi^*: \overline{\text{NE}}(T) \simeq \overline{\text{NE}}(\mathbb{S})$  by the pull-back map  $\varphi^*: N^1(T) \rightarrow N^1(\mathbb{S})$ . Hence there exists a one-to-one correspondence between the extremal rays of  $\overline{\text{NE}}(T)$  and  $\overline{\text{NE}}(\mathbb{S})$ . Then  $\varphi^*(\mathbb{R}_{\geq 0}[F]) = \mathbb{R}_{\geq 0}[f]$ , since there exists a finite morphism  $h: C \rightarrow C'$  such that  $\pi_T \circ \varphi = h \circ \pi_{\mathbb{S}}$ . Hence  $\varphi^*(\mathbb{R}_{\geq 0}[F]) = \mathbb{R}_{\geq 0}[f]$ . Thus  $\varphi^*\Delta \equiv as_{\infty}$  for some  $a > 0$ . Now we calculate the self-intersection number of both sides. We have  $(\varphi^*\Delta)^2 = \deg \varphi \cdot \Delta^2 = \deg \varphi \cdot \deg \ell \neq 0$ , which contradicts  $s_{\infty}^2 = 0$ .  $\square$

Applying these results, we shall classify  $\mathbb{P}^1$ -bundles over elliptic curves admitting an ESP (cf. Definition 2.3).

**Proposition 5.10.** *Suppose that there exists an ESP  $S_{\bullet} = (g_n: S_n \rightarrow S_{n+1})_n$  of elliptic ruled surfaces  $S_n$ . Then, one of the following cases occurs:*

- (1) *There exists an integer  $k \leq \infty$  such that*
  - (a) *For any  $i \leq k$ ,  $S_i \simeq \mathbb{P}_{C_i}(\mathcal{O}_{C_i} \oplus \ell_i)$  for some torsion line bundle  $\ell_i \in \text{Pic}^0(C_i)$  on an elliptic curve  $C_i$ .*
  - (b) *For any  $i > k$ ,  $S_i \simeq \mathbb{P}_{C_i}(\mathcal{E}_i)$ , where  $\mathcal{E}_i$  is a stable vector bundle of rank two and degree one on an elliptic curve  $C_i$ .*
- (2) *For any  $i$ ,  $S_i \simeq \mathbb{P}_{C_i}(\mathcal{O}_{C_i} \oplus \mathcal{L}_i)$  for some non-torsion line bundle  $\mathcal{L}_i$  of degree 0 on an elliptic curve  $C_i$ .*
- (3)  *$S_i$  is isomorphic to the Atiyah surface  $\mathbb{S}$  for any  $i$ .*

*Proof.* By Proposition 4.1, any  $S_n$  is a  $\mathbb{P}^1$ -bundle associated to a semi-stable vector bundle of rank two on an elliptic curve. Hence all  $S_n$  is isomorphic to one of the following 4 types:

- type (a):  $\mathbb{P}_{C_n}(\mathcal{E}_n)$ , where  $\mathcal{E}_n$  is a stable vector bundle of rank two and degree one on an elliptic curve  $C_n$ .
- type (b): the Atiyah surface  $\mathbb{S}$ .
- type (c):  $\mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \ell_n)$ , where  $\ell_n \in \text{Pic}^0(C_n)$  is a torsion line bundle on an elliptic curve  $C_n$ .
- type (d):  $\mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$ , where  $\mathcal{L}_n \in \text{Pic}^0(C_n)$  is a line bundle of infinite order on an elliptic curve  $C_n$ .

By considering a truncated sequence  $S_n \xrightarrow{g_n} S_{n+1} \xrightarrow{g_{n+1}} \cdots$  starting from  $S_n$  for each  $n$ , it is sufficient to show the claim for  $i \geq 0$ .

- (1) Suppose that  $S_0$  is of type (a). Then, by Proposition 5.6 and Corollary 5.7,  $S_n$  is of

type (a) for any  $n > 0$ .

(2) Suppose that  $S_0$  is of type (b). Then, by Proposition 5.6 and Corollary 5.7,  $S_n$  is of type (b) for any  $n > 0$ .

(3) Suppose that  $S_0$  is of type (c). Then, the claim (1) holds by Proposition 5.6 and Corollary 5.7,

(4) Suppose that  $S_0$  is of type (d). Then, by Proposition 5.6 and Corollary 5.7,  $S_n$  is of type (d) for any  $n > 0$ .  $\square$

Now, we devote the rest of this section to describe the structure of the set of the surjective morphisms between certain elliptic ruled surfaces; those which are associated to decomposable vector bundles of rank 2 and degree 0 on elliptic curves and with only two disjoint sections. First, we start with an auxiliary result concerning morphisms of elliptic curves.

**Lemma 5.11.** *Let  $h, g: C \rightarrow C'$  be two surjective morphisms between elliptic curves  $C$  and  $C'$ . Suppose that  $h^*\ell \simeq g^*\ell$  for a line bundle  $\ell$  of degree zero on  $C'$ , where  $\ell \in \text{Pic}^0(C')$  is of infinite order. Then  $h = g$  up to translation under the group law of the elliptic curve  $C'$ .*

*Proof.* Since  $\ell \in \text{Pic}^0(C')$ ,  $\ell$  is invariant under translation (cf. [32]). Since any surjective morphism from  $C$  to  $C'$  can be expressed as a composite of a group homomorphism and a translation, we may assume that both  $g$  and  $h$  are group homomorphisms. If  $h - g: C \rightarrow C'$  is surjective, then Proposition 4.7 implies that  $\ell$  is of finite order, which contradicts the assumption.  $\square$

Let  $C$  (resp.  $C'$ ) be an elliptic curve and  $\ell$  (resp.  $\mathcal{L}$ ) a line bundle of degree zero on  $C$  (resp.  $C'$ ) which is of infinite order in  $\text{Pic}^0(C)$  (resp.  $\text{Pic}^0(C')$ ). Let  $\pi_S: S := \mathbb{P}_C(\mathcal{O}_C \oplus \ell) \rightarrow C$  (resp.  $\pi_T: T := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}) \rightarrow C'$ ) be the  $\mathbb{P}^1$ -bundle over  $C$  (resp.  $C'$ ) associated to a decomposable vector bundle of rank 2 and degree 0. We describe the structure of the set  $\text{Mor}_{\text{surj}}(S, T)$  consisting of surjective morphism  $\varphi: S \rightarrow T$  from  $S$  to  $T$ . Let  $s$  and  $s'$  (resp.  $t$  and  $t'$ ) be the sections of  $\pi_S$  (resp.  $\pi_T$ )  $C$  corresponding to the first and second projections  $\mathcal{O}_C \oplus \ell \rightarrow \ell$  and  $\mathcal{O}_C \oplus \ell \rightarrow \mathcal{O}_C$  (resp.  $\mathcal{O}_{C'} \oplus \mathcal{L} \rightarrow \mathcal{L}$  and  $\mathcal{O}_{C'} \oplus \mathcal{L} \rightarrow \mathcal{O}_{C'}$ ). By the universality of the Albanese map, there exists a finite morphism  $h: C \rightarrow C'$  such that  $h \circ \pi_S = \pi_T \circ \varphi$ .

Since  $\rho(S) = \rho(T) = 2$ , Proposition 3.1 implies that  $\varphi$  is a finite morphism and the push-forward map  $\varphi_*: N_1(S) \simeq N_1(T)$  (resp. the pull-back map  $\varphi^*: N^1(T) \simeq N^1(S)$ ) gives a one-to-one correspondence between  $\overline{\text{NE}}(S)$  and  $\overline{\text{NE}}(T)$ . They are two-dimensional closed convex polyhedral cones spanned by the numerical equivalence classes of a general fiber of  $\pi_S$  (resp.  $\pi_T$ ) and  $s$  (resp.  $t$ ). Then, applying the same method as in the proof of Proposition 5.6, we see that  $\varphi^{-1}(t) \subset s \cup s'$  and  $\varphi^{-1}(t') \subset s \cup s'$ . Since  $\varphi$  is a surjective morphism, one of the following cases occur:

- $\varphi^{-1}(t) = s$  and  $\varphi^{-1}(t') = s'$  or
- $\varphi^{-1}(t) = s'$  and  $\varphi^{-1}(t') = s$ .

In the former (resp. latter) case, we have  $\varphi^*t \sim \alpha s$  (resp.  $\varphi^*t \sim \beta s$ ) for some positive integer  $\alpha$  (resp.  $\beta$ ). Since  $\mathcal{N}_{s/S} \simeq \ell$ ,  $\mathcal{N}_{s'/S} \simeq \ell^{\otimes -1}$ , and  $\mathcal{N}_{t/T} \simeq \mathcal{L}$ ,  $\mathcal{N}_{t'/T} \simeq \mathcal{L}^{\otimes -1}$ , it follows that  $h^*\mathcal{L} \simeq \ell^{\otimes n}$  in  $\text{Pic}^0(C)$  for some integer  $n$  in both cases. Such an integer  $n$  is uniquely determined and non-zero, since both  $\ell \in \text{Pic}^0 C$  and  $\mathcal{L} \in \text{Pic}^0 C'$  are of infinite order. Thus, according to the discussions made as above, we have the following description of  $h$  in terms of line bundles  $\mathcal{L}$  and  $\ell$ :

**Lemma 5.12.** *Let  $S, T$  and  $h$  as above. Then,  $h^* \mathcal{L} \simeq \ell^{\otimes n}$  for a unique non-zero integer  $n$ .*

By replacing  $\ell$  by  $\ell^{-1}$ , we may assume that  $n > 0$ . Let  $\widetilde{T} := T \times_C C \rightarrow C$  be the pull-back of  $\pi_T: T \rightarrow C'$  by  $h: C \rightarrow C'$ . Then by construction, there exists an isomorphism

$$\widetilde{T} \simeq \mathbb{P}_C(\mathcal{O}_C \oplus h^* \mathcal{L}) \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell^{\otimes n})$$

over  $C$ . By the universality of the fiber product, there exists a natural morphism  $\psi: S \rightarrow \widetilde{T}$  over  $C$  so that the following commutative diagram holds:

$$\begin{array}{ccccc} S & \xrightarrow{\psi} & \widetilde{T} & \xrightarrow{\rho} & T \\ \pi_S \downarrow & & \pi_{\widetilde{T}} \downarrow & & \pi_T \downarrow \\ C & \xrightarrow{\text{id}_C} & C & \xrightarrow{h} & C' \end{array}$$

Clearly, the natural morphism  $\rho: \widetilde{T} \rightarrow T$  is a finite étale covering.

**Lemma 5.13.** *If  $n > 1$ , then the map  $\psi$  is of degree  $n$  on each fiber of  $\pi_S$  and ramifies only along the two disjoint sections  $s, s'$  of  $\pi_S$  with ramification index  $n$ .*

*Proof.* Since  $n > 1$ , the finite morphism  $\psi$  is a ramified covering of degree  $n$ . Note that  $K_S \sim -2s + \pi_S^* \ell$ ,  $K_{\widetilde{T}} \sim -2\widetilde{t} + \pi_{\widetilde{T}}^*(\ell^{\otimes n})$  and  $\psi^* \widetilde{t} \equiv k s$  for some positive integer  $k$ . Since  $K_S \sim \psi^* K_{\widetilde{T}} + R_\psi$  for the ramification divisor  $R_\psi$  of  $\psi$ , we see that  $R_\psi \equiv \beta s$  for some integer  $\beta \geq 0$ . Applying the same argument as in the proof of Proposition 5.6,  $R_\psi$  is supported on  $s \sqcup s'$ , which is a union of two disjoint sections of  $\pi_S$ .  $\square$

**REMARK 5.14.** Conversely, we show that if  $h^* \mathcal{L} \simeq \ell^{\otimes n}$ , then there exists a finite surjective morphism  $\varphi: S \rightarrow T$ . Since there exists a natural inclusion

$$\mathcal{O}_C \oplus h^* \mathcal{L} \hookrightarrow \mathcal{O}_C \oplus \ell \oplus \cdots \oplus \ell^n = \text{Sym}^n(\mathcal{O}_C \oplus \ell),$$

there exists a natural projection  $p: \mathbb{P}_C(\text{Sym}^n(\mathcal{O}_C \oplus \ell)) \cdots \rightarrow \widetilde{T} := \mathbb{P}_C(\mathcal{O}_C \oplus h^* \mathcal{L})$ . Moreover, there exists a Veronese embedding

$$i: S := \mathbb{P}_C(\mathcal{O}_C \oplus \ell) \hookrightarrow \mathbb{P}_C(\text{Sym}^n(\mathcal{O}_C \oplus \ell)).$$

Then the composite map  $\rho \circ p \circ i: S \rightarrow T$  gives a finite surjective morphism from  $S$  to  $T$ .

Now, we define the set  $M$  as follows:

$$M := \{(n, v) \in \mathbb{Z} \times \text{Hom}_{\text{group}}(C, C') \mid v^* \mathcal{L} \simeq \ell^{\otimes n}\}.$$

Here by  $\text{Hom}_{\text{group}}(C, C')$ , we denote the set of group homomorphisms from  $C$  to  $C'$ . Then  $M \neq \emptyset$ , since  $(0, 0) \in M$ . Now the structure of the set  $M$  is described as follows:

**Proposition 5.15.** *We may endow  $M$  with the structure of an abelian group such that*

- (1) *the first projection  $p: M \rightarrow \mathbb{Z}$  is an injective group homomorphism.*
- (2) *There exists a unique element  $(d_0, v_0) \in M$  such that any  $(n, v) \in M$  can be expressed as  $n = d_0 k$  and  $v = \mu_k \circ v_0 = v_0 \circ \mu_k$  for some integer  $k \in \mathbb{Z}$ , where  $\mu_k$  denotes multiplication map by  $k$  on an elliptic curve.*

*Proof.* First, we show that  $M$  is an abelian group.

- (1) Let  $(n, v), (m, u) \in M$  be arbitrary elements. Then  $v^* \mathcal{L} \simeq \ell^{\otimes n}, u^* \simeq \ell^{\otimes m}$ . Since  $(v + u)^* \mathcal{L} \simeq v^* \mathcal{L} \otimes u^* \mathcal{L}$  by [32, p.75, (ii)], we infer that  $(n + m, v + u) \in M$ . Thus  $M$  is closed under multiplication.
- (2) The unit element of  $M$  is  $(0, 0)$ .
- (3) The inverse element of  $(n, v) \in M$  is  $(-n, -v)$ .

Clearly,  $p$  is a group homomorphism. Next we show that  $p$  is injective. Suppose that  $(n, v), (n, u) \in M$  for some integer  $n \in \mathbb{Z}$  and group homomorphisms  $v, u: C \rightarrow C'$ . Then  $v^* \mathcal{L} \simeq \ell^{\otimes n} \simeq u^* \mathcal{L}$ . By Lemma 5.11, we infer that  $v = u$ . Thus  $p$  is injective and  $M \simeq p(M)(C \setminus \mathbb{Z})$  is a cyclic group. Let  $(d_0, v_0) \in M$  (where  $d_0 > 0$ ) be its generator. Then it satisfies the required property.  $\square$

The following describes the structure of a finite surjective morphism  $\varphi: S \rightarrow T$  and is a key ingredient for the proof of Theorem 10.1.

**Proposition 5.16.** *Let  $\ell$  (resp.  $\mathcal{L}$ ) be a line bundle of degree zero on an elliptic curve  $C$  (resp.  $C'$ ) and  $\pi_S: S := \mathbb{P}_C(\mathcal{O}_C \oplus \ell) \rightarrow C$  (resp.  $\pi_T: T := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}) \rightarrow C'$ ) an associated  $\mathbb{P}^1$ -bundle. Suppose that both  $\ell \in \text{Pic}^0(C)$  and  $\mathcal{L} \in \text{Pic}^0(C')$  are of infinite order and there exists some finite surjective morphism from  $S$  to  $T$ . Then there exists a unique non-zero group homomorphism  $v_0: C \rightarrow C'$  such that  $v_0^* \mathcal{L} \simeq \ell^{\otimes d_0}$  for an integer  $d_0$  and that the following properties hold:*

- (1) *Let  $\varphi: S \rightarrow T$  be an arbitrary finite surjective morphism and  $h: C \rightarrow C'$  the finite morphism induced by the universality of the Albanese map. Then  $h^* \mathcal{L} \simeq \ell^{\otimes n}$  for a unique integer  $n$  which is a multiple of  $d_0$ . Furthermore, there exists multiplication mapping  $\mu_k: C \rightarrow C$  by a positive integer  $k := n(d_0)^{-1}$  such that  $h = v_0 \circ \mu_k$  up to translation under the group structure of  $C'$ .*
- (2) *Let  $\pi_{\widetilde{T}}: \widetilde{T} := T \times_{C'} C \rightarrow C$  be the  $\mathbb{P}^1$ -bundle obtained by the pull-back of  $\pi_T: T \rightarrow C'$  by  $h: C \rightarrow C'$ . Let  $\psi: S \rightarrow \widetilde{T}$  be the natural morphism. Then the map  $\psi$  is of degree  $n$  restricted to each fiber of  $\pi_S$  and ramifies only along the two disjoint sections of  $\pi_S$  with ramification index  $n$ . Moreover, for a general fiber  $F$  of  $\pi_T$ , the inverse image  $\varphi^{-1}(F)$  has at least  $k^2 = \deg h \cdot (\deg v_0)^{-1}$  connected components.*

$$\begin{array}{ccccccc}
 S & \xrightarrow{\psi} & \widetilde{T} & \longrightarrow & \widetilde{T}' & \longrightarrow & T \\
 \pi_S \downarrow & & \pi_{\widetilde{T}} \downarrow & & \pi_{\widetilde{T}'} \downarrow & & \pi_T \downarrow \\
 C & \xlongequal{\quad} & C & \xrightarrow{\mu_k} & C & \xrightarrow{v_0} & C'
 \end{array}$$

Proof. The finite morphism  $h: C \rightarrow C'$  can be uniquely expressed as  $h = T_\alpha \circ v$ , where  $T_\alpha: C' \simeq C'$  is a translation by  $\alpha \in C'$  and  $v: C \rightarrow C'$  is a Lie group homomorphism. Since  $T_\alpha^* \mathcal{L} \simeq \mathcal{L}$ , there exists an isomorphism:

$$T \times_{C', T_\alpha} C' = \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus T_\alpha^* \mathcal{L}) \simeq \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}) = T.$$

Hence we may assume that  $\alpha = 0$  and  $h: C \rightarrow C'$  is a group homomorphism. By Lemma 5.12,  $h^* \mathcal{L} \simeq \ell^n$  for a unique integer  $n$ . By Proposition 5.15, there exists a unique integer  $k$  such that  $n = d_0 k$  and  $v = \mu_k \circ v_0 = v_0 \circ \mu_k$ . Thus the assertion (1) has been proved. Since  $v_0^* \mathcal{L} \simeq \ell^{\otimes d_0}$ , if we put  $\widetilde{T}' := T \times_{C', v_0} C$ , then  $\widetilde{T}' \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell^{\otimes d_0})$ . Since  $\mu_k^* \ell \simeq \ell^{\otimes k}$ ,

we see that  $v^* \mathcal{L} \simeq \mu_k^*(\ell^{\otimes d_0}) \simeq \ell^{\otimes n}$ . Hence,  $\widetilde{T} := T \times_{C,v} C'$  is isomorphic to the fiber product  $\widetilde{T}' \times_{C,\mu_k} C$ . The natural morphism  $\psi: S \rightarrow \widetilde{T}$  has already been described as above. Note that  $\deg h = k^2 \cdot \deg v_0$  and the morphism  $\mu_k$  is unramified. Hence for a general fiber  $F$  of  $\pi_T$ , the inverse image  $\varphi^{-1}(F)$  consists of at least  $k^2 = \deg \mu_k = \deg v \cdot (\deg v_0)^{-1}$  connected components. Thus the assertion (2) has been proved and we are done.  $\square$

## 6. ESPs of smooth algebraic surfaces

In this section, we study the structure of ESPs of smooth algebraic surfaces (cf. Propositions 6.5, 6.6 and 6.7). These results are necessary in Section 7 when we study the structure of an FESP of 3-folds with negative Kodaira dimension. Proposition 6.8 is a slight generalization of [9, Proposition 4.9]. Propositions 6.8 and 6.9 are devoted to the study of an ESP of curves whose ambient varieties also form an ESP. Let  $Y_\bullet$  be an FESP constructed from a given ESP  $X_\bullet$  of 3-folds by a sequence of blowing-downs of an ESP. Proposition 6.9 is quite useful when we seek out elliptic curves for the candidates of the centers of Cartesian blowing-ups  $\pi_\bullet: X_\bullet \rightarrow Y_\bullet$  (cf. Corollary 1.2).

First, we recall some basic facts about elliptic fibrations and abelian fibrations.

**DEFINITION 6.1.** (cf. [13, Definition 2.3 and Lemma 2.4]) Let  $V \rightarrow S$  be a projective fiber space from a smooth variety  $V$  to a normal variety  $S$  whose general fibers are abelian varieties. It is called a *Seifert* abelian fiber space if there exist finite surjective morphisms  $W \rightarrow V$  and  $T \rightarrow S$  satisfying the following conditions:

- (1)  $W$  and  $T$  are smooth varieties;
- (2)  $W$  is isomorphic to the normalization of  $V \times_S T$  over  $T$ ;
- (3)  $W \rightarrow V$  is étale;
- (4)  $W \rightarrow T$  is smooth.

If  $V \rightarrow S$  is a Seifert abelian fiber space, then  $V$  is a unique relative minimal model over  $S$ , since  $K_V$  is relatively numerically trivial and there are no rational curves contained in fibers. If  $S$  is compact and  $\dim(V) = \dim(S) + 1$ , then we may replace the condition (4) with that  $W \simeq E \times T$  over  $T$  for an elliptic curve  $E$ . The notion of Seifert abelian fiber space was first introduced in [35] as the name of  $Q$ -smooth abelian fibration.

The following proposition reveals a new phenomenon; There exists a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a 3-fold  $X$  with  $\kappa(X) = -\infty$  which admits no Seifert elliptic fibrations.

**Proposition 6.2.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Assume that there exists an extremal ray  $R$  of divisorial type such that the exceptional divisor  $D$  of the contraction morphism  $\text{Cont}_R$  of  $R$  is isomorphic to the Atiyah surface  $\mathbb{S}$ . Then there exists on  $X$  no Seifert elliptic fiber space structure.*

**Proof.** The proof is by contradiction. Suppose the contrary that there exists on  $X$  a Seifert elliptic fiber space structure  $\varphi: X \rightarrow S$  over a surface  $S$ . Then, a suitable finite étale Galois cover  $p: X' \rightarrow X$  of  $X$  is isomorphic to the direct product  $T \times E$  of a smooth surface  $T$  and an elliptic curve  $E$ . From Lemma 3.12, there exists an extremal ray  $R'$  of

divisorial type on  $X'$  such that  $p_*R' = R$ . Let  $D'$  be the exceptional divisor of the contraction morphism  $\text{cont}_{R'}$ . Then applying Proposition 5.6 and Lemma 5.9 to the finite étale covering  $p: D' \rightarrow D$ , we see that  $D' \simeq \mathbb{S}$ .

Let  $q: X' \rightarrow T$  be the first projection. Then by applying [11, Theorem 2], we infer that the pushforward map  $q_*: \overline{\text{NE}}(X') \rightarrow \overline{\text{NE}}(T)$  induces a one-to-one correspondence between the set of extremal rays of  $X'$  and the set of extremal rays of  $T$ . Furthermore, let  $C$  be a  $(-1)$ -curve on the surface  $T$  whose numerical class  $[C]$  spans the extremal ray  $\bar{R} := q_*R$  of  $T$ . Then  $D' = C \times E$ . On the other hand, since  $D' \simeq \mathbb{S}$ , there exists no elliptic fiber space structure on  $D'$  by Corollary 5.3. Thus a contradiction.  $\square$

The following lemma may be well-known for experts and was stated implicitly in [9, Proposition 7.1]. We shall give a proof for reader's convenience.

**Lemma 6.3.** (1) *Let  $\varphi: S \rightarrow \mathbb{P}^1$  be a minimal elliptic surface with  $\kappa(S) = 1$  and  $\chi(\mathcal{O}_S) = 0$ . Then the canonical divisor  $K_S$  of  $S$  can be expressed as  $K_S \sim_{\mathbb{Q}} \varphi^*D$  for some  $\mathbb{Q}$ -divisor  $D$  on  $\mathbb{P}^1$  with  $\deg D \geq 1/42$ .*

(2) *Let  $\psi: T \rightarrow C$  be a minimal elliptic surface over an elliptic curve  $C$  such that  $\kappa(T) = 1$  and  $\chi(\mathcal{O}_S) = 0$ . Then  $K_T \equiv \psi^*D$  for some  $\mathbb{Q}$ -divisor  $D$  on  $C$  with  $\deg D \geq 1/2$ .*

*Proof.* (1) By assumption,  $\varphi: S \rightarrow \mathbb{P}^1$  is a Seifert elliptic surface and has no singular fibers except multiple fibers  $m_i E_i$  ( $i = 1, 2, \dots, r$ ) supported on elliptic curves  $E_i$ . Then, by the canonical bundle formula of Kodaira [18, Theorem 12], we have  $K_S \sim_{\mathbb{Q}} \varphi^*D$  for a  $\mathbb{Q}$ -divisor  $D$  of degree  $A := -2 + \sum_{i=1}^r m_i^{-1}(m_i - 1)$  on  $\mathbb{P}^1$ . Then  $A > 0$ , since  $\kappa(S) = 1$ . Hence,  $r \geq 3$ . Then by an easy calculation, we see that  $A \geq 1/42$ . Thus we are done.

(2) As in (1),  $\psi: T \rightarrow C$  is a Seifert elliptic surface which has only multiple singular fibers  $m_i E_i$  ( $1 \leq i \leq r, m_i \geq 2$ ) of type  $m_i I_0$ . Then, by the canonical bundle formula of Kodaira, we have  $K_T \sim \psi^*(-\delta) + \sum_{i=1}^r (m_i - 1)E_i$  for a divisor  $\delta$  on  $C$  with  $\deg \delta = 0$ . Hence,  $K_T \equiv \psi^*D$  for an effective  $\mathbb{Q}$ -divisor  $D := \sum_{i=1}^r m_i^{-1}(m_i - 1)P_i$  on  $C$  for  $P_i := \varphi(E_i)$ . Since  $\kappa(S) = 1$ , we have  $r \geq 1$  and  $\deg D \geq 1/2$ .  $\square$

Now, we shall study the structure of ESPs of smooth algebraic surfaces.

**Proposition 6.4.** *Let  $S_{\bullet} = (g_n: S_n \rightarrow S_{n+1})_n$  be an ESP of smooth algebraic surfaces  $S_n$ . Then each surface  $S_n$  satisfies the following conditions.*

- $\chi(\mathcal{O}_{S_n}) = \chi_{\text{top}}(S_n) = 0$ .
- $K_{S_n}^2 = 0$ .
- There exist no  $(-1)$ -curves on  $S_n$ .
- $S_n$  is not of general type.

*Proof.* The first and the second assertion follow immediately from Lemma 2.1. Now we shall prove a non-existence of  $(-1)$ -curves on each  $S_n$ . Suppose that there exists a  $(-1)$ -curve  $e_n$  on  $S_n$ . Then  $e_{n+1} := g_n(e_n)$  is also a  $(-1)$ -curve on  $S_{n+1}$ , since  $g_n$  is étale. Since  $\deg g_n > 1$  and  $e_{n+1} \simeq \mathbb{P}^1$  is simply connected,  $g_n^*(e_{n+1})$  is a disjoint union of more than one  $(-1)$ -curves on  $S_n$ . This contradicts Proposition 3.1 (3). The last assertion is clear, since  $K_{S_n}^2 = 0$  and there exist no  $(-1)$ -curves on  $S_n$ .  $\square$

**Proposition 6.5.** *Let  $S_\bullet = (g_n: S_n \rightarrow S_{n+1})_n$  be an ESP of smooth algebraic surfaces  $S_n$  with  $\kappa(S_n) = 1$ . Then, there exists a Cartesian diagram*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S_n & \xrightarrow{g_n} & S_{n+1} & \longrightarrow & \cdots \\ & & \varphi_n \downarrow & & \downarrow \varphi_{n+1} & & \\ \cdots & \longrightarrow & C_n & \xrightarrow{h_n} & C_{n+1} & \longrightarrow & \cdots \end{array}$$

satisfying the following properties below;

- For each  $n$ , the Iitaka fibration  $\varphi_n: S_n \rightarrow C_n$  of  $S_n$  gives  $S_n$  a Seifert elliptic surface structure over the curve  $C_n$  and
- $h_n: C_n \rightarrow C_{n+1}$  is an isomorphism for a sufficiently large positive integer  $n$ .

Proof. Since  $\chi(\mathcal{O}_{S_n}) = 0$  and there exist no  $(-1)$ -curves on  $S_n$ , the Iitaka fibration  $\varphi_n: S_n \rightarrow C_n$  gives a Seifert elliptic surface structure, that is,  $\varphi_n$  has at most multiple singular fibers of type  $mI_0$  in the sense of Kodaira [18]. Thus  $K_{S_n} \sim_{\mathbb{Q}} \varphi_n^* H_n$  for an ample  $\mathbb{Q}$ -divisor  $H_n$  on  $C_n$ . Let  $F_n$  be a general fiber of  $\varphi_n$  for each  $n$ . Since

$$(K_{S_{n+1}}, (g_n)_* F_n) = (g_n^* K_{S_{n+1}}, F_n) = (K_{S_n}, F_n) = 0,$$

$\varphi_{n+1} \circ g_n(F_n)$  is a point on  $C_{n+1}$ . Hence by the *rigidity lemma* (cf. [21]), there exists a finite morphism  $h_n: C_n \rightarrow C_{n+1}$  such that  $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$  for all  $n$ . Then we show that  $h_n$  is an isomorphism for a sufficiently large positive integer  $n$ . Since the genus  $p_a(C_n)$  of the curve  $C_n$  decreases as  $n$  increases, there exist constants  $g \geq 0$  and  $k > 0$  such that  $p_a(C_n) = g$  for all  $n \geq k$ . If  $g \geq 2$ , then  $h_n$  ( $n \geq k$ ) is an isomorphism by Riemann-Hurwitz formula.

Next, suppose that  $g = 0$ . Then, from Lemma 6.3, we infer that for any  $n$ ,  $K_{S_n} \sim_{\mathbb{Q}} \varphi_n^*(H_n)$  for a  $\mathbb{Q}$ -divisor  $H_n$  on  $C_n$  with  $\deg H_n \geq 1/42$ . Since  $K_{S_0} \sim_{\mathbb{Q}} (g_{n-1} \circ \cdots \circ g_0)^* K_{S_n}$ , we have  $H_0 \sim_{\mathbb{Q}} (h_{n-1} \circ \cdots \circ h_0)^* H_n$ . Hence,  $\deg H_0 = \deg H_n \cdot \prod_{j=0}^{n-1} \deg h_j$ . Assume that  $\deg h_m \geq 2$  for infinitely many positive integers  $m$ . Then we have  $\deg H_n \rightarrow 0$  as  $n \rightarrow \infty$ . This contradicts the fact that  $\deg H_n \geq 1/42$  for any  $n$ . Hence  $h_n$  is an isomorphism for  $n \gg 0$ .

Finally, suppose that  $g = 1$ . Then, by Lemma 6.3,  $K_{S_n} \equiv \varphi_n^* H_n$  for some  $\mathbb{Q}$ -divisor  $H_n$  with  $\deg H_n \geq 1/2$ . Hence, if we replace  $\mathbb{Q}$ -linear equivalence relation  $\sim_{\mathbb{Q}}$  by numerically equivalence relation  $\equiv$ , completely the same method as in the case of  $g = 0$  works in our situation. Thus,  $h_n$  is an isomorphism for  $n \gg 0$ .  $\square$

**Proposition 6.6.** *Let  $S_\bullet = (g_n: S_n \rightarrow S_{n+1})_n$  be an ESP of smooth algebraic surfaces  $S_n$  with  $\kappa(S_n) = 0$ . Then, there exists an integer  $k \geq 0$  such that  $S_n$  is isomorphic to an abelian surface (resp. a hyperelliptic surface) for all  $n \leq k$  (resp. for all  $n > k$ ).*

Proof. Since  $\chi(\mathcal{O}_{S_n}) = 0$  and  $S_n$  is minimal, it follows from the classification theory of algebraic surfaces that  $S_n$  is either an abelian surface or a hyperelliptic surface. Since  $\rho(S_n) \geq \rho(S_{n+1})$ , the claim follows.  $\square$

**Proposition 6.7.** *Let  $S_\bullet = (g_n: S_n \rightarrow S_{n+1})_n$  be an ESP of smooth algebraic surfaces  $S_n$  with  $\kappa(S_n) = -\infty$ . Then, there exist an ESP  $C_\bullet = (h_n: C_n \rightarrow C_{n+1})_n$  of elliptic curves  $C_n$  and a Cartesian morphism  $\alpha_\bullet = (\alpha_n)_n: S_\bullet \rightarrow C_\bullet$  such that  $S_n$  is isomorphic to a  $\mathbb{P}^1$ -bundle  $\mathbb{P}_{C_n}(\mathcal{E}_n)$  associated with a rank two semi-stable vector bundle  $\mathcal{E}_n$  on  $C_n$ .*

*Proof.* Since there exist no  $(-1)$ -curves on  $S_n$ , each  $S_n$  is either  $\mathbb{P}^2$  or a ruled surface. Since  $g_n$  a non-isomorphic étale covering,  $S_n$  is not simply connected. Hence,  $S_n \neq \mathbb{P}^2$  for any  $n$ . Since  $\chi(\mathcal{O}_{S_n}) = 0$  and  $p_g(S_n) = 0$ ,  $q(S_n) = 1$  and  $S_n$  is an elliptic ruled surface. Thus the Albanese map  $\alpha_n: S_n \rightarrow C_n$  is a  $\mathbb{P}^1$ -bundle over an elliptic curve  $C_n$ , which is associated to a semi-stable vector bundle  $\mathcal{E}_n$  of rank 2 on  $C_n$  by Proposition 4.1. By the universality of the Albanese map, there exists a finite morphism  $h_n: C_n \rightarrow C_{n+1}$  such that  $\alpha_{n+1} \circ g_n = h_n \circ \alpha_n$ . Each  $h_n$  is étale, since  $C_n$  is an elliptic curve. Since  $g_n$  is étale and  $\mathbb{P}^1$  is simply connected,  $g_n$  is of degree one on each fiber of  $\alpha_n$ . Thus the natural morphism  $S_n \rightarrow S_{n+1} \times_{C_{n+1}} C_n$  is an isomorphism and  $\deg h_n = \deg g_n \geq 2$ .  $\square$

When we blow-up an FESP along elliptic curves to recover the original étale endomorphism  $f: X \rightarrow X$ , the following propositions are frequently used for analyzing blowing-up centers (cf. Proposition 7.8).

**Proposition 6.8.** *Let  $Z_\bullet = (v_n: Z_n \rightarrow Z_{n+1})_n$  be an ESP of smooth projective varieties  $Z_n$  of dimension  $> 1$ . Suppose that  $v_n^{-1}(C_{n+1}) = C_n$  for each irreducible curve  $C_n$  on  $Z_n$ . Then, for any  $n$ ,  $C_n$  is an elliptic curve,  $(K_{Z_n}, C_n) = 0$ , and its conormal bundle  $\mathcal{N}_{C_n/Z_n}^\vee$  is a semi-stable vector bundle of degree 0.*

*Proof.* By considering the truncated sequence  $Z_n \xrightarrow{v_n} Z_{n+1} \xrightarrow{v_{n+1}} \cdots$  starting from  $Z_n$  for each  $n$ , it is sufficient to prove the assertion in the case where  $n = 0$ .

We first prove that  $C_0$  is non-singular. Suppose that  $C_0$  is singular. Since  $v_n$  is étale,  $C_n$  is singular for any  $n$ . We set  $w_n := v_n|_{C_n}$ . Then by assumption,  $\deg w_n = \deg v_n > 1$ . Hence,  $C_\bullet := (w_n: C_n \rightarrow C_{n+1})_n$  is also an ESP of irreducible curves. Thus the number of singular points of  $C_n$  is a strictly decreasing function of  $n$ . Hence  $C_n$  is non-singular for a sufficiently large positive integer  $n$ . Since  $v_n^{-1}(C_{n+1}) = C_n$ , this is a contradiction.

Next, we show that each  $C_0$  is an elliptic curve. Since the arithmetic genus  $p_a(C_n)$  of the curve  $C_n$  is a decreasing function of  $n$ , there exist a positive integer  $k$  and a non-negative integer  $q$  such that  $p_a(C_n) = q$  for any  $n \geq k$ . If  $q \geq 2$ , then  $\deg w_n = 1$  by the Riemann-Hurwitz formula applied to  $w_n: C_n \rightarrow C_{n+1}$ . This is a contradiction. Thus  $q \leq 1$ . If  $q = 0$ , then  $C_n \cong \mathbb{P}^1$  for some  $n \gg 0$ . Since  $C_{n-1} = v_n^{-1}(C_n)$  and  $v_n$  is a non-isomorphic finite étale covering,  $C_{n-1}$  is not connected. This contradicts the assumption that  $C_{n-1}$  is irreducible. Hence  $q = 1$  and  $C_0$  is an elliptic curve.

By completely the same argument as in the proof of Proposition 1.1, (4), we see that  $\deg \mathcal{N}_n = 0$  for the normal bundle  $\mathcal{N}_n$  of  $C_n$  in  $Z_n$ . And there is induced an ESP  $D_\bullet = (D_n \rightarrow D_{n+1})_n$  of projective bundles  $D_n := \mathbb{P}_{C_n}(\mathcal{N}_n^\vee)$  associated to the conormal sheaf  $\mathcal{N}_n^\vee := \text{Hom}_{\mathcal{O}_{C_n}}(\mathcal{N}_n, \mathcal{O}_{C_n})$ . Hence,  $\mathcal{N}_0$  is semi-stable by Proposition 4.1.  $\square$

Next, we consider the case where  $Z_n$  is a smooth algebraic surface.

**Proposition 6.9.** *Let  $S_\bullet = (g_n: S_n \rightarrow S_{n+1})_n$  be an ESP of smooth algebraic surfaces  $S_n$ . Suppose that there exists an ESP  $\gamma_\bullet = (g_n|_{\gamma_n}: \gamma_n \rightarrow \gamma_{n+1})_n$  of irreducible curves  $\gamma_n$  on  $S_n$ . Then every  $\gamma_n$  is an elliptic curve satisfying  $(\gamma_n)^2 = 0$  and the following hold:*

- (1) *If  $\kappa(S_n) = 1$  for any  $n$ , then  $\varphi_n(\gamma_n)$  is a point for the Iitaka fibration  $\varphi_n: S_n \rightarrow C_n$  of  $S_n$ .*
- (2) *If any  $S_n$  is an abelian surface, then there exists an elliptic fiber bundle structure*

$\varphi_n: S_n \rightarrow C_n$  over an elliptic curve  $C_n$  such that

- $\varphi_n(\gamma_n)$  is a point on  $C_n$ , and
- $\varphi_{n+1} \circ g_n = u_n \circ \varphi_n$  for an isomorphism  $u_n: C_n \cong C_{n+1}$ .

(3) If  $S_n$  is a hyperelliptic surface, then one of the following cases holds;

(3.1) The Albanese map  $\alpha_n: S_n \rightarrow C_n$  is an elliptic fiber bundle such that

- $\alpha_n(\gamma_n)$  is a point, and
- $\alpha_{n+1} \circ g_n = u_n \circ \alpha_n$  for an isomorphism  $u_n: C_n \cong C_{n+1}$ .

(3.2)  $\text{Aut}^0(S_n)$  is an elliptic curve and the natural projection  $p_n: S_n \rightarrow \Gamma_n := S_n/\text{Aut}^0(S_n) \cong \mathbb{P}^1$  is a Seifert elliptic surface such that

- there exists a finite morphism  $v_n: \Gamma_n \rightarrow \Gamma_{n+1}$  such that  $p_{n+1} \circ g_n = v_n \circ p_n$  and  $p_n(\gamma_n)$  is a point, and
- $v_n$  is an isomorphism or  $p_n(\gamma_n)$  is a ramification point of  $v_n$ .

(3.3) • The Albanese map  $\alpha_n: S_n \rightarrow C_n$  is an elliptic fiber bundle such that  $\alpha_n(\gamma_n)$  is a point,

- $p_{n+1}: S_{n+1} \rightarrow \Gamma_{n+1} \simeq \mathbb{P}^1$  is a Seifert elliptic surface such that  $\alpha_{n+1}(\gamma_{n+1})$  is a point, and

- there exists a finite morphism  $w_n: C_n \rightarrow \Gamma_{n+1}$  such that  $p_{n+1} \circ g_n = w_n \circ \alpha_n$ .

(4) If  $\kappa(S_n) = -\infty$  for any  $n$ , then the Albanese map  $\alpha_n: S_n \rightarrow C_n$  is a  $\mathbb{P}^1$ -bundle over an elliptic curve  $C_n$  associated to a semi-stable vector bundle  $\mathcal{E}_n$  of rank two such that

- $\alpha_n(\gamma_n) = C_n$  and
- there exists a Cartesian morphism  $\alpha_\bullet: \gamma_\bullet \rightarrow C_\bullet = (h_n: C_n \rightarrow C_{n+1})_n$ .

Furthermore, one of the following hold.

(4.1) For any  $n$ ,  $S_n \simeq \mathbb{P}_{C_n}(\mathcal{E}_n)$  for a stable vector bundle  $\mathcal{E}_n$  on  $C_n$  and  $\gamma_n$  is a multi-section of  $\alpha_n$ .

(4.2) For any  $n$ ,  $S_n \simeq \mathbb{S}$  and  $\gamma_n$  is the canonical section of  $\alpha_n$ .

(4.3) For any  $n$ ,

- there exists an isomorphism  $S_n \simeq \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$  for a line bundle  $\mathcal{L}_n \in \text{Pic}^0(C_n)$  of infinite order, and
- the elliptic curve  $\gamma_n$  coincides either of the two sections of  $\alpha_n$  corresponding to the first projection  $\mathcal{O}_{C_n} \oplus \mathcal{L}_n \twoheadrightarrow \mathcal{O}_{C_n}$  or the second projection  $\mathcal{O}_{C_n} \oplus \mathcal{L}_n \twoheadrightarrow \mathcal{L}_n$ .

(4.4) For any  $n$ , there exists an isomorphism  $S_n \simeq \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$  for a line bundle  $\mathcal{L}_n \in \text{Pic}^0(C_n)$  which is of finite order.

Proof. By Proposition 6.8, every  $\gamma_n$  is an elliptic curve satisfying  $\gamma_n^2 = (K_{S_n}, \gamma_n) = 0$ . Thus by Proposition 6.4, it is sufficient to know how the curve  $\gamma_n$  is located in the surface  $S_n$ . Here we use the same notation as in Proposition 6.4.

(1) Since  $K_{S_n} \sim_{\mathbb{Q}} \varphi_n^* H_n$  for an ample  $\mathbb{Q}$ -divisor  $H_n$  on  $C_n$ , we infer that  $(H_n, (\varphi_n)_* \gamma_n) = (K_{S_n}, \gamma_n) = 0$ . Thus each  $\gamma_n$  is contained in a fiber of  $\varphi_n$ .

(2) Suppose that  $S_n$  is an abelian surface. Since  $\gamma_n \subset S_n$  is an elliptic curve, there exists an elliptic bundle structure  $\varphi_n: S_n \rightarrow C_n$  over an elliptic curve  $C_n$  such that  $p_n := \varphi_n(\gamma_n)$  is a point. By assumption,  $\varphi_{n+1} \circ g_n \circ \varphi_n^{-1}(p_n) = p_{n+1}$  is a point. Hence by the rigidity lemma, there exists a finite morphism  $u_n: C_n \rightarrow C_{n+1}$  such that  $\varphi_{n+1} \circ g_n = u_n \circ \varphi_n$ . Each  $u_n$  is étale, since  $C_n$  is an elliptic curve. By assumption,  $u_n^{-1}(p_{n+1}) = p_n$ . Hence each  $u_n$  is an isomorphism.

(3) Suppose that any  $S_n$  is a hyperelliptic surface. We recall the following basic fact:

**Fact.** Let  $S$  be a hyperelliptic surface. Then

(i)  $S$  has only two non-trivial fibrations up to isomorphism.

Type (A). The Albanese map  $\alpha_S: S \rightarrow \text{Alb}(S)$  gives  $S$  the structure of an elliptic fiber bundle over the Albanese elliptic curve  $\text{Alb}(S)$ .

Type (B).  $E := \text{Aut}^0(S)$  is an elliptic curve and the natural projection  $p_S: S \rightarrow C_S := S/E \cong \mathbb{P}^1$  is called the Fujiki quotient, which is a Seifert elliptic surface.

(ii)  $\rho(S) = 2$  and  $\overline{\text{NE}}(S) = \mathbb{R}_{\geq 0}[F_S] + \mathbb{R}_{\geq 0}[G_S]$ , where  $F_S$  (resp.  $G_S$ ) is the general fiber of  $\alpha_S$  (resp.  $p_S$ ).

Since  $\gamma_n^2 = 0$ ,  $\gamma_n$  is either a fiber of the Albanese map  $\alpha_n: S_n \rightarrow C_n$  or a fiber of the Fujiki quotient  $p_n: S_n \rightarrow \Gamma_n \cong \mathbb{P}^1$ .

By Proposition 3.1 and the above Fact, there is a one-to-one correspondence between the extremal rays of  $\overline{\text{NE}}(S_n)$  and the extremal rays of  $\overline{\text{NE}}(S_{n+1})$  under the isomorphisms  $(g_n)_*$  and  $(g_n)^*$ . Moreover, there exists a finite étale covering  $u_n: C_n \rightarrow C_{n+1}$  such that  $\alpha_{n+1} \circ g_n = u_n \circ \alpha_n$  by the universality of the Albanese map. Hence there exists a finite morphism  $v_n: \Gamma_n \rightarrow \Gamma_{n+1}$  such that  $p_{n+1} \circ g_n = v_n \circ p_n$ . The other claims follow by the assumption that  $g_n^{-1}(\gamma_{n+1}) = \gamma_n$ .

(4) We set  $\tilde{\gamma}_{n+1} := \gamma_{n+1} \times_{C_{n+1}} C_n$ . By the universality of the fiber product, there exists a unique morphism  $\rho_n: \gamma_n \rightarrow \tilde{\gamma}_{n+1}$  such that  $p_n \circ \rho_n = g_n|_{\gamma_n}$  for the natural projection  $p_n: \tilde{\gamma}_{n+1} \rightarrow \gamma_{n+1}$ . Since  $g_n^{-1}(\gamma_{n+1}) = \gamma_n$ , we infer that  $\deg g_n|_{\gamma_n} = \deg g_n = \deg h_n$ . On the other hand,  $\deg p_n = \deg h_n$ . Hence  $\deg \rho_n = 1$  and  $\rho_n$  is an isomorphism.

Suppose that any  $S_n$  is an elliptic ruled surface. Since  $\gamma_n$  is an elliptic curve,  $\alpha_n(\gamma_n) = C_n$ . If  $S_0 \simeq \mathbb{P}_{C_0}(\mathcal{F}_2)$ , then by Propositions 5.10 and 5.2,  $S_n \simeq \mathbb{P}_{C_n}(\mathcal{F}_2)$  for any  $n$  and  $\gamma_n$  is the unique section of  $\alpha_n$  corresponding to the surjection  $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_{C_n}$ . For the other case, all the assertions follow from Proposition 5.10.  $\square$

We finish this section with the following result whose proof follows immediately from Proposition 4.1 and Theorem 4.5. It will be used later in the proof of Proposition 7.10:

**Lemma 6.10.** *Let  $S_\bullet = (f_n: S_n \rightarrow S_{n+1})_n$  be an ESP of  $\mathbb{P}^1$ -bundles  $\alpha_n: S_n \rightarrow C_n$  over an elliptic curve  $C_n$ . Let  $E_i^{(n)}$  ( $i = 1, 2$ ) be two different elliptic curves on  $S_n$  with zero self-intersection number. Then  $E_1^{(n)} \cap E_2^{(n)} = \emptyset$ .*

## 7. Classifications of an FESP

In this section, we shall classify an FESP  $(Y_\bullet, R_\bullet)$  of smooth projective 3-folds of negative Kodaira dimension. Proposition 7.3 shows that for an FESP  $(Y_\bullet, R_\bullet)$ , the contraction morphism of  $R_\bullet$  induces a Cartesian morphism which is either a smooth del Pezzo fiber space  $Y_\bullet \rightarrow C_\bullet$  over an ESP  $C_\bullet$  of elliptic curves (i.e.,  $(Y_\bullet, R_\bullet)$  is of type (D)), or a conic bundle  $Y_\bullet \rightarrow S_\bullet$  over an ESP  $S_\bullet$  of smooth algebraic surfaces  $S_n$  (i.e.,  $(Y_\bullet, R_\bullet)$  is of type (C)). In the case where  $(Y_\bullet, R_\bullet)$  is of type (C), Proposition 7.5 describes the structure of  $Y_\bullet$  according as the Kodaira dimension of the base surface  $S_n$ . When we can recover the original non-isomorphic étale endomorphism  $f: X \rightarrow X$  from its FESP  $Y_\bullet$  by a sequence of blowing-ups  $\pi_\bullet: (X, f) \rightarrow Y_\bullet$  of an ESP, Proposition 7.8 and Corollary 7.9 give precise informations on the candidate of blowing-up centers. Furthermore, Proposition 7.10 shows that  $\pi_\bullet$ -exceptional locus are simple normal crossings of elliptic ruled surfaces. Proposition

7.12 is devoted to the study of an ESP of 3-folds with two extremal rays  $R_\bullet$  and  $R'_\bullet$  of different type.

First, we will use the following lemma when we study a non-isomorphic finite étale covering between conic bundles over smooth surfaces.

**Lemma 7.1.** *Let  $p: X \rightarrow S$  and  $q: Y \rightarrow T$  be conic bundles over smooth projective varieties  $S$  and  $T$  respectively. Suppose that  $q \circ f = g \circ p$  for a finite étale covering  $f: X \rightarrow Y$  and a morphism  $g: S \rightarrow T$ . Then  $g$  is also a finite étale covering and the above commutative diagram is Cartesian.*

*Proof.* By an easy dimension count, we easily see that  $g$  is a finite morphism. Hence,  $X \xrightarrow{p} S \xrightarrow{g} T$  is the Stein factorization of the composite map  $q \circ f: X \rightarrow T$ . Since each fiber of  $q$  is isomorphic to a conic in  $\mathbb{P}^2$ , it is simply connected. Since  $f$  is finite étale,  $f^{-1}(q^{-1}(t))$  is decomposed into a disjoint union of connected components. Hence  $g$  is finite unramified. Since both  $S$  and  $T$  are non-singular,  $g: S \rightarrow T$  is a finite étale covering. By the universality of the fiber product, there exists a unique morphism  $\pi: X \rightarrow \widetilde{Y} := Y \times_T S$  such that  $f = \pi \circ p$ , where  $p$  is the natural morphism. Since  $f$  is finite étale and each fiber of  $p$  is simply connected,  $f$  is of degree one on any fiber of  $p$  and  $\deg f = \deg g$ . Since  $\deg p = \deg g$ , we infer that  $\deg \pi = 1$  and  $\pi$  is an isomorphism.  $\square$

The following lemma gives a sufficient condition for a given ESP to be a fiber bundle.

**Lemma 7.2.** *Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective varieties  $X_n$ . Suppose that the following conditions are satisfied:*

- (1) *Each  $X_n$  has a fiber space structure  $\varphi_n: X_n \rightarrow E_n$  over an elliptic curve  $E_n$ .*
- (2) *There exists an ESP  $E_\bullet = (h_n: E_n \rightarrow E_{n+1})_n$  of elliptic curves  $E_n$  such that  $h_n \circ \varphi_n = \varphi_{n+1} \circ f_n$ .*
- (3) *Each general fiber of  $\varphi_n$  is simply connected.*

*Then  $\varphi_n$  is a smooth morphism for any  $n$ . Furthermore, suppose that*

- (4)  *$H^1(F_n, \Theta_{F_n}) = H^2(F_n, \Theta_{F_n}) = 0$  for a general fiber  $F_n$  of  $\varphi_n$ .*

*Then,  $\varphi_n: X_n \rightarrow E_n$  is a fiber bundle for any  $n$ .*

*Proof.* First, we show that under the assumptions (1), (2) and (3),  $\varphi_n$  is a smooth morphism. By the universality of the Albanese map, for any  $n$ , there exists a morphism  $h_n: E_n \rightarrow E_{n+1}$  such that  $\varphi_{n+1} \circ f_n = h_n \circ \varphi_n$ . Since  $\deg f_n > 1$  and general fiber of  $\varphi_n$  is simply connected, each  $h_n$  is a non-isomorphic finite étale covering. Hence there is induced an ESP  $E_\bullet = (h_n: E_n \rightarrow E_{n+1})_n$  such that  $\varphi_\bullet = (\varphi_n)_n: X_\bullet \rightarrow E_\bullet$  is a Cartesian morphism. Without loss of generality, it is sufficient to show the claim for  $n = 0$ .

Suppose that  $\varphi_0: X_0 \rightarrow E_0$  has a singular fiber over a point  $p_0 \in E_0$ . Since any  $f_n$  and  $g_n$  are finite étale,  $\varphi_n$  has a singular fiber over a point  $h_{n-1} \circ \cdots \circ h_0(p_0) \in E_n$  for any  $n > 0$ . Since  $\deg h_i \geq 2$  for any  $i$ ,  $\varphi_0: X_0 \rightarrow E_0$  has a singular fiber over a dense subset  $\bigcup_{n=0}^{\infty} (h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ h_0)(p_0)$  of  $E_0$ . This contradicts Sard's theorem. Hence  $\varphi_0$  is smooth.

Next, under the further condition (4), we show that all the fibers of  $\varphi_0$  are mutually isomorphic. We choose a point  $q_0 \in E_0$  arbitrarily. Since each  $f_i$  and  $h_i$  are étale,  $f_i|_{\varphi_i^{-1}(t)}: \varphi_i^{-1}(t) \rightarrow \varphi_{i+1}^{-1}(h_i(t))$  is an isomorphism for all  $t \in E_i$ . Since each fiber of  $\varphi_i$  is simply connected, there exists an isomorphism  $\varphi_0^{-1}(t') \cong \varphi_0^{-1}(q_0)$  for any  $n$  and  $t' \in (h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ h_0)(q_0)$ .

$h_0)(q_0)$ . Since  $\deg(h_i) \geq 2$  for each  $i$ , the set  $M := \{t \in E_0 \mid \varphi_0^{-1}(t) \cong \varphi_0^{-1}(q_0)\} \subset E_0$  contains the dense subset  $\bigcup_{n=0}^{\infty} (h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ h_0)(q_0)$  of  $E_0$ . Hence  $M$  is dense in  $E_0$ . By the condition (4), the Kuranishi space of  $\varphi_0^{-1}(q_0)$  is non-singular and  $\varphi_0^{-1}(q_0)$  is local analytically rigid. Hence, for any  $t \in E_0$ , there exists a small open neighborhood  $U(t)$  such that all the fibers  $\varphi_0^{-1}(x)$ ,  $x \in U(t)$  are isomorphic to each other. Since  $U(t) \cap M \neq \emptyset$ , we have an isomorphism  $\varphi_0^{-1}(t) \cong \varphi_0^{-1}(q_0)$  for all  $t \in E_0$ . Hence, by the theorem of Fischer–Grauert [7],  $\varphi_0: X_0 \rightarrow E_0$  is a holomorphic fiber bundle.  $\square$

Our next concern is to study the structure of an FESP constructed from a given non-isomorphic étale endomorphism  $f: X \rightarrow X$  by a sequence of blowing-downs of an ESP.

**Proposition 7.3.** *Let  $Y_{\bullet} = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an FESP of smooth projective 3-folds  $Y_n$  with  $\kappa(Y_n) = -\infty$ . Suppose that there exists an extremal ray  $R_0$  of fiber type on  $\overline{\text{NE}}(Y_0)$ . For any  $n > 0$ , we set  $R_n := (g_{n-1} \circ \cdots \circ g_0)_* R_0$  and  $R_{-n} := (g_{-n} \circ \cdots \circ g_{-1})^* R_0$ . Let  $R_{\bullet} := (R_n)_n$  be the set of extremal rays of  $\overline{\text{NE}}(Y_{\bullet})$ . Then*

- any  $R_n$  is also an extremal ray of fiber type, and
- the FESP  $(Y_{\bullet}, R_{\bullet})$  is either of type (D) or of type (C).

Furthermore, if we denote by  $\pi_n := \text{Cont}_{R_n}: Y_n \rightarrow Y'_n$  the contraction morphism associated to  $R_n$ , the structure of the FESP  $(Y_{\bullet}, R_{\bullet})$  is one of the followings;

- (1)  $(Y_{\bullet}, R_{\bullet})$  is of type (D). Then,
  - (a)  $\pi_n: Y_n \rightarrow Y'_n := C_n$  is a smooth del Pezzo fiber space over an elliptic curve  $C_n$  for any  $n$ .
  - (b) There are induced an ESP  $C_{\bullet} = (w_n: C_n \rightarrow C_{n+1})_n$  of elliptic curves and a Cartesian morphism  $\pi_{\bullet} = (\pi_n)_n: Y_{\bullet} \rightarrow C_{\bullet}$ .
- (2)  $(Y_{\bullet}, R_{\bullet})$  is of type (C). Then
  - (a)  $\pi_n: Y_n \rightarrow Y'_n := S_n$  is a conic bundle over a smooth algebraic surface  $S_n$  for any  $n$ .
  - (b) There is induced an ESP  $S_{\bullet} = (h_n: S_n \rightarrow S_{n+1})_n$  of surfaces  $S_n$  and a Cartesian morphism  $\pi_{\bullet} = (\pi_n)_n: Y_{\bullet} \rightarrow S_{\bullet}$ .
  - (c) The discriminant locus  $\Delta_n$  of  $\pi_n$  is a disjoint union of elliptic curves  $\Delta_n^{(i_n)}$  whose self-intersection number is zero.
  - (d) Let  $M_n$  be a finite set consisting of all the connected components  $\Delta_n^{(i_n)}$  of  $\Delta_n$ . Then, the map  $\mu_n: M_n \rightarrow M_{n+1}$  defined by  $\Delta_n^{(i_n)} \mapsto h_n(\Delta_n^{(i_n)})$  gives a bijection between the set  $M_n$  and  $M_{n+1}$  for  $n \gg 0$ .

*Proof.* (1) Suppose that  $R_0$  is of type (D). Then, by Proposition 3.1, the FESP  $(Y_{\bullet}, R_{\bullet})$  is of type (D). Let  $\pi_n := \text{Cont}_{R_n}: Y_n \rightarrow C_n$  be the contraction morphism associated to  $R_n$ . By Proposition 3.1, there exists a finite morphism  $w_n: C_n \rightarrow C_{n+1}$  such that  $\pi_{n+1} \circ g_n = w_n \circ \pi_n$ . Since general fiber of  $\pi_n$  is a del Pezzo surface and simply connected, and  $g_n$  is a non-isomorphic étale covering, we have  $\deg w_n \geq 2$  for any  $n$ . Since the genus  $p_a(C_n)$  of the curve  $C_n$  decreases as  $n$  increases, there exists a non-negative integer  $g$  such that  $p_a(C_n) = g$  for  $n \gg 0$ . Applying Lemma 2.1, we see that  $0 = \chi(\mathcal{O}_{Y_n}) (= 1 - q(Y_n) + h^{2,0}(Y_n) - p_g(Y_n))$ . Since  $p_g(Y_n) = 0$ , we have  $p_a(C_n) = q(Y_n) = 1 + h^{2,0}(Y_n) \geq 1$ . In particular,  $g \geq 1$ . Suppose that  $g \geq 2$ . Then, by the Riemann–Hurwitz formula,  $w_n$  is an isomorphism for  $n \gg 0$ , which derives a contradiction. Hence  $g = 1$  and any  $C_n$  is an elliptic curve.

Since  $-K_{Y_{n,t}}$  is ample for any  $Y_{n,t} := \pi_n^{-1}(t)$ , Lemma 7.2 shows that  $\pi_n: Y_n \rightarrow C_n$  is a smooth del Pezzo fibration over an elliptic curve  $C_n$ . Since any rational curve on  $Y_n$  is contained in fibers of  $\pi_n$ ,  $\pi_n: Y_n \rightarrow C_n$  is a maximally rationally connected fibration and also gives the Albanese map of  $Y_n$ .

(2) Suppose that  $R_0$  is of type (C). Then, by Proposition 3.1, the FESP  $(Y_\bullet, R_\bullet)$  is of type (C) and the contraction morphism  $\pi_n := \text{Cont}_{R_n}: Y_n \rightarrow S_n$  is a conic bundle over a smooth surface  $S_n$ . By Proposition 3.1 and Lemma 7.1, there exists a non-isomorphic finite étale covering  $h_n: S_n \rightarrow S_{n+1}$  such that  $h_n \circ \pi_n = \pi_{n+1} \circ g_n$ . Hence, the structure of  $S_n$  is classified into 3 types by Proposition 6.4. Let  $\Delta_n$  be the discriminant locus of  $\pi_n$ . Suppose that  $\Delta_n \neq \emptyset$  for some  $n$ . Then, by [4],  $\Delta_n$  is a simple normal crossing divisor of  $S_n$  and  $\text{Sing } \Delta_n$  coincides with the set  $\{s \in S_n \mid \pi_n^{-1}(s) \text{ is non-reduced}\}$ .

We first show that  $\text{Sing } \Delta_n = \emptyset$ . It is sufficient to prove the case where  $n = 0$  without loss of generality. The proof is by contradiction. Assume the contrary. Since  $g_n, h_n$  are finite étale and each fiber of  $\pi_n$  is simply connected, we have an isomorphism  $Y_n \cong Y_{n+1} \times_{S_{n+1}} S_n$ . Hence  $h_n^{-1}(\Delta_{n+1}) = \Delta_n$  and  $h_n^{-1}(\text{Sing } \Delta_{n+1}) = \text{Sing } \Delta_n$  for each  $n$ . We take a point  $q_0 \in \text{Sing } \Delta_0$  arbitrarily. Then the set  $M := \bigcup_{n=0}^{\infty} (h_n \circ \dots \circ h_0)^{-1}(h_n \circ \dots \circ h_0)(q_0)$  is contained in  $\text{Sing } \Delta_0$ . Since  $\deg h_i \geq 2$  for each  $i$ ,  $M$  is an infinite set. Since  $\text{Sing } \Delta_0$  is a finite set, a contradiction is derived. Hence  $\Delta_0$  is non-singular. Next, we show that each connected component  $\Delta_n^{(i_n)}$  is an elliptic curve whose self intersection number is zero. Let  $M_n$  denote the finite set consisting of irreducible components of  $\Delta_n$ . From the above consideration, the natural map  $\mu_n: M_n \rightarrow M_{n+1}$  is a surjection between finite sets. Since the cardinality  $|M_n|$  decreases as  $n$  increases, there exists a constant  $k$  such that  $|M_n| = k$  for  $n \gg 0$ . Hence  $\mu_n$  is a bijection for  $n \gg 0$ . Then, by considering a truncated étale sequence, we may assume that any  $\mu_n$  is bijective. Hence, by Proposition 6.9, each  $\Delta_n^{(i_n)}$  is an elliptic curve whose self-intersection number equals 0.  $\square$

**REMARK 7.4.** In Proposition 7.3, suppose that  $(Y_\bullet, R_\bullet)$  is an FESP of type (D) constructed from a non-isomorphic étale endomorphism  $f: X \rightarrow X$  by a sequence of blowing-downs of an ESP. Then  $\pi_n: Y_n \rightarrow C$  is a del Pezzo fiber bundle over the Albanese elliptic curve  $C$  of  $X$ . This fact will be proved in our subsequent article, Part III.

**Proposition 7.5.** *Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an FESP constructed from smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$  by a sequence of blowing-downs of an ESP. Let  $R_\bullet = (R_n)_n$  be the set of extremal rays of fiber type on  $\overline{\text{NE}}(Y_\bullet)$  such that  $(g_n)_* R_n = R_{n+1}$  for any  $n$ . Suppose that the FESP  $(Y_\bullet, R_\bullet)$  is of type (C). Then one of the following cases occurs. (Hereafter, we use the same notation as in Proposition 7.3.)*

- (1) For any  $n$ ,  $\kappa(S_n) = 1$ . In this case, for any  $n$ ,
  - (a) the Iitaka fibration  $\varphi_n: S_n \rightarrow C_n$  gives  $S_n$  a Seifert elliptic surface structure.
  - (b) There exists a finite morphism  $v_n: C_n \rightarrow C_{n+1}$  such that  $\varphi_{n+1} \circ h_n = v_n \circ \varphi_n$  and  $v_n$  is an isomorphism for all  $n \gg 0$ .
  - (c) The discriminant locus  $\Delta_n$  of  $\pi_n$  is non-singular and each connected component is an elliptic curve which is some fiber of  $\varphi_n$ .
- (2) For any  $n$ ,  $\kappa(S_n) = 0$ . Then there exists an integer  $k$  such that  $S_n$  is an abelian surface (resp. a hyperelliptic surface) for all  $n \leq k$  (resp. for all  $n > k$ ).
  - (a) Suppose that any  $S_n$  is an abelian surface. Then, there exists an elliptic bundle

structure  $\alpha_n: S_n \rightarrow E_n$  over an elliptic curve  $E_n$  such that  $\alpha_n(\Delta_n)$  are points on  $E_n$  and  $\alpha_{n+1} \circ h_n = u_n \circ \alpha_n$  for an isomorphism  $u_n: E_n \cong E_{n+1}$  for  $n \gg 0$ .

(b) Suppose that any  $S_n$  is a hyperelliptic surface. Then, there exist either an elliptic bundle structure  $\alpha_n: S_n \rightarrow E_n$  over an elliptic curve  $E_n$  such that  $\alpha_n(\Delta_n)$  are points, or a Seifert elliptic surface structure  $p_n: S_n \rightarrow \mathbb{P}^1$  over  $\mathbb{P}^1$  such that  $p_n(\Delta_n)$  are points.

(3) For any  $n$ ,  $\kappa(S_n) = -\infty$ . In this case, for any  $n$ ,

(a) the Albanese map  $\alpha_n: S_n \rightarrow C_n$  induce an ESP  $C_\bullet = (h_n: C_n \rightarrow C_{n+1})_n$  of elliptic curves  $C_n$  and a Cartesian morphism  $\alpha_\bullet = (\alpha_n)_n: S_\bullet \rightarrow C_\bullet$ .

(b)  $S_n \simeq \mathbb{P}_{E_n}(\mathcal{E}_n)$  for a semi-stable vector bundle  $\mathcal{E}_n$  of rank 2 on  $C_n$ .

(c)  $\Delta_n$  is non-singular and each of its connected component is an elliptic curve mapped onto  $C_n$  by  $\alpha_n$ .

Proof. All the assertions follow immediately by Propositions 6.9, 6.4, 6.5, 6.6, 6.7, and 7.3.  $\square$

**DEFINITION 7.6.** Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an FESP of smooth projective 3-folds  $Y_n$  with  $\kappa(Y_n) = -\infty$ . Let  $R_\bullet = (R_n)_n$  be the set of extremal rays  $R_n$  of fiber type on  $\overline{\text{NE}}(Y_n)$  such that  $(g_n)_*R_n = R_{n+1}$  for any  $n$ . Suppose that the FESP  $(Y_\bullet, R_\bullet)$  is of type (C). Then  $(Y_\bullet, R_\bullet)$  is called of type  $(C_1)$  (resp.  $(C_0)$  and  $(C_{-\infty})$ ) if the condition (1) (resp. (2) and (3)) in Proposition 7.5 is satisfied.

**REMARK 7.7.** Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_0)$  constructed from  $(X, f)$  by a sequence of blowing-downs of an ESP. Let  $\pi_n: Y_n \rightarrow S_n$  be a conic bundle, here we use the same notation as in Proposition 7.5 (2). Since  $q(X) = q(Y_n) = q(S_n)$ , one of the following possibilities can occur:

- Any  $S_n$  is an abelian surface.
- Any  $S_n$  is a hyperelliptic surface.

When we recover the given non-isomorphic étale endomorphism  $f: X \rightarrow X$  from its FESP by a sequence of blowing-ups of an ESP along elliptic curves, the next proposition gives some perspective on the blowing-up centers.

**Proposition 7.8.** Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an FESP constructed from an ESP  $X_\bullet$  of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$  by a sequence of blowing-downs of an ESP. Let  $R_\bullet = (R_n)_n$  be the set of extremal rays of fiber type on  $\overline{\text{NE}}(Y_n)$  such that  $(g_n)_*R_n = R_{n+1}$  and  $\varphi_n$  the contraction morphism associated to  $R_n$ . Suppose that there exists an elliptic curve  $E_n$  on  $Y_n$  such that  $g_n^{-1}(E_{n+1}) = E_n$  for any  $n$ . Then one of the following cases occurs.

- (1) The FESP  $(Y_\bullet, R_\bullet)$  is of type (D). Then we have  $\varphi_n(E_n) = C_n$ , where  $\varphi_n: Y_n \rightarrow C_n$  is a smooth del Pezzo fiber space over an elliptic curve  $C_n$ .
- (2) The FESP  $(Y_\bullet, R_\bullet)$  is of type  $(C_0)$  or  $(C_1)$ . Then  $\varphi_n: Y_n \rightarrow S_n$  is a conic bundle over a Seifert elliptic surface  $\alpha_n: S_n \rightarrow C_n$  with  $\kappa(S_n) \geq 0$  and  $\varphi_n(E_n)$  is an elliptic curve which is some fiber of  $\alpha_n$ .
- (3) The FESP  $(Y_\bullet, R_\bullet)$  is of type  $(C_{-\infty})$ . Then  $\varphi_n: Y_n \rightarrow S_n$  is a conic bundle over a surface  $S_n$ . Furthermore,  $\Gamma_n := \varphi_n(E_n)$  is an elliptic curve such that  $(\Gamma_n)^2 = 0$  and

$\alpha_n(\Gamma_n) = C_n$ , where  $\alpha_n: S_n \rightarrow C_n$  is a  $\mathbb{P}^1$ -bundle over an elliptic curve  $C_n$ .

*Proof.* (1) Since  $\varphi_n: Y_n \rightarrow C_n$  is the Albanese map of  $Y_n$ , there exists a morphism  $h_n: C_n \rightarrow C_{n+1}$  such that  $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$ . The proof is by contradiction. Suppose that  $p_n := \varphi_n(E_n)$  is a point on  $C_n$  for some  $n$ . Then, for the elliptic curve  $E_{n+1} = g_n(E_n)$ ,  $p_{n+1} := \pi_{n+1}(E_{n+1})$  is also a point on  $C_{n+1}$ . By assumption,  $E_n$  is contained in all the connected component of  $g_n^{-1} \circ \varphi_{n+1}^{-1}(p_{n+1}) = \varphi_n^{-1} \circ h_n^{-1}(p_{n+1})$ . Since  $\deg h_n = \deg f_n > 1$  and  $h_n$  is étale,  $E_n$  is not connected. Thus a contradiction is derived, since  $E_n$  is irreducible.

(2), (3) In both cases, there exists a finite étale covering  $h_n: S_n \rightarrow S_{n+1}$  such that  $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$  and  $\deg h_n = \deg g_n > 1$ . Then  $\Gamma_n := \varphi_n(E_n)$  is an irreducible curve on  $S_n$  for each  $n$ , since all the irreducible components of any fiber of  $\varphi_n$  are rational curves. By construction, there exists a Cartesian morphism  $\varphi_\bullet = (\varphi_n)_n: Y_\bullet \rightarrow S_\bullet := (h_n: S_n \rightarrow S_{n+1})_n$ . Applying Lemma 2.6 to the above diagram, we infer that  $\Gamma_n = h_n^{-1}(\Gamma_{n+1})$ . Hence, by Proposition 6.8,  $\Gamma_n$  is an elliptic curve with  $\Gamma_n^2 = 0$ . The last claim in (3) is derived from Proposition 6.9.  $\square$

Combining Lemma 2.6, Propositions 1.1 and 7.8, we obtain:

**Corollary 7.9.** *Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Let  $(Y_\bullet, R_\bullet)$  be an FESP constructed from  $X_\bullet$  by a sequence of blowing-downs of an ESP and set  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  ( Here, we use the same notation as in Corollary 1.2 and Proposition 7.3. ) Let  $C_n^{(i)}$  be the elliptic curve which is the center of the blowing-up  $\pi_n^{(i-1)}: X_n^{(i-1)} \rightarrow X_n^{(i)}$ . Then,  $\gamma_n^{(i)} := \pi_n^{(k-1)} \circ \dots \circ \pi_n^{(i)}(C_n^{(i)})$  is an elliptic curve on  $Y_n$  such that  $\gamma_\bullet^{(i)} = (g_n: \gamma_n^{(i)} \rightarrow \gamma_{n+1}^{(i)})_n$  is an ESP of elliptic curves. Furthermore, the following hold:*

- (1) *If  $(Y_\bullet, R_\bullet)$  is of type (D), then  $\varphi_n(\gamma_n^{(i)}) = C_n$ .*
- (2) *If  $(Y_\bullet, R_\bullet)$  is of type  $(C_0)$  or  $(C_1)$ , then  $\Delta_n^{(i)} := \varphi_n(\gamma_n^{(i)})$  is some fiber of  $\alpha_n: S_n \rightarrow C_n$  such that  $\Delta_\bullet = (h_n: \Delta_n \rightarrow \Delta_{n+1})_n$  is an ESP of elliptic curves.*
- (3) *If  $(Y_\bullet, R_\bullet)$  is of type  $(C_{-\infty})$ , then  $\Delta_n^{(i)} := \varphi_n(\gamma_n^{(i)})$  is an elliptic curve on  $S_n$  with self-intersection number 0 and dominates  $C_n$ . Furthermore,  $\Delta_\bullet = (h_n: \Delta_n \rightarrow \Delta_{n+1})_n$  is an ESP of elliptic curves.*

The existence of a non-isomorphic étale endomorphism imposes very strong conditions on the  $\pi$ -exceptional locus for the Cartesian blowing-up  $\pi_\bullet: X_\bullet \rightarrow Y_\bullet$ , as the following shows:

**Proposition 7.10.** *Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type (C) constructed from  $X_\bullet$  by a sequence of blowing-downs of an ESP  $\pi_\bullet: X_\bullet \rightarrow Y_\bullet$ . Suppose that  $\pi_\bullet$  is not an isomorphism. Suppose furthermore that for  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$ , the contraction morphism  $\varphi_n := \text{Cont}_{R_n}: Y_n \rightarrow S_n$  is a  $\mathbb{P}^1$ -bundle for any  $n$ . Let  $\Delta_n$  be the discriminant locus of  $\psi_n := \varphi_n \circ \pi_n: X \rightarrow S_n$ . Then the following hold:*

- $\psi_n$  is an equi-dimensional  $\mathbb{P}^1$ -fiber space.
- $\Delta_n$  is non-singular and any of its irreducible component  $\Delta_{n,i}$  is an elliptic curve.
- For any  $i$ ,  $\psi_n^{-1}(\Delta_{n,i})$  is a simple normal crossing divisor and any of its irreducible components is a  $\mathbb{P}^1$ -bundle over an elliptic curve associated to a semi-stable vector bundle.

Proof. By construction of an FESP, there exists the following Cartesian morphisms of ESPs;

$$X_{\bullet} = X_{\bullet}^{(0)} \longrightarrow \cdots \longrightarrow X_{\bullet}^{(i)} \xrightarrow{\pi_{\bullet}^{(i)}} X_{\bullet}^{(i+1)} \longrightarrow \cdots \longrightarrow Y_{\bullet} = X_{\bullet}^{(k)} \xrightarrow{\varphi_{\bullet}} S_{\bullet},$$

in which the following are satisfied for any  $n$  and  $i$ .

- $\pi_n^{(i)}: X_n^{(i-1)} \rightarrow X_n^{(i)}$  is (the inverse of) the blowing-up along an elliptic curve  $C_n^{(i)}$  on  $X_n^{(i)}$ , where  $X_n^{(0)} := X$  and  $X_n^{(k)} := Y_n$ .
- $X_{\bullet}^{(i)} = (f_n^{(i)}: X_n^{(i)} \rightarrow X_{n+1}^{(i)})_n$  is an ESP.
- $Y_{\bullet} = (g_n: Y_n \rightarrow Y_{n+1})_n$  is an FESP, where  $g_n := f_n^{(k)}$ .
- $(g_n^{(i)})^{-1}(C_{n+1}^{(i)}) = C_n^{(i)}$ .
- $\pi_n = \pi_n^{(k-1)} \circ \cdots \circ \pi_n^{(0)}$ .

Now we shall describe the centers of the succession of blowing-ups  $\pi_n^{(i)}$ . By Lemma 6.10, Proposition 6.9 and Corollary 7.9,  $\Delta_n$  is non-singular and any irreducible component of  $\Delta_n$  is an elliptic curve. If we set  $\gamma_n^{(k)} := \varphi_n(C_n^{(k)})$ , then  $\gamma_n^{(k)}$  is an elliptic curve on  $S_n$  with zero self-intersection number by Corollary 7.9. Then the surface  $T_n^{(k)} := \varphi_n^{-1}(\gamma_n^{(k)})$  is a  $\mathbb{P}^1$ -bundle over  $\gamma_n^{(k)}$  which contains  $C_n^{(k)}$ . Since  $u_n^{-1}(\gamma_{n+1}^{(k)}) = \gamma_n^{(k)}$ , we infer that  $g_n^{-1}(T_{n+1}^{(k)}) = T_n^{(k)}$  for any  $n$ . Thus there is induced an ESP  $T_{\bullet}^{(k)} := (g_n: T_n^{(k)} \rightarrow T_{n+1}^{(k)})_n$  of elliptic ruled surfaces. Then by Proposition 4.1, each  $T_n^{(k)}$  is a  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{E}_n^{(k)})$  associated to a semi-stable vector bundle  $\mathcal{E}_n^{(k)}$  of rank two on  $\gamma_n^{(k)}$ . Furthermore,  $C_n^{(k)}$  is an elliptic curve with zero self-intersection number on  $T_n^{(k)}$ .

If we set  $\gamma_n^{(k-1)} := \varphi_n \circ \pi_n^{(k-1)}(C_n^{(k-1)})$ , then by the same argument as above,  $\gamma_n^{(k-1)}$  is an elliptic curve on  $S_n$  with zero self-intersection number. By Proposition 7.3,  $S_n$  is a  $\mathbb{P}^1$ -bundle associated to a semi-stable vector bundle of rank 2 on an elliptic curve  $C$ . Hence applying Lemma 6.10, we infer that  $\gamma_n^{(k-1)} \cap \gamma_n^{(k)} = \emptyset$  or  $\gamma_n^{(k-1)} = \gamma_n^{(k)}$ . If  $\gamma_n^{(k-1)} \cap \gamma_n^{(k)} = \emptyset$ , then  $C_n^{(k-1)} \cap T_n^{(k)} = \emptyset$  and there is nothing to prove. Next, suppose that  $\gamma_n^{(k-1)} = \gamma_n^{(k)}$ . Then  $D_n^{(k)} := \pi_n^{(k-1)}(C_n^{(k-1)})$  is contained in  $T_n^{(k)}$ . Since  $C_{\bullet}^{(k-1)} := (C_n^{(k-1)} \rightarrow C_{n+1}^{(k-1)})_n$  is an ESP,  $D_{\bullet}^{(k)} := (f_n^{(k-1)}: D_n^{(k)} \rightarrow D_{n+1}^{(k)})_n$  is also an ESP by Lemma 2.6. Hence by Proposition 6.9,  $D_n^{(k)}$  is an elliptic curve with zero self-intersection number on  $T_n^{(k)}$ . Thus Proposition 6.10 yields that  $C_n^{(k)} \cap D_n^{(k)} = \emptyset$  or  $D_n^{(k)} = C_n^{(k)}$ . If  $C_n^{(k)} \cap D_n^{(k)} = \emptyset$ , then clearly the proper transform of  $T_n^{(k)}$  and  $\pi_n^{(k-1)} \circ \pi_n^{(k-2)}$ -exceptional divisors are all elliptic ruled surfaces crossing normally. Next suppose that  $D_n^{(k)} = C_n^{(k)}$ . Then  $C_n^{(k-1)}$  is contained in the  $\pi_n^{(k-1)}$ -exceptional divisor  $E_n^{(k-1)}$  and is an elliptic curve with zero self-intersection number. Let  $T_n'^{(k)}$  be the proper transform of  $T_n^{(k)}$  by  $\pi_n^{(k-1)}$  and  $\Delta_n^{(k)} := E_n^{(k-1)} \cap T_n'^{(k)}$  the intersection curve. Since  $(f_n^{(k-1)})^{-1}(E_{n+1}^{(k-1)}) = E_n^{(k-1)}$  and  $(f_n^{(k-1)})^{-1}(T_{n+1}'^{(k)}) = T_n'^{(k)}$ ,  $\Delta_{\bullet}^{(k)} := (f_n^{(k-1)}: \Delta_n^{(k)} \rightarrow \Delta_{n+1}^{(k)})_n$  is an ESP. Hence  $\Delta_n^{(k)}$  is an elliptic curve on  $E_n^{(k)}$  with zero self-intersection number. Hence Lemma 6.10 yields that  $C_n^{(k-1)} \cap \Delta_n^{(k)} = \emptyset$  or  $C_n^{(k-1)} = \Delta_n^{(k)}$ . Then by the same argument as before, we see that the proper transform of  $T_n^{(k)}$  and  $\pi_n^{(k-1)} \circ \pi_n^{(k-2)}$ -exceptional divisors are all elliptic ruled surfaces crossing normally. Continuing the same argument as above, we can prove the proposition.  $\square$

REMARK 7.11. As pointed out by the referee, Theorem C of the reference [40] gives a general structure theorem for a uniruled manifold  $X$  with étale endomorphisms, in particular, one can immediately obtain a natural lower-dimensional variety  $Z$  which is the base of the MRC fibration and admits a quasi-étale endomorphism. In our case,  $\dim X = 3$ ,  $Z$  is smooth

with  $\dim Z = 1$  or  $2$  and the MRC fibration  $X \rightarrow Z$  is a *morphism*. In view of this result, in the case where  $\dim X$  is general, it would be more natural to study separately the case where  $\dim Z = 1, 2$ .

The next propositions give some description of a non-isomorphic étale endomorphism  $f: X \rightarrow X$  which has extremal rays of different type.

**Proposition 7.12.** *Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Suppose that on  $\overline{\text{NE}}(X_\bullet)$  there exist two extremal rays  $R_\bullet = (R_n)_n$  and  $R'_\bullet = (R'_n)_n$  such that  $(X_\bullet, R_\bullet)$  and  $(X_\bullet, R'_\bullet)$  are of different types. Then, the structure of  $X_\bullet$  is as follows:*

- (1) *The pair  $(X_\bullet, R_\bullet)$  is an FESP of type  $(C_\infty)$ , i.e., the contraction morphism  $\varphi_n := \text{Cont}_{R_n}: X_n \rightarrow S_n$  associated to  $R_n$  is a conic bundle over a surface  $S_n$ , which is a  $\mathbb{P}^1$ -bundle  $\alpha_n: S_n \rightarrow C_n$  over an elliptic curve  $C_n$ .*
- (2) *Another  $(X, R'_\bullet)$  is of divisorial type, i.e., the contraction morphism  $\pi_n := \text{Cont}_{R'_n}: X_n \rightarrow X'_n$  associated to  $R'_n$  is (the inverse of) the blowing-up of a smooth projective 3-fold  $X'_n$  along an elliptic curve  $\Gamma_n \subset X'_n$ .*
- (3)  *$X'_n$  has the structure of a smooth del Pezzo fiber space  $g_n: X'_n \rightarrow C_n$  over  $C_n$  such that  $g_n \circ \pi_n = \alpha_n \circ \varphi_n$ , which is an extremal contraction.*
- (4)  *$\varphi_n(D_n) = S_n$  for the  $\pi_n$ -exceptional divisor  $D_n$ .*

*Proof.* First, suppose that  $(X_\bullet, R'_\bullet)$  is of divisorial type and  $(Y_\bullet, R_\bullet)$  is of type (D) respectively. Then each fiber of the contraction morphism  $\pi_n := \text{Cont}_{R'_n}: X_n \rightarrow X'_n$  is either a point or  $\mathbb{P}^1$ . It is mapped to a point by the contraction morphism  $\varphi_n := \text{Cont}_{R_n}: X_n \rightarrow C_n$ , since  $C_n$  is an elliptic curve by Proposition 3.6. Hence, by the rigidity lemma, there exists a morphism  $\psi_n: X'_n \rightarrow C_n$  with  $\varphi_n = \psi_n \circ \pi_n$ . This contradicts  $\rho(X_n/C_n) = 1$ , since each fiber of  $\psi_n$  is positive-dimensional. By the same way as above, we can prove that the case where  $(X_\bullet, R'_\bullet)$  is of type (C) and  $(X_\bullet, R_\bullet)$  is of type (D) cannot occur.

Hence the only possibility is that  $(X_\bullet, R'_\bullet)$  is of divisorial type and  $(X_\bullet, R_\bullet)$  is of type (C) respectively. Let  $\varphi_n := \text{Cont}_{R_n}: X_n \rightarrow S_n$  (resp.  $\pi_n := \text{Cont}_{R'_n}: X_n \rightarrow X'_n$ ) be the contraction morphism associated to  $R_n$  (resp.  $R'_n$ ). If  $\kappa(S_n) \geq 0$ , then by Proposition 6.4,  $S_n$  contains no rational curves. Hence each fiber of  $\pi_n$  is mapped to a point by  $\varphi_n$ . Thus there exists a morphism  $\psi_n: X'_n \rightarrow S_n$  such that  $\psi_n \circ \pi_n = \varphi_n$ . This contradicts the fact that  $\rho(X_n/S_n) = 1$ . Hence  $\kappa(S_n) = -\infty$  and by Proposition 6.4,  $S_n$  is a  $\mathbb{P}^1$ -bundle  $\alpha_n: S_n \rightarrow C_n$  over an elliptic curve  $C_n$ . Hence, by the same reason as above, there exists a morphism  $g_n: X'_n \rightarrow C_n$  such that  $\alpha_n \circ \varphi_n = g_n \circ \pi_n$ . Since both  $\pi_n$  and  $\varphi_n$  are extremal contractions, we have  $\rho(X_n) = \rho(S_n) + 1 = 3$  and  $\rho(X'_n) = \rho(X_n) - 1 = 2$ . Since  $\rho(X'_n) = \rho(C_n) + 1$  and  $K_{X'_n}$  is not  $g_n$ -nef,  $g_n$  is also an extremal contraction and is a smooth del Pezzo fiber space over an elliptic curve  $C_n$  by Proposition 7.3. Thus the assertion (3) is verified.

$\pi_n: X_n \rightarrow X'_n$  is (the inverse of) the blowing-up along an elliptic curve  $\Gamma_n \subset X'_n$  and let  $D_n$  be an exceptional divisor of  $\pi_n$ . Suppose that  $g_n(\Gamma_n)$  is a point  $Q_n$  on  $C_n$ . Then  $D_n$  is contained in the surface  $(\alpha_n \circ \varphi_n)^{-1}(Q_n)$ , which is a conic bundle over  $\mathbb{P}^1$  by Proposition 7.3. This contradicts the fact that  $D_n \rightarrow \Gamma_n$  is an elliptic ruled surface. Hence  $g_n(\Gamma_n) = C_n$ . Since  $R_n \neq R'_n$  and  $C_n$  is an elliptic curve, any fiber  $\gamma$  of the  $\mathbb{P}^1$ -bundle  $\pi_n|_{D_n}: D_n \rightarrow \Gamma_n$  is not contracted to a point on  $S_n$  by  $\varphi_n$  but contracted to a point by  $\alpha_n \circ \varphi_n$ . Hence  $\varphi_n(\gamma) = F$  for some fiber  $F$  of  $\alpha_n$ . If  $\varphi_n(D_n) \neq S_n$ , then  $\varphi_n(D_n) = F$ , since  $\gamma \subset D_n$  and  $(F \subset) \varphi_n(D_n)$  is

irreducible. Hence we have  $C_n = g_n(\Gamma_n) = g_n \circ \pi_n(D_n) = \alpha_n \circ \varphi_n(D_n) = \alpha_n(F)$ , which is a point. Thus a contradiction is derived and the assertion (4) is proved.  $\square$

The following can be proved similarly.

**Proposition 7.13.** *Let  $X_\bullet = (f_n: X_n \rightarrow X_{n+1})_n$  be an ESP of smooth projective 3-folds  $X_n$  with  $\kappa(X_n) = -\infty$ . Suppose that there exist an FESP  $(Y_\bullet, R_\bullet)$  of type (C) and another FESP  $(Z_\bullet, R'_\bullet)$  of type (D) constructed from  $X$  by a sequence of blowing-downs of an ESP. Then  $(Y_\bullet, R_\bullet)$  is of type  $(C_{-\infty})$ .*

**REMARK 7.14.** If  $S$  is a two-pointed blown-up of  $\mathbb{P}^2$ , then  $S$  is also a one point blown-up of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $X := S \times E$  be the direct product of  $S$  and an elliptic curve  $E$ . Let  $n \geq 2$  be a positive integer and  $\mu_n: E \rightarrow E$  the multiplication by  $n$  mapping. Then,  $f := \text{id}_S \times \mu_n: X \rightarrow X$  is a non-isomorphic étale endomorphism of  $X$ . After a succession of divisorial contractions  $X \rightarrow \mathbb{P}^2 \times E$ , there is induced an FESP  $\text{id}_{\mathbb{P}^2} \times \mu_n: \mathbb{P}^2 \times E \rightarrow \mathbb{P}^2 \times E$ , where the second projection  $\mathbb{P}^2 \times E \rightarrow E$  gives an extremal contraction of type (D).

Furthermore, after another divisorial contraction  $X \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \times E$ , there is induced another FESP  $\text{id}_{\mathbb{P}^1 \times \mathbb{P}^1} \times \mu_n: (\mathbb{P}^1 \times \mathbb{P}^1) \times E \rightarrow (\mathbb{P}^1 \times \mathbb{P}^1) \times E$  of type  $(C_{-\infty})$ , where the projection  $pr_1 \times \text{id}_E: (\mathbb{P}^1 \times \mathbb{P}^1) \times E \rightarrow \mathbb{P}^1 \times E$  gives an extremal contraction.

## 8. Finiteness of extremal rays

Our description of the FESP eventually turns out to be the most effective. We can extend these ideas to show finiteness of extremal rays of  $\overline{NE}(X)$  in the case where there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_1)$  or  $(C_0)$  on an FESP constructed from  $X_\bullet$  by a sequence of blowing-downs of an ESP. (cf. Theorem 1.4). Corollary 8.1 shows that in this case, the MMP works compatibly with étale endomorphisms and we can take  $Y_\bullet$  to be a constant FESP. Theorem 8.6 shows the finiteness of extremal rays of fiber type for a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of a 3-fold  $X$  with  $\kappa(X) = -\infty$ .

First we begin with a proof of Theorem 1.4.

**Proof of Theorem 1.4.** By Proposition 3.1, it is sufficient to prove the theorem for  $n = 0$ . Applied Propositions 7.3, 7.5 to the assumption, there exist Cartesian morphisms of ESPs,

$$X_\bullet \rightarrow Y_\bullet \rightarrow S_\bullet,$$

such that the following hold:

- $\pi_\bullet = (\pi_n)_n: X_\bullet \rightarrow Y_\bullet$  is a succession of Cartesian blowing-ups along elliptic curves.
- $S_\bullet = (h_n: S_n \rightarrow S_{n+1})_n$  is an ESP of smooth algebraic surfaces  $S_n$  with  $\kappa(S_n) = 0$  for any  $n$  or  $\kappa(S_n) = 1$  for any  $n$ .
- $\varphi_\bullet = (\varphi_n)_n: Y_\bullet \rightarrow S_\bullet$  is a conic bundle.
- Any  $S_n$  has a Seifert elliptic surface structure  $\alpha_n: S_n \rightarrow C_n$ .

Take an arbitrary extremal ray  $R_0$  on  $X_0$  which is of divisorial type. For any  $n > 0$ , we set  $R_n := (f_n \circ \cdots \circ f_0)_* R_0$  and  $R_{-n} := (f_{-1} \circ \cdots \circ f_{-n})^* R_0$ . Then, for any  $n$ ,  $R_n$  is also an extremal ray of divisorial type on  $X_n$ , whose contraction morphism  $\psi_n := \text{Cont}_{R_n}: X_n \rightarrow X'_n$  is (the inverse of) the blowing-up of a smooth projective 3-fold  $X'_n$  along an elliptic curve  $\Gamma_n \subset X'_n$ . By Proposition 3.1, there are induced an ESP  $X'_\bullet := (f'_n: X'_n \rightarrow X'_{n+1})_n$  of  $X'_\bullet$  and an ESP  $\Gamma_\bullet = (f_n|_{\Gamma_n}: \Gamma_n \rightarrow \Gamma_{n+1})_n$  of  $\Gamma_n$  such that  $\psi_{n+1} \circ f_n = f'_n \circ \psi_n$ . Let  $D_n := \text{Exc}(\psi_n)$  be the

$\psi_n$ -exceptional divisor. Since  $S_n$  contains no rational curves, each fiber of  $\psi_n|_D: D_n \rightarrow \Gamma_n$  is mapped to a point by the morphism  $\varphi_n \circ \pi_n: X_n \rightarrow S_n$ . Hence there exists a morphism  $v_n: X'_n \rightarrow S_n$  such that  $\varphi_n \circ \pi_n = v_n \circ \psi_n$ . By construction, we also have  $h_n \circ v_n = v_{n+1} \circ f'_n$ . To sum up, there exist Cartesian morphisms of ESPs;

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\pi_\bullet} & Y_\bullet \\ \psi_\bullet \downarrow & & \downarrow \varphi_\bullet \\ X'_\bullet & \xrightarrow[v_\bullet]{} & S_\bullet, \end{array}$$

where  $v_\bullet = (v_n)_n$  and  $\psi_\bullet = (\psi_n)_n$  is the contraction morphism of the extremal rays  $R_\bullet = (R_n)_n$ .

By Propositions 7.3 and 7.8, any fiber of  $\varphi_n \circ \pi_n: X_n \rightarrow S_n$  is a union of rational curves and there exists a finite set  $T_n \subset C_n$  such that  $\varphi_n \circ \pi_n$  is a  $\mathbb{P}^1$ -bundle over  $S_n \setminus \alpha_n^{-1}(T_n)$ . In particular,  $\varphi_n \circ \pi_n$  is equi-dimensional. If  $v_n(\Gamma_n) \subset S_n$  is a point, then the  $\psi_n$ -exceptional divisor  $D_n$  is contained in  $(\varphi_n \circ \pi_n)^{-1}(v_n(\Gamma_n))$ , which derives a contradiction. Hence, any  $\Delta_n := v_n(\Gamma_n)$  is an irreducible curve on  $S_n$ . Since  $f_n'^{-1}(\Gamma_{n+1}) = \Gamma_n$ , applying Lemma 2.6 to the above diagram, we infer that  $h_n^{-1}(\Delta_{n+1}) = \Delta_n$ . Thus there are induced an ESP  $\Delta_\bullet = (h_n|_{\Delta_n}: \Delta_n \rightarrow \Delta_{n+1})_n$  of  $\Delta_n$  and a Cartesian morphism  $v_\bullet|_{\Gamma_\bullet} = (v_n|_{\Gamma_n})_n: \Gamma_\bullet \rightarrow \Delta_\bullet$ . Hence, by Proposition 7.3, any  $\Delta_n$  is an elliptic curve and coincides with some fiber of  $\alpha_n: S_n \rightarrow C_n$ . In particular, if we consider the case for  $n = 0$ , then  $\varphi_0 \circ \pi_0(D_0) = v_0 \circ \psi_0(D_0) = v_0(\Gamma_0)$  is contained in some fiber of  $\alpha_0: S_0 \rightarrow C_0$ . By the composition of the morphisms  $\beta_0 := \alpha_0 \circ \varphi_0 \circ \pi_0: X_0 \rightarrow C_0$ , we regard  $X_0$  as a fiber space over the curve  $C_0$ . Then  $\beta_0$  has a singular fiber over a finite subset  $T_0 \subset C_0$ , smooth outside  $T_0$  and its smooth fiber is an elliptic ruled surface. Note that the extremal ray  $R_0$  is spanned by a fiber  $\gamma_0$  of the ruling  $\psi_0|_{D_0}: D_0 \rightarrow \Gamma_0$  of the exceptional divisor  $D_0 := \text{Exc}(\psi_0)$  and we have  $(K_{X_0}, \gamma_0) = -1$ . On the other hand, we have  $(K_{X_0}, \gamma_t) = -2$  for any point  $t \in C_0 \setminus T_0$  and any fiber  $\gamma_t$  of the ruling of the elliptic ruled surface  $\beta_0^{-1}(t) \rightarrow \alpha_0^{-1}(t)$ . Hence  $D_0$  is an irreducible component of some singular fiber of  $\beta_0$ . Note that there exist at most finitely many irreducible components of the singular fiber  $\beta_0^{-1}(T)$ . Thus the possibility of  $D_0$  (and hence that of  $R$ ) is finite and there exist only finitely many extremal rays of divisorial type on  $\overline{\text{NE}}(X_0)$ .  $\square$

As a fairly easy consequence of Theorem 1.4 and Proposition 3.8, we can state the following result; Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_0)$  or type  $(C_1)$  constructed from a given non-isomorphic étale endomorphism  $f: X \rightarrow X$  by a sequence of blowing-downs of an ESP. Then we can run the minimal model program compatibly with étale endomorphisms and obtain a constant FESP.

**Corollary 8.1.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y'_\bullet, R_\bullet)$  of type  $(C_0)$  or of type  $(C_1)$  constructed from  $(X, f)$  by a sequence of blowing-downs of an ESP. Then, by replacing  $f$  by its suitable power  $f^k$  ( $k > 0$ ), we have a Cartesian morphism of constant ESP*

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, h),$$

such that the following conditions are satisfied:

- (1)  $\pi: X \rightarrow Y$  is a succession of blowing-ups of a smooth 3-fold  $Y$  along elliptic curves.
- (2)  $\varphi: Y \rightarrow S$  is a conic bundle over a smooth surface  $S$  with  $\kappa(S) = 0$  or  $1$ .

- (3) The composite map  $\varphi \circ \pi: X \rightarrow S$  gives a MRC (maximally rationally connected) fibration of  $X$ .

In particular, we can take  $Y_\bullet$  to be a constant FESP of  $X_\bullet$ .

The following shows the uniqueness of the  $\mathbb{P}^1$ -fiber space structure on  $X$ .

**Corollary 8.2.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_1)$  or  $(C_0)$  constructed from  $(X, f)$  by a sequence of blowing-downs of an ESP. Let  $\psi: X \rightarrow T$  be an arbitrary  $\mathbb{P}^1$ -fiber space structure on  $X$ . Then there exists an isomorphism  $u: T \simeq S$  such that  $u \circ \psi = \varphi \circ \pi$ . In particular,  $\psi$  is equi-dimensional and unique up to isomorphism.*

*Proof.* We use the same notation as in Corollary 8.1. Since  $\varphi \circ \pi: X \rightarrow S$  is a MRC-fibration and  $S$  contains no rational curves, there exists a morphism  $u: T \rightarrow S$  with  $u \circ \varphi = \varphi \circ \pi$ . Since  $\varphi \circ \pi$  is equi-dimensional and each fiber of  $\varphi \circ \pi$  is connected,  $u$  is of degree one, hence an isomorphism by Zariski's main theorem.  $\square$

**REMARK 8.3.** The constant FESP  $Y_\bullet = (Y, g)$  is not uniquely determined by the given étale endomorphism  $f: X \rightarrow X$ . Using *elementary transformations*, we shall construct such an example. Let  $\mathbb{S}$  be the Atiyah surface over an elliptic curve  $E$  and  $C$  a smooth curve of genus  $g(C) \geq 1$ . For a positive integer  $n > 1$ , let  $\mu_n: E \rightarrow E$  be multiplication by  $n$ . Then with the aid of Proposition 4.13, we see that  $f: \mathbb{S} \simeq \mathbb{S} \times_{E, \mu_n} E \rightarrow \mathbb{S}$  gives an étale endomorphism of  $\mathbb{S}$  of degree  $n^2 > 1$ . If we set  $Y := \mathbb{S} \times C$ , then  $q: Y \rightarrow C \times E$  is a  $\mathbb{P}^1$ -bundle and  $g := f \times \text{id}_C: Y \rightarrow Y$  gives a non-isomorphic étale endomorphism of  $Y$ . Thus we have the following Cartesin morphism of constant ESP;

$$(Y, g) \xrightarrow{q} (C \times E, \text{id}_C \times \mu_n).$$

Let  $s_\infty$  be the canonical section of  $\mathbb{S}$ . We fix a point  $o \in C$  and take an elliptic curve  $\gamma := s_\infty \times \{o\}$  on  $Y$ . Now we shall perform an elementary transformation to  $Y$  along  $\gamma$ . Let  $\pi: X := \text{Bl}_\gamma(Y) \rightarrow Y$  be the blowing-up of  $Y$  along  $\gamma$ . Note that  $g^{-1}(\gamma) = \gamma$  by construction. Hence by Lemma 3.3,  $g: Y \rightarrow Y$  can be lifted to a non-isomorphic étale endomorphism  $f: X \rightarrow X$ . Since  $N_{\gamma/Y} \simeq \mathcal{O}_\gamma \oplus \mathcal{O}_\gamma$ , the exceptional divisor  $\text{Exc}(\pi)$  is isomorphic to the product  $\gamma \times \mathbb{P}^1$ . Let  $\mathbb{S}'$  be the proper transform of the surface  $\mathbb{S} \times \{o\}$  by  $\pi$ . Then there exists a divisorial contraction  $\pi': X \rightarrow Y'$  which contracts the divisor  $\mathbb{S}'$  to an elliptic curve  $\gamma'$ . Since  $f^{-1}(\mathbb{S}') = \mathbb{S}'$ , there exists a non-isomorphic étale endomorphism  $g': Y' \rightarrow Y'$  such that  $\pi' \circ f = g' \circ \pi'$ . By the natural projection  $p': Y' \rightarrow C$ ,  $Y'$  can be regarded as a smooth fiber space over  $C$  and put  $Y'_t := p'^{-1}(t)$  for  $t \in C$ .

Then

$$Y'_t \simeq \begin{cases} \mathbb{S}, & t \neq o, \\ \mathbb{P}^1 \times E, & t = o \end{cases},$$

and thus a jumping phenomenon occurs. Both  $Y_\bullet := (Y, g)$  and  $Y'_\bullet := (Y', g')$  are constant FESPs constructed from the original constant ESP  $X_\bullet := (X, f)$  by blowing-downs of an ESP.

In the case where  $f: X \rightarrow X$  is a non-isomorphic étale endomorphism of a smooth projective 3-fold with  $\kappa(X) \geq 0$ , we can say more about a type of extremal rays of  $\overline{\text{NE}}(X)$ .

**Proposition 8.4.** (cf. [9, Proposition 4.6 and Theorem 4.8 ]) *Let  $f: X \rightarrow X$  be a non-isomorphic surjective endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) \geq 0$ . If  $K_X$  is not nef, then the following assertions hold:*

- (1) *The set of extremal rays  $R$  is a finite set and  $f$  induces a permutation of them.*
- (2) *The contraction morphism  $\text{Contr}_R: X \rightarrow X'$  associated to any extremal ray  $R$  of  $\overline{\text{NE}}(X)$  is a divisorial contraction which is (the inverse of) the blowing-up along an elliptic curve  $C$  on  $X'$ .*
- (3) *Let  $E$  be the exceptional divisor of  $\text{Contr}_R$ . Then  $E \cong \mathbb{P}_C(\mathcal{O}_C \oplus \ell)$  for some torsion line bundle  $\ell \in \text{Pic}^0(C)$ .*

*Proof.* The assertions (1) and (2) have already been proved in [9, Proposition 4.6 and Theorem 4.8]. Thus it is enough to prove only the assertion (3). By Remark 3.2,  $E$  is isomorphic to either the Atiyah surface  $\mathbb{S}$  or  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell)$  for an elliptic curve  $C$  and a line bundle  $\ell \in \text{Pic}^0(C)$ . If we replace  $f$  by its suitable power  $f^k$  ( $k > 0$ ), we may assume that  $f^{-1}(E) = E$ . Hence, the restriction of  $f$  to  $E$  gives a non-isomorphic étale endomorphism  $f|_E: E \rightarrow E$ . Then, in the first case,  $\ell \in \text{Pic}^0(C)$  is of finite order by Proposition 4.8.

Next, we show that the case where  $E \simeq \mathbb{S}$  cannot occur. Applying [13, Main Theorem], we infer that there exists a finite étale Galois covering  $\rho: \widetilde{X} \rightarrow X$  such that there exists on  $\widetilde{X}$  an abelian scheme structure  $\varphi: \widetilde{X} \rightarrow T$  over a variety  $T$  with  $\dim T \leq 2$ . Then Lemma 3.12 shows the existence of an extremal ray  $\widetilde{R}(\subset \overline{\text{NE}}(\widetilde{X}))$  of divisorial type such that  $\rho_*\widetilde{R} = R$ . Let  $\widetilde{E}$  be the exceptional divisor of the contraction morphism  $\text{Contr}_{\widetilde{R}}$ . Then  $\rho^{-1}(E) = \widetilde{E}$  and  $\widetilde{E}$  is isomorphic to the Atiyah surface  $p: \mathbb{S} \rightarrow \widetilde{C}$  over an elliptic curve  $\widetilde{C}$  by Lemma 4.12.

First, suppose that  $\dim T = 0$ . Then  $\widetilde{X}$  is an abelian 3-fold and contains no rational curves. Thus a contradiction is derived. Next, suppose that  $\dim T = 2$ . Assume that some fiber of  $p: \widetilde{E} \rightarrow \widetilde{C}$  is mapped to a point by  $\varphi|_{\widetilde{E}}: \widetilde{E} \rightarrow T$ . Then by the rigidity lemma, there exists a morphism  $u: \widetilde{C} \rightarrow T$  such that  $u \circ p = \varphi|_{\widetilde{E}}$ . This contradicts the assumption that  $p$  is a  $\mathbb{P}^1$ -bundle and  $\varphi$  is an elliptic bundle. Hence any fiber of  $p$  is mapped to a curve by  $\varphi|_{\widetilde{E}}$  and  $\dim \varphi(\widetilde{E}) \geq 1$ . Suppose that  $\dim \varphi(\widetilde{E}) = 1$  and let  $F$  be any fiber of the Stein factorization of  $\varphi|_{\widetilde{E}}: \widetilde{E} \rightarrow T$ . Then  $p(F) = \widetilde{C}$  and  $F^2 = 0$ . Thus applying Proposition 5.2, we infer that  $F$  equals the canonical section  $s_\infty$  and hence  $\widetilde{E} = s_\infty$ , which derives a contradiction. Suppose that  $\varphi(\widetilde{E}) = T$ . Then  $\kappa(T) = -\infty$ , which contradicts the fact that  $\kappa(T) = \kappa(\widetilde{X}) = \kappa(X) \geq 0$ .

Finally, suppose that  $\dim T = 1$ . Then applying the same argument as above, each fiber of  $p$  is mapped onto  $T$  by  $\varphi$ . Hence  $T \simeq \mathbb{P}^1$ , and  $\kappa(\widetilde{X}) = \kappa(T) = -\infty$ , which again contradicts the fact that  $\kappa(\widetilde{X}) = \kappa(X) \geq 0$ .  $\square$

Here we insert some remark related with a new phenomenon in the case of  $\kappa(X) = -\infty$ .

**REMARK 8.5.** Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . We encounter a new phenomenon which does not appear in the case of  $\kappa(X) \geq 0$ . In Remark 8.3, we have constructed such an example of a 3-fold  $X$  which enjoys the following properties;

- There exists an extremal ray  $R$  of divisorial type.
- The exceptional divisor  $E$  of the contraction morphism  $\text{Contr}_R: X \rightarrow X'$  is the Atiyah surface  $\mathbb{S}$ .

There exists no Seifert elliptic fiber space structure on  $X$ , as Proposition 6.2 shows.

Next, using ideas of the proof of Theorem 3.10, we can show the finiteness of extremal rays which are of *fiber type*.

**Theorem 8.6.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Then there exist at most finitely extremal rays which are of fiber type on  $\overline{NE}(X)$ .*

*Proof.* Choose an extremal ray  $R(\subset \overline{NE}(X))$  of fiber type arbitrarily and fix it. Then for any extremal ray  $R'(\neq R)$  of fiber type, both FESPs  $(X, R)$  and  $(X, R')$  are of type (D) or of type (C) by Proposition 7.12.

First, we show that the case where both FESPs  $(X, R)$  and  $(X, R')$  are of type (D) cannot occur. Suppose the contrary. Then by Proposition 7.3, the contraction morphism  $\pi := \text{Cont}_R: X \rightarrow C$  (resp.  $\pi' := \text{Cont}_{R'}: X \rightarrow C'$ ) is a smooth del Pezzo fiber space over an elliptic curve  $C$  (resp.  $C'$ ). Hence any fiber of  $\pi$  (resp.  $\pi'$ ) is covered by rational curves and contracted to a point by  $\pi'$  (resp.  $\pi$ ). Thus by the rigidity lemma, there exists an isomorphism  $v: C \simeq C'$  such that  $\pi' = v \circ \pi$ . Hence  $R = R'$ , which is a contradiction.

Hence, we may consider the latter case. We first show that both FESPs  $(X, R)$  and  $(X, R')$  are of type  $(C_{-\infty})$ . Suppose the contrary that the FESP  $(X, R)$  is of type  $(C_0)$  or type  $(C_1)$ . Let  $\pi := \text{Cont}_R: X \rightarrow S$  (resp.  $\pi' := \text{Cont}_{R'}: X \rightarrow S'$ ) be the contraction morphism associated to  $R$  (resp.  $R'$ ). Then by Proposition 7.3, the surface  $S$  contains no rational curves. Hence any fiber of  $\pi'$  is mapped to a point by  $\pi$ . Thus by the rigidity lemma, there exists a morphism  $u: S' \rightarrow S$  with  $\pi = u \circ \pi'$ . Since any fiber of  $\pi$  is connected,  $u$  is of degree one and hence an isomorphism by Zariski's main theorem. Thus  $R = R'$ , which derives a contradiction. Hence by Proposition 7.3, both  $S$  and  $S'$  are elliptic ruled surfaces. In particular,  $\rho(S) = \rho(S') = 2$ . Now, we follow the arguments of [15]. Let  $\{H_1, H_2\}$  be a set of ample divisors of  $S$  such that  $\{\text{cl}(H_1), \text{cl}(H_2)\}$  is a basis of  $N^1(S)$ . Since  $R \neq R'$ , we infer that  $(\pi^*H_i) \cdot R' > 0$  for all  $i = 1, 2$ . Then there exists a positive rational number  $\alpha$  such that  $\pi^*(H_1 - \alpha H_2) \cdot R' = 0$ . Hence if we set  $D := H_1 - \alpha H_2$ , then  $\pi^*(D) \sim \pi'^*(D')$  for some Cartier divisor  $D'$  on  $S'$ .

We show that  $D^2 = 0$ . The proof is by contradiction. Suppose the contrary. Then the product  $\pi^*(D)^2$  in  $N_1(X)$  is numerically equivalent to  $\delta Z$  for a non-zero effective 1-cycle  $Z$  on  $X$  and for a non-zero rational number  $\delta := D^2 \neq 0$ . Hence  $\pi^*(L)Z = \delta^{-1}\pi^*(LD^2) = 0$  and  $\pi'^*(L')Z = \delta^{-1}\pi'^*(L'D'^2) = 0$  for any Cartier divisor  $L$  on  $S$  and any Cartier divisor  $L'$  on  $S'$ . Thus the numerical equivalence class  $\text{cl}(Z)$  is contained in  $R \cap R'$ . Hence  $R = R'$ , which contradicts the assumption.

We consider the algebraic equation  $(H_1 - zH_2)^2 = 0$  of degree two on  $z$ , which depends only on  $H_1$  and  $H_2$ . It has at most two roots, one of which equals  $\alpha$ . For such  $\alpha$ , consider the real 2-dimensional vector space  $F := \pi^*(\text{cl}(H_1 - \alpha H_2))^\perp \subset N_1(X)$ . By construction, we see that  $R' \subset F$ . Since there exists at most two extremal rays contained in the real 2-dimensional vector subspace  $F$ , there exist at most 4 extremal rays  $R'$  which is of fiber type and different from  $R$ .  $\square$

**REMARK 8.7.** (1) Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold with  $\kappa(X) = -\infty$ . Then, with the aid of Theorem 8.6, we infer that the number of extremal rays of  $X$  is finite if and only if the number of extremal rays of  $X$  of divisorial type is finite.

(2) Let  $E$  be an elliptic curve and  $\mu: E \rightarrow E$  multiplication by  $n(> 1)$ . Then the product

$X := E \times \mathbb{P}^1 \times \mathbb{P}^1$  admits a non-isomorphic étale endomorphism  $f: X \rightarrow X$  of  $X$  defined by  $f := \mu_n \times \text{id}_{\mathbb{P}^1} \times \text{id}_{\mathbb{P}^1}$ . There exist exactly two extremal rays on  $X$ . The projections  $p_{1,2}: X \rightarrow E \times \mathbb{P}^1$  and  $p_{1,3}: X \rightarrow E \times \mathbb{P}^1$  are all extremal contractions.

### 9. Some reductions of a conic bundle to a $\mathbb{P}^1$ -bundle

Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type (C) constructed from  $X$  by a sequence of blowing-downs of an ESP. In this section, we show that  $f: X \rightarrow X$  can be lifted to a non-isomorphic étale endomorphism  $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ , where  $\tilde{X}$  is a suitable finite étale covering of  $X$  and is obtained from a  $\mathbb{P}^1$ -bundle by successive blowing-ups along elliptic curves. We follow some arguments by Mori and Mukai (cf. [30]) used in the classification of Fano 3-folds.

**Proposition 9.1.** *Let  $f: Y \rightarrow Y$  be a finite étale covering of a smooth projective 3-fold  $Y$  with  $\kappa(Y) = -\infty$ . Suppose that;*

- (1) *There exists a conic bundle  $\varphi: Y \rightarrow S$  over a product surface  $S := C \times E$  of a smooth curve  $C$  and an elliptic curve  $E$ .*
- (2)  *$\varphi$  is not smooth and the discriminant locus  $\Delta_\varphi$  of  $\varphi$  is contained in fibers of the first projection  $p_1: S \rightarrow C$ .*
- (3) *There exists a Lie group homomorphism  $\alpha: E \rightarrow E$  such that  $g \circ \varphi = \varphi \circ f$  for an étale endomorphism  $g := \text{id}_C \times \alpha: S \rightarrow S$ .*

*Then, by replacing  $f$  by a suitable power  $f^k$  ( $k > 0$ ), there exist a finite étale Galois covering  $\psi: \tilde{Y} \rightarrow Y$  and a commutative diagram*

$$\begin{array}{ccc}
 \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
 \pi \downarrow & & \downarrow \pi \\
 M & \xrightarrow{\bar{f}} & M \\
 q \downarrow & & \downarrow q \\
 \tilde{S} & \xrightarrow{\tilde{g}} & \tilde{S} \\
 \downarrow & & \downarrow \\
 C & \xlongequal{\quad} & C
 \end{array}$$

*satisfying the following conditions:*

- (1)  $\tilde{f}: \tilde{Y} \rightarrow \tilde{Y}$  is an étale endomorphism such that  $f \circ \psi = \psi \circ \tilde{f}$ .
- (2)  $\pi: \tilde{Y} \rightarrow M$  is a succession of blowing-ups of a smooth projective 3-fold  $M$  along elliptic curves.
- (3)  $\bar{f}: M \rightarrow M$  is an étale endomorphism.
- (4)  $q: M \rightarrow \tilde{S}$  is a  $\mathbb{P}^1$ -bundle over a surface  $\tilde{S}$  which is a copy of  $S$ .
- (5)  $\tilde{g}: \tilde{S} \rightarrow \tilde{S}$  is an étale endomorphism of  $\tilde{S} \simeq S$  such that  $\tilde{g} := \text{id}_C \times \alpha$ .

**Proof.** By assumption, the discriminant loci  $\Delta_\varphi$  consist of a disjoint union of elliptic curves  $\Delta_1, \dots, \Delta_r$  which are fibers of  $p_1$  such that  $g^{-1}(\Delta_i) = \Delta_i$  for any  $i$ .

**Step 1.** Suppose that  $\varphi$  is not an extremal contraction. Then, by [30, Proposition 4.8],  $\varphi^{-1}(\Delta_i)$  is reducible for some  $\Delta_i$ . Let  $\varphi^{-1}(\Delta_i) = E_i^{(1)} + E_i^{(2)}$  be an irreducible decomposition. Then, by [30, Proposition 4.9],  $\varphi|_{E_i^{(j)}}: E_i^{(j)} \rightarrow \Delta_i$  is a  $\mathbb{P}^1$ -bundle over  $\Delta_i$  with fiber  $\ell_i^{(j)}$  for each  $j = 1, 2$ . Replacing  $f$  by  $f^2$ , we may assume that  $f^{-1}(E_i^{(j)}) = E_i^{(j)}$  for any  $j = 1, 2$ . Then,  $-K_Y - E_i^{(j)}$  is  $\varphi$ -nef for each  $j$  and

$$\overline{\text{NE}}(Y/S) \cap (-K_Y - E_i^{(j)})^\perp = \mathbb{R}_{\geq 0} [\ell_i^{(j')}]$$

for  $j' \in \{1, 2\} \setminus j$ . Hence, each  $[\ell_i^{(j)}]$  spans an extremal ray  $R_i^{(j)}$ . Let  $\pi_i^{(j)} := \text{Cont}_{R_i^{(j)}}: Y \rightarrow Y_i^{(j)}$  be the contraction morphism associated to  $R_i^{(j)}$ . Then, there exists a conic bundle  $\varphi_i^{(j)}: Y_i^{(j)} \rightarrow S$  such that  $\varphi = \varphi_i^{(j)} \circ \pi_i^{(j)}$ . Since  $f_*[\ell_i^{(j)}] = [\ell_i^{(j)}]$  for any  $j$ , there exists an étale endomorphism  $f_i^{(j)}: Y_i^{(j)} \rightarrow Y_i^{(j)}$  such that  $f_i^{(j)} \circ \pi_i^{(j)} = \pi_i^{(j)} \circ f$ . Thus we have obtained the following commutative diagram which we are looking for:

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y \\ \pi_i^{(j)} \downarrow & & \downarrow \pi_i^{(j)} \\ Y_i^{(j)} & \xrightarrow{f_i^{(j)}} & Y_i^{(j)} \\ \varphi_i^{(j)} \downarrow & & \downarrow \varphi_i^{(j)} \\ S & \xrightarrow{g} & S \\ p_1 \downarrow & & \downarrow p_1 \\ C & \xlongequal{\quad} & C \end{array}$$

**Step 2.** Suppose that  $\varphi$  is an extremal contraction. Then, by [30], each  $\varphi^{-1}(\Delta_i)$  is irreducible. Let  $e \in E$  be the zero element of the elliptic curve  $E$  and put  $o_i := (\{e\} \times C) \cap \Delta_i$ . Let  $S$  be the set of two irreducible components of  $\varphi^{-1}(o_i)$ . Then, an analytic continuation along a loop  $\gamma \in \pi_1(\Delta_i, o_i)$  induces a permutation of  $S$ . Hence, there exists a monodromy representation  $\chi_i: \pi_1(\Delta_i, o_i) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ , which is surjective, since  $\varphi^{-1}(\Delta_i)$  is irreducible. Let  $\widetilde{\Delta}_i \rightarrow \Delta_i$  be an étale double covering associated to  $\text{Ker } \chi_i$ . By the composition of a suitable isogeny  $\Delta_i \rightarrow \widetilde{\Delta}_i$ , we may assume that  $\Delta_i \rightarrow \widetilde{\Delta}_i$  is multiplication by  $m_i$ . Let

$$\rho_i := \text{id}_C \times \mu_{m_i}: S_i = C \times \Delta_i \rightarrow S = C \times E$$

be a finite étale covering. Since  $\mu_{m_i}$  commutes with  $\alpha: \Delta_i \rightarrow \Delta_i$ ,  $g$  can be lifted to an étale endomorphism  $\widetilde{g} := \text{id}_C \times \alpha: S_i \rightarrow S_i$ . Furthermore,  $\widetilde{Y}_i := Y_i \times_S S_i \rightarrow Y$  is also a finite étale covering and there exists a conic bundle  $\widetilde{\varphi}_i: \widetilde{Y}_i \rightarrow S$ . Since  $\varphi \circ f = g \circ \varphi$  and  $\widetilde{g}$  is a lift of  $g$ , there is induced an étale endomorphism  $\widetilde{f}_i: \widetilde{Y}_i \rightarrow \widetilde{Y}_i$  which is a lift of  $f: Y \rightarrow Y$ . By construction,  $\widetilde{\Delta}_i := \rho_i^{-1}(\Delta_i)$  is reducible. Hence, by Step 1, there exists a blowing-up  $\pi_i: \widetilde{Y}_i \rightarrow \widetilde{Y}_i'$  along an elliptic curve on a conic bundle  $\widetilde{\varphi}_i: \widetilde{Y}_i' \rightarrow S_i$ . Moreover, if we replace  $f$  by  $f^2$ ,  $\widetilde{f}_i$  descends to an étale endomorphism  $\widetilde{f}_i': \widetilde{Y}_i' \rightarrow \widetilde{Y}_i'$ .

In the general case, suppose that  $\varphi^{-1}(\Delta_i)$  is reducible for all  $1 \leq i \leq k$ , ( $k \leq r$ ) and irreducible otherwise. Let  $\mu: E \rightarrow E$  be a multiplication mapping which factors through  $\mu_{m_i}: \Delta_i \rightarrow \Delta_i$  for all  $i > k$ . If we take the base change  $\widetilde{Y} = Y \times_{E, \mu} E$  and perform the same

procedure as above, then the proof of proposition is finished.  $\square$

REMARK 9.2. Proposition 9.1 holds even if  $f$  is an isomorphism.

In the case where there exists an FESP of type  $(C_1)$ , we encounter the situation produced in Proposition 9.1 with the aid of Proposition 7.3 and Corollary 8.1.

**Theorem 9.3.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_1)$  constructed from  $X$  by a sequence of blowing-downs of an ESP. Then, by replacing  $f$  by a suitable power  $f^k$  ( $k > 0$ ), there exist a finite étale Galois covering  $\rho: \tilde{X} \rightarrow X$  and a Cartesian morphism of constant ESPs*

$$(\tilde{X}, \tilde{f}) \xrightarrow{\pi'} (M, f') \xrightarrow{q} (\tilde{S}, \tilde{h}),$$

satisfying the following conditions:

- (1)  $\pi': \tilde{X} \rightarrow M$  is a succession of blowing-ups of a smooth projective 3-fold  $M$  along elliptic curves and  $f': M \rightarrow M$  is a non-isomorphic étale endomorphism.
- (2)  $q: M \rightarrow \tilde{S}$  is a  $\mathbb{P}^1$ -bundle over the product surface  $\tilde{S} := \tilde{C} \times E$  of a smooth curve  $\tilde{C}$  and an elliptic curve  $E$ .
- (3)  $\tilde{h} := \text{id}_{\tilde{C}} \times \alpha: \tilde{S} \rightarrow \tilde{S}$  for a Lie group homomorphism  $\alpha: E \rightarrow E$ .
- (4) By  $q$ , the centers of the blowing-up  $\pi'$  are mapped onto the fibers of the first projection  $p: \tilde{S} \rightarrow \tilde{C}$ .

Proof. By Corollary 8.1, replacing  $f$  by a suitable power  $f^k$  ( $k > 0$ ), we may assume that the conclusions of Corollary 8.1 are satisfied. Since  $S$  is a minimal algebraic surface with  $\kappa(S) = 1$  and  $e(S) = 0$ , the Itaka fibration  $\psi: S \rightarrow C$  gives a Seifert elliptic surface structure. By [40], an étale endomorphism  $h: S \rightarrow S$  induces a finite automorphism of the base curve  $C$ . Hence, replacing  $f$  by a suitable power  $f^\ell$  ( $\ell > 0$ ), we may assume that  $\psi \circ h = \psi$ . Thus we have the following Cartesian morphism of constant ESP:

$$(X, f) \xrightarrow{\pi} (Y, g) \xrightarrow{\varphi} (S, h),$$

where  $\pi$  is a sequence of equivariant blowing-downs and  $\varphi$  is a conic bundle. By [13, Theorem 2.24], the irreducible component of the fixed point locus  $\text{Fix}(h)$  of  $h$  is a finite étale covering of  $C$ . Hence, by [13, Lemma 2.25], we can take a finite étale covering  $C' \rightarrow C$  such that the normalization  $S'$  of the fiber product  $S \times_C C'$  satisfies the following;

- (i)  $S'$  is isomorphic over  $C'$  to the direct product  $C' \times E$  of a smooth curve  $C'$  of genus  $p_a(C') \geq 2$  and an elliptic curve  $E$ .
- (ii) There exists a non-isomorphic étale endomorphism  $\tilde{h}: S' \rightarrow S'$  which is a lift of  $h: S \rightarrow S$  such that  $\tilde{h} = \text{id}_{C'} \times \mu_\alpha$  for a non-isomorphic Lie group homomorphism  $\mu_\alpha: E \rightarrow E$ .

Then, if we put  $\bar{X} := X \times_C C'$  and  $\bar{Y} = Y \times_C C'$ , the following conditions are satisfied;

- (a)  $\bar{X}$  and  $\bar{Y}$  are both non-singular and there exists a birational morphism  $\bar{\pi}: \bar{X} \rightarrow \bar{Y}$  which is a succession of blowing-ups along elliptic curves.
- (b) There exist isomorphisms  $\bar{X} \cong X \times_S S'$  and  $\bar{Y} \cong Y \times_S S'$ .
- (c)  $\bar{\varphi}: \bar{Y} \rightarrow S'$  is a conic bundle.

- (d) There exists a non-isomorphic étale endomorphism  $\bar{f}: \bar{X} \rightarrow \bar{X}$  (resp.  $\bar{g}: \bar{Y} \rightarrow \bar{Y}$ ) which is a lift of  $f: X \rightarrow X$  (resp.  $g: Y \rightarrow Y$ .)

By Proposition 7.3, the discriminant locus  $\text{disc}(\bar{\varphi})$  is a disjoint union of elliptic curves supported in fibers of the first projection  $S' \cong C' \times E \rightarrow C'$ . Hence, by Proposition 9.1, if we replace  $f$  by a suitable power  $f^k$  ( $k > 0$ ), the following conditions are satisfied.

- (A) There exists a finite étale Galois covering  $\tilde{Y} \rightarrow \bar{Y}$  and a non-isomorphic étale endomorphism  $\tilde{g}: \tilde{Y} \rightarrow \tilde{Y}$  which is a lift of  $\bar{g}: \bar{Y} \rightarrow \bar{Y}$ .  
 (B) There exists a succession of blowing-ups  $\tau: \tilde{Y} \rightarrow M$  of a smooth projective 3-fold  $M$  along elliptic curves and there induced a non-isomorphic étale endomorphism  $f': M \rightarrow M$ .  
 (C)  $M$  is a  $\mathbb{P}^1$ -bundle over a surface  $\tilde{S}$  which is a copy of  $S'$ .  
 (D) By a non-isomorphic étale endomorphism  $f': M \rightarrow M$ , there induced a non-isomorphic étale endomorphism  $\tilde{h} = \text{id}_C \times \alpha: \tilde{S} \rightarrow \tilde{S}$ .

Thus we are done.  $\square$

**REMARK 9.4.** We give such an example to illustrate our idea. Let  $\tilde{C} \rightarrow C$  be an étale double covering of a smooth curve  $C$  with  $g(C) \geq 2$  and  $i: \tilde{C} \simeq \tilde{C}$  an involution such that  $\tilde{C}/\langle i \rangle \simeq C$ . Take a line bundle  $\ell$  of order two on an elliptic curve  $E$  and set  $p: S := \mathbb{P}_E(\mathcal{O}_E \oplus \ell) \rightarrow E$ . Let  $s_k$  ( $k = 0, 1$ ) be two disjoint sections of  $p$  corresponding to a surjection  $\mathcal{O}_E \oplus \ell \twoheadrightarrow \mathcal{O}_E$  (resp.  $\mathcal{O}_E \oplus \ell \twoheadrightarrow \ell$ ). Then by [22, Theorem 2.(4)], there exists a relative automorphism  $u \in \text{Aut}(S/E)$  of order two such that  $u(s_k) = s_{1-k}$  for  $k = 0, 1$ . Then  $j := i \times u$  defines a free involutive automorphism of  $\tilde{C} \times S$  and let  $Y := \tilde{C} \times S / \langle j \rangle$  be its quotient. For an odd integer  $n > 1$ , let  $\mu_n: E \rightarrow E$  be multiplication by  $n$ . Since  $\mu_n^* \ell \sim \ell$  by [32, p75, (ii)], there is induced a non-isomorphic étale endomorphism  $g: S \simeq S \times_{E, \mu_n} E \rightarrow S$ . We define a finite étale endomorphism  $\psi$  of  $\tilde{C} \times S$  by  $\psi := \text{id}_{\tilde{C}} \times g$ . Since  $g \circ j = j \circ g$ ,  $\psi$  commutes with the involution  $j$  of  $\tilde{C} \times S$  and descends to a non-isomorphic étale endomorphism  $\varphi: Y \rightarrow Y$  over  $C$ . By the natural projection  $Y \rightarrow T := C \times E \rightarrow \tilde{C}/\langle i \rangle \simeq C$ ,  $Y$  is a  $\mathbb{P}^1$ -bundle over  $T$  and is also a fiber bundle over  $C$  whose fiber is isomorphic to  $S$ .

The same method applies to other situation. We consider the case where there exists an FESP of type  $(C_0)$  constructed from  $X$  by a sequence of blowing-downs of an ESP.

**Theorem 9.5.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_0)$  constructed from  $X$  by a sequence of blowing-downs of an ESP. Then, by replacing  $f$  by a suitable power  $f^k$  ( $k > 0$ ), there exists a finite étale Galois covering  $\rho: \tilde{X} \rightarrow X$  and the Cartesian morphism of constant ESPs*

$$(\tilde{X}, \tilde{f}) \xrightarrow{\pi'} (M, f') \xrightarrow{q} (A, h'),$$

which satisfies the following properties;

- (1) *There exists a succession of blowing-ups  $\pi': \tilde{X} \rightarrow M$  of a smooth projective 3-fold  $M$  along elliptic curves such that  $\pi' \circ \tilde{f} = f' \circ \pi'$  for some non-isomorphic étale endomorphism  $f': M \rightarrow M$ .*  
 (2)  *$q: M \rightarrow A$  is a  $\mathbb{P}^1$ -bundle over an abelian surface  $A$ .*

- (3) If  $\pi'$  is not an isomorphism, then  $A$  can be chosen to be a direct product  $E \times E'$  of elliptic curves  $E$  and  $E'$  such that  $h' = \text{id}_E \times \mu_\alpha$  for some Lie group homomorphism  $\mu_\alpha: E' \rightarrow E'$ .
- (4) The centers of the blowing-up  $\pi'$  are mapped onto the fibers of the first projection  $A := E \times E' \rightarrow E$ .

Proof. By Corollary 8.1 and Propositions 7.3, 3.8, there exists a constant FESP  $Y_\bullet := (Y, g)$ . Then there exist Cartesian morphisms of constant ESPs

$$X_\bullet := (X, f) \xrightarrow{\pi} Y_\bullet := (Y, g) \xrightarrow{\varphi} S_\bullet := (S, h)$$

such that the following conditions are satisfied;

- (1)  $\pi$  is a succession of blowing-ups along elliptic curves on  $Y$ .  
 (2)  $\varphi: Y \rightarrow S$  is a conic bundle.  
 (3)  $S$  is isomorphic to an abelian surface or a hyperelliptic surface.

**Step 1.** First we show that the proof can be reduced to the case where  $S$  is an abelian surface. Suppose that  $S$  is a hyperelliptic surface. Then some finite étale Galois cover  $p: \tilde{S} \rightarrow S$  of  $S$  is isomorphic to an abelian surface  $A$ . By [40], there exists a non-isomorphic étale endomorphism  $\tilde{h}: A \rightarrow A$  which is a lift of  $h: S \rightarrow S$ . Let  $\tilde{X} := X \times_S A$  (resp.  $\tilde{Y} := Y \times_S A$ ) be the pull-back of  $\varphi \circ \pi: X \rightarrow S$  (resp.  $\varphi: Y \rightarrow S$ ) by the étale morphism  $p: A \rightarrow S$ . Then the natural projection  $\tilde{X} \rightarrow X$  (resp.  $\tilde{Y} \rightarrow Y$ ) is a finite étale Galois covering. Thus there exist Cartesian morphisms of constant ESP

$$\tilde{X}_\bullet := (\tilde{X}, \tilde{f}) \xrightarrow{\tilde{\pi}} \tilde{Y}_\bullet := (\tilde{Y}, \tilde{g}) \xrightarrow{\tilde{\varphi}} A_\bullet := (A, \tilde{h})$$

such that the following conditions are satisfied:

- (1)  $\tilde{\pi}$  is a succession of blowing-ups along elliptic curves on a smooth 3-fold  $\tilde{Y}$ .  
 (2)  $\tilde{f}$  (resp.  $\tilde{g}$ ) is a lift of  $f$  (resp.  $g$ ).

Thus we may assume that  $S = A$  is an abelian surface.

**Step 2.** Then by Propositions 7.3 and 7.8, the set  $M := \varphi(\text{Exc}(\pi)) \cup \text{disc}(\varphi)$  is a disjoint union of finitely many elliptic curves on  $S$ . Moreover, there is induced a permutation of the finite set  $M$  by the étale endomorphism  $h: A \rightarrow A$ . Hence if we replace an endomorphism  $f$  by its suitable power  $f^k$  ( $k > 0$ ), we may assume that  $h^{-1}(\Delta_i) = \Delta_i$  for any connected component  $\Delta_i$  of  $M$ . Thus the abelian surface  $A$  is an elliptic bundle  $p: A \rightarrow E$  over an elliptic curve  $E$  so that  $p \circ h = p$ . We have the following Cartesian morphisms of constant ESP;

$$X_\bullet := (X, f) \xrightarrow{\pi} Y_\bullet := (Y, g) \xrightarrow{\varphi} A_\bullet := (A, h).$$

Applying Proposition 9.1, we can finish the proof in the same way as that of Theorem 9.3.  $\square$

Suppose that there exists an FESP of type  $(C_{-\infty})$  constructed from  $X$  by a sequence of blowing-downs of an ESP. Note that we cannot necessarily obtain another *constant* FESP, since there may exist infinitely many extremal rays of  $\text{NE}(X)$ .

**Theorem 9.6.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Suppose that there exists an FESP  $(Y_\bullet, R_\bullet)$  of type*

$(C_{-\infty})$  constructed from  $X$  by a sequence of blowing-downs of an ESP. Then, there exists a finite étale Galois covering  $\rho: \widetilde{X} \rightarrow X$  which satisfies the following properties:

- (1) There exists a non-isomorphic étale endomorphism  $\widetilde{X} \rightarrow \widetilde{X}$  such that  $\rho \circ \widetilde{f} = f \circ \rho$ .
- (2)  $Z_{\bullet} = (g_n: Z_n \rightarrow Z_{n+1})_n$  is an FESP constructed from  $\widetilde{X}_{\bullet} = (\widetilde{X}, \widetilde{f})$  by a sequence of blowing-downs  $\pi_{\bullet} = (\pi_n)_n: \widetilde{X}_{\bullet} \rightarrow Z_{\bullet}$  of an ESP.
- (3)  $Z_n$  is a  $\mathbb{P}^1$ -bundle  $\varphi_n: Z_n \rightarrow S_n$  over a smooth algebraic surface  $S_n$ .
- (4) There exists an ESP  $S_{\bullet} = (h_n: S_n \rightarrow S_{n+1})_n$  such that  $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$ .
- (5)  $S_n$  is a  $\mathbb{P}^1$ -bundle  $\alpha_n: S_n \rightarrow C$  over the Albanese elliptic curve  $C$  of  $X$ .
- (6) The composite map  $\alpha_n \circ \varphi_n \circ \pi_n: X \rightarrow C$  is independent of  $n$  and coincides with the Albanese map  $\text{Alb}_X: X \rightarrow C$  of  $X$ .
- (7) There exists a non-isomorphic Lie group homomorphism  $u: C \rightarrow C$  such that  $u \circ \alpha_n = \alpha_{n+1} \circ h_n$ .
- (8)  $S := S_0$  is isomorphic to either the Atiyah surface  $\mathbb{S}$  or  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_0)$  for some  $\ell_0 \in \text{Pic}^0(C)$ . Moreover, in the former case, we have  $S_n \simeq \mathbb{S}$  for any  $n$ . In the latter case, any  $S_n$  is isomorphic to the  $\mathbb{P}^1$ -bundle  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$  for some  $\ell_n \in \text{Pic}^0(C)$ .

Proof. Set  $Y_{\bullet} = (g_n: Y_n \rightarrow Y_{n+1})_n$  and  $R_{\bullet} = (R_n)_n$ , where  $(g_n)_*R_n = R_{n+1}$  for any  $n$ . Let  $\pi_{\bullet} = (\pi_n)_n: X_{\bullet} \rightarrow Y_{\bullet}$  be the Cartesian blowing-up. By assumption, each  $\varphi_n: Y_n \rightarrow S_n$  is a conic bundle over a smooth surface  $S_n$  and there exists an ESP  $S_{\bullet} = (h_n: S_n \rightarrow S_{n+1})_n$  such that  $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$ . Moreover, by the Albanese map  $\alpha_n: S_n \rightarrow C_n$ ,  $S_n$  is a  $\mathbb{P}^1$ -bundle over an elliptic curve  $C_n$  and there exists a finite étale morphism  $u_n: C_n \rightarrow C_{n+1}$  such that  $u_n \circ \alpha_n = \alpha_{n+1} \circ h_n$ . For any  $n$ , the composite map  $\psi_n := \alpha_n \circ \varphi_n \circ \pi_n: X \rightarrow C_n$  is isomorphic to the Albanese map  $\text{Alb}_X$  of  $X$  and also gives the MRC fibration of  $X$ . Hence any  $C_n$  is isomorphic to the Albanese elliptic curve  $C$  of  $X$  and there exists an isomorphism  $v_n: C_n \simeq C_0$  such that  $\psi_0 = v_n \circ \psi_n$  for any  $n$ . Thus, if we replace  $\alpha_n$  (resp.  $\psi_n$ ) by the composite  $v_n \circ \alpha_n$  (resp.  $v_n \circ \psi_n$ ), we may assume that  $\psi_n: X \rightarrow C$  is independent of  $n$  (i.e.,  $\psi_n \equiv \psi$ ) and coincides with the Albanese map  $\text{Alb}_X$  of  $X$ . By the universality of the Albanese map of  $X$ , the map  $u_n: C \rightarrow C$  is independent of  $n$ , and  $u := u_n$  satisfies  $\psi \circ f = u \circ \psi$  and  $\alpha_{n+1} \circ h_n = u \circ \alpha_n$ . Since  $\deg u = \deg h_n = \deg g_n = \deg f > 1$ ,  $u: C \rightarrow C$  has a fixed point  $P$ . Hence  $C$  is endowed with an abelian group structure with  $P$  as the unit element and  $u$  is a group homomorphism. By Propositions 4.1 and 4.4, there exists a multiplication map  $\mu_k: C \rightarrow C$  by some integer  $k$  such that the pull-back  $\widetilde{S}_0 := S_0 \times_{C, \mu_k} C$  is isomorphic to either the Atiyah surface  $\mathbb{S}$  or  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell'_0)$  for some  $\ell'_0 \in \text{Pic}^0(C)$ . We set  $\widetilde{X} := X \times_{C, \mu_k} C$ . Since  $\mu_k \circ h = h \circ \mu_k$ , there exists a non-isomorphic étale endomorphism  $\widetilde{f}: \widetilde{X} \rightarrow \widetilde{X}$  which is a lift of  $f: X \rightarrow X$ . Hence if we replace  $X$  by  $\widetilde{X}$  (resp.  $f$  by  $\widetilde{f}$ ), we may assume from the beginning that either  $S_0 = \mathbb{S}$  or  $S_0 \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell'_0)$  for some  $\ell'_0 \in \text{Pic}^0(C)$ . By Proposition 7.3, the discriminant locus  $\Delta_{(\psi_0)}$  of  $\psi_0$  is a disjoint union of elliptic curves, i.e.,  $\Delta_{(\psi_0)} = \bigsqcup_i \Delta_i$  such that  $\alpha_0(\Delta_i) = C$  for all  $i$ . If  $S_0 \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell')$  (resp.  $\mathbb{S}$ ), then any elliptic curve  $\Delta_i$  is a multi-section of  $\alpha_0$  (resp. the canonical section  $s_{\infty}$ ). Hence we can apply the same method as in the proof of Proposition 9.1. Then there exists some multiplication mapping  $\mu_m: C \rightarrow C$  for some integer  $m$  such that the following conditions are satisfied.

- If we set  $\widetilde{X} := X \times_{C, \mu_m} C$ , then there exists a non-isomorphic étale endomorphism  $\widetilde{f}: \widetilde{X} \rightarrow \widetilde{X}$  which is a lift of  $f: X \rightarrow X$ .
- $\pi_0: \widetilde{X} \rightarrow Z_0$  is a succession of blowing-ups of a smooth projective 3-fold  $Z_0$  along elliptic curves.

- $\varphi_0: Z_0 \rightarrow S_0$  is a  $\mathbb{P}^1$ -bundle over an elliptic ruled surface  $S_0$ .
- $S_0$  is isomorphic to either  $\mathbb{S}$  or  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_0)$  for some  $\ell_0 \in \text{Pic}^0(C)$ .

By Corollary 1.2, we can construct the FESP  $Z_\bullet := (g_n: Z_n \rightarrow Z_{n+1})_n$  of  $f: X \rightarrow X$ . Then applying Proposition 7.3, we infer that  $Z_\bullet$  is an FESP of type (C). There exists a  $\mathbb{P}^1$ -bundle structure  $\varphi_n: Z_n \rightarrow S_n$  over an elliptic ruled surface  $S_n$  for any  $n$ , since  $g_n$  is étale and  $\varphi_0: Z_0 \rightarrow S_0$  is a  $\mathbb{P}^1$ -bundle. The assertion (8) follows from Proposition 5.10. Thus all the properties in the statement are satisfied.  $\square$

## 10. Possibilities of the base surface of type $(C_{-\infty})$

In this section, as an illustration of how an *étaleness assumption* of a given endomorphism  $f: X \rightarrow X$  is used, we consider the case where there exists an FESP  $(Y_\bullet, R_\bullet)$  of type  $(C_{-\infty})$  constructed from  $(X, f)$  by a sequence of blowing-downs of an ESP, i.e., for  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$ , each  $\varphi_n: Y_n \rightarrow S_n$  is a conic bundle over an elliptic ruled surface  $S_n$ . Combining Propositions 4.8, 5.15 and 5.16, we show that the base space  $S_n$  is of very limited type. Theorem 10.1 below provides a key to many of the results in our subsequent articles. It is also related to the following basic fact concerning the existence of endomorphisms of elliptic ruled surfaces (cf. Corollary 4.2, Remark 4.3, Proposition 4.8):

- Any  $\mathbb{P}^1$ -bundle  $S$  over an elliptic curve  $C$  admits a non-isomorphic surjective endomorphism.
- Let  $\mathcal{L}$  be a line bundle on an elliptic curve  $C$ . Then the elliptic ruled surface  $S := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$  admits an ESP  $S_\bullet = (g_n: S_n \rightarrow S_{n+1})_n$  of elliptic ruled surfaces over  $C$  such that  $S_0 = S$  if and only if  $\deg \mathcal{L} = 0$ .
- $S$  admits a non-isomorphic étale endomorphism if and only if  $\mathcal{L} \in \text{Pic}(C)$  is torsion.

One of the applications of Theorem 1.5 is to give classifications of 3-folds admitting an FESP of type (D), which will be considered in our subsequent article; Part III. Now, we shall start the proof of the following which is a variant of Theorem 1.5.

**Theorem 10.1.** *Let  $f: X \rightarrow X$  be a non-isomorphic étale endomorphism of a smooth projective 3-fold  $X$  with  $\kappa(X) = -\infty$ . Let  $Y_\bullet = (g_n: Y_n \rightarrow Y_{n+1})_n$  be an FESP and  $\pi_\bullet = (\pi_n)_n: X_\bullet := (X, f) \rightarrow Y_\bullet$  a succession of Cartesian blowing-ups along elliptic curves. Suppose that there exists an ESP  $S_\bullet = (u_n: S_n \rightarrow S_{n+1})_n$  of smooth algebraic surfaces  $S_n$  and a Cartesian morphism  $\varphi_\bullet = (\varphi_n)_n: Y_\bullet \rightarrow S_\bullet$  such that the following hold:*

- (1)  $\varphi_n: Y_n \rightarrow S_n$  is a  $\mathbb{P}^1$ -bundle for any  $n$ .
- (2) Any  $S_n$  is isomorphic to a  $\mathbb{P}^1$ -bundle  $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$  for a line bundle  $\ell_n$  of degree 0 on the Albanese elliptic curve  $C$  of  $X$ .

*Then  $\ell_n \in \text{Pic}^0(C)$  is of finite order for any  $n$ .*

*Proof.* By definition, we have a Cartesian morphism of ESPs

$$(X, f) \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet = (\alpha_n)} (C, h)$$

in which the following are satisfied for any  $n$ .

- $\pi_n$  is a succession of Cartesian blowing-ups along elliptic curves,
- $\varphi_n$  is a  $\mathbb{P}^1$ -bundle,
- the Albanese map  $\alpha_n: S_n \rightarrow C$  gives  $S_n$  a  $\mathbb{P}^1$ -bundle structure over  $C$ ,

- $h$  is a non-isomorphic group homomorphism of an elliptic curve  $C$ , and
- the composite map  $\alpha_n \circ \varphi_n \circ \pi_n: X \rightarrow C$  is independent of  $n$  and coincides with the Albanese map  $\alpha_X: X \rightarrow C$  of  $X$ .

We may assume that  $n = 0$  without loss of generality. The proof is by contradiction. Suppose the contrary that  $S_0 \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell_0)$  for  $\ell_0 \in \text{Pic}^0(C)$  which is of infinite order. Then, by Proposition 5.10, we see that  $\ell_n \in \text{Pic}^0(C)$  is of infinite order for *any*  $n$ .

**Step 1:** First, we show that there exists some extremal ray of divisorial type on  $X$ . Suppose the contrary. Then  $\varphi := \varphi_0: X \simeq Y_0 \rightarrow S := S_0$  is an extremal contraction giving a  $\mathbb{P}^1$ -bundle structure. Hence, applying Theorem 3.10 and Proposition 7.3 (2), there is induced a non-isomorphic étale endomorphism  $u: S \rightarrow S$  such that  $\varphi \circ f^k = u \circ \varphi$  for some  $k > 0$ . Thus by Proposition 4.8,  $\ell_0 \in \text{Pic}^0(C)$  is of finite order, which contradicts the assumption.

**Step 2:** Let  $D_0^{(n)}$  (resp.  $D_\infty^{(n)}$ ) be the section of  $\alpha_n: S_n \rightarrow C$  corresponding to the first projection  $\mathcal{O}_C \oplus \ell_n \rightarrow \ell_n$  (resp. the second projection  $\mathcal{O}_C \oplus \ell_n \rightarrow \mathcal{O}_C$ ). By replacing  $\ell_n$  by  $\ell_n^{\otimes -1}$  if necessary, we may assume that  $u_n^{-1}(D_0^{(n+1)}) = D_0^{(n)}$  and  $u_n^{-1}(D_\infty^{(n+1)}) = D_\infty^{(n)}$  for all  $n$ . Then by Corollary 7.9, the  $\mathbb{P}^1$ -fiber space  $\psi_n := \varphi_n \circ \pi_n: X \rightarrow S_n$  over  $S_n$  is a  $\mathbb{P}^1$ -bundle outside  $D_0^{(n)} \cup D_\infty^{(n)}$ . Proposition 7.10 yields that all the irreducible components of  $\Delta_i$  of  $\psi_0^{-1}(D_0 \sqcup D_\infty)$  are elliptic ruled surfaces and cross normally with each other.

We show that there exists an elliptic ruled surface  $\Delta(\subset X)$  such that  $\psi_0(\Delta)$  coincides with either of the two disjoint sections of  $\alpha_0$ , and  $\psi_n(\Delta) = S_n$  for  $n \gg 0$ .

**Lemma 10.2.** *Let  $D_0 \sqcup D_\infty$  be a union of two disjoint sections of the  $\mathbb{P}^1$ -bundle  $\alpha_0: S_0 := \mathbb{P}_C(\mathcal{O}_C \oplus \ell_0) \rightarrow C$  defined as above. Then, there exists an irreducible component  $\Delta(\subset \psi_0^{-1}(D_0 \sqcup D_\infty))$  which is isomorphic to an elliptic ruled surface of the form  $\mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L})$ , where  $\mathcal{L} \in \text{Pic}^0(C')$  of infinite order on an elliptic curve  $C'$ .*

*Proof.* By Proposition 7.10, all the irreducible components  $\Delta_i$  of  $\psi_0^{-1}(D_0 \sqcup D_\infty)$  are elliptic ruled surfaces which cross normally. Furthermore, they are either the proper transform of  $\varphi_0^{-1}(D_0 \sqcup D_\infty)$  or the  $\pi_0$ -exceptional divisors. By Proposition 1.1,  $f^{-n}(\Delta_i)$  is irreducible for any  $n \geq 0$  and there exists an ESP  $\Delta_{i,\bullet} = (f: \Delta_{i,n} \rightarrow \Delta_{i,n+1})_n$  of elliptic ruled surfaces  $\Delta_{i,n} := f^n(\Delta_i)$  ( $n \in \mathbb{Z}$ ). Then by Proposition 4.1, we see that any  $\Delta_i$  isomorphic over  $C'$  to an elliptic ruled surface of the form  $\mathbb{P}_{C'}(\mathcal{E})$  for a semi-stable vector bundle  $\mathcal{E}$  such that  $\mathcal{E}$  is either indecomposable of rank 2 or isomorphic to  $\mathcal{O}_{C'} \oplus \mathcal{L}_i$  for a line bundle  $\mathcal{L}_i$  of degree 0 on  $C'$ .

Suppose that for *any*  $i$ ,  $\Delta_i$  is isomorphic to  $\mathbb{P}_{C'}(\mathcal{E})$  for an *indecomposable* vector bundle  $\mathcal{E}$ , or  $\Delta_i \simeq \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}_i)$  for some *torsion* line bundle  $\mathcal{L}_i \in \text{Pic}^0(C')$  and we shall derive a contradiction. Since  $f: X \rightarrow X$  is a finite étale covering,  $\Delta_{i,n}$  is also an elliptic ruled surface of the same type as  $\Delta_i$  for any  $n$  by Proposition 5.10. Hence, by Proposition 5.6 and Corollary 5.7,  $\psi_k(\Delta_{i,\ell})$  is an irreducible curve on  $S_k$  for any  $k, \ell \in \mathbb{Z}$ . For any  $m \in \mathbb{Z}$ , let  $\Delta_{i,\bullet}[m]$  be the ESP which is a shift of  $\Delta_{i,\bullet}$  by  $m$ , i.e.,  $\Delta_{i,k}[m] = \Delta_{i,k+m}$  for any  $k$  and  $\Delta_{i,\bullet}[m] = (f|_{\Delta_{i,k+m}}: \Delta_{i,k+m} \rightarrow \Delta_{i,k+m+1})_k$ . Since  $f: X \rightarrow X$  is an *endomorphism*, there exist the following Cartesian morphisms of ESPs;

$$\begin{array}{ccc} \Delta_{i,\bullet}[m] & \xrightarrow{i_\bullet} & X_\bullet \\ \downarrow \psi_\bullet = \varphi_\bullet \circ \pi_\bullet & & \\ S_\bullet & & \end{array}$$

Applied Lemma 2.6 to this Cartesian diagram, there is induced an ESP  $\psi_\bullet(\Delta_{i,\bullet}[m])$  of elliptic curves such that the inclusion  $\psi_\bullet(\Delta_{i,\bullet}[m]) \hookrightarrow S_\bullet$  is Cartesian. Set  $\Gamma_{i,m} := \psi_0(\Delta_{i,m})$ . Then by Proposition 6.9 (4.3), we have  $\Gamma_{i,m} = D_\infty$  or  $D_0$  for any  $m$ . Thus for any  $m$  and  $i$ , the surface  $\Delta_{i,m} := f^m(\Delta_i)$  coincides with some of the irreducible components of  $\psi_0^{-1}(D_0 \sqcup D_\infty)$ , whose number is finite. Hence for any  $m > 0$ , the  $m$ -th power  $f^m: X \rightarrow X$  gives rise to a permutation of the finite set  $M$  consisting of all the irreducible components of  $\psi_0^{-1}(D_0 \sqcup D_\infty)$ . Hence, a suitable power  $f^k: X \rightarrow X$  of  $f$  gives rise to the identity permutation of the set  $M$ . In particular, all the irreducible components of  $\pi_0$ -exceptional divisors are preserved by  $f^k$ . Thus the MMP works compatibly with étale endomorphisms and there exists a constant ESP  $Y_\bullet := (Y, g)$  of  $X_\bullet := (X, f^k)$ . Then by Theorem 3.10 and Proposition 7.3, there exists a non-isomorphic étale endomorphism  $u_0: S_0 \rightarrow S_0$  such that  $\psi_0 \circ f^{kr} = u_0 \circ \psi_0$  for some  $r > 0$ . Hence by Proposition 4.8,  $\ell_0 \in \text{Pic}^0(C)$  is of finite order. Thus a contradiction is derived and we are done.  $\square$

**Lemma 10.3.** *Let  $\Delta$  be as in Lemma 10.2. Then,*

- (1)  $\psi_n(\Delta) = S_n$  for a sufficiently large positive integer  $n$ .
- (2)  $\psi_0(f^{-n}(\Delta)) = S_0$  for a sufficiently large positive integer  $n$ .

*Proof.* We use the same notation as in the proof of Lemma 10.2. First we shall prove the assertion (2). The proof is by contradiction. Suppose that for infinitely many positive integer  $n$ ,  $\gamma_n := \psi_0(f^{-n}(\Delta))$  is an irreducible curve. Then  $\gamma_n = D_0$  or  $D_\infty$  by the same argument as in the proof of Lemma 10.2. Hence, the irreducible surface  $f^{-n}(\Delta)$  is contained in  $\psi_0^{-1}(D_0 \sqcup D_\infty)$  for infinitely many  $n$ . Thus, there exists positive integers  $p < q$  such that  $f^{-p}(\Delta) = f^{-q}(\Delta)$ . Since  $T = f^{q-p}(T)$  for the elliptic ruled surface  $T := f^{-q}(\Delta)$  and  $f^{-1}(f^{n+1}(\Delta)) = f^n(\Delta)$  for  $n$ ,  $f^{q-p}|_T: T \rightarrow T$  defines a non-isomorphic étale endomorphism of  $T$ . Hence by Proposition 4.8,  $T \simeq \mathbb{P}_{C''}(\mathcal{O}_{C''} \oplus \ell'')$  for some line bundle  $\ell'' \in \text{Pic}^0(C'')$  of finite order on an elliptic curve  $C''$ . This contradicts Proposition 5.10. Thus the assertion (2) has been proved. Since there exists a Cartesian morphism of ESPs  $\psi_\bullet: X_\bullet \rightarrow S_\bullet$ , the assertions (1) and (2) are equivalent. Thus we are done.  $\square$

**Step 3:** We are at the final stage of our proof. Using Lemma 10.3 and Proposition 5.16, we describe the structure of the surjective morphism  $\psi_n|_\Delta: \Delta \rightarrow S_n$  ( $n \gg 0$ ) in terms of line bundles of degree zero on elliptic curves. Let  $o \in C$  be a zero element. Then we show that the intersection  $\Delta \cap \alpha_X^{-1}(o)$  has an infinite number of connected components, which derives a contradiction.

Now let us continue the proof. By Lemma 10.3, there exists a positive integer  $n_0$  such that  $\psi_n(\Delta) = S_n$  for all  $n \geq n_0$ , where  $\Delta = \mathbb{P}_{C'}(\mathcal{O} \oplus \mathcal{L})$  and  $S_n = \mathbb{P}_C(\mathcal{O} \oplus \ell_n)$ . Let  $\alpha_\Delta: \Delta \rightarrow C'$  be the Albanese map of  $\Delta$ . Then there exists a finite surjective morphism  $v_n: C' \rightarrow C$  such that  $\alpha_n \circ \psi_n|_\Delta = v_n \circ \alpha_\Delta$ . Since  $\alpha_n \circ \psi_n|_\Delta = \alpha_X|_\Delta$ , the composite  $\Delta \xrightarrow{\alpha_\Delta} C' \xrightarrow{v_n} C$  is the Stein factorization of the morphism  $\alpha_X|_\Delta: \Delta \rightarrow C$  and  $v_n$  is independent of  $n$ . Hereafter, we denote  $v_n$  by  $v$ . By Lemma 5.12, there exists a non-zero integer  $a_n$  such that  $v^* \ell_n \simeq \mathcal{L}^{\otimes a_n}$  for all  $n \geq n_0$ .

**Lemma 10.4.** *The sequence  $\{a_n\}_{n \geq n_0}$  is unbounded. i.e.,*

$$\sup_{n \geq n_0} |a_n| = \infty.$$

Proof. Assume the contrary. Then there exists an integer  $b$  such that  $a_n = b$  for infinitely many  $n$ 's. In particular, there exist integers  $p < q$  such that  $v^* \ell_p \simeq \mathcal{L}^{\otimes b} \simeq v^* \ell_q$ . Since  $\ell_n \simeq h^* \ell_{n+1}$  for any  $n$ , we infer that  $\ell_p \simeq (h^{q-p})^* \ell_q$ . Thus we have  $(h^{q-p} \circ v)^* \ell_q \simeq v^* \ell_q$ . Since  $\ell_q \in \text{Pic}^0(C)$  is of infinite order, Lemma 5.11 yields that  $h^{q-p} \circ v = v$  up to translation under the group law of the elliptic curve  $C$ . Thus a contradiction is derived, since  $\deg h > 1$ .  $\square$

Let  $\alpha_\Delta: \Delta \rightarrow C'$  be the Albanese map of  $\Delta$ . By replacing  $\ell_n$  by  $\ell_n^{\otimes -1}$  if necessary, we may assume that  $\ell_n \simeq h^* \ell_{n+1}$  for any  $n$ . By the universality of the Albanese map, there is induced the following commutative diagram:

$$\begin{array}{ccccccc}
 \Delta & \xrightarrow{\psi_{n_0}} & S_{n_0} & \xrightarrow{u_{n_0}} & S_{n_0+1} & \xrightarrow{u_{n_0+1}} & \cdots \xrightarrow{u_{n-1}} S_n \\
 \alpha_\Delta \downarrow & & \alpha_{n_0} \downarrow & & \alpha_{n_0+1} \downarrow & & \alpha_n \downarrow \\
 C' & \xrightarrow{v} & C & \xrightarrow{h} & C & \xrightarrow{h} & \cdots \xrightarrow{h} C.
 \end{array}$$

Then we infer that

$$\mathcal{L}^{\otimes a_{n_0}} \simeq v^* \ell_{n_0} \simeq (h^{n-n_0} \circ v)^* \ell_n,$$

for any  $n \geq n_0$ . Note that by Lemma 5.12, the integer  $a_{n_0}$  is uniquely determined by the finite surjective morphism  $u_{n-1} \circ \cdots \circ u_{n_0} \circ \psi_{n_0}|_\Delta: \Delta \rightarrow S_n$  between elliptic ruled surfaces.

For each integer  $n \geq n_0$ , define a set  $M_n$  by

$$M_n := \{(k_n, \beta_n) \in \mathbb{Z} \times \text{Hom}_{\text{group}}(C', C) \mid \beta_n^* \ell_n \simeq \mathcal{L}^{\otimes k_n}\}.$$

Then by Proposition 5.15, the set  $M_n$  can be endowed with the structure of an abelian group such that the first projection  $p_n: M_n \rightarrow \mathbb{Z}$  is an injective group homomorphism. Thus  $p_n(M_n)$  is an additive subgroup of  $\mathbb{Z}$  generated by a unique positive integer  $\nu_n$ .

Then we have  $(a_{n_0}, h^{n-n_0} \circ v) \in M_n$  and the integer  $a_{n_0}$  is divisible by the integer  $\nu_n$  for each  $n \geq n_0$ . Hence we see that the set  $B := \{\nu_n \mid n \geq n_0\}$  is a finite set and  $\text{Sup } B < \infty$ . Furthermore, by construction,  $\psi_n(\Delta) = S_n$  and there exists a finite étale morphism  $v: C' \rightarrow C$  such that  $v \circ \alpha_\Delta = \alpha_n \circ \psi_n|_\Delta$  and  $v^* \ell_n \simeq \mathcal{L}^{\otimes a_n}$  for any  $n \geq n_0$ . Hence we infer that  $(a_n, v) \in M_n$  and the integer  $a_n$  is a multiple of  $\nu_n$ . If we put  $b_n := a_n / \nu_n$ , then we have  $\text{Sup}\{b_n \mid n \geq n_0\} = \infty$  by Lemma 10.4. Let  $F_n$  be the fiber of the Albanese map  $\alpha_n: S_n \rightarrow C$  over the zero element  $o \in C$ . Since  $\alpha_X = \alpha_n \circ \psi_n$  and  $\psi_n = \varphi_n \circ \pi_n$  for any  $n$ , we have  $\alpha_X^{-1}(o) = \psi_n^{-1}(F_n)$ . Hence

$$(2) \quad \Delta \cap \alpha_X^{-1}(o) = (\psi_n|_\Delta)^{-1}(F_n)$$

and the number of the connected components of the left hand side of (2) is independent of  $n$ . On the other hand, by Propositions 5.15 and 5.16, the number of the connected components of the right hand side of (2) is more than or equal to  $b_n^2$ , which is unbounded as  $n$  varies. Thus a contradiction is derived and the proof of Theorem 10.1 has been done.  $\square$

As a deeper application of Theorem 10.1, we shall prove Theorem 1.5.

Proof of Theorem 1.5. It follows from Lemma 4.6 that the property whether  $\ell_n \in \text{Pic}^0(C)$  is of finite order is invariant under a finite étale base change  $\tilde{C} \rightarrow C$ . Hence applying Theorem 9.6, we can reduce to the case where  $\varphi_n: Y_n \rightarrow S_n$  is a  $\mathbb{P}^1$ -bundle for any  $n$ . Thus

Theorem 10.1 yields the claim.  $\square$

**REMARK 10.5.** Theorem 10.1 does not hold without the assumption that the endomorphism  $f: X \rightarrow X$  is *étale*. We shall give such an example. Let  $C$  be an elliptic curve and  $\mathcal{L} \in \text{Pic}^0(C)$  a line bundle of infinite order. Let  $S := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$  be an elliptic ruled surface and  $\Delta_1, \Delta_2$  mutually disjoint sections of  $\alpha_S: S \rightarrow C$  corresponding to the first projection  $\mathcal{O}_C \oplus \mathcal{L} \twoheadrightarrow \mathcal{O}_C$  and the second projection  $\mathcal{O}_C \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$  respectively. Then by [38, Proposition 5],  $S$  admits a non-isomorphic endomorphism  $\psi: S \rightarrow S$  which is ramified only along  $\Delta_1 \cup \Delta_2$ , and  $\varphi^{-1}(\Delta_i) = \Delta_i$  for each  $i$ . Let  $Y := S \times \mathbb{P}^1$  be the product variety. Then  $Y$  admits a non-isomorphic endomorphism  $g := \psi \times \text{id}_{\mathbb{P}^1}: Y \rightarrow Y$ . Let  $E_i$  be the elliptic curve on  $Y$  defined by  $E_i := \Delta_i \times \{0\}$  and  $X_i := \text{Bl}_{E_i}(Y)$  the blowing-up of  $Y$  along  $E_i$ . Then  $g^{-1}(E_i) = E_i$  for each  $i = 1, 2$ . Hence by the universality of the blowing-up (cf. the proof of Lemma 3.3),  $g: Y \rightarrow Y$  lifts to a non-isomorphic *ramified* endomorphism  $f_i: X_i \rightarrow X_i$  of  $X_i$ . In other words, the endomorphism  $f: Y \rightarrow Y$  is constructed from  $f_i: X_i \rightarrow X_i$  by an equivariant blowing-down and  $Y \rightarrow S$  is a trivial  $\mathbb{P}^1$ -bundle over  $S$ . Since  $\mathcal{L} \in \text{Pic}^0(C)$  is a *non-torsion* line bundle, the conclusion as in Theorem 1.5 does not hold.

As an immediate application of Theorem 10.1, we see that if there exists an FESP of type (C) constructed from the given variety by a sequence of blowing-downs of an ESP, then up to finite étale covering we can reduce the FESP to a  $\mathbb{P}^1$ -bundle over an elliptic ruled surface of very limited type.

**Corollary 10.6.** *Under the same assumption as in Theorem 9.6, up to finite étale covering, there exists an FESP  $Y_\bullet$  of  $X_\bullet := (X, f)$  such that in the assertion (8),  $\ell_n \in \text{Pic}(C)$  is a torsion line bundle for any  $n$ , and in particular  $S_0 \simeq C \times \mathbb{P}^1$ .*

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