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ÉTALE ENDOMORPHISMS OF 3-FOLDS. I

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Abstract

This paper is the first part of our project towards classifications of smooth projective 3-folds X with $\kappa(X) = -\infty$ admitting a non-isomorphic étale endomorphism. We can prove that for any extremal ray R of divisorial type, the contraction morphism $\pi_R\colon X \to X'$ associated to R is the blowing-up of a smooth 3-fold X' along an elliptic curve. The difficulty is that there may exist infinitely many extremal rays on X. Thus we introduce the notion of an 'ESP' which is an infinite sequence of non-isomorphic finite étale coverings of 3-folds with constant Picard number. We can run the minimal model program ('MMP') with respect to an ESP and obtain the 'FESP' $Y_\bullet$ of (X, f) which is a distinguished ESP with *extremal rays of fiber type* (cf. Definition 3.6). We first classify $Y_\bullet$ and then blow-up $Y_\bullet$ along elliptic curves to recover the original X. The finiteness of extremal rays of $\overline{\mathrm{NE}}(X)$ is verified in certain cases (cf. Theorem 1.4). We encounter a new phenomenon showing that our étaleness assumption is related with torsion line bundles on an elliptic curve (cf. Theorem 1.5).

Contents

1. Introduction

1.1. Main results. We work over the field $\mathbb{C}$ of complex numbers. An endomorphism of a projective variety X means a morphism (holomorphic map) from X to itself. The study of surjective endomorphisms of a given variety X, such as the complex projective space $\mathbb{P}^n$ or a K3 surface, is a chief concern of study in complex dynamical systems. On the other hand, it is also related to the classification of projective manifolds X admitting a non-

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isomorphic surjective endomorphism, which is our deep concern. This is the first part of a series of articles which study structures of smooth projective 3-folds X with negative Kodaira dimension *admitting a non-isomorphic étale endomorphism*. The main purpose of this paper is to develop a strategy for classifications of such 3-folds X; i.e., the application of the MMP to endomorphisms (cf. Proposition 1.1, Corollary 1.2), finiteness of extremal rays (cf. Theorem 1.4), and clarifications of new phenomena in the case of $\kappa(X) = -\infty$ (cf. Theorem 1.5, Remark 1.3).

Our slogan towards classifications is to apply the *minimal model program* ('MMP' for short) to the 'ESP' and construct its 'FESP' which is the counterpart of the minimal model in the case of $\kappa(X) \geq 0$. Compared with the case of $\kappa(X) \geq 0$, an extremal ray R of $\overline{\mathrm{NE}}(X)$ may not necessarily be preserved by a suitable power f^k ($k > 0$), which is one of the serious troubles. We first explain the notions of an ESP and an FESP (cf. Definitions 2.3, 2.4, and 3.6). Let us consider the following infinite sequence; $Z_\bullet = (v_n\colon Z_n \to Z_{n+1})_{n\in\mathbb{Z}}$ such that

- any Z_n is a smooth projective variety,
- any v_n is a non-isomorphic finite étale covering, and
- the Picard number $\rho(Z_n)$ is constant.

Then we say that the above sequence $Z_\bullet$ is an *étale sequence of constant Picard number* (ESP, for short). A variety Z_n appearing in an ESP $Z_\bullet$ has strong conditions. For example:

- $\chi(\mathcal{O}_{Z_n}) = \chi_{top}(Z_n) = K_{Z_n}^{\dim Z_n} = 0$ for any n (cf. Lemma 2.1).
- If $Z_n \simeq \mathbb{P}_{C_n}(\mathcal{E}_n)$ for a vector bundle $\mathcal{E}_n$ on an elliptic curve C_n for any n, then $\mathcal{E}_n$ is *semi-stable* (cf. Proposition 4.1).

Furthermore, we say that an ESP $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ is an *ESP of fiber type* (FESP, for short) if any Y_n ($n \in \mathbb{Z}$) is a smooth projective 3-fold with an extremal ray of *fiber type*. Even if we consider a non-isomorphic étale endomorphism, it is more natural to consider an ESP, on which the MMP works well. Thus we introduce a new notion. We say that an ESP (resp. an FESP) $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ is a '*constant ESP*' (resp. '*constant FESP*') if there exists a non-isomorphic étale endomorphism $g\colon Y \to Y$ such that $Y_n = Y$ and $g_n = g$ for any n (cf. Definitions 2.4, 3.6). We denote it by $Y_\bullet = (Y, g)$.

Now we state one of our main results.

Proposition 1.1. *Let $Y_\bullet = (f_n\colon Y_n \to Y_{n+1})_n$ be an ESP of smooth projective 3-folds Y_n. Suppose that the canonical bundle K_{Y_0} of Y_0 is not nef. Let R_0 be an extremal ray of $\overline{\mathrm{NE}}(Y_0)$ such that the contraction morphism $\pi_0 := \mathrm{Cont}_{R_0}\colon Y_0 \to Z_0$ associated to R_0 is a birational morphism. If we put $R_i := (f_{i-1} \circ \cdots \circ f_0)_*(R_0)$ and $R_{-i} := (f_{-1} \circ \cdots \circ f_{-i})^* R_0$ for each $i > 0$, then the following assertions hold for any integer n:*

1. *R_n is an extremal ray of $\overline{\mathrm{NE}}(Y_n)$ and the contraction morphism $\pi_n := \mathrm{Cont}_{R_n}\colon Y_n \to Z_n$ associated to R_n is a divisorial contraction, and is (the inverse of) the blowing-up of a smooth projective 3-fold Z_n along an elliptic curve C_n on Z_n.*
2. *There exists an ESP $Z_\bullet = (g_n\colon Z_n \to Z_{n+1})_n$ of smooth projective 3-folds Z_n and a Cartesian morphism $\pi_\bullet := (\pi_n)_n\colon Y_\bullet \to Z_\bullet$, i.e., $\pi_{n+1} \circ f_n = g_n \circ \pi_n$.*
3. *$g_n^{-1}(C_{n+1}) = C_n$ and $C_\bullet = (g_n|_{C_n}\colon C_n \to C_{n+1})_n$ is an ESP of elliptic curves.*
4. *The normal bundle $\mathcal{N}_n$ of C_n in Z_n is a semi-stable vector bundle of rank 2 and degree 0. Moreover, $\mathcal{N}_n \simeq (g_n|_{C_n})^*\mathcal{N}_{n+1}$.*

$$\begin{array}{ccccccc}
X_\bullet : \cdots & \longrightarrow & X_n & \xrightarrow{f_n} & X_{n+1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \pi_n^{(0)}} & & \big\downarrow{\scriptstyle \pi_{n+1}^{(0)}} & & \\
 & & \vdots & & \vdots & & \\
 & & \big\downarrow & & \big\downarrow & & \\
X_\bullet^{(i)} : \cdots & \longrightarrow & X_n^{(i)} & \xrightarrow{f_n^{(i)}} & X_{n+1}^{(i)} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \pi_n^{(i)}} & & \big\downarrow{\scriptstyle \pi_{n+1}^{(i)}} & & \\
X_\bullet^{(i+1)} : \cdots & \longrightarrow & X_n^{(i+1)} & \xrightarrow{f_n^{(i+1)}} & X_{n+1}^{(i+1)} & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \\
 & & \vdots & & \vdots & & \\
 & & \big\downarrow & & \big\downarrow & & \\
X_\bullet^{(k)} : \cdots & \longrightarrow & X_n^{(k)} & \xrightarrow{f_n^{(k)}} & X_{n+1}^{(k)} & \longrightarrow & \cdots
\end{array}$$

Fig. 1. FESP

Corollary 1.2 (Construction of an FESP). *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$. Then there exists a Cartesian diagram as in Figure 1 which satisfies the following conditions:*

(1) *For all $0 \le i \le k$, the i-th row $X_\bullet^{(i)} = (f_n^{(i)}\colon X_n^{(i)} \to X_{n+1}^{(i)})_n$ is an ESP.*
(2) *For all n and $1 \le i < k$, $\pi_n^{(i-1)}\colon X_n^{(i-1)} \to X_n^{(i)}$ is (the inverse of) the blowing-up along an elliptic curve $C_n^{(i)}$ on $X_n^{(i)}$.*
(3) *$(f_n^{(i)})^{-1}(C_{n+1}^{(i)}) = C_n^{(i)}$ for all n and $1 \le i \le k$. That is, $C_\bullet^{(i)} := (f_n^{(i)}\colon C_n^{(i)} \to C_{n+1}^{(i)})_n$ is an ESP of elliptic curves and the inclusion $C_\bullet^{(i)} \hookrightarrow X_\bullet^{(i)}$ is a Cartesian morphism.*
(4) *The normal bundle $\mathcal{N}_n^{(i)}$ of $C_n^{(i)}$ in $X_n^{(i)}$ is a semi-stable vector bundle of rank 2 and degree 0. Moreover, $(f_n^{(i)}|_{C_n^{(i)}})^*(\mathcal{N}_{n+1}^{(i)}) \simeq \mathcal{N}_n^{(i)}$.*
(5) *On the bottom k-th row $X_\bullet^{(k)} = (f_n^{(k)}\colon X_n^{(k)} \to X_{n+1}^{(k)})_n$, all the extremal rays of $\overline{\mathrm{NE}}\,(X_n^{(k)})$ are of fiber type. In particular, $X_\bullet^{(k)}$ is an FESP (cf.* Definition 3.6).

In Corollary 1.2, we call the vertical sequence

$$X_n \xrightarrow{\pi_n^{(0)}} \cdots \longrightarrow X_n^{(i)} \xrightarrow{\pi_n^{(i)}} X_n^{(i+1)} \longrightarrow \cdots \longrightarrow X_n^{(k)}$$

a sequence of blowing-downs of an ESP. Furthermore, $X_\bullet^{(k)}$ is called an FESP constructed from $X_\bullet$ by a sequence of blowing-downs of an ESP (cf. Definition 3.7). In this paper, by *an extremal ray R* of a smooth projective variety X, we always mean a *K_X-negative extremal ray* of $\overline{\mathrm{NE}}(X)$. Furthermore, $\mathcal{F}_r$ denotes a unique indecomposable vector bundle of rank r and degree 0 on an elliptic curve E with $\Gamma(E, \mathcal{F}_r) \neq 0$ (cf. Theorem 4.11). We are particularly interested in the case of $r = 2$. We call the $\mathbb{P}^1$-bundle $\mathbb{P}_E(\mathcal{F}_2)$ over E associated with $\mathcal{F}_2$ the 'Atiyah surface' and denote it by $\mathbb{S}$ (cf. Definition 5.1). It is worthwhile to note that we can construct an example of a non-isomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$ such that the following hold (cf. Remark 8.3);

- There exists an extremal ray R of $\overline{\mathrm{NE}}(X)$ such that the exceptional divisor of the contraction morphism $\mathrm{Cont}_R\colon X \to X'$ is isomorphic to $\mathbb{S}$.
- No finite étale covering $\widetilde{X}$ of X is isomorphic to the product of a uniruled surface and an elliptic curve.

Thus in the case of a non-isomorphic étale endomorphism $f\colon X \to X$ with $\kappa(X) = -\infty$, the same conclusion as in the case of $\kappa(X) \geq 0$ (cf. [9], [13]) *does not hold*. This is one of the new phenomena appearing for an étale endomorphism in the case of $\kappa(X) = -\infty$.

Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then we can apply Corollary 1.2 to a constant ESP (X, f) and construct an FESP $Y_\bullet$. Let $\mathcal{R}$ be the set of extremal rays of X. If $\mathcal{R}$ is a finite set, then we can take $Y_\bullet$ as a constant FESP (Y, g) induced from a non-isomorphic étale endomorphism $g\colon Y \to Y$ (cf. Proposition 3.8). The difficulty is that $\mathcal{R}$ *may be an infinite set and it is not clear if one of them is f-invariant. Thus the MMP does not necessarily work compatibly with étale endomorphisms* (cf. Remark 3.9). Thus we shall introduce the notion of an 'FESP' and study the structure of an endomorphism $f\colon X \to X$ through its FESP $Y_\bullet$. The notions of an ESP and an FESP first appeared in [9], [10]. In particular, the author [10] studied the structure of an ESP $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ of smooth projective 3-folds X_n with $\kappa(X_n) = 2$, in which a counterpart of an FESP was called a '*modified minimal reduction*'. M. Aprodu, S. Kebekus and T. Peternell [1] also tried to study a constant ESP (X, f) induced from a non-isomorphic étale endomorphism $f\colon X \to X$ of a 3-fold X with $\kappa(X) = -\infty$. In our Corollary 1.2, the condition (4) is new (cf. Proposition 8.4 and Remark 8.3). Notably, if $X_\bullet = (X, f)$ is a constant ESP induced from a non-isomorphic étale endomorphism $f\colon X \to X$, then we can say more. It is one of the charateristic features of étale endomorphisms.

Remark 1.3. (1) In the assertion (4) of Corollary 1.2, let $E_n^{(i)}$ be the $\pi_n^{(i-1)}$-exceptional divisor. Then each $E_n^{(i)}$ is isomorphic to either the Atiyah surface $\mathbb{S}$ or $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}_n^{(i)})$, where $\mathcal{L}_n^{(i)}$ is a line bundle of degree zero on $C_n^{(i)}$.

(2) Furthermore, suppose that $X_\bullet$ is a constant ESP (X, f) induced from a non-isomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then, each $\mathcal{L}_n^{(i)}$ is a *torsion* line bundle on $C_n^{(i)}$. In particular, let R be an arbitrary extremal ray of divisorial type on X and E_R the exceptional divisor of the contraction morphism $\pi_R := \mathrm{Cont}_R\colon X \to Z$ associated to R. Then E_R is isomorphic to either the Atiyah surface $\mathbb{S}$, or $\mathbb{P}_C(\mathcal{O} \oplus \mathcal{L})$ for a *torsion* line bundle $\mathcal{L}$ on an elliptic curve C.

It is remarkable that Remark 1.3 (2) holds even if the set $\mathcal{R}$ of extremal rays of $\overline{\mathrm{NE}}(X)$ is an infinite set. It will be proved in our subsequent Part III article as an application of Theorem 1.5 below.

The following theorem describes the finiteness of $\mathcal{R}$ in certain cases:

Theorem 1.4. *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$ and $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ an FESP constructed from $X_\bullet$ by a sequence of blowing-downs of an ESP. Let R_n be an extremal ray of fiber type on $\overline{\mathrm{NE}}(Y_n)$ such that $(g_n)_* R_n = R_{n+1}$ for any n and set $R_\bullet := (R_n)_n$. Suppose that $(Y_\bullet, R_\bullet)$ is of type (C_1) or (C_0). Then for any n, there exist at most finitely many extremal rays of divisorial type on $\overline{\mathrm{NE}}(X_n)$.*

Now, we shall explain briefly our terminology. Let $X_\bullet = (f_n : X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$. Then there exists an FESP $Y_\bullet = (g_n : Y_n \to Y_{n+1})_n$ constructed from $X_\bullet$ by a sequence of blowing-downs of an ESP (cf. Corollary 1.2). Take an extremal ray R_0 of fiber type on $\overline{\mathrm{NE}}(Y_0)$ arbitrarily. For each $n > 0$, we set $R_n := (g_{n-1} \circ \cdots \circ g_0)_*(R_0)$ and $R_{-n} := (g_{-1} \circ \cdots \circ g_{-n})^*(R_0)$. Then, any R_n is an extremal ray of fiber type on Y_n. Furthermore, with the aid of classifications of extremal rays of 3-folds due to Mori [29], we see that any R_n is of the same type C or of the same type D in the sense of [29].

Note that an FESP $Y_\bullet$ is *not uniquely determined* for a given étale endomorphism $f : X \to X$ (cf. Remark 8.3). Let $R_\bullet = (R_n)_n$ be the set of extremal rays of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$. Let $(Y_\bullet, R_\bullet)$ be the pair of $Y_\bullet$ and $R_\bullet$. Then we say that the FESP $(Y_\bullet, R_\bullet)$ is of type (C) if the extremal contraction $\varphi_n := \mathrm{Cont}_{R_n} : Y_n \to S_n$ associated to R_n is a conic bundle over a smooth algebraic surface S_n (cf. Definition 3.6) for any n. Then by Propositions 3.1, 6.4 and Lemma 7.1, we see that a suitable finite étale covering $\widetilde{S_n}$ of S_n is isomorphic to one of the following:

- the direct product $B \times E$ of a smooth curve B of $g(B) \geq 2$ and an elliptic curve E (i.e., the FESP $(Y_\bullet, R_\bullet)$ is called of type (C_1)), or
- an abelian surface (i.e., the FESP $(Y_\bullet, R_\bullet)$ is called of type (C_0)), or
- a $\mathbb{P}^1$-bundle over an elliptic curve C associated to a semi-stable vector bundle of rank two on C (i.e., the FESP $(Y_\bullet, R_\bullet)$ is called of type ($\mathrm{C}_{-\infty}$)).

Next, we say that the FESP $(Y_\bullet, R_\bullet)$ is of type (D) if for any n, the contraction morphism $\varphi_n := \mathrm{Cont}_{R_n} : Y_n \to C_n$ associated to R_n is a del Pezzo fiber space over a curve C_n (cf. Definition 3.6). Then in this case, φ_n is smooth and C_n is an elliptic curve for any n (cf. Proposition 7.3).

Now we shall state another main result concerning *torsion line bundles* on an elliptic curve.

Theorem 1.5 (Torsion theorem. I). *Let $f : X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $Y_\bullet = (g_n : Y_n \to Y_{n+1})$ be an FESP constructed from the constant ESP (X, f) by a sequence of blowing-downs of an ESP. Let R_n be an extremal ray of fiber type on $\overline{\mathrm{NE}}(Y_n)$ such that $(g_n)_* R_n = R_{n+1}$ for any n and set $R_\bullet = (R_n)_n$. Suppose that the FESP $(Y_\bullet, R_\bullet)$ is of type ($C_{-\infty}$), i.e., each Y_n is a conic bundle over an elliptic ruled surface S_n. Furthermore, suppose that any S_n is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a line bundle ℓ_n of degree 0 on the Albanese elliptic curve C of X. Then $\ell_n \in \mathrm{Pic}^0(C)$ is of finite order for any n.*

Theorem 1.5 shows that the base surface S_n of the conic bundle Y_n is of very limited type; A suitable finite étale covering $\widetilde{S_n}$ of S_n is isomorphic to either the Atiyah surface $\mathbb{S}$ over C or the direct product $C \times \mathbb{P}^1$, where C is the Albanese elliptic curve of X (cf. Corollary 10.6). Note that the above fact does not hold without the assumption that the non-isomorphic endomorphism $f : X \to X$ is *étale* (cf. Remark 10.5). Furthermore, it does not hold if we replace the étale endomorphism $f : X \to X$ by the general ESP $X_\bullet = (f_n : X_n \to X_{n+1})_n$ (cf. Remark 3.2). Though Theorem 1.5 seems to be very technical at first glace, it distinguishes the structure of an FESP obtained from a non-isomorphic étale endomorphism $f : X \to X$ from an FESP obtained from a general ESP $X_\bullet$. Using Theorem 1.5, we can study the structure of a non-isomorphic étale endomorphism $f : X \to X$ and especially the set $\mathcal{R}$ of extremal rays of $\overline{\mathrm{NE}}(X)$ through its FESP $Y_\bullet$ in great detail.

Now, we go back to the construction of an FESP. Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. If there exist at most finitely many extremal rays, then we can replace f by its suitable power $f^k (k > 0)$ so that $f_* R = R$ for any extremal ray R of $\overline{\mathrm{NE}}(X)$. Thus the MMP works compatibly with étale endomorphisms and by a succession of equivariant blowing-downs $\pi_\bullet : X_\bullet = (X, f) \to Y_\bullet = (Y, g)$, we obtain a constant FESP $Y_\bullet$ induced from a non-isomorphic étale endomorphism $g: Y \to Y$. For example, this holds true in the case where there exists an FESP of type (C_1) or of type (C_0) constructed from $f: X \to X$ by a sequence of blowing-downs of an ESP (cf. Theorem 1.4 and Corollary 8.1). Furthermore, even if there exist infinitely many extremal rays of $\overline{\mathrm{NE}}(X)$, it is sufficient to find an extremal ray R such that $(f^k)_* R = R$ for some $k > 0$. Running the MMP again by the same method as above, we obtain a constant FESP $Y_\bullet = (Y, g)$. Fortunately, *in most cases, these nice conditions concerning extremal rays are satisfied*; Actually applying Theorem 1.5 to the case where there exists an FESP $(Y_\bullet, R_\bullet)$ of type $(C_{-\infty})$ or of type (D), we can show the following:

Fact: Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then we can find a f^k-invariant extremal ray R of $\overline{\mathrm{NE}}(X)$ for some $k > 0$ and run the MMP compatibly with étale endomorphisms except the following case: There exists a finite étale covering $\widetilde{X} \to X$ of X such that $\widetilde{X}$ is isomorphic to the product $S \times E$ of a rational surface S and an elliptic curve E.

We shall show this in our subsequent articles. In summary, our **strategy** is as follows:

(1) (Construction of an FESP): Using Corollary 1.2, we shall first construct an FESP $Y_\bullet$ of a non-isomorphic étale endomorphism $f: X \to X$ by a sequence of blowing-downs of an ESP.

(2) (Classifications of an FESP): Using results in Section 7, Theorems 1.5, 10.1 and Corollary 10.6, we shall study the structure of the FESP $Y_\bullet$ roughly according as the type of the pair $(Y_\bullet, R_\bullet)$.

(3) (Finiteness of extremal rays of $\overline{\mathrm{NE}}(X)$): Applied Theorems 1.4, 1.5 to the FESP $(Y_\bullet, R_\bullet)$, we shall find a f^k-invariant extremal ray R of $\overline{\mathrm{NE}}(X)$ for some $k > 0$. This is possible in most cases as stated in the above **Fact**. For example, in the case where $(F_\bullet, R_\bullet)$ is of type (C_1) or (C_0) (resp. type $(C_{-\infty})$), we can apply Theorem 1.4 (resp. Theorem 1.5).

(4) (Construction of a constant FESP): If (3) holds true, then we shall again run the MMP compatibly with étale endomorphisms and obtain another constant FESP (Z, υ) induced from a non-isomorphic étale endomorphism $\upsilon: Z \to Z$.

(5) (Study the structure of a constant FESP): We shall again study the structure of the constant FESP (Z, υ) according as its type.

(6) (Blowing-ups of an FESP): Using Lemma 3.3, we shall find a υ-invarant elliptic curve E on Z and perform an equivariant blowing-ups along E to recover the original endomorphism $f: X \to X$. The structure of X can be studied in great detail.

In this Part I article, we shall mainly focus our attention to the topics (1), (2). The other topics (3), (4), (5), (6) will be studied in our subsequent articles.

1.2. Announcement of results in our subsequent articles. Now we shall announce the main results of our subsequent articles.

Classification results:

Let X be a smooth projective 3-fold with $\kappa(X) = -\infty$ admitting a non-isomorphic étale endomorphism. Then up to finite étale covering, X satisfies one of the following 6 conditions:

(1) $X \simeq S \times E$ for an elliptic curve E and a uniruled algebraic surface S.
(2) X is a $\mathbb{P}^1$-bundle over an abelian surface.
(3) X is obtained by a succession of blowing-ups along elliptic curves from a $\mathbb{P}^1$-bundle Y over the product $S := C \times E$, where C is a curve of genus $g(C) \geq 1$ and E is an elliptic curve.
(4) X is a $\mathbb{P}^2$-bundle over an elliptic curve C.
(5) X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over an elliptic curve C.
(6) The Albanese map $\alpha\colon X \to C$ is a *fiber bundle* over the Albanese elliptic curve C of X whose fiber is a blown-up of a Hirzeburch surface.

To be more precise, 'up to finite étale covering' means the following; There exist a finite étale covering $\widetilde{X} \to X$ of X and a non-isomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of some power $f^k (k > 0)$ of the original non-isomorphic étale endomorphism $f\colon X \to X$ such that $\widetilde{X}$ satisfies one of the 6 conditions listed as above.

We note that these 6 classes are not necessarily mutually disjoint from each other. For example, the product of the Hirzebruch surface $\mathbb{F}_1$ and an elliptic curve satisfies both of the conditions (1) and (6). Comparing with classification of 3-folds with non-negative Kodaira dimension admitting non-isomorphic étale endomorphisms (cf. [9], [13]), our classification is not so simple and not strong enough. We mainly give a necessary condition for the existence of such varieties and state a sufficient condition in some special cases. As an application, combinig Theorem 1.5 with our classification result, we shall prove the fact stated in Remark 1.3.

Contents of Part II: Applied Corollary 1.2 and Theorem 1.4, we can run the MMP compatibly with étale endomorphisms. We shall study the structure of X satisfying the condition (2) and (3) as above:

Classification in the case (3): In this case, there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_1) or (C_0) constructed from X by blowing-downs. Theorem 1.4 shows that there exists at most finitely many extremal rays of $\overline{\mathrm{NE}}(X)$. Thus if we replace f by its suitable power f^k $(k > 0)$, we can choose $Y_\bullet$ as a *constant* FESP (Y, g) induced from a non-isomorphic étale endomorphism $g\colon Y \to Y$ of a smooth projective 3-fold Y with $\kappa(Y) = -\infty$. Applied the results in Sections $4, 7, 9$, we shall first study the structure of Y. Next, performing a succession of equivariant blowing-ups of Y along elliptic curves (cf. Lemma 3.3 and Corollary 7.9), we shall recover the original endomorphism $f\colon X \to X$ from its FESP Y. Thus, up to finite étale covering, X satisfies one of the following conditions:

(i) X is isomorphic to the direct product of a uniruled surface and an elliptic curve.
(ii) $X \simeq Y$ and X is a $\mathbb{P}^1$-bundle over an abelian surface.
(iii) Y is a fiber bundle over a smooth curve B of genus $g(B) \geq 1$ whose fiber is the Atiyah surface $\mathbb{S}$ over an elliptic curve E.
(iv) There exists a smooth morphism $\psi\colon Y \to B$ such that
   - the general fiber of ψ is isomorphic to the Atiyah surface $\mathbb{S}$ over an elliptic curve E and,

- the special fiber of ψ is isomorphic to the direct product $E \times \mathbb{P}^1$.

Furthermore, in both cases (iii) and (iv), X is a fiber bundle over E.

Contents of Part III:

Combining Theorem 1.5 with Atiyah's theory of vector bundles on an elliptic curve (cf. [2]), we shall study the structure of X in the case where there exist an FESP $(Y_\bullet, R_\bullet)$ of type (D) constructed from X by a sequence of blowing-downs of an ESP. Then up to finite étale covering, we can choose the FESP $Y_\bullet = (g_n \colon Y_n \to Y_{n+1})_n$ as a $\mathbb{P}^2$-bundle, or a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over the Albanese elliptic curve C of X. Applying Theorems 1.5 and 10.1, we can show:

(i) If Y_n is a $\mathbb{P}^2$-bundle over C, then $Y_n \simeq \mathbb{P}_C(\mathcal{E}_n)$ for a semi-stable vector bundle $\mathcal{E}_n$ of rank 3 on C. Furthermore, $\mathcal{E}_n$ is isomorphic to one of the following; a stable bundle, $\mathcal{F}_2 \oplus \mathcal{P}_n$, or $\mathcal{O}_C \oplus \mathcal{M}_n \oplus \mathcal{N}_n$, where $\mathcal{P}_n, \mathcal{M}_n$ and $\mathcal{N}_n$ are all *torsion* line bundles on C. We show that up to finite étale covering, X satisfies the condition (4), (6) or (1) respectively.

(ii) If Y_n is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over C, then up to finite étale covering, Y_n is isomorphic to one of the following; $C \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{S} \times \mathbb{P}^1$, or $\mathbb{S} \times_C \mathbb{S}$ for the Atiyah surface $\mathbb{S}$.

From these data, we can always find a f^k-invariant extremal ray R of $\overline{\mathrm{NE}}(X)$ for some $k > 0$ and run again the MMP compatibly with étale endomorphisms except only two cases where $Y_n \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{M}_n \oplus \mathcal{N}_n)$ or $\widetilde{Y_n} \simeq C \times \mathbb{P}^1 \times \mathbb{P}^1$ for a suitable finite étale covering $\widetilde{Y_n}$ of Y_n. Then after a succession of equivariant blowing-ups along elliptic curves (cf. Lemma 3.3, Corollary 7.9 and Proposition 7.10), we can recover the structure of the original endomorphism $f \colon X \to X$. We show that X satisfies the condition (1), (4), (5) or (6) respectively.

What is new in this article: Although our techniques of studying ESPs are new, many arguments appearing in the proofs of the various classification results have already appeared in our older papers on endomorphisms of threefolds and surfaces (cf. [9], [10], [12], [13], [14]). Since our new results seem to be hidden behind these classical arguments, we shall briefly describe *'what is new in this article'*.

- (Construction of an FESP): Proposition 1.1 and Corollary 1.2 have been already stated implicitely (cf. [9], [13]) in the special case where $X_\bullet = (X, f)$ is a constant ESP introduced from a non-isomorphic étale endomorphism $f \colon X \to X$. The assertions in Proposition 1.1 (4) and in Corollary 1.2 (4) are new which relate with Propositions 4.1 and 6.8.
- (Classifications of an FESP): In Propositions 7.3, 7.5, and 7.8, classifications of an FESP of 3-folds with negative Kodaira dimension are treated. Its proofs are built upon classifications of an ESP of smooth algebraic surfaces in Section 6 which essentially appeared in [9], [10]. On the other hand, some of its proofs are simplified.
- (Finiteness of extremal rays): Theorems 1.4 and 8.6 show finiteness of extremal rays in certain cases.
- (New phenomena): Theorems 1.5, 10.1, Corollary 10.6, Remark 1.3 and Proposition 3.4 are related with *torsion* line bundles on an elliptic curve and distinguish an FESP constructed by an *étale endomorphism* from that of a general ESP. Propositions 6.2, 8.4 and Remarks 8.3, 8.5 are related with the Atiyah surface $\mathbb{S}$ in the ESPs.

Organization of this article. In Section 2, we recall fundamental facts and techniques concerning endomorphisms. Moreover, we introduce the notions of an ESP (cf. Definition 2.3) and a *constant* ESP (cf. Definition 2.4). We show that very strong conditions are imposed on such varieties appearing in an ESP (cf. Lemma 2.1 and Proposition 2.2).

In Section 3, we shall prove Proposition 1.1 which summarizes fundamental results of extremal rays on smooth projective 3-folds X appearing in an ESP. In Definition 3.6, we shall introduce a notion called an FESP. Applying Proposition 1.1, we shall give a proof of Corollary 1.2 which shows the existence of an FESP $Y_\bullet$ constructed from the given étale endomorphism $f\colon X \to X$ by a sequence of blowing-downs of an ESP (cf. Definition 3.7). Furthermore, we show that if there exist finitely many extremal rays of $\overline{\mathrm{NE}}(X)$, then we can take $Y_\bullet$ as a constant FESP (cf. Proposition 3.8). Remark 3.9 (3) shows that the FESP $Y_\bullet$ is not uniquely determined.

In Section 4, we study the structure of an elliptic ruled surface admitting a non-isomorphic étale endomorphism. Proposition 4.1 shows that for a vector bundle $\mathcal{E}$ on an elliptic curve C, if $\mathbb{P}_C(\mathcal{E})$ appears in an ESP of projective bundles, then $\mathcal{E}$ is semi-stable. Furthermore, we recall Atiyah's vector bundle $\mathcal{F}_r$ on an elliptic curve (cf. [2]) and study the structure of its associated projective bundles from the viewpoint of an ESP (cf. Proposition 4.13).

In Section 5, we consider the structure of surjective morphisms between elliptic ruled surfaces which are not necessarily endomorphisms. Proposition 5.10 gives the structure of elliptic ruled surfaces appearing in an ESP. Proposition 5.5 describes endomorphisms of the Atiyah surface $\mathbb{S}$. Proposition 5.16 is applied to the proof of Theorem 10.1.

In Section 6, we first recall basic facts about elliptic fibrations and abelian fibrations. In Proposition 6.4, we shall study the structure of an ESP of algebraic surfaces. Next, we consider a pair of ESPs $(C_\bullet, Z_\bullet)$ such that $C_\bullet$ is an ESP of elliptic curves and the inclusion map $i_\bullet\colon C_\bullet \to Z_\bullet$ is a Cartesian morphism of ESPs. Proposition 6.8 shows that the normal bundle of each elliptic curve appearing in the ESP $C_\bullet$ is a *semi-stable* vector bundle of degree zero. Proposition 6.9 classifies such a pair $(C_\bullet, S_\bullet)$ in the case where $S_\bullet$ is an ESP of algebraic surfaces. Proposition 6.2 (cf. Remark 8.5) treats some new phenomenon of a non-isomorphic étale endomorphism $f\colon X \to X$ with $\kappa(X) = -\infty$ such that X has no Seifert elliptic fiber space structure.

In Section 7, we shall classify the FESP $(Y_\bullet, R_\bullet)$ constructed from an ESP $X_\bullet$ of smooth projective 3-folds with negative Kodaira dimension. They are classified into two types according as the extremal contraction of $R_\bullet$ gives a conic bundle structure (type (C)) or a del Pezzo fibration (type (D)) (cf. Proposition 7.3). Corollary 7.9 and Proposition 7.10 describe the blowing-up centers when we recover the original endomorphism $f\colon X \to X$ from its FESP $Y_\bullet$ by a sequence of blowing-ups of an ESP.

In Section 8, we shall prove Theorems 1.4 and 8.6 concerning finiteness of extremal rays on an ESP of 3-folds in certain cases. In Remark 8.3, we shall give an example of a non-isomorphic étale endomorphism $f\colon X \to X$ such that an FESP $(Y_\bullet, R_\bullet)$ of type (C_1) or (C_0) is not uniquely determined. Proposition 8.4 and Remark 8.5 describe a new phenomenon concerning extremal rays.

In Section 9, we shall prove some technical lemma (cf. Theorems 9.3, 9.5 and 9.6) in the case where there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C) constructed from an endomorphism $f\colon X \to X$. We introduce a method for reducing the FESP $Y_\bullet$ of conic bundles to an FESP of $\mathbb{P}^1$-bundles over smooth algebraic surfaces.

In Section 10, based on the results of Sections 5 and 9, we shall prove Theorems 10.1 and 1.5 which are one of our main results.

Section Dependency. The contents of each section are related in the following diagram;

$$\begin{tikzcd}
 & & & 5 \arrow[r] & 9 \arrow[r] & 10 \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[u] \arrow[d] & & \\
 & & & 6 \arrow[r] & 7 \arrow[r] & 8
\end{tikzcd}$$

2. Preliminaries

2.1. Notation. In this paper, we work over the complex number field $\mathbb{C}$.

A variety means a reduced and irreducible complex algebraic scheme. A projective variety is a complex variety embedded in a projective space, and a quasi-projective variety is a Zariski open subset of a projective variety. By a smooth projective n-fold, we mean a nonsingular projective variety of dimension n. By an endomorphism $f\colon X \to X$, we mean a morphism from a projective variety X to itself.

The following symbols are used for a variety X.

K_X: the canonical divisor of X.

T_X: the tangent bundle of X.

$\kappa(X)$: the Kodaira dimension of X.

$c_i(X)$: the i-th Chern class of X.

$b_i(X)$: the i-th Betti number of X.

$\chi(\mathcal{O}_X)$: the Euler–Poincaré characteristic of the structure sheaf $\mathcal{O}_X$.

$\chi_{\mathrm{top}}(X)$: the topological Euler characteristic of X.

$\mathrm{Sing}(X)$: the singular locus of X.

$\mathrm{Aut}(X)$: the algebraic group of automorphisms of X.

$\mathrm{Aut}^0(X)$: the identity component of $\mathrm{Aut}(X)$.

$N_1(X)$:= $(\{1\text{-cycles on } X\}/\equiv)\otimes_{\mathbb{Z}} \mathbb{R}$, where $\equiv$ means a numerical equivalence.

$N^1(X)$:= $(\{\text{Cartier divisors on } X\}/\equiv)\otimes_{\mathbb{Z}} \mathbb{R}$, where $\equiv$ means a numerical equivalence.

$\mathrm{NE}(X)$: the smallest convex cone in $N_1(X)$ containing all effective 1-cycles.

$\overline{\mathrm{NE}}(X)$: the Kleiman-Mori cone of X, i.e., the closure of $\mathrm{NE}(X)$ in $N_1(X)$ for the metric topology.

$\mathrm{Nef}(X)$: the smallest closed convex cone in $N^1(X)$ containing all nef divisors.

$\rho(X) := \dim_{\mathbb{R}} N_1(X)$, the Picard number of X.

$[C]$: the numerical equivalence of a 1-cycle C.

$\mathrm{cl}(D)$: the numerical equivalence class of a Cartier divisor D.

$\sim_{\mathbb{Q}}$: the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-divisors of X.

$g(C)$: the genus of a smooth curve C.

$\mathrm{Hom}_{\mathrm{group}}(C, C')$: the set of group homomorphisms from an elliptic curve C to C'.

For an endomorphism $f\colon X \to X$ and an integer $k > 0$, f^k stands for the k-times composite $k \circ \cdots \circ f$ of f.

Extremal rays: For a smooth projective variety X, an extremal ray R means a K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$, i.e., a 1-dimensional face of $\overline{\mathrm{NE}}(X)$ with $K_X R < 0$. A extremal ray R defines a proper surjective morphism with connected fibers $\mathrm{Cont}_R\colon X \to Y$ such that, for an irreducible curve $C \subset X$, $\mathrm{Cont}_R(C)$ is a point if and only if $[C] \in R$. This is called the contraction morphism associated to R.

Fibrations: A proper surjective morphism $\pi\colon X \to S$ is called a fibration or a fiber space if X and S are normal projective varieties and π has a connected fiber. The closed subset $\Delta_\pi := \{s \in S \,|\, \pi$ is not smooth at some point of $\pi^{-1}(s)\}$ is called the discriminant locus of π.

Elliptic fibrations: A fibration $f\colon X \to S$ is called an elliptic fibration or an elliptic fiber space if the general fibers of f are elliptic curves.

ESP: An étale sequence $X_\bullet = (f_n\colon X_n \to X_{n+1})_{n\in\mathbb{Z}}$ of constant Picard number ('ESP', for short) is an infinite sequence

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

of smooth projective varieties X_n such that each f_n is a non-isomorphic finite étale covering and each Picard number $\rho(X_n)$ is constant. For simplicity, we often omit the subscript $n \in \mathbb{Z}$ and denote it by $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$.

$\mathbb{P}^1$**- fiber space:** A fibration $g\colon Y \to T$ is called a $\mathbb{P}^1$-fiber space if the general fibers of g are isomorphic to the complex projective line $\mathbb{P}^1$.

2.2. Basic facts and techniques. In this section, we invoke some fundamental facts and techniques related with finite étale coverings and étale endomorphisms which will be frequently used in later Sections. For the proofs, the reader can consult [9].

Lemma 2.1. *Suppose that there exists an infinite sequence*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

of smooth projective k-folds such that each f_n is a non-isomorphic finite étale covering. Then $\chi(\mathcal{O}_{X_n}) = \chi_{\mathrm{top}}(X_n) = (K_{X_n})^k = 0$ for any n.

The existence of a non-isomorphic étale endomorphism imposes on the variety X strong conditions as the following proposition shows.

Proposition 2.2. (*cf.* [9, Lemma 2.3, Proposition 2.6]) *Let $f\colon X \to X$ be a surjective endomorphism of a smooth projective m-fold X. Then,*

(1) *f is a finite morphism.*
(2) *If $\kappa(X) \geq 0$, then f is an étale endomorphism.*
(3) *If $\kappa(X) = \dim X$, then f is an automorphism.*
(4) *If f is a non-isomorphic finite étale endomorphism, then*
   - $\chi(\mathcal{O}_X) = \chi_{\mathrm{top}}(X) = K_X^m = 0$ *and,*
   - *the fundamental group $\pi_1(X)$ of X is an infinite group.*

Now we introduce an important notion called an 'ESP' which includes as a special case the non-isomorphic étale endomorphisms.

Definition 2.3. Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an infinite sequence of finite coverings between smooth projective k-folds X_n. Then we say that the sequence $X_\bullet$ is an étale sequence of constant Picard number ('ESP' for short) if the following conditions are satisfied:

(1) The Picard number $\rho(X_n)$ is independent of n.

(2) Any f_n is a non-isomorphic finite étale covering.

DEFINITION 2.4. Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective varieties.

(1) If there exists a smooth projective variety X such that $X_n = X$ for any n, then we say that $X_\bullet$ is a 'stable ESP'.

(2) Furthermore, if $X_n = X$ and $f_n = f$ for any n for a non-isomorphic étale endomorphism $f\colon X \to X$ of a smooth projective variety X, then we say that $X_\bullet$ is a 'constant ESP' induced by $f\colon X \to X$ and denote it by $X_\bullet = (X, f)$.

DEFINITION 2.5. Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ and $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be ESP's. Then by a Cartesian morphism $\varphi_\bullet = (\varphi_n)_n\colon X_\bullet \to Y_\bullet$, we mean that the following Cartesian diagram holds for any n:

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\varphi_n"'] & X_{n+1} \arrow[d, "\varphi_{n+1}"] \\ Y_n \arrow[r, "g_n"'] & Y_{n+1}. \end{tikzcd}$$

The next lemma, which is stated implicitly in the papers [9] and [13], will provide the key to many of the results in later Sections.

Lemma 2.6. *Let $\lambda\colon V \to S$ and $\nu\colon W \to T$ be fiber spaces between normal varieties, i.e., λ and ν are proper surjective morphisms with connected fibers. Furthermore, suppose that $g\colon V \to W$ and $u\colon S \to T$ are finite surjective morphisms with the following commutative diagram:*

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\lambda"'] & W \arrow[d, "\nu"] \\ S \arrow[r, "u"'] & T . \end{tikzcd}$$

Suppose that $g^{-1}(B) = A$ for irreducible subvarieties $A(\subset V)$ and $B(\subset W)$. If the above commutative diagram is Cartesian, then $\lambda(A) = u^{-1}(\nu(B))$.

Proof. By definition, we have $u(\lambda(A)) = \nu(g(A)) = \nu(B)$. For the inverse image Γ of $\nu(B)$ by u, let us consider the fiber product $\widetilde{\Gamma} := \Gamma \times_T B$. Since $u(\Gamma) = \nu(B)$, by the universality, there exists a morphism $\sigma\colon \widetilde{\Gamma} \to V$ such that

- $g \circ \sigma = p_2$ for the second projection $p_2\colon \widetilde{\Gamma} \to B$, and
- $\lambda \circ \sigma = p_1$ for the first projection $p_1\colon \widetilde{\Gamma} \to \Gamma$.

Again, by the universality, there exists a morphism $\rho\colon \widetilde{\Gamma} \to V \times_W B$ such that the following commutative diagram holds:

$$\begin{tikzcd}
V \times_W B \arrow[r, equal] & V \times_W B \arrow[r] \arrow[d] & B \arrow[d] \\
\widetilde{\Gamma} \arrow[u, "\rho"] \arrow[r, "\sigma"] \arrow[d, "p_1"'] & V \arrow[r, "g"] \arrow[d, "\lambda"] & W \arrow[d, "\nu"] \\
\Gamma \arrow[r] & S \arrow[r, "u"] & T.
\end{tikzcd}$$

Hence, Γ is contained in the image of the composite map $V \times_W B \to V \xrightarrow{\lambda} S$. Note that by construction, the image of the natural projection $V \times_W B \to V$ is $g^{-1}(B) = A$. Hence we infer that $\Gamma \subset \lambda(A)$. □

3. Extremal contractions and a construction of an FESP

In this section, we shall apply the minimal model program (MMP) to a non-isomorphic étale endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) = -\infty$. One of the serious troubles is that there may exist infinitely many extremal rays of divisorial type on X. Compared with our previous results [9], [13], it is unknown whether there exists an extremal ray R of $\overline{\mathrm{NE}}(X)$ which is preserved by an endomorphism f of X. Hence, in general, we cannot run the MMP compatibly with étale endomorphisms. Thus, even if our concern is the study of endomorphisms, it is necessary to consider an ESP of 3-folds all of which are birationally equivalent to X. Using Proposition 1.1 and Theorem 3.10, we shall apply the MMP to a non-isomorphic étale endomorphism $f\colon X \to X$ and reduce the study of $f\colon X \to X$ to that of an *ESP of fiber type* (called an 'FESP' for short) which is not necessarily an endomorphism. We shall explain this reduction. Actually, we will consider a slightly more general set-up and give a proof of Corollary 1.2.

First, we recall some basic properties of non-isomorphic surjective endomorphisms from [9]. We shall use the standard terminology and results of the MMP as in [29] and [21]. We have proved the following results related to the extremal rays in [9].

Proposition 3.1. (*cf.* [9, Proposition 4.2 and 4.12]) *Let $f\colon Y \to X$ be a finite surjective morphism between smooth projective n-folds with $\rho(X) = \rho(Y)$. Then the following assertions hold:*

(1) *The push-forward map $f_*\colon \mathrm{N}_1(Y) \to \mathrm{N}_1(X)$ is an isomorphism and $f_*\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.*
(2) *Let $f_*\colon \mathrm{N}^1(Y) \to \mathrm{N}^1(X)$ be the map induced from the push-forward map $D \mapsto f_*D$ of divisors D. Then the dual $f^*\colon \mathrm{N}_1(X) \to \mathrm{N}_1(Y)$ (called the pullback map) is an isomorphism and $f^*\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$.*
(3) *If f is étale and the canonical divisor K_X is not nef, then there is a one-to-one correspondence between the set of extremal rays of X and the set of extremal rays of Y under the isomorphisms f_* and f^*.*
(4) *Under the same assumption as in* (3)*, let $\phi\colon X \to X'$ be the contraction morphism Cont_R associated to an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ and let $\psi\colon Y \to Y'$ be the contraction morphism associated to the extremal ray f^*R. Then there exists a finite surjective morphism $f'\colon Y' \to X'$ and the Cartesian diagram below such that $f^{-1}(\mathrm{Exc}(\varphi)) = \mathrm{Exc}(\psi)$ and $f'^{-1}(\phi(\mathrm{Exc}(\varphi))) = \psi(\mathrm{Exc}(\psi))$:*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ Y' & \xrightarrow{f'} & X' . \end{array}$$

Moreover, ϕ is a birational (resp. divisorial) contraction if and only if ψ is a birational (resp. divisorial) contraction.

Proposition 3.1 is applied to the following fundamental result on extremal rays of a smooth projective 3-fold, which is proved in [9] except the assertion (4).

Proof of Proposition 1.1. By Proposition 3.1, for any n,

- R_n is an extremal ray,
- there exists a finite morphism $g_n\colon Z_n \to Z_{n+1}$ such that $\pi_{n+1} \circ f_n = g_n \circ \pi_n$ and,
- $f_n^{-1}(E_n) = E_{n+1}$ for the exceptional divisor E_n of π_n.

In [29], extremal divisorial contractions of smooth projective 3-folds are classified into 5 types. In our situation, we shall exclude 4 cases where a prime divisor is contracted to a point. Suppose that E_n is contracted to a point by $\pi_n := \mathrm{Cont}_{R_n}$ for some $n > 0$. Then by [29], E_n is isomorphic to $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, or a singular quadric surface and is simply-connected. Since $(f_{n-1})^{-1}(E_n) = E_{n-1}$ and f_{n-1} is a non-isomorphic étale covering, E_{n-1} is not connected. Thus a contradiction is derived. Hence $\pi_n(E_n)$ is not a point for any n. Thus by [29], $\pi\colon Y_n \to Z_n$ is the (inverse of the) blowing-up of a smooth projective 3-fold Z_n along a smooth curve C_n for any n. By the purity of branch loci, g_n is also a finite étale covering and hence $Y_n \cong Y_{n+1} \times_{Z_{n+1}} Z_n$ and $\deg g_n = \deg f_n > 1$ for each n. Thus, by Porposition 3.1, $g_n^{-1}(C_{n+1}) = C_n$ for any n. Thus there is induced an ESP $C_\bullet = (w_n\colon C_n \to C_{n+1})_n$ of smooth curves C_n, where we put $w_n := g_n|_{C_n}$. Hence C_n is an elliptic curve for any n.

Finally, we shall prove the assertion (4). It is sufficient to show the claim for $n = 0$. First, we show that $\deg \mathcal{N}_0 = 0$. For the defining ideal $\mathcal{I}_n$ of C_n, we have $g_n^*(\mathcal{I}_{n+1}^j) = \mathcal{I}_n^j$ for any j and n, since g_n is étale. Hence there exists an isomorphism $(w_n)^*\mathcal{N}_{n+1} \cong \mathcal{N}_n$ for the normal bundle $\mathcal{N}_n$ of C_n in Z_n. Since $K_{Z_n} \sim g_n^* K_{Z_{n+1}}$, by the projection formula we have

$$(K_{Z_n}, C_n) = (K_{Z_{n+1}}, (g_n)_*(C_n)) = (\deg g_n)\,(K_{Z_{n+1}}, C_{n+1})$$

for any n. Hence for any $n > 0$,

$$(K_{Z_0}, C_0) = (K_{Z_n}, C_n) \prod_{i=0}^{n-1} \deg g_i \,.$$

Then $(K_{Z_0}, C_0) = 0$, since it is divisible by infinitely many positive integers. Hence, $\deg \mathcal{N}_0 = -(K_{Z_0}, C_0) = 0$.

The semi-stability of $\mathcal{N}_0$ will be proved by contradiction. Suppose that $\mathcal{N}_0$ is unstable. Since $(w_n)^*\mathcal{N}_{n+1} \simeq \mathcal{N}_n$, any $\mathcal{N}_n$ is unstable. Since $\mathcal{N}_n$ is an unstable vector bundle of degree zero on an elliptic curve C_n, $\mathcal{N}_n \simeq \mathcal{L}_n \oplus \mathcal{V}_n$ for line bundles $\mathcal{L}_n$ of degree $d_n > 0$ and $\mathcal{V}_n$ of degree $-d_n < 0$ on C_n. Since $\mathcal{L}_0 \simeq (w_{n-1} \circ \cdots \circ w_1 \circ w_0)^*\mathcal{L}_n$, we have the following equality for $n > 0$:

$$d_0 = d_n \prod_{i=0}^{n-1} \deg w_i \,.$$

Since $\deg w_i = \deg g_i = \deg f_i \geq 2$ for all i, we see that $0 < d_n < 1$ for a sufficiently positive integer n. This contradicts the fact that $d_n \in \mathbb{Z}$. Hence $\mathcal{N}_0$ is semi-stable. □

REMARK 3.2. Let E_n be the exceptional divisor of π_n. Then, by Proposition 1.1, (4) and Atiyah's classification (cf. [2]), E_n is isomorphic to either the Atiyah surface $\mathbb{S}$ or $\mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \ell_n)$ for some line bundle ℓ_n of degree zero on C_n. In fact, all these types can occur. However, in the case where $Y_\bullet$ is a constant ESP (Y, f) induced from a non-isomorphic étale endomorphism $f\colon Y \to Y$, the case where $\ell_n \in \operatorname{Pic}^0(C_n)$ is of infinite order cannot occur (cf. Theorems 10.1 and 1.3). We shall prove these facts in our subsequent article; Part III.

The following asserts the existence of '*Cartesian blowing-ups*'.

Lemma 3.3. *Let $V_\bullet = (g_n\colon V_n \to V_{n+1})_n$ be an ESP of 3-folds V_n. Suppose that there exists an ESP $C_\bullet = (g_n|_{C_n}\colon C_n \to C_{n+1})_n$ of elliptic curves on $V_\bullet$ such that the inclusion morphism $i_\bullet\colon C_\bullet \to V_\bullet$ is Cartesian. Let $\pi_n\colon W_n := \mathrm{Bl}_{C_n}(V_n) \to V_n$ be the blowing-up of V_n along C_n for any n. Then there is induced an ESP $W_\bullet = (f_n\colon W_n \to W_{n+1})_n$ of W_n such that $\pi_\bullet = (\pi_n)_n\colon W_\bullet \to V_\bullet$ is a Cartesian morphism.*

Proof. By the universality of the blowing-up, there exists a unique morphism $f_n\colon W_n \to W_{n+1}$ such that $\pi_{n+1} \circ f_n = g_n \circ \pi_n$ for any n. Since $g_n^{-1}(C_{n+1}) = C_n$, we see that $f_n^* E_{n+1} = E_n$ for the π_n-exceptional divisors E_n for any n. Hence $K_{W_n} \sim f_n^* K_{W_{n+1}}$ and f_n is a finite étale covering. Since $\rho(W_n) = \rho(V_n) + 1$ is constant, $W_\bullet := (f_n\colon W_n \to W_{n+1})_n$ is also an ESP of W_n. By construction, $\pi_\bullet$ is a Cartesian morphism. □

The following is concerned with some new phenomenon related with Remark 1.3.

Proposition 3.4. *There exists an example of an ESP $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ of 3-folds X_n with extremal rays $R_\bullet = (R_n)_n$ of $\overline{\mathrm{NE}}(X_\bullet)$ such that the following hold:*

(1) *$R_\bullet$ is of divisorial type and $(f_n)_* R_n = R_{n+1}$ for any n.*
(2) *Let D_n be the exceptional divisor of the contraction morphism $\mathrm{Cont}_{R_n}\colon X_n \to Y_n$ associated to R_n and $D_\bullet := (f_n|_{D_n}\colon D_n \to D_{n+1})_n$ an induced ESP of D_n. Then any D_n is isomorphic to the elliptic ruled surface $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{M}_n^{\otimes -1})$ associated to a line bundle $\mathcal{M}_n \in \operatorname{Pic}^0(E)$ of infinite order on a fixed elliptic curve E.*
(3) *$X_n \not\cong X_m$ for any $n \neq m$. In particular, $X_\bullet$ is not a constant ESP induced from a non-isomorphic étale endomorphism.*

Proof. **Step 1:** Let E be an elliptic curve and we fix two line bundles $\mathcal{L}, \mathcal{M}$ on E. Suppose that both $\mathcal{L}$ and $\mathcal{M}$ are of infinite order in $\operatorname{Pic}^0 E \cong E$ and there exist no pair of integers $(m, n) \neq (0, 0)$ such that $\mathcal{L}^{\otimes m} \cong \mathcal{M}^{\otimes n}$. We fix a positive integer $k > 1$. Note that $\operatorname{Pic}^0 E$ is divisible by any positive integer. Thus there exists a sequence $\{\mathcal{L}_n\}_n$ (resp. $\{\mathcal{M}_n\}_n$) of line bundles of degree zero on E such that $\mathcal{L}_0 \cong \mathcal{L}$ and $\mathcal{L}_{n+1}^{\otimes k} \cong \mathcal{L}_n$ (resp. $\mathcal{M}_0 \cong \mathcal{M}$ and $\mathcal{M}_{n+1}^{\otimes k} \cong \mathcal{M}_n$) for all n. For any n, let $\mathcal{E}_n := \mathcal{O}_E \oplus \mathcal{L}_n \oplus \mathcal{M}_n$ be the decomposable vector bundle of rank 3 and degree 0 on E. Let $\alpha_n\colon Y_n := \mathbb{P}_E(\mathcal{E}_n) \to E$ be the associated $\mathbb{P}^2$-bundle over E. For the multiplication map $\mu_k\colon E \to E$ by k, there exist isomorphisms $\mu_k^* \mathcal{L}_{n+1} \cong \mathcal{L}_{n+1}^{\otimes k} \cong \mathcal{L}_n$ and $\mu_k^* \mathcal{M}_{n+1} \cong \mathcal{M}_{n+1}^{\otimes k} \cong \mathcal{M}_n$. Hence there exists an isomorphism

$$\psi_n\colon Y_n \cong \widetilde{Y_{n+1}} := Y_{n+1} \times_{E, \mu_k} E\,.$$

Let $p_n\colon \widetilde{Y_n} \to Y_n$ be the first projection. Then the composite map $g_n := p_{n+1} \circ \psi_n\colon Y_n \to Y_{n+1}$ is a non-isomorphic finite étale covering of degree $k^2 (\geq 4)$ for all n. Thus we have constructed an ESP $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ of $\mathbb{P}^2$-bundles over E. Note that $Y_q \not\cong Y_p$ for any $q > p$ as an abstract variety.

(The reason is as follows: Suppose the contrary that there exists an isomorphism $v_{q,p}\colon Y_q \simeq Y_p$. Then the composite map $v_{q,p} \circ g_{q-1} \circ \cdots g_p\colon Y_p \to Y_p$ gives a non-isomorphic étale endomorphism of Y_p. Then applying Proposition 4.7 and Remark 1.3, we see that both $\mathcal{L}_p, \mathcal{M}_p \in \mathrm{Pic}^0(E)$ are of finite order. Thus a contradiction is derived. The details will be given in our subsequent part III article.)

Step 2: Let $s_n\colon E \to Y_n$ be a holomorphic section of α_n corresponding to a surjection $\mathcal{E}_n \twoheadrightarrow \mathcal{L}_n$ and put $C_n := s_n(E)$. Let $\pi_n\colon X_n := \mathrm{B}\ell_{C_n}(Y_n) \to Y_n$ be the blowing-up of Y_n along C_n. Since $g_n^{-1}(C_{n+1}) = C_n$ by construction, $C_\bullet = (g_n|_{C_n}\colon C_n \to C_{n+1})_n$ is an ESP of elliptic curves such that the inclusion morphism $i_\bullet\colon C_\bullet \to Y_\bullet$ is Cartesian. Hence by Lemma 3.3, there exists an ESP $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ such that $\pi_\bullet = (\pi_n)_n\colon X_\bullet \to Y_\bullet$ is the Cartesian blowing-up along $C_\bullet$. Then the normal bundle $\mathcal{N}_{C_n/Y_n} \cong \mathcal{O}_E \oplus \mathcal{M}_n$ is a decomposable, semi-stable vector bundle of rank 2 and degree 0 on E. By construction, we see that the π_n-exceptional divisor D_n is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}_E(\mathcal{O}_E \oplus \mathcal{M}_n^{\otimes -1})$ over E, where $\mathcal{M}_n \in \mathrm{Pic}^0(E)$ is of infinite order for all n. By applying the same argument as in the proof of Theorems 1.5, 10.1 and Remark 1.3, we see that $X_n \not\cong X_m$ for any $n \neq m$. (The details will be proved in our next article, Part III). □

We will now apply Propositions 3.1 and 1.1 to a non-isomorphic, finite étale endomorphism $f\colon X \to X$ of a 3-fold X with $\kappa(X) = -\infty$.

Proposition 3.5. *Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an ESP of smooth projective 3-folds Y_n with negative Kodaira dimension. Then the following are equivalent:*

(1) *For any integer n, there exist no extremal rays whose contraction morphisms are birational.*
(2) *For some integer n, there exist no extremal rays whose contraction morphisms are birational.*
(3) *For any integer n, there exist no extremal rays whose contraction morphisms are birational and contract an irreducible divisor to an elliptic curve.*
(4) *For any extremal ray R_n of $\overline{\mathrm{NE}}(Y_n)$ such that $(g_n)_* R_n = R_{n+1}$, the contraction morphism $\varphi_n := \mathrm{Cont}_{R_n}\colon Y_n \to W_n$ induces a Cartesian morphism $\varphi_\bullet = (\varphi_n)_n\colon Y_\bullet \to W_\bullet$ such that either of the following hold.*
    - *φ_n is a conic bundle over a smooth algebraic surface W_n for any n.*
    - *φ_n is a del Pezzo fibration over a smooth curve W_n for any n.*

Proof. (1) ⇒ (2): Trivial.

(2) ⇒ (1) follows immediately from Proposition 3.1.

(1) ⇒ (3): Trivial.

(3) ⇒ (1) follows immediately from Proposition 1.1.

(4) ⇒ (1): Trivial.

(1) ⇒ (4): Any Y_n is not a Fano manifold, since Fano manifolds are simply connected and g_n is a non-isomorphic finite étale covering. Hence the assertion follows immediately from Proposition 3.1 (4) and the classification of extremal rays of smooth projective 3-folds

by Mori [29]. □

Accordingly, we make the following definition:

Definition 3.6. (1) Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an ESP of smooth projective 3-folds with negative Kodaira dimension. Then $Y_\bullet$ is called an *ESP of fiber type* ('FESP' for short), if there exists an extremal ray R_n of fiber type on any $\overline{\mathrm{NE}}(Y_n)$, that is, the contraction morphism $\mathrm{Cont}_{R_n}\colon Y_n \to W_n$ associated to R_n is a Mori fiber space, i.e., $\dim W_n < \dim Y_n$ for any n.

(2) Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an FESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$. Let $R_\bullet = (R_n)_n$ be an extremal ray of fiber type on $\overline{\mathrm{NE}}(X_n)$ such that $(f_n)_*R_n = R_{n+1}$ for any n. Then
- the pair $(Y_\bullet, R_\bullet)$ is called of type (C) if $\varphi_n := \mathrm{Cont}_{R_n}\colon X_n \to S_n$ is a conic bundle over a smooth algebraic surface S_n and,
- the pair $(Y_\bullet, R_\bullet)$ is called of type (D) if $\varphi_n := \mathrm{Cont}_{R_n}\colon X_n \to C_n$ is a del Pezzo fibration over a curve C_n.

(3) Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an FESP. Suppose that there exists a non-isomorphic étale endomrphism $g\colon Y \to Y$ of a smooth projective 3-fold Y such that $Y_n = Y$ and $g_n = g$ for any n. Then we say that $Y_\bullet$ is a '*constant FESP*' induced from $g\colon Y \to Y$. We denote it by $Y_\bullet = (Y, g)$.

Using Proposition 1.1, we are ready to prove Corollary 1.2.

Proof of Corollary 1.2. Since $\kappa(X_0) = -\infty$, X_0 is uniruled and K_{X_0} is not nef by the 3-dimensional abundance theorem (cf. [19], [26], [27]). Hence there exists an extremal ray. Suppose that there exists *no* extremal ray of divisorial type on $\overline{\mathrm{NE}}\,(X_0)$. Then $X_\bullet$ itself is already an FESP and there is nothing to prove. Hereafter, we may assume that there exists some extremal ray R_0 of *divisorial* type on $\overline{\mathrm{NE}}\,(X_0)$. Let us choose such a ray R_0 arbitrarily and put $R_n := (f_{n-1} \circ \cdots \circ f_0)_*R_0$ and $R_{-n} := (f_{-1} \circ \cdots \circ f_{-n})^*R_0$ for any $n > 0$. Then, by Proposition 1.1, any R_n is also an extremal ray of divisorial type on $\overline{\mathrm{NE}}\,(X_n)$ and the contraction morphism $\pi_n^{(0)} := \mathrm{Cont}_{R_n}\colon X_n \to X_n^{(1)}$ is the blowing-up of a smooth projective 3-fold $X_n^{(1)}$ along an elliptic curve $C_n^{(1)} (\subset X_n^{(1)})$. Moreover, it induces the ESP $X_\bullet^{(1)} := (f_n^{(1)}\colon X_n^{(1)} \to X_{n+1}^{(1)})_n$ of smooth projective 3-folds $X_n^{(1)}$ such that
- $\pi_{n+1}^{(0)} \circ f = f_n^{(1)} \circ \pi_n^{(0)}$ and,
- $(f_n^{(1)})^{-1}(C_{n+1}^{(1)}) = C_n^{(1)}$.

Namely, there exists a Cartesian blowing-up $\pi_\bullet^{(0)}\colon X_\bullet^{(0)} \to X_\bullet^{(1)}$ along an ESP $C_\bullet^{(1)} := (f_n^{(1)}|_{C_n^{(1)}}\colon C_n^{(1)} \to C_{n+1}^{(1)})_n$ of elliptic curves.

If there exists no extremal ray of divisorial type on $X_0^{(1)}$, then we stop here. Otherwise, we repeat the same procedure as above. We again apply Proposition 1.1 to the new ESP $X_\bullet^{(1)}$ and consider the contraction morphism $\pi_n^{(1)}\colon X_n^{(1)} \to X_n^{(2)}$. In this way, for each positive integer n, we have successive contractions of extremal rays $X := X_n^{(0)} \to X_n^{(1)} \to X_n^{(2)} \to \cdots$ with a strictly decreasing sequence $\rho(X) > \rho(X_n^{(1)}) > \cdots$ of Picard numbers. Note that no flipping contractions can occur in our situation and $X_n^{(i)}$ is nonsingular for all n and $0 \le i \le k$. Thus, there exists some positive integer k such that $X_n^{(k)}$ has no extremal rays of divisorial type for any n. Furthermore, they form an FESP $X_\bullet^{(k)} := (f_n^{(k)}\colon X_n^{(k)} \to X_{n+1}^{(k)})_n$.

To sum up, we have the Cartesian diagram in Figure 1 such that the following hold;

- For each n and $1 \leq i < k$, $\pi_n^{(i-1)}\colon X_n^{(i-1)} \to X_n^{(i)}$ is (the inverse of) the blowing-up along an elliptic curve $C_n^{(i)}$ on $X_n^{(i)}$.
- For $0 \leq i \leq k$, $X_\bullet^{(i)} := (f_n^{(i)}\colon X_n^{(i)} \to X_{n+1}^{(i)})_n$ is an ESP.
- $(f_n^{(i)})^{-1}(C_{n+1}^{(i)}) = C_n^{(i)}$ for all $1 \leq i \leq k$. That is, $C_\bullet^{(i)} := (f_n^{(i)}|_{C_n^{(i)}}\colon C_n^{(i)} \to C_{n+1}^{(i)})_n$ is an ESP of elliptic curves and the inclusion $C_\bullet^{(i)} \hookrightarrow X_\bullet^{(i)}$ is a Cartesian morphism.
- The bottom row $X_\bullet^{(k)} = (f_n^{(k)}\colon X_n^{(k)} \to X_{n+1}^{(k)})_n$ is an FESP.

Thus, after finitely many divisorial contractions, we have obtained an FESP $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$, where we put $Y_n := X_n^{(k)}$ and $g_n := f_n^{(k)}$ for any n. All the assertions except (4) follow by construction. The assertion (4) follows from Proposition 1.1 (4). □

Definition 3.7. In Corollary 1.2, the vertical sequence

$$X_n \xrightarrow{\pi_n^{(0)}} \cdots \longrightarrow X_n^{(i)} \xrightarrow{\pi_n^{(i)}} X_n^{(i+1)} \longrightarrow \cdots \longrightarrow X_n^{(k)}$$

is called a sequence of blowing-downs of an ESP. Furthermore, we call $X_\bullet^{(k)}$ the FESP constructed from $X_\bullet$ by a sequence of blowing-downs of an ESP. In other words, $X_\bullet$ is reconstructed from the FESP $X_\bullet^{(k)}$ by a sequence of blowing-ups of an ESP.

In particular, we consider the special case where $f\colon X \to X$ is a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. First, suppose that the constant ESP $X_\bullet := (X, f)$ is already an FESP, i.e., for *some* extremal ray R of X, the contraction morphism $\pi := \mathrm{Cont}_R\colon X \to X'$ is *not birational*. Then we do not need to consider the reduction any more. By Proposition 3.1 and [29], the FESP $(X_\bullet, R_\bullet)$ is of type either (C) or (D). By Theorem 3.10, $(f^k)_*R = R$ for some positive integer k. Thus, if we replace f by the power f^k, we may assume that $f_*R = R$. Hence by Proposition 3.5, if $(X_\bullet, R_\bullet)$ is of type (C) (resp. (D)), then the conic bundle $\pi\colon (X, f) \to (S, g)$ (resp. the del Pezzo fibration $\pi\colon (X, f) \to (C, h)$) gives the Cartesian morphism of constant ESPs. Thus in this case, the MMP works compatibly with étale endomorphisms.

Next, suppose that $X_\bullet = (X, f)$ is not an FESP, i.e., for *any* K_X-neative extremal ray R of X, the contraction morphism $\mathrm{Cont}_R\colon X \to X'$ is *birational*. Then we can apply the above procedure to the constant ESP $X_\bullet := (X, f)$. Thus, by Corollary 1.2, after finitely many blowing-downs of an ESP, we can obtain an FESP $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ as above. The following proposition gives a sufficient condition for the existence of a constant FESP (cf. Definition 3.6).

Proposition 3.8. *Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a 3-fold X and $X_\bullet := (X, f)$ a constant ESP. Suppose that there exist finitely many extremal rays of divisorial type on X. Then by replacing f by a suitable power $f^k (k > 0)$, there exist a constant ESP $Y_\bullet = (Y, g)$ and a Cartesian morphism $\pi_\bullet\colon X_\bullet \to Y_\bullet$ induced from a divisorial contraction $\pi\colon X \to Y$.*

Proof. By Proposition 3.1 (3), f induces a permutation of the finite set $\mathcal{R}$ consisting of extremal rays of divisorial type. Hence replacing f by a suitable power $f^k (k > 0)$, f induces an identity permutation of $\mathcal{R}$. Thus applying Proposition 1.1, the proof is finished. □

Suppose that there exist finitely many extremal rays of divisorial type on Y. Then applying Proposition 3.8 again to the constant ESP $Y_\bullet$, we can obtain a constant ESP $Z_\bullet = (Z, h)$ and a Cartesian morphism $\pi'_\bullet\colon Y_\bullet \to Z_\bullet$ induced from a divisorial contraction $\pi'\colon Y \to Z$. If there exist only finitely many extremal rays of divisorial type in each ESP, we can continue this procedure and eventually obtain a constant FESP $W_\bullet = (W, v)$ induced by a non-isomorphic étale endomorphism $v\colon W \to W$.

REMARK 3.9. (1) Suppose that $f\colon X \to X$ is a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) \geq 0$. Then there exist only finitely many extremal rays of $\overline{\mathrm{NE}}(X)$. Hence by Proposition 3.8, there exists a constant FESP $Y_\bullet = (Y, g)$. Furthermore, Y is a smooth and a unique minimal model of X. $Y_\bullet$ is called a *minimal reduction* in [9], [13].

(2) On the contrary, in the case where $\kappa(X) = -\infty$, the trouble is that there exists a smooth projective 3-fold X with infinitely many extremal rays of divisorial type and admits a non-isomorphic étale endomorphism. For example (cf. [15], Remark A.9), let S be a rational elliptic surface with global sections whose Mordell-Weil rank is positive. It is obtained as 9-points blowing-up of $\mathbb{P}^2$. We regard S as an elliptic curve C_K defined over the function field K of the base curve. Since S is relatively minimal, the translation mapping $C_K \to C_K$ given by the non-torsion section γ induces a relative automorphism $g\colon S \simeq S$ over C, which is of infinite order. Let $X := S \times E$ be the product variety of S and an elliptic curve E. Since γ is a (-1)-curve of S, the curve $\gamma \times \{o\}$ for a point $o \in E$ spans the extremal ray R of divisorial type. If we denote by $\mu_k\colon E \to E$ multiplication by $k > 1$, then the product mapping $f := g \times \mu_k\colon X \to X$ gives a non-isomorphic étale endomorphism of X. By construction, $(f^k)_* R \neq R$ and $(f^k)^* R \neq R$ for any positive integer k. Furthermore, we can take an FESP of the constant ESP $X_\bullet = (X, f)$ as $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$, where $Y_n \simeq \mathbb{P}^2 \times E$ for any n. However, the étale endomorphism g_n depends on n, since the contraction morphism $\pi_n\colon X \to Y_n$ depends on n. Thus $Y_\bullet$ is *not* a constant FESP, but is a stable FESP (cf. Definition 2.4).

(3) If $f\colon X \to X$ is a non-isomorphic étale endomorphism with $\kappa(X) = -\infty$, then an FESP $Y_\bullet$ of the constant ESP (X, f) is not uniquely determined (cf. Remark 7.14).

The following theorem is useful for studying the structure of an étale endomorphism of a smooth projective variety which has a structure of a Mori fiber space.

Theorem 3.10 (Fujimoto, Nakayama [15])**.** *Let X be a normal projective variety defined over an algebraically closed field of characteristic zero such that X has only log-terminal singularities. Let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal ray such that $K_X R < 0$ and the associated contraction morphism* Cont_R *is a fibration to a lower dimensional variety. Then, for any surjective endomorphism $f\colon X \to X$, there exists a positive integer k such that $(f^k)_*(R) = R$ for the automorphism $(f^k)_*\colon \mathrm{N}_1(X) \cong \mathrm{N}_1(X)$ induced from the power $f^k = f \circ \cdots \circ f$.*

REMARK 3.11. Theorem 3.10 holds for any positive dimensional log terminal variety X and for any surjective endomorphisms f including automorphisms and finite ramified coverings of X. However, we cannot allow the case where Cont_R is a birational morphism. An easy counterexample is given in [15], Remark A.9.

In proving theorems concerning finiteness of extremal rays, the following fact is often useful.

Lemma 3.12. *Let $\varphi\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X. Suppose that there exists a finite étale covering $p\colon X' \to X$. Then*

(1) *for any extremal ray R of divisorial type on X, there exists some extremal ray R' of divisorial type on X' such that $p_*R' = R$.*

(2) *Furthermore, suppose that there exists a non-isomorphic étale endomorphism $\varphi'\colon X' \to X'$ such that $p \circ \varphi' = \varphi \circ p$. Then the contraction morphism $\mathrm{Cont}_{R'}\colon X' \to W'$ is (the inverse of) the blowing-up of a smooth projective 3-fold W' along an elliptic curve C' on W'.*

Proof. Let $R(\subset \overline{\mathrm{NE}}(X))$ be any extremal ray of divisorial type and $\pi := \mathrm{Cont}_R\colon X \to Y$ the contraction morphism associated to R. Then by Proposition 1.1, π is the inverse of the blowing-up of Y along an elliptic curve. Let e be any extremal rational curve on X whose numerical class $[e]$ spans R. Let e' be one of the connected components of $p^{-1}(e)$. Since $K_{X'} \sim p^*K_X$, we infer by the projection formula that $(K_{X'}, e') = (K_X, e) = -1 < 0$. For an ample divisor H' of X', we take a sufficiently small positive real number $\epsilon > 0$ such that $(K_{X'} + \epsilon H, e') < 0$. Then, by the cone theorem [29], we have: $\overline{\mathrm{NE}}\,(X') = \overline{\mathrm{NE}}_{K_{X'}+\epsilon H \geq 0}\,(X') + \sum_{j=1}^{r} R_j$, where R_j is an extremal ray spanned by a numerical equivalence class $[e_j]$ of an extremal rational curve e_j on X'. Hence $e' \equiv c' + \sum_{j=1}^{r} a_j e_j$, where $[c'] \in \overline{\mathrm{NE}}_{K_{X'}+\epsilon H \geq 0}(X')$, and $a_j \geq 0$ for any j. By construction, we have $a_i > 0$ for some i. Let $\psi := \pi \circ p\colon X' \to Y$ be the composite map. Then $0 = \psi_* e' \equiv \psi_* c' + \sum_{j=1}^{r} a_j \psi_* e_j$. Hence $\psi_*(e_i) \equiv 0$. Thus the curve $p(e_i)$ on X is contracted to a point by $\pi\colon X \to Y$, hence its numerical equivalence class $[p(e_i)]$ spans the extremal ray R on X. Thus $p_*R_i = R$. We set $R' := R_i$.

Next, we show that the extremal ray R' is of divisorial type. Assume the contrary. Let $\pi'\colon X' \to Y'$ be the contraction morphism. Then there exists a finite morphism $p'\colon Y' \to Y$ such that $p' \circ \pi' = \pi \circ p$. Since $\dim Y' \leq 2$, this contradicts the finiteness of p'. Hence R' is of divisorial type and the assertion (1) has been proved. The assertion (2) follows immediately from the assertion (1) and Proposition 1.1. □

4. Projective bundles over an elliptic curve

In this section, we shall study projective bundles X over an elliptic curve C which admit non-isomorphic étale endomorphisms $f\colon X \to X$. We can show that such X is associated to a semi-stable vector bundle on C. A key ingredient is the following Proposition 4.1 which relates the semi-stability of vector bundles on an elliptic curve with the existence of non-isomorphic étale endomorphisms of associated projective bundles. As an immediate application, we shall relate torsion line bundles on an elliptic curve with an elliptic ruled surface admitting a non-isomorphic étale endomorphism (cf. Proposition 4.8). Furthermore, using Atiyah's vector bundle $\mathcal{F}_r$ ([2]) on an elliptic curve, we shall construct concrete examples of such projective bundles (cf. Proposition 4.13). We also remark that the étaleness assumption is essentially used in our paper (cf. Remark 4.3).

Proposition 4.1. *Let $r > 1$ be a fixed integer. For each integer n, let $\mathcal{E}_n$ be a vector bundle of rank r on an elliptic curve C_n and $\varphi_n\colon X_n := \mathbb{P}_{C_n}(\mathcal{E}_n) \to C_n$ the associated $\mathbb{P}^{r-1}$-bundle. Suppose that there exists an ESP $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ of X_n. Then $\mathcal{E}_n$ is semi-stable for all n.*

Proof. By considering the following truncated sequence

$$X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots$$

starting from X_n for each n, it is sufficient to show that $\mathcal{E}_0$ is semi-stable without loss of generality. Assuming that $\mathcal{E}_0$ is unstable, we derive a contradiction. Since φ_n gives the Albanese map of X_n, there exists a unique finite étale covering $h_n \colon C_n \to C_{n+1}$ with $h_n \circ \varphi_n = \varphi_{n+1} \circ f_n$ for each n. Since f_n is étale and $\mathbb{P}^{r-1}$ is simply connected, f_n is of degree one on each fiber of φ_n and $\deg h_n = \deg f_n > 1$. Thus, there exists the following Cartesian morphism of ESPs;

$$\varphi_\bullet = (\varphi_n)_n \colon X_\bullet \to C_\bullet := (h_n \colon C_n \to C_{n+1})_n .$$

In particular, there exists an isomorphism $h_n^* \mathcal{E}_{n+1} \cong \mathcal{E}_n \otimes \ell_n$ for some line bundle ℓ_n on C_n. Since the semi-stability of a vector bundle is preserved after taking a tensor product with a line bundle or pulling-back under a finite étale base change, $\mathcal{E}_n$ is unstable for all $n \geq 0$. For a sufficiently large positive integer n, take a maximal destabilizing subbundle $\mathcal{G}_n$ of $\mathcal{E}_n$ with $r' := \operatorname{rank} \mathcal{G}_n < r$. Then there exists an isomorphism $\mathcal{E}_0 \cong (h_{n-1} \circ \cdots \circ h_0)^* \mathcal{E}_n \otimes \mathcal{L}_0^{-1}$ for some line bundle $\mathcal{L}_0$ on C. Now we take a coherent subsheaf $\mathcal{G}_0 := (h_{n-1} \circ \cdots \circ h_0)^* \mathcal{G}_n \otimes \mathcal{L}_0^{-1}$ of $\mathcal{E}_0$. Then

$$\mu(\mathcal{G}_0) + \deg \mathcal{L}_0 = \mu(\mathcal{G}_n) \cdot \prod_{i=0}^{n-1} \deg h_i ,$$

for the slope $\mu(\mathcal{G}_0)(:= \deg \mathcal{G}_0 / r')$. Similarly,

$$\mu(\mathcal{E}_0) + \deg \mathcal{L}_0 = \mu(\mathcal{E}_n) \cdot \prod_{i=0}^{n-1} \deg h_i ,$$

for the slope $\mu(\mathcal{E}_0)$ of $\mathcal{E}_0$. Hence, we have an equality:

$$\mu(\mathcal{G}_0) - \mu(\mathcal{E}_0) = (\mu(\mathcal{G}_n) - \mu(\mathcal{E}_n)) \cdot \prod_{i=0}^{n-1} \deg h_i .$$

Since $\mu(\mathcal{G}_n) > \mu(\mathcal{E}_n)$, $\mathcal{G}_0$ is unstable. Furthermore, due to the uniqueness of the Harder-Narashimhan filtration, $\mathcal{G}_0$ is also a maximal destabilizing subsheaf of $\mathcal{E}_0$ and does not depend on the choice of n. By definition, the positive rational number $\mu(\mathcal{G}_n) - \mu(\mathcal{E}_n)$ is uniformly bounded below by the constant $(rr')^{-1} > r^{-2}$, which is independent of n. Since $\deg h_i \geq 2$ for each i, we have the following inequality for $n \gg 0$;

$$\mu(\mathcal{G}_0) - \mu(\mathcal{E}_0) \geq 2^n / r^2 .$$

Thus a contradiction is derived. □

The following corollary is obtained immediately.

Corollary 4.2. *Let $S_\bullet = (f_n \colon S_n \to S_{n+1})_n$ be an ESP of relatively minimal elliptic ruled surfaces S_n. Suppose that $S_n = \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$ for an invertible sheaf $\mathcal{L}_n$ on an elliptic curve C_n for each n. Then $\deg(\mathcal{L}_n) = 0$ for any n.*

Remark 4.3. (1) Proposition 4.1 does not hold if we consider an infinite sequence of bounded above

$$\cdots \longrightarrow X_n \xrightarrow{g_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{g_1} X_0$$

of non-isomorphic finite étale coverings of projective bundles over elliptic curves. We shall give such an example. Let C be an elliptic curve with $o \in C$ as the identity element and $\mu_N \colon C \to C$ the multiplication map by an integer $N(>1)$. Let $\mathcal{O}_C([o])$ be the invertible sheaf associated to the prime divisor $[o]$ consisting of o. Then $\mu_N^*\mathcal{O}_C([o]) \cong \mathcal{O}_C([o])^{\otimes N^2}$. If we put $\mathcal{L}_n := (\mu_{nN})^*\mathcal{O}([o])$, then the direct sum $\mathcal{E}_n := \mathcal{O}_C \oplus \mathcal{L}_n$ is an unstable vector bundle on C and $\mu_N^*\mathcal{E}_n \cong \mathcal{E}_{n+1}$. Let $S_n := \mathbb{P}(\mathcal{E}_n)$ be a $\mathbb{P}^1$-bundle over $C_n := C$ associated to $\mathcal{E}_n$. Then for each n, there exists an isomorphism $S_{n+1} \cong S_n \times_{C,\,\mu_N} C$ and the natural projection $g_n \colon S_{n+1} \to S_n$ gives a finite étale covering of degree $N^2(>1)$.

(2) Proposition 4.1 does not hold without the assumption that f_n is finite étale. In fact, by [38, Propsition 5], any $\mathbb{P}^1$-bundle over an elliptic curve admits non-isomorphic surjective endomorphisms. In particular, let $S := \mathbb{P}_C(\mathcal{E})$ be a $\mathbb{P}^1$-bundle associated with an unstable vector bundle $\mathcal{E}$ on an elliptic curve C. Then S admits a surjective endomorphism $f \colon S \to S$ which is a finite ramified covering. Thus, there exists an infinite sequence of non-isomorphic finite ramified coverings between elliptic ruled surfaces:

$$\cdots \xrightarrow{f} S \xrightarrow{f} S \xrightarrow{f} \cdots .$$

The follolwing result will be used later.

Proposition 4.4. (*cf.* [39], [44], [46]) *Let $\pi \colon X := \mathbb{P}_C(\mathcal{E}) \to C$ be a $\mathbb{P}^1$-bundle over an elliptic curve C associated to a stable vector bundle $\mathcal{E}$ of rank two. Then the following hold:*

(1) *The anti-pluricanonical system $|-2K_X|$ is base point free and defines a Seifert elliptic fibration* (*cf.* Definition 6.1) *$\Phi \colon X \to \mathbb{P}^1$ such that $\mathcal{O}_X(-2K_X) \simeq \Phi^*\mathcal{O}(1)$. Furthermore, Φ has three multiple singular fibers of type ${}_2I_0$.*

(2) *Let $\mu_2 \colon C \to C$ be a multiplication mapping by two. Then the pull-back $X \times_{C,\mu_2} C$ is isomorphic over C to the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times C$.*

We recall the following theorem due to Miyaoka concerning a numerical characterization for a semistable vector bundle on curves.

Theorem 4.5 (Miyaoka [25])**.** *Let $\mathcal{E}$ be a vector bundle of rank two on a smooth projective curve C and $\pi \colon X := \mathbb{P}_C(\mathcal{E}) \to C$ the associated $\mathbb{P}^1$-bundle. Then the following conditions are equivalent:*

(1) *$\mathcal{E}$ is semistable.*

(2) *The relative anti-canonical divisor $-K_{X/C} := -K_X + \pi^*K_C$ is nef.*

(3) $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}[-K_{X/C}] + \mathbb{R}_{\geq 0}[F]$, *where F is a fiber of π.*

(4) $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[-K_{X/C}] + \mathbb{R}_{\geq 0}[F]$.

(5) *Every effective divisor on X is nef.*

Now, we need the following preliminary result.

Lemma 4.6. *Let $\varphi\colon Y \to X$ be a surjective morphism of normal projective varieties X and Y. Suppose that $\varphi^*\mathcal{L} \cong \mathcal{O}_Y$ for a line bundle $\mathcal{L}$ on X. Then $\mathcal{L}$ is of finite order in* $\mathrm{Pic}(X)$. *Furthermore, if φ is generically finite, then* $\mathrm{ord}(\mathcal{L})|\deg\varphi$.

Proof. With the aid of the Stein factorization, the claim can be reduced to the following two cases:

- The case where $\varphi\colon Y \to X$ is a fiber space.
- The case where φ is a finite morphism.

(1) In the former case, since $\varphi_*\mathcal{O}_Y \simeq \mathcal{O}_X$, it follows immediately by the projection formula that $\mathcal{L} \simeq \mathcal{O}_X$.

(2) In the latter case, there exists some Cartier divisor D of X such that $\mathcal{L} \simeq \mathcal{O}_X(D)$. Then by assumption, we infer that $\varphi^*D \sim 0$. We regard $\mathcal{L}$ as a Cartier divisor and consider the push-forward map $\varphi_*\colon N^1(Y) \to N^1(X)$ of Cartier divisors. Since $\varphi_*\varphi^*(D) \sim (\deg\varphi)\, D$, it follows that $(\deg\varphi)\, D \sim 0$ and $\mathcal{L}$ is of finite order in $\mathrm{Pic}(X)$. In particular, we see that $\mathrm{ord}(\mathcal{L})|\deg\varphi$. □

As an application of this technique, we can now state the following result:

Proposition 4.7. *Let A be an m-dimensional abelian variety, X an m-dimensional projective variety and $f, g\colon X \twoheadrightarrow A$ surjective morphisms. Suppose that*

- *$f - g\colon X \to A$ is a surjective morphism and;*
- *$f^*\mathcal{L} \simeq g^*\mathcal{L}$ for a line bundle $\mathcal{L} \in$* $\mathrm{Pic}^0(A)$.

Then $\mathcal{L}$ is of finite order in $\mathrm{Pic}^0(A)$.

Proof. By [32, p75, (ii)], we have $(f - g)^*\mathcal{L} \simeq \mathcal{O}_X$. Then applying Lemma 4.6, we infer that $\mathcal{L}$ is of finite order in $\mathrm{Pic}(A)$. □

We are ready to prove the following fundamental result.

Proposition 4.8. *Let $g\colon S \to S$ be a non-isomorphic étale endomorphism of a relatively minimal elliptic ruled surface S. Suppose that $S = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ on an elliptic curve C. Then $\mathcal{L} \in$* $\mathrm{Pic}\,(C)$ *is of finite order.*

Proof. Let $\varphi\colon S \to C$ be the Albanese map of S, which gives S a $\mathbb{P}^1$-bundle structure over the elliptic curve C. Then there exists an étale endomorphism $h\colon C \to C$ of C with $\varphi \circ g = h \circ \varphi$. Since g is étale and $\mathbb{P}^1$ is simply connected, there exists an isomorphism

$$S := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \cong S \times_{C,h} C \simeq \mathbb{P}(\mathcal{O}_C \oplus h^*\mathcal{L})$$

over C and $\deg h = \deg g$. Thus there exists an isomorphism

$$\mathcal{O}_C \oplus \mathcal{L} \cong (\mathcal{O}_C \oplus h^*\mathcal{L}) \otimes \ell$$

for some invertible sheaf ℓ on C. If $h^*\mathcal{L} \simeq \mathcal{O}_C$, then $\mathrm{ord}(\mathcal{L})|\deg h$ by Lemma 4.6. Suppose that $h^*\mathcal{L} \not\simeq \mathcal{O}_C$. Then by [3, Theorem 1], this is possible if and only if

- $\ell \cong \mathcal{O}_C$ and $h^*\mathcal{L} \cong \mathcal{L}$, or
- $\ell \cong \mathcal{L}$ and $h^*\mathcal{L} \cong \mathcal{L}^{-1}$.

Then since $\deg h > 1$, we see that $\deg \mathcal{L} = 0$. Since both $h - \mathrm{id}_C$ and $h + \mathrm{id}_C$ are surjective endomorphisms of C, Proposition 4.7 implies that $\mathcal{L} \in \mathrm{Pic}^0(C)$ is of finite order. □

The following follows immediately from Lemma 4.6 and the proof of Proposition 4.8.

Corollary 4.9. *Under the same notation as in the proof of* Proposition 4.8, $\mathrm{ord}(\mathcal{L})$ *divides* $\deg(h + \mathrm{id}_C), \deg(h - \mathrm{id}_C)$, *or* $\deg(h)$.

REMARK 4.10. Proposition 4.8 does not hold without the assumption that an endomorphism $f \colon X \to X$ is étale. For example, let $\mathcal{L}$ be a line bundle of degree zero on an elliptic curve C which is of *infinite* order in $\mathrm{Pic}^0(C)$. Let $S := \mathbb{P}_C(\mathcal{E})$ be a $\mathbb{P}^1$-bundle over C associated to a decomposable vector bundle $\mathcal{E} := \mathcal{O}_C \oplus \mathcal{L}$ of rank two and degree zero on C. Then, by [38], S admits a non-isomorphic surjective endomorphism $f \colon S \to S$, which gives a finite *ramified* covering of S.

The following is a well-known fact (cf. [42]).

Fact:

(1) Let $\mathcal{F}$ and $\mathcal{F}'$ be two semi-stable vector bundles over an elliptic curve C with slope μ and μ' respectively. If $\mu < \mu'$, then $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{F}, \mathcal{F}') = 0$.

(2) An arbitrary unstable vector bundle of finite rank on an elliptic curve is a direct sum of semi-stable bundles.

(3) Any indecomposable vector bundle on an elliptic curve is semi-stable.

Now we quote some deep results due to Atiyah concerning vector bundles on an elliptic curve which we need throughout our articles. For a proof, the reader may consult [2].

Theorem 4.11 (Atiyah [2])**.** *Let C be an elliptic curve. Then the following hold:*

(1) *There exists on C an indecomposable vector bundle $\mathcal{F}_r$ of rank r and degree* 0 *with $H^0(C, \mathcal{F}_r) \neq 0$, unique up to isomorphism. Moreover, $\mathcal{F}_r$ is defined inductively as follows; $\mathcal{F}_1 = \mathcal{O}_C$ and $\mathcal{F}_r$ is the unique non-trivial extension of $\mathcal{F}_{r-1}$ by $\mathcal{O}_C$.*

(2) *Let F be an indecomposable vector bundle on C of rank r and degree* 0*. Then $F \cong \mathcal{F}_r \otimes \mathcal{L}$ for a unique line bundle $\mathcal{L}$ of degree* 0 *on C such that $\mathcal{L}^{\otimes r} \cong \det F$.*

(3) *The bundle $\mathcal{F}_r$ is self-dual, that is, $\mathcal{F}_r^\vee \simeq \mathcal{F}_r$, where $\mathcal{F}_r^\vee := \mathcal{H}om_{\mathcal{O}_C}(\mathcal{F}_r, \mathcal{O}_C)$.*

The following is an immediate consequence of Theorem 4.11.

Lemma 4.12. *Let $h \colon C \to C$ be a non-isomorphic étale endomorphism of an elliptic curve C. Then $h^*\mathcal{F}_r \simeq \mathcal{F}_r$ for any $r > 0$.*

Proof. By construction, the vector bundle $\mathcal{F}_r$ is constructed as a successive extension of trivial line bundles by the following non-split exact sequence;

$$(E_r) :\ 0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow 0\,.$$

We use induction on r. Since $\mathcal{F}_1 \simeq \mathcal{O}_C$, the claim is trivial for $r = 1$. Assume that the claim holds true for $r - 1$. Let $\delta_r\ (\neq 0)$ be the non-trivial extension class in $\mathrm{Ext}^1(\mathcal{F}_{r-1}, \mathcal{O}_C) \simeq H^1(C, \mathcal{F}_{r-1}^\vee) \simeq \mathbb{C}$ corresponding to the non-split exact sequence (E_r). Then pulling-back (E_r) by $h \colon C \to C$, we obtain the following exact sequence:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow h^*\mathcal{F}_r \longrightarrow h^*\mathcal{F}_{r-1} \longrightarrow 0. \tag{1}$$

Thanks to the projection formula and the finiteness of h, we infer that

$$\mathrm{Ext}^1(h^*\mathcal{F}_{r-1}, \mathcal{O}_C) \simeq H^1(C, h^*\mathcal{F}_{r-1}^\vee) \simeq H^1(C, h_*h^*\mathcal{F}_{r-1}^\vee) \simeq H^1(C, \mathcal{F}_{r-1}^\vee \otimes h_*\mathcal{O}_C)\,.$$

On the other hand, the trace map $\mathrm{Tr}_{C/C}\colon h_*\mathcal{O}_C \to \mathcal{O}_C$ gives rise to a splitting of the natural inclusion $\mathcal{O}_C \hookrightarrow h_*\mathcal{O}_C$. Hence $\mathcal{F}_{r-1}^\vee \to \mathcal{F}_{r-1}^\vee \otimes h_*\mathcal{O}_C$ likewise splits. Thus $\mathrm{Ext}^1(\mathcal{F}_{r-1}, \mathcal{O}_C) \simeq H^1(C, \mathcal{F}_{r-1}^\vee)$ embeds as a direct summand of $\mathrm{Ext}^1(h^*\mathcal{F}_{r-1}, \mathcal{O}_C)$. Hence the natural homomorphism $h^*\colon \mathrm{Ext}^1(\mathcal{F}_r, \mathcal{O}_C) \to \mathrm{Ext}^1(h^*\mathcal{F}_r, \mathcal{O}_C)$ is injective. Thus $h^*\delta_r \neq 0$ and the exact sequence (1) does not split. By assumption, we have $h^*\mathcal{F}_{r-1} \simeq \mathcal{F}_{r-1}$. Since $H^0(C, \mathcal{F}_{r-1}) \neq 0$ and $H^0(C, h^*\mathcal{F}_{r-1}) \neq 0$, we infer that $h^*\mathcal{F}_r \simeq \mathcal{F}_r$ by Theorem 4.11. □

The following proposition gives a sufficient condition for a projective bundle over an elliptic curve to have a non-isomorphic étale endomorphism.

Proposition 4.13. *For a positive integer $r > 1$, let $\varphi\colon Y := \mathbb{P}_C(\mathcal{F}_r) \to C$ be a $\mathbb{P}^{r-1}$-bundle associated to the vector bundle $\mathcal{F}_r$ on an elliptic curve C. Then Y admits a non-isomorphic étale endomorphism.*

Proof. For any positive integer $k\,(> 1)$, let $\mu_k\colon C \to C$ be multiplication by k. Then by Lemma 4.12, there exists an isomorphism

$$\widetilde{Y} := Y \times_{C,\mu_k} C \simeq \mathbb{P}_C(\mu_k^*\mathcal{F}_m) \simeq \mathbb{P}_C(\mathcal{F}_m) =: Y\,.$$

Hence the natural morphism $Y \simeq \widetilde{Y} \to Y$ gives an étale endomorphism of Y of degree $k^2\,(> 1)$. □

5. Surjective morphisms between elliptic ruled surfaces

In this section, we shall study the structure of surjective morphisms between $\mathbb{P}^1$-bundles over elliptic curves which are not necessarily endomorphisms. Propositions 5.6, 5.15 and 5.16 are our main results, which may be new and are necessary for use in the proof of Theorem 10.1. Our methods of proof are quite elementary and based on the one-to-one correspondence between the Kleiman-Mori cones of the source variety and target variety. Proposition 5.5 is devoted to the study of endomorphisms of the 'Atiyah surface'. Combining all the results stated as above, we shall study ESPs of elliptic ruled surfaces in Proposition 5.10. First we recall some remarkable properties of Atiyah's vector bundle $\mathcal{F}_2$. Hereafter, we shall use the following terminology.

Definition 5.1. Let $\mathcal{F}_2$ be an indecomposable semi-stable locally free sheaf of rank 2 and degree 0 on an elliptic curve C (cf. Theorem 4.11). Let $\pi\colon \mathbb{S} := \mathbb{P}_C(\mathcal{F}_2) \to C$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{F}_2$ and s_∞ the section of π corresponding to a surjection $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_C$. Then we call $\mathbb{S}$ an 'Atiyah surface' (over C) and s_∞ the 'canonical section' of π.

The following seems to be well-known.

Proposition 5.2. *Let $\pi\colon \mathbb{S} \to C$ be the Atiyah surface. Then the following hold:*
(1) *The canonical section s_∞ is a unique irreducible curve on $\mathbb{S}$ such that its self-intersection number is zero and is not contained in any fiber of π.*
(2) $\mathrm{h}^0(\mathbb{S}, \mathcal{O}_{\mathbb{S}}(a\,s_\infty)) = 1$ *for any positive integer a.*

Corollary 5.3. *There exists no elliptic fiber space structure on the Atiyah surface $\mathbb{S}$.*

Proof. Suppose that there exists on $\mathbb{S}$ an elliptic surface structure $\rho\colon \mathbb{S} \to \Delta$. Since $K_{\mathbb{S}} \sim -2s_\infty$ and $s_\infty^2 = 0$, the anti-canonical divisor $-K_{\mathbb{S}}$ of $\mathbb{S}$ is nef and the intersection number $(K_{\mathbb{S}}, \gamma)$ is an even integer for every irreducible curve γ on $\mathbb{S}$. Hence ρ is a relatively minimal elliptic fibration. Clearly, we have $(K_{\mathbb{S}}, F) = F^2 = 0$ for a general fiber F of ρ. Then either $s_\infty \cap F = \emptyset$ or $s_\infty = F$, since $s_\infty^2 = 0$. Then Proposition 5.2 implies that $F = s_\infty$. Thus $\mathbb{S} = s_\infty$, which derives a contradiction. □

REMARK 5.4. By Proposition 4.13, the Atiyah surface $\mathbb{S}$ admits a non-isomorphic étale endomorphism. (cf. [38])

One basic fact worth mentioning is the following.

Proposition 5.5. *Let $\pi\colon \mathbb{S} \to C$ be the Atiyah surface. Then every surjective endomorphism $\varphi\colon \mathbb{S} \to \mathbb{S}$ is a finite étale covering satisfying $\varphi^*(s_\infty) = s_\infty$.*

Proof. Since the pull-back map $\varphi^*\colon N_1(\mathbb{S}) \to N_1(\mathbb{S})$ induces an isomorphism of the Kleiman-Mori cone $\overline{\mathrm{NE}}(\mathbb{S})$, which is a 2-dimensional closed convex polyhedral cone. Hence φ^* induces a permutation of the two extremal rays of $\overline{\mathrm{NE}}(\mathbb{S})$. Since π is the Albanese map of $\mathbb{S}$, there exists an étale endomorphism $h\colon C \to C$ with $\pi \circ \varphi = h \circ \pi$ by the universality of the Albanese map. Hence both extremal rays $\mathbb{R}_{\geq 0}\,[s_\infty]$ and $\mathbb{R}_{\geq 0}\,[f]$ are preserved by φ^*. We have $K_{\mathbb{S}} \sim \varphi^* K_{\mathbb{S}} + R_\varphi$, where R_φ is an effective divisor called the ramification divisor of φ. Then $K_{\mathbb{S}} \equiv -2\,s_\infty$, since $K_{\mathbb{S}} \sim -2L + \pi^*(K_C + \det \mathcal{F}_2)$ for the tautological divisor $L := \mathcal{O}_{\mathbb{P}(\mathcal{F}_2)}(1)$. Hence $[R_\varphi]$ belongs to the ray $\mathbb{R}_{\geq 0}[s_\infty]$.

The proof is by contradiction. Suppose the contrary that $R_\varphi \neq 0$ and let $R_\varphi = \sum_i a_i\, D_i$ be an irreducible decomposition. Since $[s_\infty]$ spans the extremal ray of $\overline{\mathrm{NE}}(\mathbb{S})$, we have $[D_i] \in \mathbb{R}_{\geq 0}[s_\infty]$ for all i. Then, $D_i = s_\infty$ for all i by Proposition 5.2. Hence we have $R_\varphi = b\, s_\infty$ for some positive integer b and φ is unramified over $\mathbb{S} \setminus s_\infty$. Applying Lemma 4.12, there exists an isomorphism $\mathbb{S} \times_{h,C} C \cong \mathbb{P}_C(h^*\mathcal{F}_2) \cong \mathbb{P}_C(\mathcal{F}_2) =: \mathbb{S}$. Since h is étale, the natural morphism $\psi\colon \mathbb{S} \to \mathbb{S} \times_{h,C} C \simeq \mathbb{S}$ is also unramified over $\mathbb{S} \setminus s_\infty$. For any fiber f of π, set $p := \pi(f)$ and $F := \varphi(f)$. Then $F = \pi^{-1}(h(p))$ is a disjoint union of fibers of π and f is one of its connected components. We have : $K_S \sim \varphi^* K_S + b s_\infty$. The restriction of this to the fiber f of π gives the following equality:

$$K_f \sim (\varphi|_f)^* K_F + b s_\infty|_f\ .$$

Hence the restriction of the morphism ψ to f gives a finite ramified covering from $\mathbb{P}^1$ to itself branched over only one point $s_\infty \cap f$. Thus a contradiction is derived. Hence φ is a finite étale covering and there exists an isomorphism $\psi\colon \mathbb{S} \cong \mathbb{S} \times_{h,C} C$. Since $\varphi^*[s_\infty] \in \mathbb{R}_{\geq 0}[s_\infty]$, Proposition 5.2 implies that $\varphi^{-1}(s_\infty) = s_\infty$. □

We prove some simple propositions that provide the key to the proof of Theorem 1.5. First, we study the structure of a surjective morphism ψ between $\mathbb{P}^1$-bundles over elliptic curves which are associated to semi-stable vector bundles of rank 2. Note that ψ is not necessarily an endomorphism.

Proposition 5.6. *There exist no surjective morphisms from one to another among the following three $\mathbb{P}^1$-bundles S_i over an elliptic curve C_i ($1 \leq i \leq 3$).*

(1) $\pi_1 : S_1 = \mathbb{P}_{C_1}(\mathcal{O}_{C_1} \oplus \ell) \to C_1$ *for a line bundle* ℓ *of degree* 0 *on* C_1*, where* $\ell \in \mathrm{Pic}^0(C_1)$ *is torsion.*

(2) $\pi_2 : S_2 = \mathbb{P}_{C_2}(\mathcal{O}_{C_2} \oplus \mathcal{L}) \to C_2$ *for a line bundle* $\mathcal{L}$ *of degree* 0 *on* C_2*, where* $\mathcal{L} \in \mathrm{Pic}^0(C_2)$ *is of infinite order.*

(3) $\pi_3 : S_3 = \mathbb{S} \to C_3$ *is the Atiyah surface.*

Proof. Let s_i be the section of π_i ($1 \le i \le 3$) corresponding to a surjection $\mathcal{O}_{C_1} \oplus \ell \twoheadrightarrow \ell$ (resp. $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$ and $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_{C_3}$). It is well-known that $\rho(S_i) = 2$ and the Kleiman-Mori cone $\overline{\mathrm{NE}}(S_i)$ is a two-dimensional closed convex polyhedral cone spanned by the numerical equivalence classes $[f_i]$ and $[s_i]$ of a fiber f_i and a section s_i of π_i, i.e., $\overline{\mathrm{NE}}(S_i) = \mathbb{R}_{\ge 0}[f_i] + \mathbb{R}_{\ge 0}[s_i]$. The proof is by contradiction.

(1) Suppose the contrary that there exists a surjective morphism $\varphi: S_1 \twoheadrightarrow S_2$. Since $\rho(S_i) = 2$ for all i, by Proposition 3.1, there is induced an isomorphism $\varphi_* : \overline{\mathrm{NE}}(S_1) \to \overline{\mathrm{NE}}(S_2)$ of the Kleiman-Mori cone by the push-forward map $\varphi_* : N_1(S_1) \to N_1(S_2)$. Hence there exists a one-to-one correspondence between the extremal rays of $\overline{\mathrm{NE}}(S_i)$'s. Then $\varphi_*(\mathbb{R}_{\ge 0}[f_1]) = \mathbb{R}_{\ge 0}[f_2]$, since there exists a finite morphism $h: C_1 \to C_2$ such that $\pi_2 \circ \varphi = h \circ \pi_1$ by the universality of the Albanese map. Hence $\varphi_*(\mathbb{R}_{\ge 0}[s_1]) = \mathbb{R}_{\ge 0}[s_2]$. Since $\ell \in \mathrm{Pic}^0(C_1)$ is of finite order, there exists a finite étale covering $\widetilde{S_1}$ of S_1 which is isomorphic to the trivial $\mathbb{P}^1$-bundle $C_1 \times \mathbb{P}^1 \to C_1$. Thus all the irreducible curves Γ on S_1 which are numerically equivalent to s_1 form a positive-dimensional algebraic family. Let s_2' be another section of π_2 corresponding to a surjection $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$. Since $\mathcal{L} \in \mathrm{Pic}(C_2)$ is of infinite order, s_2' is the unique irreducible curve which is numerically equivalent to s_2. (The reason is as follows: Let $D(\neq s_2, s_2')$ be another irreducible curve such that $D \equiv s_2 \equiv s_2'$. Since $(D, s_2) = (D, s_2') = 0$, D, s_2 and s_2' are three disjoint sections of π_2. Hence $\pi_2 : S_2 \to C_2$ is a trivial $\mathbb{P}^1$-bundle and a contradiction is derived.) Hence $\varphi(\Gamma) = s_2$ or s_2'. Since Γ moves and sweeps out S_1, $\varphi(S_1) \subset s_2 \cup s_2'$ and φ is not surjective. Thus a contradiction.

(2) Suppose the contrary that there exists a surjective morphism $\psi: S_2 \twoheadrightarrow S_1$. Then, by Proposition 3.1, the pull-back map ψ^* gives an isomorphism $\psi^* : \overline{\mathrm{NE}}(S_1) \cong \overline{\mathrm{NE}}(S_2)$ of the Kleiman-Mori cone. By the similar method as in (1), we can show that $\psi^{-1}(\Gamma) = s_2 \cup s_2'$ for all the irreducible curve Γ on S_1 which is numerically equivalent to s_1. Since Γ moves and sweeps out S_1, we infer that $\psi^{-1}(S_1) = s_2 \cup s_2'$. This contradicts the fact that ψ is a map.

(3) Suppose the contrary that there exists a surjective morphism $g: S_1 \twoheadrightarrow S_3$. Then, by Proposition 5.2, the canonical section s_∞ is the unique irreducible curve on S_3 which is numerically equivalent to s_∞. By a similar method as in (1), we can show that $g(\Gamma) = s_\infty$ for all the irreducible curves Γ on S_1 which are numerically equivalent to s_1. Since Γ moves and sweeps out S_1, we infer that $g(S_1) = s_\infty$. This contradicts the assumption that g is surjective.

(4) Suppose the contrary that there exists a surjective morphism $u: S_3 \twoheadrightarrow S_1$. Then, by a similar method as in (1), we can show that $u^{-1}(\Gamma) = s_\infty$ for all the irreducible curves Γ on S_1, which sweeps out S_1. Hence $u^{-1}(S_1) = s_\infty$ and thus a contradiction.

(5) Suppose the contrary that there exists a surjective morphism $v: S_2 \twoheadrightarrow S_3$. Then, by the universality of the Albanese map, there exists a finite covering $h: C_2 \to C_3$ with $\pi_3 \circ v = h \circ \pi_2$. By a similar method as in (1), we can show that $v_*(\mathbb{R}_{\ge 0}[s_2]) = \mathbb{R}_{\ge 0}([s_3])$. Then $K_{S_2} \sim v^* K_{S_3} + R_v$ for the ramification divisor R_v of v. Since $K_{S_2} \equiv -2\, s_2, K_{S_3} \equiv -2\, s_3$, we have $R_v \equiv a s_2$ for some non-negative integer $a \ge 0$ and the branched divisor B_v of v is supported on s_3 by Proposition 5.2. If $a = 0$, then v is finite étale and of degree one on each

fiber of π_2. Hence, by Lemma 4.12, there exists an isomorphism $S_2 \cong S_3 \times_{C_3} C_2 \cong S_3 = \mathbb{S}$, which derives a contradiction. If $a > 0$, then by Proposition 5.2, v is unramified outside $v^{-1}(s_3)$. By the same reason as in the proof of (1), we infer that $v^{-1}(s_3) = s_2 \sqcup s_2'$, where s_2, s_2' are disjoint two sections of π_2 correspondiong to surjections $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{O}_{C_2}$ and $\mathcal{O}_{C_2} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$, respectively. For each fiber f of π_3, $f \setminus s_3 \cong \mathbb{C}$ is simply connected. Hence the inverse image $v^{-1}(f \setminus s_3)$ is a disjoint union of copies of $\mathbb{C}$. However, by the above remark, $v^{-1}(f \setminus s_3) = v^{-1}(f) \setminus (s_2 \sqcup s_2')$ is a disjoint union of copies of $\mathbb{C}^*$. Thus a contradiction.

(6) Suppose the contrary that there exists a surjective morphism $w\colon S_3 \twoheadrightarrow S_2$. Then, by a similar method as in (4), we can show that $w^{-1}(s_2) = w^{-1}(s_2') = s_3$. This contradicts the assumption that w is a map. □

We employ the same notation as in Proposition 5.6.

Corollary 5.7. *Let C' be an elliptic curve and*

$$0 \longrightarrow \mathcal{O}_{C'} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{C'}([p]) \longrightarrow 0$$

a non-split extension for a point $p \in C'$. Let $\pi'\colon V := \mathbb{P}_{C'}(\mathcal{E}) \to C'$ be the $\mathbb{P}^1$-bundle over C' associated with the stable vector bundle $\mathcal{E}$. Then

(1) *There exist no surjective morphisms $V \twoheadrightarrow S_3$ and $S_3 \twoheadrightarrow V$.*
(2) *There exist no surjective morphisms $V \twoheadrightarrow S_2$ and $S_2 \twoheadrightarrow V$.*
(3) *There exist no finite étale coverings from V to S_1.*

Proof. (1) Suppose that there exists a surjective morphism $\varphi\colon V \twoheadrightarrow S_3$. By Proposition 4.4, there exists an étale double covering $p\colon \widetilde{V} \to V$ and an isomorphism $\widetilde{V} \cong \mathbb{P}^1 \times C''$ for some elliptic curve C''. Then, $\varphi \circ p$ is a surjective morphism from $\widetilde{V}$ to S_3, which contradicts Proposition 5.6. Conversely, suppose that there exits a surjective morphism $\psi\colon S_3 \twoheadrightarrow V$. Let $\widetilde{S_3}$ be the connected component of the fiber product $S_3 \times_V \widetilde{V}$ and $q\colon \widetilde{S_3} \to \widetilde{V}$ the natural projection. Then by Lemma 4.12, the surface $\widetilde{S_3}$ is isomorphic to the Atiyah surface $\mathbb{S}$ and this contradicts Proposition 5.6.

(2) Suppose that there exists a surjective morphism $f\colon V \twoheadrightarrow S_2$. Then, $f \circ p\colon \widetilde{V} \twoheadrightarrow S_2$ gives a surjective morphism from $\widetilde{V}$ to S_2, which contradicts Proposition 5.6. Conversely, suppose that there exists a surjective morphism $g\colon S_2 \twoheadrightarrow V$. Let $\widetilde{S_2}$ be the connected component of the fiber product $S_2 \times_V \widetilde{V}$ and $q\colon \widetilde{S_2} \to \widetilde{V} \simeq \mathbb{P}^1 \times C''$ the natural projection. Then there exist an elliptic curve D and an isomorphism $\widetilde{S_2} \simeq \mathbb{P}_D(\mathcal{O}_D \oplus \ell')$ for some non-torsion line bundle $\ell' \in \mathrm{Pic}^0(D)$. Hence a contradiction is derived by Proosition 5.6.

(3) Suppose that there exists a finite étale covering $u\colon V \to S_1$. Then there exists a finite morphism $h\colon C' \to C_1$ with $\pi_1 \circ u = h \circ \pi'$. Since u is étale, u is of degree one restricted to each fiber of π' and there exists an isomorphism $V \simeq S_1 \times_{C_1} C' \simeq \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus h^*\ell)$. Since $\mathcal{E}$ is indecomposable, we have a contradiction. □

REMARK 5.8. Corollary 5.7 (3) does not necessarily hold without the assumption that the morphism is étale. In fact, using Proposition 4.4, we can construct an example of a finite ramified covering $\rho\colon V \to S_1$ of degree 4.

There is also an analogous statement for elliptic ruled surfaces which are associated to unstable vector bundles of rank 2 on elliptic curves.

Lemma 5.9. *Let $\pi\colon \mathbb{S} \to C$ be the Atiyah surface and $T := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \ell)$ a $\mathbb{P}^1$-bundle, where ℓ is a line bundle on an elliptic curve C' with $\deg \ell \neq 0$. Furthermore, set $U := \mathbb{P}_{C''}(\mathcal{O}_{C''} \oplus \mathcal{L})$ for a torsion line bundle $\mathcal{L} \in \mathrm{Pic}^0(C'')$ on an elliptic curve C''. Then the followings hold:*

(1) *There exist no surjective morphisms $\mathbb{S} \twoheadrightarrow T$ and $T \twoheadrightarrow \mathbb{S}$.*

(2) *There exist no surjective morphisms $\mathbb{S} \twoheadrightarrow U$ and $U \twoheadrightarrow \mathbb{S}$.*

Proof. Since the proof of (2) is completely analogue to that of (1), we shall only give a proof of (1). Let $\Delta(\subset T)$ be the section of $\pi_T : T \to C$ corresponding to the second proection $\mathcal{O}_C \oplus \ell \twoheadrightarrow \ell$ and F a general fiber of π_T. It is well-known that $\rho(T) = 2$ and the Kleiman-Mori cone $\overline{\mathrm{NE}}(T)$ is a 2-dimensional, closed convex polyhedral cone spanned by the numerical equivalence classes $[F]$ and $[\Delta]$; i.e., $\overline{\mathrm{NE}}(T) = \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[\Delta]$.

(1) Suppose that there exists a surjective morphism $\varphi\colon \mathbb{S} \twoheadrightarrow T$. Since $\rho(\mathbb{S}) = \rho(T) = 2$, by Proposition 3.1, there is induced an isomorphism $\varphi^*\colon \overline{\mathrm{NE}}(T) \simeq \overline{\mathrm{NE}}(\mathbb{S})$ by the pull-back map $\varphi^*\colon N^1(T) \to N^1(\mathbb{S})$. Hence there exists a one-to-one correspondence between the extremal rays of $\overline{\mathrm{NE}}(T)$ and $\overline{\mathrm{NE}}(\mathbb{S})$. Then $\varphi^*(\mathbb{R}_{\geq 0}[F]) = \mathbb{R}_{\geq 0}[f]$, since there exists a finite morphism $h\colon C \to C'$ such that $\pi_T \circ \varphi = h \circ \pi_S$. Hence $\varphi^*(\mathbb{R}_{\geq 0}[F]) = \mathbb{R}_{\geq 0}[f]$. Thus $\varphi^*\Delta \equiv a s_\infty$ for some $a > 0$. Now we calculate the self-intersection number of both sides. We have $(\varphi^*\Delta)^2 = \deg \varphi \cdot \Delta^2 = \deg \varphi \cdot \deg \ell \neq 0$, which contradicts $s_\infty^2 = 0$. □

Applying these results, we shall classify $\mathbb{P}^1$-bundles over elliptic curves admitting an ESP (cf. Definition 2.3).

Proposition 5.10. *Suppose that there exists an ESP $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ of elliptic ruled surfaces S_n. Then, one of the following cases occurs:*

(1) *There exists an integer $k \leq \infty$ such that*
   (a) *For any $i \leq k$, $S_i \simeq \mathbb{P}_{C_i}(\mathcal{O}_{C_i} \oplus \ell_i)$ for some torsion line bundle $\ell_i \in \mathrm{Pic}^0(C_i)$ on an elliptic curve C_i.*
   (b) *For any $i > k$, $S_i \simeq \mathbb{P}_{C_i}(\mathcal{E}_i)$, where $\mathcal{E}_i$ is a stable vector bundle of rank two and degree one on an elliptic curve C_i.*

(2) *For any i, $S_i \simeq \mathbb{P}_{C_i}(\mathcal{O}_{C_i} \oplus \mathcal{L}_i)$ for some non-torsion line bundle $\mathcal{L}_i$ of degree 0 on an elliptic curve C_i.*

(3) *S_i is isomorphic to the Atiyah surface $\mathbb{S}$ for any i.*

Proof. By Proposition 4.1, any S_n is a $\mathbb{P}^1$-bundle associated to a semi-stable vector bundle of rank two on an elliptic curve. Hence all S_n is isomorphic to one of the following 4 types:

- type (a): $\mathbb{P}_{C_n}(\mathcal{E}_n)$, where $\mathcal{E}_n$ is a stable vector bundle of rank two and degree one on an elliptic curve C_n.
- type (b): the Atiyah surface $\mathbb{S}$.
- type (c): $\mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \ell_n)$, where $\ell_n \in \mathrm{Pic}^0(C_n)$ is a torsion line bundle on an elliptic curve C_n.
- type (d): $\mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$, where $\mathcal{L}_n \in \mathrm{Pic}^0(C_n)$ is a line bundle of infinite order on an elliptic curve C_n.

By considering a truncated sequence $S_n \xrightarrow{g_n} S_{n+1} \xrightarrow{g_{n+1}} \cdots$ starting from S_n for each n, it is sufficient to show the claim for $i \geq 0$.

(1) Suppose that S_0 is of type (a). Then, by Proposition 5.6 and Corollary 5.7, S_n is of

type (a) for any $n > 0$.

(2) Suppose that S_0 is of type (b). Then, by Proposition 5.6 and Corollary 5.7, S_n is of type (b) for any $n > 0$.

(3) Suppose that S_0 is of type (c). Then, the claim (1) holds by Proposition 5.6 and Corollary 5.7,

(4) Suppose that S_0 is of type (d). Then, by Proposition 5.6 and Corollary 5.7, S_n is of type (d) for any $n > 0$. □

Now, we devote the rest of this section to describe the structure of the set of the surjective morphisms between certain elliptic ruled surfaces; those which are associated to decomposable vector bundles of rank 2 and degree 0 on elliptic curves and with only two disjoint sections. First, we start with an auxiliary result concerning morphisms of elliptic curves.

Lemma 5.11. *Let $h, g\colon C \twoheadrightarrow C'$ be two surjective morphisms between elliptic curves C and C'. Suppose that $h^*\ell \simeq g^*\ell$ for a line bundle ℓ of degree zero on C', where $\ell \in \mathrm{Pic}^0(C')$ is of infinite order. Then $h = g$ up to translation under the group law of the elliptic curve C'.*

Proof. Since $\ell \in \mathrm{Pic}^0(C')$, ℓ is invariant under translation (cf. [32]). Since any surjective morphism from C to C' can be expressed as a composite of a group homomorphism and a translation, we may assume that both g and h are group homomorphisms. If $h - g\colon C \to C'$ is surjective, then Proposition 4.7 implies that ℓ is of finite order, which contradicts the assumption. □

Let C (resp. C') be an elliptic curve and ℓ (resp. $\mathcal{L}$) a line bundle of degree zero on C (resp. C') which is of infinite order in $\mathrm{Pic}^0(C)$ (resp. $\mathrm{Pic}^0(C')$). Let $\pi_S\colon S := \mathbb{P}_C(\mathcal{O}_C \oplus \ell) \to C$ (resp. $\pi_T\colon T := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}) \to C'$) be the $\mathbb{P}^1$-bundle over C (resp. C') associated to a decomposable vector bundle of rank 2 and degree 0. We describe the structure of the set $\mathrm{Mor}_{surj}(S, T)$ consisting of surjective morphism $\varphi\colon S \twoheadrightarrow T$ from S to T. Let s and s' (resp. t and t') be the sections of π_S (resp. π_T) C corresponding to the first and second projections $\mathcal{O}_C \oplus \ell \twoheadrightarrow \ell$ and $\mathcal{O}_C \oplus \ell \twoheadrightarrow \mathcal{O}_C$ (resp. $\mathcal{O}_{C'} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$ and $\mathcal{O}_{C'} \oplus \mathcal{L} \twoheadrightarrow \mathcal{O}_{C'}$). By the universality of the Albanese map, there exists a finite morphism $h\colon C \to C'$ such that $h \circ \pi_S = \pi_T \circ \varphi$.

Since $\rho(S) = \rho(T) = 2$, Proposition 3.1 implies that φ is a finite morphism and the push-forward map $\varphi_*\colon N_1(S) \simeq N_1(T)$ (resp. the pull-back map $\varphi^*\colon N^1(T) \simeq N^1(S)$) gives a one-to-one correspondence between $\overline{\mathrm{NE}}(S)$ and $\overline{\mathrm{NE}}(T)$. They are two-dimensional closed convex polyhedral cones spanned by the numerical equivalence classes of a general fiber of π_S (resp. π_T) and s (resp. t). Then, applying the same method as in the proof of Proposition 5.6, we see that $\varphi^{-1}(t) \subset s \cup s'$ and $\varphi^{-1}(t') \subset s \cup s'$. Since φ is a surjective morphism, one of the following cases occur:

- $\varphi^{-1}(t) = s$ and $\varphi^{-1}(t') = s'$ or
- $\varphi^{-1}(t) = s'$ and $\varphi^{-1}(t') = s$.

In the former (resp. latter) case, we have $\varphi^* t \sim \alpha s$ (resp. $\varphi^* t \sim \beta s$) for some positive integer α (resp. β). Since $\mathcal{N}_{s/S} \simeq \ell, \mathcal{N}_{s'/S} \simeq \ell^{\otimes -1}$, and $\mathcal{N}_{t/T} \simeq \mathcal{L}, \mathcal{N}_{t'/T} \simeq \mathcal{L}^{\otimes -1}$, it follows that $h^*\mathcal{L} \simeq \ell^{\otimes n}$ in $\mathrm{Pic}^0(C)$ for some integer n in both cases. Such an integer n is uniquely determined and non-zero, since both $\ell \in \mathrm{Pic}^0 C$ and $\mathcal{L} \in \mathrm{Pic}^0 C'$ are of infinite order. Thus, according to the discussions made as above, we have the following description of h in terms of line bundles $\mathcal{L}$ and ℓ:

Lemma 5.12. *Let S, T and h as above. Then, $h^*\mathcal{L} \simeq \ell^{\otimes n}$ for a unique non-zero integer n.*

By replacing ℓ by ℓ^{-1}, we may assume that $n > 0$. Let $\widetilde{T} := T \times_{C'} C \to C$ be the pull-back of $\pi_T\colon T \to C'$ by $h\colon C \to C'$. Then by construction, there exists an isomorphism

$$\widetilde{T} \simeq \mathbb{P}_C(\mathcal{O}_C \oplus h^*\mathcal{L}) \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell^{\otimes n})$$

over C. By the universality of the fiber product, there exists a natural morphism $\psi\colon S \to \widetilde{T}$ over C so that the following commutative diagram holds:

$$\begin{array}{ccccc} S & \xrightarrow{\psi} & \widetilde{T} & \xrightarrow{\rho} & T \\ {\scriptstyle \pi_S}\downarrow & & {\scriptstyle \pi_{\widetilde{T}}}\downarrow & & {\scriptstyle \pi_T}\downarrow \\ C & \xrightarrow{\mathrm{id}_C} & C & \xrightarrow{h} & C' \end{array}$$

Clearly, the natural morphism $\rho\colon \widetilde{T} \to T$ is a finite étale covering.

Lemma 5.13. *If $n > 1$, then the map ψ is of degree n on each fiber of π_S and ramifies only along the two disjoint sections s, s' of π_S with ramification index n.*

Proof. Since $n > 1$, the finite morphism ψ is a ramified covering of degree n. Note that $K_S \sim -2\,s + \pi_S^*\ell, K_{\widetilde{T}} \sim -2\widetilde{t} + \pi_{\widetilde{T}}^*(\ell^{\otimes n})$ and $\psi^*\widetilde{t} \equiv k\,s$ for some positive integer k. Since $K_S \sim \psi^* K_{\widetilde{T}} + R_\psi$ for the ramification divisor R_ψ of ψ, we see that $R_\psi \equiv \beta\,s$ for some integer $\beta \geq 0$. Applying the same argument as in the proof of Proposition 5.6, R_ψ is supported on $s \bigsqcup s'$, which is a union of two disjoint sections of π_S. □

REMARK 5.14. Conversely, we show that if $h^*\mathcal{L} \simeq \ell^{\otimes n}$, then there exists a finite surjective morphism $\varphi\colon S \to T$. Since there exists a natural inclusion

$$\mathcal{O}_C \oplus h^*\mathcal{L} \hookrightarrow \mathcal{O}_C \oplus \ell \oplus \cdots \oplus \ell^n = \mathrm{Sym}^n(\mathcal{O}_C \oplus \ell)\,,$$

there exists a natural projection $p\colon \mathbb{P}_C(\mathrm{Sym}^n(\mathcal{O}_C \oplus \ell)) \dashrightarrow \widetilde{T} := \mathbb{P}_C(\mathcal{O}_C \oplus h^*\,\mathcal{L})$. Moreover, there exists a Veronese embedding

$$i\colon S := \mathbb{P}_C(\mathcal{O}_C \oplus \ell) \hookrightarrow \mathbb{P}_C(\mathrm{Sym}^n(\mathcal{O}_C \oplus \ell))\,.$$

Then the composite map $\rho \circ p \circ i\colon S \to T$ gives a finite surjective morphism from S to T.

Now, we define the set M as follows:

$$M := \{(n, v) \in \mathbb{Z} \times \mathrm{Hom}_{\mathrm{group}}(C, C') | v^*\mathcal{L} \simeq \ell^{\otimes n}\}\,.$$

Here by $\mathrm{Hom}_{\mathrm{group}}(C, C')$, we denote the set of group homomorphisms from C to C'. Then $M \neq \emptyset$, since $(0, 0) \in M$. Now the structure of the set M is described as follows:

Proposition 5.15. *We may endow M with the structure of an abelian group such that*

(1) *the first projection $p\colon M \to \mathbb{Z}$ is an injective group homomorphism.*

(2) *There exists a unique element $(d_0, v_0) \in M$ such that any $(n, v) \in M$ can be expressed as $n = d_0 k$ and $v = \mu_k \circ v_0 = v_0 \circ \mu_k$ for some integer $k \in \mathbb{Z}$, where μ_k denotes multiplication map by k on an elliptic curve.*

Proof. First, we show that M is an abelian group.

(1) Let $(n,v),(m,u) \in M$ be arbitrary elements. Then $v^*\mathcal{L} \simeq \ell^{\otimes n}, u^* \simeq \ell^{\otimes m}$. Since $(v+u)^*\mathcal{L} \simeq v^*\mathcal{L} \otimes u^*\mathcal{L}$ by [32, p.75, (ii)] , we infer that $(n+m, v+u) \in M$. Thus M is closed under multiplication.
(2) The unit element of M is $(0,0)$.
(3) The inverse element of $(n,v) \in M$ is $(-n,-v)$.

Clearly, p is a group homomorphism. Next we show that p is injective. Suppose that $(n,v),(n,u) \in M$ for some integer $n \in \mathbb{Z}$ and group homomorphisms $v,u\colon C \to C'$. Then $v^*\mathcal{L} \simeq \ell^{\otimes n} \simeq u^*\mathcal{L}$. By Lemma 5.11, we infer that $v = u$. Thus p is injective and $M \simeq p(M)(\subset \mathbb{Z})$ is a cyclic group. Let $(d_0, v_0) \in M$ (where $d_0 > 0$) be its generator. Then it satisfies the required property. □

The following describes the structure of a finite surjective morphism $\varphi\colon S \twoheadrightarrow T$ and is a key ingredient for the proof of Theorem 10.1.

Proposition 5.16. *Let ℓ (resp. $\mathcal{L}$) be a line bundle of degree zero on an elliptic curve C (resp. C') and $\pi_S : S := \mathbb{P}_C(\mathcal{O}_C \oplus \ell) \to C$ (resp. $\pi_T : T := \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}) \to C'$) an associated $\mathbb{P}^1$-bundle. Suppose that both $\ell \in \operatorname{Pic}^0(C)$ and $\mathcal{L} \in \operatorname{Pic}^0(C')$ are of infinite order and there exists some finite surjective morphism from S to T. Then there exists a unique non-zero group homomorphism $v_0\colon C \to C'$ such that $v_0^*\mathcal{L} \simeq \ell^{\otimes d_0}$ for an integer d_0 and that the the following properties hold:*

(1) *Let $\varphi\colon S \to T$ be an arbitrary finite surjective morphism and $h\colon C \to C'$ the finite morphism induced by the universality of the Albanese map. Then $h^*\mathcal{L} \simeq \ell^{\otimes n}$ for a unique integer n which is a multiple of d_0. Furthermore, there exists multiplication mapping $\mu_k\colon C \to C$ by a positive integer $k := n\,(d_0)^{-1}$ such that $h = v_0 \circ \mu_k$ up to translation under the group structure of C'.*
(2) *Let $\pi_{\widetilde{T}}\colon \widetilde{T} := T \times_{C'} C \to C$ be the $\mathbb{P}^1$-bundle obtained by the pull-back of $\pi_T : T \to C'$ by $h\colon C \to C'$. Let $\psi\colon S \to \widetilde{T}$ be the natural morphism. Then the map ψ is of degree n restricted to each fiber of π_S and ramifies only along the two disjoint sections of π_S with ramification index n. Moreover, for a general fiber F of π_T, the inverse image $\varphi^{-1}(F)$ has at least $k^2 = \deg h \cdot (\deg v_0)^{-1}$ connected components.*

$$\begin{array}{ccccccc}
S & \xrightarrow{\psi} & \widetilde{T} & \longrightarrow & \widetilde{T}' & \longrightarrow & T \\
{\scriptstyle \pi_S}\downarrow & & {\scriptstyle \pi_{\widetilde{T}}}\downarrow & & {\scriptstyle \pi_{\widetilde{T}'}}\downarrow & & {\scriptstyle \pi_T}\downarrow \\
C & = & C & \xrightarrow[\mu_k]{} & C & \xrightarrow[v_0]{} & C'
\end{array}$$

Proof. The finite morphism $h\colon C \to C'$ can be uniquely expressed as $h = T_\alpha \circ v$, where $T_\alpha\colon C' \simeq C'$ is a translation by $\alpha \in C'$ and $v\colon C \to C'$ is a Lie group homomorphism. Since $T_\alpha^*\mathcal{L} \simeq \mathcal{L}$, there exists an isomorphism:

$$T \times_{C',T_\alpha} C' = \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus T_\alpha^*\mathcal{L}) \simeq \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}) = T\ .$$

Hence we may assume that $\alpha = 0$ and $h\colon C \to C'$ is a group homomorphism. By Lemma 5.12, $h^*\mathcal{L} \simeq \ell^n$ for a unique integer n. By Proposition 5.15, there exists a unique integer k such that $n = d_0\,k$ and $v = \mu_k \circ v_0 = v_0 \circ \mu_k$. Thus the assertion (1) has been proved. Since $v_0^*\mathcal{L} \simeq \ell^{\otimes d_0}$, if we put $\widetilde{T}' := T \times_{C',v_0} C$, then $\widetilde{T}' \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell^{\otimes d_0})$. Since $\mu_k^*\ell \simeq \ell^{\otimes k}$, we see that $\upsilon^*\mathcal{L} \simeq \mu_k^*(\ell^{\otimes d_0}) \simeq \ell^{\otimes n}$. Hence, $\widetilde{T} := T \times_{C,\upsilon} C'$ is isomorphic to the fiber product $\widetilde{T}' \times_{C,\mu_k} C$. The natural morphism $\psi\colon S \to \widetilde{T}$ has already been described as above. Note that $\deg h = k^2 \cdot \deg \upsilon_0$ and the morphism μ_k is unramified. Hence for a general fiber F of π_T, the inverse image $\varphi^{-1}(F)$ consists of at least $k^2 = \deg \mu_k = \deg \upsilon \cdot (\deg \upsilon_0)^{-1}$ connected components. Thus the assertion (2) has been proved and we are done. □

6. ESPs of smooth algebraic surfaces

In this section, we study the structure of ESPs of smooth algebraic surfaces (cf. Propositions 6.5, 6.6 and 6.7). These results are necessary in Section 7 when we study the structure of an FESP of 3-folds with negative Kodaira dimension. Proposition 6.8 is a slight generalization of [9, Proposition 4.9]. Propositions 6.8 and 6.9 are devoted to the study of an ESP of curves whose ambient varieties also form an ESP. Let $Y_\bullet$ be an FESP constructed from a given ESP $X_\bullet$ of 3-folds by a sequence of blowing-downs of an ESP. Proposition 6.9 is quite useful when we seek out elliptic curves for the candidates of the centers of Cartesian blowing-ups $\pi_\bullet\colon X_\bullet \to Y_\bullet$ (cf. Corollary 1.2).

First, we recall some basic facts about elliptic fibrations and abelian fibrations.

Definition 6.1. (cf. [13, Definition 2.3 and Lemma 2.4]) Let $V \to S$ be a projective fiber space from a smooth variety V to a normal variety S whose general fibers are abelian varieties. It is called a *Seifert* abelian fiber space if there exist finite surjective morphisms $W \to V$ and $T \to S$ satisfying the following conditions:

(1) W and T are smooth varieties;
(2) W is isomorphic to the normalization of $V \times_S T$ over T;
(3) $W \to V$ is étale;
(4) $W \to T$ is smooth.

If $V \to S$ is a Seifert abelian fiber space, then V is a unique relative minimal model over S, since K_V is relatively numerically trivial and there are no rational curves contained in fibers. If S is compact and $\dim(V) = \dim(S)+1$, then we may replace the condition (4) with that $W \simeq E\times T$ over T for an elliptic curve E. The notion of Seifert abelian fiber space was first introduced in [35] as the name of Q-smooth abelian fibration.

The following proposition reveals a new phenomenon; There exists a non-isomorphic étale endomorphism $f\colon X \to X$ of a 3-fold X with $\kappa(X) = -\infty$ which admits no Seifert elliptic fibrations.

Proposition 6.2. *Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Assume that there exists an extremal ray R of divisorial type such that the exceptional divisor D of the contraction morphism* Cont_R *of R is isomorphic to the Atiyah surface $\mathbb{S}$. Then there exists on X no Seifert ellptic fiber space structure.*

Proof. The proof is by contradiction. Suppose the contrary that there exists on X a Seifert elliptic fiber space structure $\varphi\colon X \to S$ over a surface S. Then, a suitable finite finite étale Galois cover $p\colon X' \to X$ of X is isomorphic to the direct product $T\times E$ of a smooth surface T and an elliptic curve E. From Lemma 3.12, there exists an extremal ray R' of divisorial type on X' such that $p_*R' = R$. Let D' be the exceptional divisor of the contraction morphism $\mathrm{cont}_{R'}$. Then applying Proposition 5.6 and Lemma 5.9 to the finite étale covering $p\colon D' \to D$, we see that $D' \simeq \mathbb{S}$.

Let $q\colon X' \to T$ be the first projection. Then by applying [11, Theorem 2], we infer that the pushfoward map $q_*\colon \overline{\mathrm{NE}}(X') \to \overline{\mathrm{NE}}(T)$ induces a one-to-one correspondence between the set of extremal rays of X' and the set of extremal rays of T. Furthermore, let C be a (-1)-curve on the surface T whose numerical class $[C]$ spans the extremal ray $\overline{R} := q_*R$ of T. Then $D' = C \times E$. On the other hand, since $D' \simeq \mathbb{S}$, there exists no elliptic fiber space structure on D' by Corollary 5.3. Thus a contradiction. □

The following lemma may be well-known for experts and was stated implicitely in [9, Proposition 7.1]. We shall give a proof for reader's convenience.

Lemma 6.3. (1) *Let $\varphi\colon S \to \mathbb{P}^1$ be a minimal elliptic surface with $\kappa(S) = 1$ and $\chi(\mathcal{O}_S) = 0$. Then the canonical divisor K_S of S can be expressed as $K_S \sim_{\mathbb{Q}} \varphi^*D$ for some $\mathbb{Q}$-divisor D on $\mathbb{P}^1$ with* $\deg D \geq 1/42$.

(2) *Let $\psi\colon T \to C$ be a minimal elliptic surface over an elliptic curve C such that $\kappa(T) = 1$ and $\chi(\mathcal{O}_S) = 0$. Then $K_T \equiv \psi^*D$ for some $\mathbb{Q}$-divisor D on C with* $\deg D \geq 1/2$.

Proof. (1) By assumption, $\varphi\colon S \to \mathbb{P}^1$ is a Seifert elliptic surface and has no singular fibers except multiple fibers m_iE_i $(i = 1, 2, \cdots, r)$ supported on elliptic curves E_i. Then, by the canonical bundle formula of Kodaira [18, Theorem 12], we have $K_S \sim_{\mathbb{Q}} \varphi^*D$ for a $\mathbb{Q}$-divisor D of degree $A := -2 + \sum_{i=1}^{r} m_i^{-1}(m_i - 1)$ on $\mathbb{P}^1$. Then $A > 0$, since $\kappa(S) = 1$. Hence, $r \geq 3$. Then by an easy calculation, we see that $A \geq 1/42$. Thus we are done.

(2) As in (1), $\psi\colon T \to C$ is a Seifert elliptic surface which has only multiple singular fibers m_iE_i $(1 \leq i \leq r, m_i \geq 2)$ of type ${}_mI_0$. Then, by the canonical bundle formula of Kodaira, we have $K_T \sim \psi^*(-\delta) + \sum_{i=1}^{r}(m_i - 1)E_i$ for a divisor δ on C with $\deg \delta = 0$. Hence, $K_T \equiv \psi^*D$ for an effective $\mathbb{Q}$-divisor $D := \sum_{i=1}^{r} m_i^{-1}(m_i - 1)P_i$ on C for $P_i := \varphi(E_i)$. Since $\kappa(S) = 1$, we have $r \geq 1$ and $\deg D \geq 1/2$. □

Now, we shall study the structure of ESPs of smooth algebraic surfaces.

Proposition 6.4. *Let $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ be an ESP of smooth algebraic surfaces S_n. Then each surface S_n satisfies the following conditions.*

- $\chi(\mathcal{O}_{S_n}) = \chi_{\mathrm{top}}(S_n) = 0$.
- $K_{S_n}^2 = 0$.
- *There exist no (-1)- curves on S_n.*
- *S_n is not of general type.*

Proof. The first and the second assertion follow immediately from Lemma 2.1. Now we shall prove a non-existence of (-1)-curves on each S_n. Suppose that there exists a (-1)-curve e_n on S_n. Then $e_{n+1} := g_n(e_n)$ is also a (-1)-curve on S_{n+1}, since g_n is étale. Since $\deg g_n > 1$ and $e_{n+1} \simeq \mathbb{P}^1$ is simply connected, ${g_n}^*(e_{n+1})$ is a disjoint union of more than one (-1)-curves on S_n. This contradicts Proposition 3.1 (3). The last assertion is clear, since ${K_{S_n}}^2 = 0$ and there exist no (-1)-curves on S_n. □

Proposition 6.5. *Let $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ be an ESP of smooth algebraic surfaces S_n with $\kappa(S_n) = 1$. Then, there exists a Cartesian diagram*

$$\begin{CD} \cdots @>>> S_n @>g_n>> S_{n+1} @>>> \cdots \\ @. @V\varphi_n VV @VV\varphi_{n+1}V @. \\ \cdots @>>> C_n @>>h_n> C_{n+1} @>>> \cdots \end{CD}$$

satisfying the following properties below;

- *For each n, the Iitaka fibration $\varphi_n\colon S_n \to C_n$ of S_n gives S_n a Seifert elliptic surface structure over the curve C_n and*
- *$h_n\colon C_n \to C_{n+1}$ is an isomorphism for a sufficiently large positive integer n.*

Proof. Since $\chi(\mathcal{O}_{S_n}) = 0$ and there exist no (-1)-curves on S_n, the Iitaka fibration $\varphi_n\colon S_n \to C_n$ gives a Seifert elliptic surface structure, that is, φ_n has at most multiple singular fibers of type ${}_mI_0$ in the sense of Kodaira [18]. Thus $K_{S_n} \sim_{\mathbb{Q}} \varphi_n^* H_n$ for an ample $\mathbb{Q}$-divisor H_n on C_n. Let F_n be a general fiber of φ_n for each n. Since

$$(K_{S_{n+1}}, (g_n)_* F_n) = (g_n^* K_{S_{n+1}}, F_n) = (K_{S_n}, F_n) = 0,$$

$\varphi_{n+1} \circ g_n(F_n)$ is a point on C_n. Hence by the *rigidity lemma* (cf. [21]), there exists a finite morphism $h_n\colon C_n \to C_{n+1}$ such that $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$ for all n. Then we show that h_n is an isomorphism for a sufficiently large positive integer n. Since the genus $p_a(C_n)$ of the curve C_n decreases as n increases, there exist constants $g \geq 0$ and $k > 0$ such that $p_a(C_n) = g$ for all $n \geq k$. If $g \geq 2$, then h_n $(n \geq k)$ is an isomorphism by Riemann-Hurwitz formula.

Next, suppose that $g = 0$. Then, from Lemma 6.3, we infer that for any n, $K_{S_n} \sim_{\mathbb{Q}} \varphi_n^*(H_n)$ for a $\mathbb{Q}$-divisor H_n on C_n with $\deg H_n \geq 1/42$. Since $K_{S_0} \sim_{\mathbb{Q}} (g_{n-1} \circ \cdots \circ g_0)^* K_{S_n}$, we have $H_0 \sim_{\mathbb{Q}} (h_{n-1} \circ \cdots \circ h_0)^* H_n$. Hence, $\deg H_0 = \deg H_n \cdot \prod_{j=0}^{n-1} \deg h_j$. Assume that $\deg h_m \geq 2$ for infinitely many positive integers m. Then we have $\deg H_n \to 0$ as $n \to \infty$. This contradicts the fact that $\deg H_n \geq 1/42$ for any n. Hence h_n is an isomorphism for $n \gg 0$.

Finally, suppose that $g = 1$. Then, by Lemma 6.3, $K_{S_n} \equiv \varphi_n^* H_n$ for some $\mathbb{Q}$-divisor H_n with $\deg H_n \geq 1/2$. Hence, if we replace $\mathbb{Q}$-linear equivalence relation $\sim_{\mathbb{Q}}$ by numerically equivalence relation $\equiv$, completely the same method as in the case of $g = 0$ works in our situation. Thus, h_n is an isomorphism for $n \gg 0$. □

Proposition 6.6. *Let $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ be an ESP of smooth algebraic surfaces S_n with $\kappa(S_n) = 0$. Then, there exists an integer $k \geq 0$ such that S_n is isomorphic to an abelian surface (resp. a hyperelliptic surface) for all $n \leq k$ (resp. for all $n > k$).*

Proof. Since $\chi(\mathcal{O}_{S_n}) = 0$ and S_n is minimal, it follows from the classification theory of algebraic surfaces that S_n is either an abelian surface or a hyperelliptic surface. Since $\rho(S_n) \geq \rho(S_{n+1})$, the claim follows. □

Proposition 6.7. *Let $S_\bullet = (g_n\colon S_n \to S_{n+1})_n$ be an ESP of smooth algebraic surfaces S_n with $\kappa(S_n) = -\infty$. Then, there exist an ESP $C_\bullet = (h_n\colon C_n \to C_{n+1})_n$ of elliptic curves C_n and a Cartesian morphism $\alpha_\bullet = (\alpha_n)_n\colon S_\bullet \to C_\bullet$ such that S_n is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}_{C_n}(\mathcal{E}_n)$ associated with a rank two semi-stable vector bundle $\mathcal{E}_n$ on C_n.*

Proof. Since there exist no (-1)-curves on S_n, each S_n is either $\mathbb{P}^2$ or a ruled surface. Since g_n a non-isomorphic étale covering, S_n is not simply connected. Hence, $S_n \neq \mathbb{P}^2$ for any n. Since $\chi(\mathcal{O}_{S_n}) = 0$ and $p_g(S_n) = 0$, $q(S_n) = 1$ and S_n is an elliptic ruled surface. Thus the Albanese map $\alpha_n \colon S_n \to C_n$ is a $\mathbb{P}^1$-bundle over an elliptic curve C_n, which is associated to a semi-stable vector bundle $\mathcal{E}_n$ of rank 2 on C_n by Proposition 4.1. By the universality of the Albanese map, there esists a finite morphism $h_n \colon C_n \to C_{n+1}$ such that $\alpha_{n+1} \circ g_n = h_n \circ \alpha_n$. Each h_n is étale, since C_n is an elliptic curve. Since g_n is étale and $\mathbb{P}^1$ is simply connected, g_n is of degree one on each fiber of α_n. Thus the natural morphism $S_n \to S_{n+1} \times_{C_{n+1}} C_n$ is an isomorphism and $\deg h_n = \deg g_n \geq 2$. □

When we blow-up an FESP along elliptic curves to recover the original étale endomorphism $f \colon X \to X$, the following propositions are frequently used for analyzing blowing-up centers (cf. Proposition 7.8).

Proposition 6.8. *Let $Z_\bullet = (v_n \colon Z_n \to Z_{n+1})_n$ be an ESP of smooth projective varieties Z_n of dimension > 1. Suppose that $v_n^{-1}(C_{n+1}) = C_n$ for each irreducible curve C_n on Z_n. Then, for any n, C_n is an elliptic curve, $(K_{Z_n}, C_n) = 0$, and its conormal bundle $\mathcal{N}^\vee_{C_n/Z_n}$ is a semi-stable vector bundle of degree* 0.

Proof. By considering the truncated sequence $Z_n \xrightarrow{v_n} Z_{n+1} \xrightarrow{v_{n+1}} \cdots$ starting from Z_n for each n, it is sufficient to prove the assertion in the case where $n = 0$.

We first prove that C_0 is non-singular. Suppose that C_0 is singular. Since v_n is étale, C_n is singular for any n. We set $w_n := v_n|_{C_n}$. Then by assumption, $\deg w_n = \deg v_n > 1$. Hence, $C_\bullet := (w_n \colon C_n \to C_{n+1})_n$ is also an ESP of irreducible curves. Thus the number of singular points of C_n is a strictly decreasing function of n. Hence C_n is non-singular for a sufficiently large positive integer n. Since $v_n^{-1}(C_{n+1}) = C_n$, this is a contradiction.

Next, we show that each C_0 is an elliptic curve. Since the arithmetic genus $p_a(C_n)$ of the curve C_n is a decreasing function of n, there exist a positive integer k and a non-negative integer q such that $p_a(C_n) = q$ for any $n \geq k$. If $q \geq 2$, then $\deg w_n = 1$ by the Rieman-Hurwitz formula applied to $w_n \colon C_n \to C_{n+1}$. This is a contradiction. Thus $q \leq 1$. If $q = 0$, then $C_n \cong \mathbb{P}^1$ for some $n \gg 0$. Since $C_{n-1} = v_n^{-1}(C_n)$ and v_n is a non-isomorphic finite étale covering, C_{n-1} is not connected. This contradicts the assumption that C_{n-1} is irreducible. Hence $p = 1$ and C_0 is an elliptic curve.

By completely the same argument as in the proof of Propositon 1.1, (4), we see that $\deg \mathcal{N}_n = 0$ for the normal bundle $\mathcal{N}_n$ of C_n in Z_n. And there is induced an ESP $D_\bullet = (D_n \to D_{n+1})_n$ of projective bundles $D_n := \mathbb{P}_{C_n}(\mathcal{N}_n^\vee)$ associated to the conormal sheaf $\mathcal{N}_n^\vee := \mathcal{H}om_{\mathcal{O}_{C_n}}(N_n, \mathcal{O}_{C_n})$. Hence, $\mathcal{N}_0$ is semi-stable by Proposition 4.1. □

Next, we consider the case where Z_n is a smooth algebraic surface.

Proposition 6.9. *Let $S_\bullet = (g_n \colon S_n \to S_{n+1})_n$ be an ESP of smooth algebraic surfaces S_n. Suppose that there exists an ESP $\gamma_\bullet = (g_n|_{\gamma_n} \colon \gamma_n \to \gamma_{n+1})_n$ of irreducible curves γ_n on S_n. Then every γ_n is an elliptic curve satisfying $(\gamma_n)^2 = 0$ and the following hold:*

(1) *If $\kappa(S_n) = 1$ for any n, then $\varphi_n(\gamma_n)$ is a point for the Iitaka fibration $\varphi_n \colon S_n \to C_n$ of S_n.*

(2) *If any S_n is an abelian surface, then there exists an elliptic fiber bundle structure*

$\varphi_n\colon S_n \to C_n$ over an elliptic curve C_n such that

- *$\varphi_n(\gamma_n)$ is a point on C_n, and*
- *$\varphi_{n+1} \circ g_n = u_n \circ \varphi_n$ for an isomorphism $u_n\colon C_n \cong C_{n+1}$.*

(3) *If S_n is a hyperelliptic surface, then one of the following cases holds;*

(3.1) *The Albanese map $\alpha_n\colon S_n \to C_n$ is an elliptic fiber bundle such that*

- *$\alpha_n(\gamma_n)$ is a point, and*
- *$\alpha_{n+1} \circ g_n = u_n \circ \alpha_n$ for an isomorphism $u_n\colon C_n \cong C_{n+1}$.*

(3.2) $\mathrm{Aut}^0(S_n)$ *is an elliptic curve and the natural projection $p_n\colon S_n \to \Gamma_n := S_n/\mathrm{Aut}^0(S_n) \cong \mathbb{P}^1$ is a Seifert elliptic surface such that*

- *there exists a finite morphism $v_n\colon \Gamma_n \to \Gamma_{n+1}$ such that $p_{n+1} \circ g_n = v_n \circ p_n$ and $p_n(\gamma_n)$ is a point, and*
- *v_n is an isomorphism or $p_n(\gamma_n)$ is a ramification point of v_n.*

(3.3)
- *The Albanese map $\alpha_n\colon S_n \to C_n$ is an elliptic fiber bundle such that $\alpha_n(\gamma_n)$ is a point,*
- *$p_{n+1}\colon S_{n+1} \to \Gamma_{n+1} \simeq \mathbb{P}^1$ is a Seifert elliptic surface such that $\alpha_{n+1}(\gamma_{n+1})$ is a point, and*
- *there exists a finite morphism $w_n\colon C_n \to \Gamma_{n+1}$ such that $p_{n+1} \circ g_n = w_n \circ \alpha_n$.*

(4) *If $\kappa(S_n) = -\infty$ for any n, then the Albanese map $\alpha_n\colon S_n \to C_n$ is a $\mathbb{P}^1$-bundle over an elliptic curve C_n associated to a semi-stable vector bundle $\mathcal{E}_n$ of rank two such that*

- *$\alpha_n(\gamma_n) = C_n$ and*
- *there exists a Cartesian morphism $\alpha_\bullet\colon \gamma_\bullet \to C_\bullet = (h_n\colon C_n \to C_{n+1})_n$.*

Furthermore, one of the following hold.

(4.1) *For any n, $S_n \simeq \mathbb{P}_{C_n}(\mathcal{E}_n)$ for a stable vector bundle $\mathcal{E}_n$ on C_n and γ_n is a multi-section of α_n.*

(4.2) *For any n, $S_n \simeq \mathbb{S}$ and γ_n is the canonical section of α_n.*

(4.3) *For any n,*

- *there exists an isomorphism $S_n \simeq \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$ for a line bundle $\mathcal{L}_n \in \mathrm{Pic}^0(C_n)$ of infinite order, and*
- *the elliptic curve γ_n coincides either of the two sections of α_n corresponding to the first projection $\mathcal{O}_{C_n} \oplus \mathcal{L}_n \twoheadrightarrow \mathcal{O}_{C_n}$ or the second projection $\mathcal{O}_{C_n} \oplus \mathcal{L}_n \twoheadrightarrow \mathcal{L}_n$.*

(4.4) *For any n, there exists an isomorphism $S_n \simeq \mathbb{P}_{C_n}(\mathcal{O}_{C_n} \oplus \mathcal{L}_n)$ for a line bundle $\mathcal{L}_n \in \mathrm{Pic}^0(C_n)$ which is of finite order.*

Proof. By Proposition 6.8, every γ_n is an elliptic curve satisfying $\gamma_n^2 = (K_{S_n}, \gamma_n) = 0$. Thus by Proposition 6.4, it is sufficient to know how the curve γ_n is located in the surface S_n. Here we use the same notation as in Proposition 6.4.

(1) Since $K_{S_n} \sim_{\mathbb{Q}} \varphi_n^* H_n$ for an ample $\mathbb{Q}$-divisor H_n on C_n, we infer that $(H_n, (\varphi_n)_*\gamma_n) = (K_{S_n}, \gamma_n) = 0$. Thus each γ_n is contained in a fiber of φ_n.

(2) Suppose that S_n is an abelian surface. Since $\gamma_n \subset S_n$ is an elliptic curve, there exists an elliptic bundle structure $\varphi_n\colon S_n \to C_n$ over an elliptic curve C_n such that $p_n := \varphi_n(\gamma_n)$ is a point. By assumption, $\varphi_{n+1} \circ g_n \circ \varphi_n^{-1}(p_n) = p_{n+1}$ is a point. Hence by the rigidity lemma, there exists a finite morphism $u_n\colon C_n \to C_{n+1}$ such that $\varphi_{n+1} \circ g_n = u_n \circ \varphi_n$. Each u_n is étale, since C_n is an elliptic curve. By assumption, $u_n^{-1}(p_{n+1}) = p_n$. Hence each u_n is an isomorphism.

(3) Suppose that any S_n is a hyperelliptic surface. We recall the following basic fact:

Fact. Let S be a hyperelliptic surface. Then

(i) S has only two non-trivial fibrations up to isomorphism.

Type (A). The Albanese map $\alpha_S : S \to \mathrm{Alb}(S)$ gives S the structure of an elliptic fiber bundle over the Albanese elliptic curve $\mathrm{Alb}(S)$.

Type (B). $E := \mathrm{Aut}^0(S)$ is an elliptic curve and the natural projection $p_S : S \to C_S := S/E \cong \mathbb{P}^1$ is called the Fujiki quotient, which is a Seifert elliptic surface.

(ii) $\rho(S) = 2$ and $\overline{\mathrm{NE}}(S) = \mathbb{R}_{\geq 0}[F_S] + \mathbb{R}_{\geq 0}[G_S]$, where F_S (resp. G_S) is the general fiber of α_S (resp. p_S).

Since $\gamma_n^2 = 0$, γ_n is either a fiber of the Albanese map $\alpha_n : S_n \to C_n$ or a fiber of the Fujiki quotient $p_n : S_n \to \Gamma_n \cong \mathbb{P}^1$.

By Proposition 3.1 and the above Fact, there is a one-to-one correspondence between the extremal rays of $\overline{\mathrm{NE}}(S_n)$ and the extremal rays of $\overline{\mathrm{NE}}(S_{n+1})$ under the isomorphisms $(g_n)_*$ and $(g_n)^*$. Moreover, there exists a finite étale covering $u_n : C_n \to C_{n+1}$ such that $\alpha_{n+1} \circ g_n = u_n \circ \alpha_n$ by the universality of the Albanese map. Hence there exists a finite morphism $v_n : \Gamma_n \to \Gamma_{n+1}$ such that $p_{n+1} \circ g_n = v_n \circ p_n$. The other claims follow by the assumption that $g_n^{-1}(\gamma_{n+1}) = \gamma_n$.

(4) We set $\widetilde{\gamma}_{n+1} := \gamma_{n+1} \times_{C_{n+1}} C_n$. By the universality of the fiber product, there exists a unique morphism $\rho_n : \gamma_n \to \widetilde{\gamma}_{n+1}$ such that $p_n \circ \rho_n = g_n|_{\gamma_n}$ for the natural projection $p_n : \widetilde{\gamma}_{n+1} \to \gamma_{n+1}$. Since $g_n^{-1}(\gamma_{n+1}) = \gamma_n$, we infer that $\deg g_n|_{\gamma_n} = \deg g_n = \deg h_n$. On the other hand, $\deg p_n = \deg h_n$. Hence $\deg \rho_n = 1$ and ρ_n is an isomorphism.

Suppose that any S_n is an elliptic ruled surface. Since γ_n is an elliptic curve, $\alpha_n(\gamma_n) = C_n$. If $S_0 \simeq \mathbb{P}_{C_0}(\mathcal{F}_2)$, then by Propositions 5.10 and 5.2, $S_n \simeq \mathbb{P}_{C_n}(\mathcal{F}_2)$ for any n and γ_n is the unique section of α_n corresponding to the surjection $\mathcal{F}_2 \twoheadrightarrow \mathcal{O}_{C_n}$. For the other case, all the assertions follow from Proposition 5.10. □

We finish this section with the following result whose proof follows immediately from Proposition 4.1 and Theorem 4.5. It will be used later in the proof of Proposition 7.10:

Lemma 6.10. *Let $S_\bullet = (f_n : S_n \to S_{n+1})_n$ be an ESP of $\mathbb{P}^1$-bundles $\alpha_n : S_n \to C_n$ over an elliptic curve C_n. Let $E_i^{(n)}$ $(i = 1, 2)$ be two different elliptic curves on S_n with zero self-intersection number. Then $E_1^{(n)} \cap E_2^{(n)} = \emptyset$.*

7. Classifications of an FESP

In this section, we shall classify an FESP $(Y_\bullet, R_\bullet)$ of smooth projective 3-folds of negative Kodaira dimension. Proposition 7.3 shows that for an FESP $(Y_\bullet, R_\bullet)$, the contraction morphism of $R_\bullet$ induces a Cartesian morphism which is either a smooth del Pezzo fiber space $Y_\bullet \to C_\bullet$ over an ESP $C_\bullet$ of elliptic curves (i.e., $(Y_\bullet, R_\bullet)$ is of type (D)), or a conic bundle $Y_\bullet \to S_\bullet$ over an ESP $S_\bullet$ of smooth algebraic surfaces S_n (i.e., $(Y_\bullet, R_\bullet)$ is of type (C)). In the case where $(Y_\bullet, R_\bullet)$ is of type (C), Proposition 7.5 describes the structure of $Y_\bullet$ according as the Kodaira dimension of the base surface S_n. When we can recover the original non-isomorphic étale endomorphism $f : X \to X$ from its FESP $Y_\bullet$ by a sequence of blowing-ups $\pi_\bullet : (X, f) \to Y_\bullet$ of an ESP, Proposition 7.8 and Corollary 7.9 give precise informations on the candidate of blowing-up centers. Furthermore, Proposition 7.10 shows that $\pi_\bullet$-exceptional locus are simple normal crossings of elliptic ruled surfaces. Proposition

7.12 is devoted to the sudy of an ESP of 3-folds with two extremal rays $R_\bullet$ and $R'_\bullet$ of different type.

First, we will use the following lemma when we study a non-isomorphic finite étale covering between conic bundles over smooth surfaces.

Lemma 7.1. *Let $p\colon X \to S$ and $q\colon Y \to T$ be conic bundles over smooth projective varieties S and T respectively. Suppose that $q \circ f = g \circ p$ for a finite étale covering $f\colon X \to Y$ and a morphism $g\colon S \to T$. Then g is also a finite étale covering and the above commutative diagram is Cartesian.*

Proof. By an easy dimension count, we easily see that g is a finite morphism. Hence, $X \xrightarrow{p} S \xrightarrow{g} T$ is the Stein factorization of the composite map $q \circ f\colon X \to T$. Since each fiber of q is isomorphic to a conic in $\mathbb{P}^2$, it is simply connected. Since f is finite étale, $f^{-1}(q^{-1}(t))$ is decomposed into a disjoint union of connected components. Hence g is finite unramified. Since both S and T are non-singular, $g\colon S \to T$ is a finite étale covering. By the universality of the fiber product, there exists a unique morphism $\pi\colon X \to \widetilde{Y} := Y \times_T S$ such that $f = \pi \circ p$, where p is the natural morphism. Since f is finite étale and each fiber of p is simply connected, f is of degree one on any fiber of p and $\deg f = \deg g$. Since $\deg p = \deg g$, we infer that $\deg \pi = 1$ and π is an isomorphism. □

The following lemma gives a sufficient condition for a given ESP to be a fiber bundle.

Lemma 7.2. *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective varieties X_n. Suppose that the following conditions are satisfied:*

(1) *Each X_n has a fiber space structure $\varphi_n\colon X_n \to E_n$ over an elliptic curve E_n.*

(2) *There exists an ESP $E_\bullet = (h_n\colon E_n \to E_{n+1})_n$ of elliptic curves E_n such that $h_n \circ \varphi_n = \varphi_{n+1} \circ f_n$.*

(3) *Each general fiber of φ_n is simply connected.*

Then φ_n is a smooth morphism for any n. Furthermore, suppose that

(4) *$H^1(F_n, \Theta_{F_n}) = H^2(F_n, \Theta_{F_n}) = 0$ for a general fiber F_n of φ_n.*

Then, $\varphi_n\colon X_n \to E_n$ is a fiber bundle for any n.

Proof. First, we show that under the assumptions (1), (2) and (3), φ_n is a smooth morphism. By the universality of the Albanese map, for any n, there exists a morphism $h_n\colon E_n \to E_{n+1}$ such that $\varphi_{n+1} \circ f_n = h_n \circ \varphi_n$. Since $\deg f_n > 1$ and general fiber of φ_n is simply connected, each h_n is a non-isomorphic finite étale covering. Hence there is induced an ESP $E_\bullet = (h_n\colon E_n \to E_{n+1})_n$ such that $\varphi_\bullet = (\varphi_n)_n\colon X_\bullet \to E_\bullet$ is a Cartesian morphism. Without loss of generality, it is sufficient to show the claim for $n = 0$.

Suppose that $\varphi_0\colon X_0 \to E_0$ has a singular fiber over a point $p_0 \in E_0$. Since any f_n and g_n are finite étale, φ_n has a singular fiber over a point $h_{n-1} \circ \cdots h_0(p_0) \in E_n$ for any $n > 0$. Since $\deg h_i \geq 2$ for any i, $\varphi_0\colon X_0 \to E_0$ has a singular fiber over a dense subset $\bigcup_{n=0}^{\infty}(h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ h_0)(p_0)$ of E_0. This contradicts Sard's theorem. Hence φ_0 is smooth.

Next, under the further condition (4), we show that all the fibers of φ_0 are mutually isomorphic. We choose a point $q_0 \in E_0$ arbitrarily. Since each f_i and h_i are étale, $f_i|_{\varphi_i^{-1}(t)}\colon \varphi_i^{-1}(t) \to \varphi_{i+1}^{-1}(h_i(t))$ is an isomorphism for all $t \in E_i$. Since each fiber of φ_i is simply connected, there exists an isomorphism $\varphi_0^{-1}(t') \cong \varphi_0^{-1}(q_0)$ for any n and $t' \in (h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ$

$h_0)(q_0)$. Since $\deg(h_i) \geq 2$ for each i, the set $M := \{t \in E_0 | \varphi_0^{-1}(t) \cong \varphi_0^{-1}(q_0)\} \subset E_0$ contains the dense subset $\bigcup_{n=0}^{\infty}(h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ h_0)(q_0)$ of E_0. Hence M is dense in E_0. By the condition (4), the Kuranishi space of $\varphi_0^{-1}(q_0)$ is non-singular and $\varphi_0^{-1}(q_0)$ is local analytically rigid. Hence, for any $t \in E_0$, there exists a small open neighborhood $U(t)$ such that all the fibers $\varphi_0^{-1}(x), x \in U(t)$ are isomorphic to each other. Since $U(t) \cap M \neq \emptyset$, we have an isomorphism $\varphi_0^{-1}(t) \cong \varphi_0^{-1}(q_0)$ for all $t \in E_0$. Hence, by the theorem of Fischer–Grauert [7], $\varphi_0 \colon X_0 \to E_0$ is a holomorphic fiber bundle. □

Our next concern is to study the structure of an FESP constructed from a given non-isomorphic étale endomorphism $f \colon X \to X$ by a sequence of blowing-downs of an ESP.

Proposition 7.3. *Let $Y_\bullet = (g_n \colon Y_n \to Y_{n+1})_n$ be an FESP of smooth projective 3-folds Y_n with $\kappa(Y_n) = -\infty$. Suppose that there exists an extremal ray R_0 of fiber type on $\overline{\mathrm{NE}}(Y_0)$. For any $n > 0$, we set $R_n := (g_{n-1} \circ \cdots \circ g_0)_* R_0$ and $R_{-n} := (g_{-n} \circ \cdots \circ g_{-1})^* R_0$. Let $R_\bullet := (R_n)_n$ be the set of extremal rays of $\overline{\mathrm{NE}}(Y_\bullet)$. Then*

- *any R_n is also an extremal ray of fiber type, and*
- *the FESP $(Y_\bullet, R_\bullet)$ is either of type (D) or of type (C).*

Furthermore, if we denote by $\pi_n := \mathrm{Cont}_{R_n} \colon Y_n \to Y_n'$ the contraction morphism associated to R_n, the structure of the FESP $(Y_\bullet, R_\bullet)$ is one of the followings;

(1) *$(Y_\bullet, R_\bullet)$ is of type (D). Then,*
 (a) $\pi_n \colon Y_n \to Y_n' := C_n$ is a smooth del Pezzo fiber space over an elliptic curve C_n for any n.
 (b) There are induced an ESP $C_\bullet = (w_n \colon C_n \to C_{n+1})_n$ of elliptic curves and a Cartesian morphism $\pi_\bullet = (\pi_n)_n \colon Y_\bullet \to C_\bullet$.

(2) *$(Y_\bullet, R_\bullet)$ is of type (C). Then*
 (a) $\pi_n \colon Y_n \to Y_n' := S_n$ is a conic bundle over a smooth algebraic surface S_n for any n.
 (b) There is induced an ESP $S_\bullet = (h_n \colon S_n \to S_{n+1})_n$ of surfaces S_n and a Cartesian morphism $\pi_\bullet = (\pi_n)_n \colon Y_\bullet \to S_\bullet$.
 (c) The discriminant locus Δ_n of π_n is a disjoint union of elliptic curves $\Delta_n^{(i_n)}$ whose self-intersection number is zero.
 (d) Let M_n be a finite set consisting of all the connected components $\Delta_n^{(i_n)}$ of Δ_n. Then, the map $\mu_n \colon M_n \to M_{n+1}$ defined by $\Delta_n^{(i_n)} \mapsto h_n(\Delta_n^{(i_n)})$ gives a bijection between the set M_n and M_{n+1} for $n \gg 0$.

Proof. (1) Suppose that R_0 is of type (D). Then, by Proposition 3.1, the FESP $(Y_\bullet, R_\bullet)$ is of type (D). Let $\pi_n := \mathrm{Cont}_{R_n} \colon Y_n \to C_n$ be the contraction morphism associated to R_n. By Proposition 3.1, there exists a finite morphism $w_n \colon C_n \to C_{n+1}$ such that $\pi_{n+1} \circ g_n = w_n \circ \pi_n$. Since general fiber of π_n is a del Pezzo surface and simply connected, and g_n is a non-isomorphic étale covering, we have $\deg w_n \geq 2$ for any n. Since the genus $p_a(C_n)$ of the curve C_n decreases as n increases, there exists a non-negative integer g such that $p_a(C_n) = g$ for $n \gg 0$. Applying Lemma 2.1, we see that $0 = \chi(\mathcal{O}_{Y_n})$ $(:= 1 - q(Y_n) + h^{2,0}(Y_n) - p_g(Y_n))$. Since $p_g(Y_n) = 0$, we have $p_a(C_n) = q(Y_n) = 1 + h^{2,0}(Y_n) \geq 1$. In particular, $g \geq 1$. Suppose that $g \geq 2$. Then, by the Riemann-Hurwitz formula, w_n is an isomorphism for $n \gg 0$, which derives a contradiction. Hence $g = 1$ and any C_n is an elliptic curve.

Since $-K_{Y_{n,t}}$ is ample for any $Y_{n,t} := \pi_n^{-1}(t)$, Lemma 7.2 shows that $\pi_n\colon Y_n \to C_n$ is a smooth del Pezzo fibration over an elliptic curve C_n. Since any rational curve on Y_n is contained in fibers of π_n, $\pi_n\colon Y_n \to C_n$ is a maximally rationally connected fibration and also gives the Albanese map of Y_n.

(2) Suppose that R_0 is of type (C). Then, by Proposition 3.1, the FESP $(Y_\bullet, R_\bullet)$ is of type (C) and the contraction morphism $\pi_n := \mathrm{Cont}_{R_n}\colon Y_n \to S_n$ is a conic bundle over a smooth surface S_n. By Proposition 3.1 and Lemma 7.1, there exists a non-isomorphic finite étale covering $h_n\colon S_n \to S_{n+1}$ such that $h_n \circ \pi_n = \pi_{n+1} \circ g_n$. Hence, the structure of S_n is classified into 3 types by Proposition 6.4. Let Δ_n be the discriminant locus of π_n. Suppose that $\Delta_n \neq \emptyset$ for some n. Then, by [4], Δ_n is a simple normal crossing divisor of S_n and $\mathrm{Sing}\,\Delta_n$ coincides with the set $\{s \in S_n | \, \pi_n^{-1}(s) \text{ is non-reduced}\}$.

We first show that $\mathrm{Sing}\,\Delta_n = \emptyset$. It is sufficient to prove the case where $n = 0$ without loss of generality. The proof is by contradiction. Assume the contrary. Since g_n, h_n are finite étale and each fiber of π_n is simply connected, we have an isomorphism $Y_n \cong Y_{n+1} \times_{S_{n+1}} S_n$. Hence $h_n^{-1}(\Delta_{n+1}) = \Delta_n$ and $h_n^{-1}(\mathrm{Sing}\,\Delta_{n+1}) = \mathrm{Sing}\,\Delta_n$ for each n. We take a point $q_0 \in \mathrm{Sing}\,\Delta_0$ arbitrarily. Then the set $M := \bigcup_{n=0}^{\infty} (h_n \circ \cdots \circ h_0)^{-1}(h_n \circ \cdots \circ h_0)(q_0)$ is contained in $\mathrm{Sing}\,\Delta_0$. Since $\deg h_i \geq 2$ for each i, M is an infinite set. Since $\mathrm{Sing}\,\Delta_0$ is a finite set, a contradiction is derived. Hence Δ_0 is non-singular. Next, we show that each connected component $\Delta_n^{(i_n)}$ is an elliptic curve whose self intersection number is zero. Let M_n denote the finite set consisting of irreducible components of Δ_n. From the above consideration, the natural map $\mu_n\colon M_n \to M_{n+1}$ is a surjection between finite sets. Since the cardinality $|M_n|$ decreases as n increases, there exists a constant k such that $|M_n| = k$ for $n \gg 0$. Hence μ_n is a bijection for $n \gg 0$. Then, by considering a truncated étale sequence, we may assume that any μ_n is bijective. Hence, by Proposition 6.9, each $\Delta_n^{(i_n)}$ is an elliptic curve whose self-intersection number equals 0. □

Remark 7.4. In Proposition 7.3, suppose that $(Y_\bullet, R_\bullet)$ is an FESP of type (D) constructed from a non-isomorphic étale endomorphism $f\colon X \to X$ by a sequence of blowing-downs of an ESP. Then $\pi_n\colon Y_n \to C$ is a del Pezzo *fiber bundle* over the Albanese elliptic curve C of X. This fact will be proved in our subsequent article, Part III.

Proposition 7.5. *Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an FESP constructed from smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$ by a sequence of blowing-downs of an ESP. Let $R_\bullet = (R_n)_n$ be the set of extremal rays of fiber type on $\overline{\mathrm{NE}}(Y_\bullet)$ such that $(g_n)_* R_n = R_{n+1}$ for any n. Suppose that the FESP $(Y_\bullet, R_\bullet)$ is of type (C). Then one of the following cases occurs. (Hereafter, we use the same notation as in* Proposition 7.3.*)*

(1) *For any n, $\kappa(S_n) = 1$. In this case, for any n,*

(a) the Iitaka fibration $\varphi_n\colon S_n \to C_n$ gives S_n a Seifert elliptic surface structure.

(b) There exists a finite morphism $v_n\colon C_n \to C_{n+1}$ such that $\varphi_{n+1} \circ h_n = v_n \circ \varphi_n$ and v_n is an isomorphism for all $n \gg 0$.

(c) The discriminant locus Δ_n of π_n is non-singular and each connected component is an elliptic curve which is some fiber of φ_n.

(2) *For any n, $\kappa(S_n) = 0$. Then there exists an integer k such that S_n is an abelian surface (resp. a hyperelliptic surface) for all $n \leq k$ (resp. for all $n > k$).*

(a) Suppose that any S_n is an abelian surface. Then, there exists an elliptic bundle

structure $\alpha_n\colon S_n \to E_n$ *over an elliptic curve* E_n *such that* $\alpha_n(\Delta_n)$ *are points on* E_n *and* $\alpha_{n+1} \circ h_n = u_n \circ \alpha_n$ *for an isomorphism* $u_n\colon E_n \cong E_{n+1}$ *for* $n \gg 0$.

(b) *Suppose that any* S_n *is a hyperelliptic surface. Then, there exist either an elliptic bundle structure* $\alpha_n\colon S_n \to E_n$ *over an elliptic curve* E_n *such that* $\alpha_n(\Delta_n)$ *are points, or a Seifert elliptic surface structure* $p_n\colon S_n \to \mathbb{P}^1$ *over* $\mathbb{P}^1$ *such that* $p_n(\Delta_n)$ *are points.*

(3) *For any* n, $\kappa(S_n) = -\infty$. *In this case, for any* n,

(a) *the Albanese map* $\alpha_n\colon S_n \to C_n$ *induce an ESP* $C_\bullet = (h_n\colon C_n \to C_{n+1})_n$ *of elliptic curves* C_n *and a Cartesian morphism* $\alpha_\bullet = (\alpha_n)_n\colon S_\bullet \to C_\bullet$.

(b) $S_n \simeq \mathbb{P}_{E_n}(\mathcal{E}_n)$ *for a semi-stable vector bundle* $\mathcal{E}_n$ *of rank 2 on* C_n.

(c) Δ_n *is non-singular and each of its connected component is an elliptic curve mapped onto* C_n *by* α_n.

Proof. All the assertions follow immediately by Propositions 6.9, 6.4, 6.5, 6.6, 6.7, and 7.3. □

DEFINITION 7.6. Let $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ be an FESP of smooth projective 3-folds Y_n with $\kappa(Y_n) = -\infty$. Let $R_\bullet = (R_n)_n$ be the set of extrmal rays R_n of fiber type on $\overline{\mathrm{NE}}(Y_n)$ such that $(g_n)_*R_n = R_{n+1}$ for any n. Suppose that the FESP $(Y_\bullet, R_\bullet)$ is of type (C). Then $(Y_\bullet, R_\bullet)$ is called of type (C_1) (resp. (C_0) and ($C_{-\infty}$)) if the condition (1) (resp. (2) and (3)) in Proposition 7.5 is satisfied.

REMARK 7.7. Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_0) constructed from (X, f) by a sequence of blowing-downs of an ESP. Let $\pi_n\colon Y_n \to S_n$ be a conic bundle, here we use the same notation as in Proposition 7.5 (2). Since $q(X) = q(Y_n) = q(S_n)$, one of the following possibilities can occur:

- Any S_n is an abelian surface.
- Any S_n is a hyperelliptic surface.

When we recover the given non-isomorphic étale endomorphism $f\colon X \to X$ from its FESP by a sequence of blowing-ups of an ESP along elliptic curves, the next proposition gives some perspective on the blowing-up centers.

Proposition 7.8. *Let* $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ *be an FESP constructed from an ESP* $X_\bullet$ *of smooth projective* 3*-folds* X_n *with* $\kappa(X_n) = -\infty$ *by a sequence of blowing-downs of an ESP. Let* $R_\bullet = (R_n)_n$ *be the set of extremal rays of fiber type on* $\overline{\mathrm{NE}}(Y_n)$ *such that* $(g_n)_*R_n = R_{n+1}$ *and* φ_n *the contraction morphism associated to* R_n. *Suppose that there exists an elliptic curve* E_n *on* Y_n *such that* $g_n^{-1}(E_{n+1}) = E_n$ *for any* n. *Then one of the following cases occurs.*

(1) *The FESP* $(Y_\bullet, R_\bullet)$ *is of type* (D). *Then we have* $\varphi_n(E_n) = C_n$, *where* $\varphi_n\colon Y_n \to C_n$ *is a smooth del Pezzo fiber space over an elliptic curve* C_n,

(2) *The FESP* $(Y_\bullet, R_\bullet)$ *is of type* (C_0) *or* (C_1). *Then* $\varphi_n\colon Y_n \to S_n$ *is a conic bundle over a Seifert elliptic surface* $\alpha_n\colon S_n \to C_n$ *with* $\kappa(S_n) \geq 0$ *and* $\varphi_n(E_n)$ *is an elliptic curve which is some fiber of* α_n.

(3) *The FESP* $(Y_\bullet, R_\bullet)$ *is of type* ($C_{-\infty}$). *Then* $\varphi_n\colon Y_n \to S_n$ *is a conic bundle over a surface* S_n. *Furthermore,* $\Gamma_n := \varphi_n(E_n)$ *is an elliptic curve such that* $(\Gamma_n)^2 = 0$ *and*

$\alpha_n(\Gamma_n) = C_n$, *where* $\alpha_n : S_n \to C_n$ *is a* $\mathbb{P}^1$*-bundle over an elliptic curve* C_n.

Proof. (1) Since $\varphi_n : Y_n \to C_n$ is the Albanese map of Y_n, there exists a morphism $h_n : C_n \to C_{n+1}$ such that $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$. The proof is by contradiction. Suppose that $p_n := \varphi_n(E_n)$ is a point on C_n for some n. Then, for the elliptic curve $E_{n+1} = g_n(E_n)$, $p_{n+1} := \pi_{n+1}(E_{n+1})$ is also a point on C_{n+1}. By assumption, E_n is contained in all the connected component of $g_n^{-1} \circ \varphi_{n+1}^{-1}(p_{n+1}) = \varphi_n^{-1} \circ h_n^{-1}(p_{n+1})$. Since $\deg h_n = \deg f_n > 1$ and h_n is étale, E_n is not connected. Thus a contradiction is derived, since E_n is irreducible.

(2), (3) In both cases, there exists a finite étale covering $h_n : S_n \to S_{n+1}$ such that $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$ and $\deg h_n = \deg g_n > 1$. Then $\Gamma_n := \varphi_n(E_n)$ is an irreducible curve on S_n for each n, since all the irreducible components of any fiber of φ_n are rational curves. By construction, there exists a Cartesian morphism $\varphi_\bullet = (\varphi_n)_n : Y_\bullet \to S_\bullet := (h_n : S_n \to S_{n+1})_n$. Applying Lemma 2.6 to the above diagram, we infer that $\Gamma_n = h_n^{-1}(\Gamma_{n+1})$. Hence, by Proposition 6.8, Γ_n is an elliptic curve with $\Gamma_n^2 = 0$. The last claim in (3) is derived from Proposition 6.9. □

Combining Lemma 2.6, Propositions 1.1 and 7.8, we obtain:

Corollary 7.9. *Let* $X_\bullet = (f_n : X_n \to X_{n+1})_n$ *be an ESP of smooth projective* 3*-folds* X_n *with* $\kappa(X_n) = -\infty$. *Let* $(Y_\bullet, R_\bullet)$ *be an FESP constructed from* $X_\bullet$ *by a sequence of blowing-downs of an ESP and set* $Y_\bullet = (g_n : Y_n \to Y_{n+1})_n$ *(Here, we use the same notation as in Corollary 1.2 and Proposition 7.3.) Let* $C_n^{(i)}$ *be the elliptic curve which is the center of the blowing-up* $\pi_n^{(i-1)} : X_n^{(i-1)} \to X_n^{(i)}$. *Then,* $\gamma_n^{(i)} := \pi_n^{(k-1)} \circ \cdots \circ \pi_n^{(i)}(C_n^{(i)})$ *is an elliptic curve on* Y_n *such that* $\gamma_\bullet^{(i)} = (g_n : \gamma_n^{(i)} \to \gamma_{n+1}^{(i)})_n$ *is an ESP of elliptic curves. Furthermore, the following hold:*

(1) *If* $(Y_\bullet, R_\bullet)$ *is of type* (D), *then* $\varphi_n(\gamma_n^{(i)}) = C_n$.
(2) *If* $(Y_\bullet, R_\bullet)$ *is of type* (C_0) *or* (C_1), *then* $\Delta_n^{(i)} := \varphi_n(\gamma_n^{(i)})$ *is some fiber of* $\alpha_n : S_n \to C_n$ *such that* $\Delta_\bullet = (h_n : \Delta_n \to \Delta_{n+1})_n$ *is an ESP of elliptic curves.*
(3) *If* $(Y_\bullet, R_\bullet)$ *is of type* $(C_{-\infty})$, *then* $\Delta_n^{(i)} := \varphi_n(\gamma_n^{(i)})$ *is an elliptic curve on* S_n *with self-intersection number* 0 *and dominates* C_n. *Furthermore,* $\Delta_\bullet = (h_n : \Delta_n \to \Delta_{n+1})_n$ *is an ESP of ellipti curves.*

The existence of a non-isomorphic étale endomorphism imposes very strong conditions on the π-exceptional locus for the Cartesian blowing-up $\pi_\bullet : X_\bullet \to Y_\bullet$, as the following shows:

Proposition 7.10. *Let* $X_\bullet = (f_n : X_n \to X_{n+1})_n$ *be an ESP of smooth projective* 3*-folds* X_n *with* $\kappa(X_n) = -\infty$. *Suppose that there exists an FESP* $(Y_\bullet, R_\bullet)$ *of type* (C) *constructed from* $X_\bullet$ *by a sequence of blowing-downs of an ESP* $\pi_\bullet : X_\bullet \to Y_\bullet$. *Suppose that* $\pi_\bullet$ *is not an isomorphism. Suppose furthermore that for* $Y_\bullet = (g_n : Y_n \to Y_{n+1})_n$, *the contraction morphism* $\varphi_n := \mathrm{Cont}_{R_n} : Y_n \to S_n$ *is a* $\mathbb{P}^1$*-bundle for any* n. *Let* Δ_n *be the discriminant locus of* $\psi_n := \varphi_n \circ \pi_n : X \to S_n$. *Then the following hold:*

- ψ_n *is an equi-dimensional* $\mathbb{P}^1$*-fiber space.*
- Δ_n *is non-singular and any of its irreducible component* $\Delta_{n,i}$ *is an elliptic curve.*
- *For any* i, $\psi_n^{-1}(\Delta_{n,i})$ *is a simple normal crossing divisor and any of its irreducible components is a* $\mathbb{P}^1$*-bundle over an elliptic curve associated to a semi-stable vector bundle.*

Proof. By construction of an FESP, there exists the following Cartesian morphisms of ESPs;

$$X_\bullet = X_\bullet^{(0)} \longrightarrow \cdots \longrightarrow X_\bullet^{(i)} \xrightarrow{\pi_\bullet^{(i)}} X_\bullet^{(i+1)} \longrightarrow \cdots \longrightarrow Y_\bullet = X_\bullet^{(k)} \xrightarrow{\varphi_\bullet} S_\bullet\,,$$

in which the following are satisfied for any n and i.

- $\pi_n^{(i)}\colon X_n^{(i-1)} \to X_n^{(i)}$ is (the inverse of) the blowing-up along an elliptic curve $C_n^{(i)}$ on $X_n^{(i)}$, where $X_n^{(0)} := X$ and $X_n^{(k)} := Y_n$.
- $X_\bullet^{(i)} = (f_n^{(i)}\colon X_n^{(i)} \to X_{n+1}^{(i)})_n$ is an ESP.
- $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ is an FESP, where $g_n := f_n^{(k)}$.
- $(g_n^{(i)})^{-1}(C_{n+1}^{(i)}) = C_n^{(i)}$.
- $\pi_n = \pi_n^{(k-1)} \circ \cdots \circ \pi_n^{(0)}$.

Now we shall describe the centers of the succession of blowing-ups $\pi_n^{(i)}$. By Lemma 6.10, Proposition 6.9 and Corollary 7.9, Δ_n is non-singular and any irreducible component of Δ_n is an elliptic curve. If we set $\gamma_n^{(k)} := \varphi_n(C_n^{(k)})$, then $\gamma_n^{(k)}$ is an elliptic curve on S_n with zero self-intersection number by Corollary 7.9. Then the surface $T_n^{(k)} := \varphi_n^{-1}(\gamma_n^{(k)})$ is a $\mathbb{P}^1$-bundle over $\gamma_n^{(k)}$ which contains $C_n^{(k)}$. Since $u_n^{-1}(\gamma_{n+1}^{(k)}) = \gamma_n^{(k)}$, we infer that $g_n^{-1}(T_{n+1}^{(k)}) = T_n^{(k)}$ for any n. Thus there is induced an ESP $T_\bullet^{(k)} := (g_n\colon T_n^{(k)} \to T_{n+1}^{(k)})_n$ of elliptic ruled surfaces. Then by Proposition 4.1, each $T_n^{(k)}$ is a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}_n^{(k)})$ associated to a semi-stable vector bundle $\mathcal{E}_n^{(k)}$ of rank two on $\gamma_n^{(k)}$. Furthermore, $C_n^{(k)}$ is an elliptic curve with zero self-intersection number on $T_n^{(k)}$.

If we set $\gamma_n^{(k-1)} := \varphi_n \circ \pi_n^{(k-1)}(C_n^{(k-1)})$, then by the same argument as above, $\gamma_n^{(k-1)}$ is an elliptic curve on S_n with zero self-intersection number. By Proposition 7.3, S_n is a $\mathbb{P}^1$-bundle associated to a semi-stable vector bundle of rank 2 on an elliptic curve C. Hence applying Lemma 6.10, we infer that $\gamma_n^{(k-1)} \cap \gamma_n^{(k)} = \emptyset$ or $\gamma_n^{(k-1)} = \gamma_n^{(k)}$. If $\gamma_n^{(k-1)} \cap \gamma_n^{(k)} = \emptyset$, then $C_n^{(k-1)} \cap T_n^{(k)} = \emptyset$ and there is nothing to prove. Next, suppose that $\gamma_n^{(k-1)} = \gamma_n^{(k)}$. Then $D_n^{(k)} := \pi_n^{(k-1)}(C_n^{(k-1)})$ is contained in $T_n^{(k)}$. Since $C_\bullet^{(k-1)} := (C_n^{(k-1)} \to C_{n+1}^{(k-1)})_n$ is an ESP, $D_\bullet^{(k)} := (f_n^{(k-1)}\colon D_n^{(k)} \to D_{n+1}^{(k)})_n$ is also an ESP by Lemma 2.6. Hence by Proposition 6.9, $D_n^{(k)}$ is an elliptic curve with zero self-intersection number on $T_n^{(k)}$. Thus Proposition 6.10 yields that $C_n^{(k)} \cap D_n^{(k)} = \emptyset$ or $D_n^{(k)} = C_n^{(k)}$. If $C_n^{(k)} \cap D_n^{(k)} = \emptyset$, then clearly the proper transform of $T_n^{(k)}$ and $\pi_n^{(k-1)} \circ \pi_n^{(k-2)}$-exceptional divisors are all elliptic ruled surfaces crossing normally. Next suppose that $D_n^{(k)} = C_n^{(k)}$. Then $C_n^{(k-1)}$ is contained in the $\pi_n^{(k-1)}$-exceptional divisor $E_n^{(k-1)}$ and is an elliptic curve with zero self-intersection number. Let $T_n'^{(k)}$ be the proper transform of $T_n^{(k)}$ by $\pi_n^{(k-1)}$ and $\Delta_n^{(k)} := E_n^{(k-1)} \cap T_n'^{(k)}$ the intersection curve. Since $(f_n^{(k-1)})^{-1}(E_{n+1}^{(k-1)}) = E_n^{(k-1)}$ and $(f_n^{(k-1)})^{-1}(T'^{(k)}_{n+1}) = T'^{(k)}_n$, $\Delta_\bullet^{(k)} := (f_n^{(k-1)}\colon \Delta_n^{(k)} \to \Delta_{n+1}^{(k)})_n$ is an ESP. Hence $\Delta_n^{(k)}$ is an elliptic curve on $E_n^{(k)}$ with zero self-intersection number. Hence Lemma 6.10 yields that $C_n^{(k-1)} \cap \Delta_n^{(k)} = \emptyset$ or $C_n^{(k-1)} = \Delta_n^{(k)}$. Then by the same argument as before, we see that the proper transform of $T_n^{(k)}$ and $\pi_n^{(k-1)} \circ \pi_n^{(k-2)}$-exceptional divisors are all elliptic ruled surfaces ccrossing normally. Continuing the same argument as above, we can prove the proposition. □

Remark 7.11. As pointed out by the referee, Theorem. C of the reference [40] gives a general structure theorem for a uniruled manifold X with étale endomorphisms, in particular, one can immediately obtain a natural lower- dimensional variety Z which is the base of the MRC fibration and admits a quiasi-étale endomorphism. In our case, $\dim X = 3$, Z is *smooth* with $\dim Z = 1$ or 2 and the MRC fibration $X \to Z$ is a *morphism*. In view of this result, in the case where $\dim X$ is general, it woud be more natural to study seperately the case where $\dim Z = 1, 2$.

The next propositions give some description of a non-isomorphic étale endomorphism $f\colon X \to X$ which has extremal rays of different type.

Proposition 7.12. *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$. Suppose that on $\overline{\mathrm{NE}}(X_\bullet)$ there exist two extremal rays $R_\bullet = (R_n)_n$ and $R'_\bullet = (R'_n)_n$ such that $(X_\bullet, R_\bullet)$ and $(X_\bullet, R'_\bullet)$ are of different types. Then, the structure of $X_\bullet$ is as follows:*

(1) *The pair $(X_\bullet, R_\bullet)$ is an FESP of type $(C_{-\infty})$, i.e., the contraction morphism $\varphi_n := \mathrm{Cont}_{R_n}\colon X_n \to S_n$ associated to R_n is a conic bundle over a surface S_n, which is a $\mathbb{P}^1$-bundle $\alpha_n\colon S_n \to C_n$ over an elliptic curve C_n.*
(2) *Another $(X, R'_\bullet)$ is of divisorial type, i.e., the contraction morphism $\pi_n := \mathrm{Cont}_{R'_n}\colon X_n \to X'_n$ associated to R'_n is (the inverse of) the blowing-up of a smooth projective 3-fold X'_n along an elliptic curve $\Gamma_n \subset X'_n$.*
(3) *X'_n has the structure of a smooth del Pezzo fiber space $g_n\colon X'_n \to C_n$ over C_n such that $g_n \circ \pi_n = \alpha_n \circ \varphi_n$, which is an extremal contraction.*
(4) *$\varphi_n(D_n) = S_n$ for the π_n-exceptional divisor D_n.*

Proof. First, suppose that $(X_\bullet, R'_\bullet)$ is of divisorial type and $(Y_\bullet, R_\bullet)$ is of type (D) respectively. Then each fiber of the contraction morphism $\pi_n := \mathrm{Cont}_{R'_n}\colon X_n \to X'_n$ is either a point or $\mathbb{P}^1$. It is mapped to a point by the contraction morphism $\varphi_n := \mathrm{Cont}_{R_n}\colon X_n \to C_n$, since C_n is an elliptic curve by Proposition 3.6. Hence, by the rigidity lemma, there exists a morphism $\psi_n\colon X'_n \to C_n$ with $\varphi_n = \psi_n \circ \pi_n$. This contradicts $\rho(X_n/C_n) = 1$, since each fiber of ψ_n is positive-dimensional. By the same way as above, we can prove that the case where $(X_\bullet, R'_\bullet)$ is of type (C) and $(X_\bullet, R_\bullet)$ is of type (D) cannot occur.

Hence the only possibility is that $(X_\bullet, R'_\bullet)$ is of divisorial type and $(X_\bullet, R_\bullet)$ is of type (C) respectively. Let $\varphi_n := \mathrm{Cont}_{R_n}\colon X_n \to S_n$ (resp. $\pi_n := \mathrm{Cont}_{R'_n}\colon X_n \to X'_n$) be the contraction morphism associated to R_n (resp. R'_n). If $\kappa(S_n) \geq 0$, then by Proposition 6.4, S_n contains no rational curves. Hence each fiber of π_n is mapped to a point by φ_n. Thus there exists a morphism $\psi_n\colon X'_n \to S_n$ such that $\psi_n \circ \pi_n = \varphi_n$. This contradicts the fact that $\rho(X_n/S_n) = 1$. Hence $\kappa(S_n) = -\infty$ and by Proposition 6.4, S_n is a $\mathbb{P}^1$-bundle $\alpha_n\colon S_n \to C_n$ over an elliptic curve C_n. Hence, by the same reason as above, there exists a morphism $g_n\colon X'_n \to C_n$ such that $\alpha_n \circ \varphi_n = g_n \circ \pi_n$. Since both π_n and φ_n are extremal contractions, we have $\rho(X_n) = \rho(S_n) + 1 = 3$ and $\rho(X'_n) = \rho(X_n) - 1 = 2$. Since $\rho(X'_n) = \rho(C_n) + 1$ and $K_{X'_n}$ is not g_n-nef, g_n is also an extremal contraction and is a smooth del Pezzo fiber space over an elliptic curve C_n by Proposition 7.3. Thus the assertion (3) is verified.

$\pi_n\colon X_n \to X'_n$ is (the inverse of) the blowing-up along an elliptic curve $\Gamma_n \subset X'_n$ and let D_n be an exceptional divisor of π_n. Suppose that $g_n(\Gamma_n)$ is a point Q_n on C_n. Then D_n is contained in the surface $(\alpha_n \circ \varphi_n)^{-1}(Q_n)$, which is a conic bundle over $\mathbb{P}^1$ by Proposition 7.3. This contradicts the fact that $D_n \to \Gamma_n$ is an elliptic ruled surface. Hence $g_n(\Gamma_n) = C_n$. Since $R_n \neq R'_n$ and C_n is an elliptic curve, any fiber γ of the $\mathbb{P}^1$-bundle $\pi_n|_{D_n}\colon D_n \to \Gamma_n$ is not contracted to a point on S_n by φ_n but contracted to a point by $\alpha_n \circ \varphi_n$. Hence $\varphi_n(\gamma) = F$ for some fiber F of α_n. If $\varphi_n(D_n) \neq S_n$, then $\varphi_n(D_n) = F$, since $\gamma \subset D_n$ and $(F \subset)\varphi_n(D_n)$ is irreducible. Hence we have $C_n = g_n(\Gamma_n) = g_n \circ \pi_n(D_n) = \alpha_n \circ \varphi_n(D_n) = \alpha_n(F)$, which is a point. Thus a contradiction is derived and the assertion (4) is proved. □

The following can be proved similarly.

Proposition 7.13. *Let $X_\bullet = (f_n\colon X_n \to X_{n+1})_n$ be an ESP of smooth projective 3-folds X_n with $\kappa(X_n) = -\infty$. Suppose that there exist an FESP $(Y_\bullet, R_\bullet)$ of type (C) and another FESP $(Z_\bullet, R'_\bullet)$ of type (D) constructed from X by a sequence of blowing-downs of an ESP. Then $(Y_\bullet, R_\bullet)$ is of type $(C_{-\infty})$.*

REMARK 7.14. If S is a two-pointed blown-up of $\mathbb{P}^2$, then S is also a one point blown-up of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $X := S \times E$ be the direct product of S and an elliptic curve E. Let $n \geq 2$ be a positive integer and $\mu_n\colon E \to E$ the multiplication by n mapping. Then, $f := \mathrm{id}_S \times \mu_n\colon X \to X$ is a non-isomorphic étale endomorphism of X. After a succession of divisorial contractions $X \to \mathbb{P}^2 \times E$, there is induced an FESP $\mathrm{id}_{\mathbb{P}^2} \times \mu_n\colon \mathbb{P}^2 \times E \to \mathbb{P}^2 \times E$, where the second projection $\mathbb{P}^2 \times E \to E$ gives an extremal contraction of type (D).

Furthermore, after another divisorial contraction $X \to (\mathbb{P}^1 \times \mathbb{P}^1) \times E$, there is induced another FESP $\mathrm{id}_{\mathbb{P}^1\times\mathbb{P}^1} \times \mu_n\colon (\mathbb{P}^1 \times \mathbb{P}^1) \times E \to (\mathbb{P}^1 \times \mathbb{P}^1) \times E$ of type ($C_{-\infty}$), where the projection $pr_1 \times \mathrm{id}_E\colon (\mathbb{P}^1 \times \mathbb{P}^1) \times E \to \mathbb{P}^1 \times E$ gives an extremal contraction.

8. Finiteness of extremal rays

Our description of the FESP eventually turns out to be the most effective. We can extend these ideas to show finiteness of extremal rays of $\overline{\mathrm{NE}}(X)$ in the case where there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_1) or (C_0) on an FESP constructed from $X_\bullet$ by a sequence of blowing-downs of an ESP. (cf. Theorem 1.4). Corollary 8.1 shows that in this case, the MMP works compatibly with étale endomorphisms and we can take $Y_\bullet$ to be a constant FESP. Theorem 8.6 shows the finiteness of extremal rays of fiber type for a non-isomorphic étale endomorphism $f\colon X \to X$ of a 3-fold X with $\kappa(X) = -\infty$.

First we begin with a proof of Theorem 1.4.

Proof of Theorem 1.4. By Proposition 3.1, it is sufficient to prove the theorem for $n = 0$. Applied Propositions 7.3, 7.5 to the assumption, there exist Cartesian morphisms of ESPs,

$$X_\bullet \to Y_\bullet \to S_\bullet\,,$$

such that the following hold:

- $\pi_\bullet = (\pi_n)_n\colon X_\bullet \to Y_\bullet$ is a succession of Cartesian blowing-ups along elliptic curves.
- $S_\bullet = (h_n\colon S_n \to S_{n+1})_n$ is an ESP of smooth algebraic surfaces S_n with $\kappa(S_n) = 0$ for any n or $\kappa(S_n) = 1$ for any n.
- $\varphi_\bullet = (\varphi_n)_n\colon Y_\bullet \to S_\bullet$ is a conic bundle.
- Any S_n has a Seifert elliptic surface structure $\alpha_n\colon S_n \to C_n$.

Take an arbitrary extremal ray R_0 on X_0 which is of divisorial type. For any $n > 0$, we set $R_n := (f_n \circ \cdots \circ f_0)_* R_0$ and $R_{-n} := (f_{-1} \circ \cdots \circ f_{-n})^* R_0$. Then, for any n, R_n is also an extremal ray of divisorial type on X_n, whose contraction morphism $\psi_n := \mathrm{Cont}_{R_n}\colon X_n \to X'_n$ is (the inverse of) the blowing-up of a smooth projective 3-fold X'_n along an elliptic curve $\Gamma_n \subset X'_n$. By Proposition 3.1, there are induced an ESP $X'_\bullet := (f'_n\colon X'_n \to X'_{n+1})_n$ of X'_n and an ESP $\Gamma_\bullet = (f_n|_{\Gamma_n}\colon \Gamma_n \to \Gamma_{n+1})_n$ of Γ_n such that $\psi_{n+1} \circ f_n = f'_n \circ \psi_n$. Let $D_n := \mathrm{Exc}(\psi_n)$ be the

ψ_n-exceptional divisor. Since S_n contains no rational curves, each fiber of $\psi_n|_D\colon D_n \to \Gamma_n$ is mapped to a point by the morphism $\varphi_n \circ \pi_n\colon X_n \to S_n$. Hence there exists a morphism $v_n\colon X'_n \to S_n$ such that $\varphi_n \circ \pi_n = v_n \circ \psi_n$. By construction, we also have $h_n \circ v_n = v_{n+1} \circ f'_n$. To sum up, there exist Cartesian morphisms of ESPs;

$$\begin{tikzcd} X_\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "\psi_\bullet"'] & Y_\bullet \arrow[d, "\varphi_\bullet"] \\ X'_\bullet \arrow[r, "v_\bullet"'] & S_\bullet \end{tikzcd},$$

where $v_\bullet = (v_n)_n$ and $\psi_\bullet = (\psi_n)_n$ is the contraction morphism of the extremal rays $R_\bullet = (R_n)_n$.

By Propositions 7.3 and 7.8, any fiber of $\varphi_n \circ \pi_n\colon X_n \to S_n$ is a union of rational curves and there exists a finite set $T_n \subset C_n$ such that $\varphi_n \circ \pi_n$ is a $\mathbb{P}^1$-bundle over $S_n \setminus \alpha_n^{-1}(T_n)$. In particular, $\varphi_n \circ \pi_n$ is equi-dimensional. If $v_n(\Gamma_n) \subset S_n$ is a point, then the ψ_n-exceptional divisor D_n is contained in $(\varphi_n \circ \pi_n)^{-1}(v_n(\Gamma_n))$, which derives a contradiction. Hence, any $\Delta_n := v_n(\Gamma_n)$ is an irreducible curve on S_n. Since $f_n'^{-1}(\Gamma_{n+1}) = \Gamma_n$, applying Lemma 2.6 to the above diagram, we infer that $h_n^{-1}(\Delta_{n+1}) = \Delta_n$. Thus there are induced an ESP $\Delta_\bullet = (h_n|_{\Delta_n}\colon \Delta_n \to \Delta_{n+1})_n$ of Δ_n and a Cartesian morphism $v_\bullet|_{\Gamma_\bullet} = (v_n|_{\Gamma_n})_n\colon \Gamma_\bullet \to \Delta_\bullet$. Hence, by Proposition 7.3, any Δ_n is an elliptic curve and coincides with some fiber of $\alpha_n\colon S_n \to C_n$. In particular, if we consider the case for $n = 0$, then $\varphi_0 \circ \pi_0(D_0) = v_0 \circ \psi_0(D_0) = v_0(\Gamma_0)$ is contained in some fiber of $\alpha_0\colon S_0 \to C_0$. By the composition of the morphisms $\beta_0 := \alpha_0 \circ \varphi_0 \circ \pi_0\colon X_0 \to C_0$, we regard X_0 as a fiber space over the curve C_0. Then β_0 has a singular fiber over a finite subset $T_0 \subset C_0$, smooth outside T_0 and its smooth fiber is an elliptic ruled surface. Note that the extremal ray R_0 is spanned by a fiber γ_0 of the ruling $\psi_0|_{D_0}\colon D_0 \to \Gamma_0$ of the exceptional divisor $D_0 := \mathrm{Exc}(\psi_0)$ and we have $(K_{X_0}, \gamma_0) = -1$. On the other hand, we have $(K_{X_0}, \gamma_t) = -2$ for any point $t \in C_0 \setminus T_0$ and any fiber γ_t of the ruling of the elliptic ruled surface $\beta_0^{-1}(t) \to \alpha_0^{-1}(t)$. Hence D_0 is an irreducible component of some singular fiber of β_0. Note that there exist at most finitely many irreducible components of the singular fiber $\beta_0^{-1}(T)$. Thus the possibility of D_0 (and hence that of R) is finite and there exist only finitely many extremal rays of divisorial type on $\overline{\mathrm{NE}}(X_0)$. □

As a fairly easy consequence of Theorem 1.4 and Proposition 3.8, we can state the following result; Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_0) or type (C_1) constructed from a given non-isomorphic étale endomorphism $f\colon X \to X$ by a sequence of blowing-downs of an ESP. Then we can run the minimal model program compatibly with étale endomorphisms and obtain a constant FESP.

Corollary 8.1. *Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y'_\bullet, R_\bullet)$ of type (C_0) or of type (C_1) constructed from (X, f) by a sequence of blowing-downs of an ESP. Then, by replacing f by its suitable power f^k $(k > 0)$, we have a Cartesian morphism of constant ESP*

$$X_\bullet = (X, f) \xrightarrow{\pi} Y_\bullet = (Y, g) \xrightarrow{\varphi} S_\bullet = (S, h)\,,$$

such that the following conditions are satisfied:

(1) *$\pi\colon X \to Y$ is a succession of blowing-ups of a smooth 3-fold Y along elliptic curves.*
(2) *$\varphi\colon Y \to S$ is a conic bundle over a smooth surface S with $\kappa(S) = 0$ or 1.*

(3) *The composite map* $\varphi \circ \pi\colon X \to S$ *gives a MRC (maximally rationally connected) fibration of* X.

In particular, we can take $Y_\bullet$ *to be a constant FESP of* $X_\bullet$.

The following shows the uniqueness of the $\mathbb{P}^1$-fiber space structure on X.

Corollary 8.2. *Let* $f\colon X \to X$ *be a non-isomorphic étale endomorphism of a smooth projective* 3*-fold* X *with* $\kappa(X) = -\infty$*. Suppose that there exists an FESP* $(Y_\bullet, R_\bullet)$ *of type* (C_1) *or* (C_0) *constructed from* (X, f) *by a sequence of blowing-downs of an ESP. Let* $\psi\colon X \to T$ *be an arbitrary* $\mathbb{P}^1$*-fiber space structure on* X*. Then there exists an isomorphism* $u\colon T \simeq S$ *such that* $u \circ \psi = \varphi \circ \pi$*. In particular,* ψ *is equi-dimensional and unique up to isomorphism.*

Proof. We use the same notation as in Corollary 8.1. Since $\varphi \circ \pi\colon X \to S$ is a MRC-fibration and S contains no rational curves, there exists a morphism $u\colon T \to S$ with $u \circ \varphi = \varphi \circ \pi$. Since $\varphi \circ \pi$ is equi-dimensional and each fiber of $\varphi \circ \pi$ is connected, u is of degree one, hence an isomorphism by Zariski's main theorem. □

Remark 8.3. The constant FESP $Y_\bullet = (Y, g)$ is not uniquely determined by the given étale endomorphism $f\colon X \to X$. Using *elementary transformations*, we shall construct such an example. Let $\mathbb{S}$ be the Atiyah surface over an elliptic curve E and C a smooth curve of genus $g(C) \geq 1$. For a positive integer $n > 1$, let $\mu_n\colon E \to E$ be multiplication by n. Then with the aid of Proposition 4.13, we see that $f\colon \mathbb{S} \simeq \mathbb{S} \times_{E,\mu_n} E \to \mathbb{S}$ gives an étale endomorphism of $\mathbb{S}$ of degree $n^2 > 1$. If we set $Y := \mathbb{S} \times C$, then $q\colon Y \to C \times E$ is a $\mathbb{P}^1$-bundle and $g := f \times \mathrm{id}_C\colon Y \to Y$ gives a non-isomorphic étale endomorphism of Y. Thus we have the following Cartesin morphism of constant ESP;

$$(Y, g) \xrightarrow{q} (C \times E, \mathrm{id}_C \times \mu_n)\,.$$

Let s_∞ be the canonical section of $\mathbb{S}$. We fix a point $o \in C$ and take an elliptic curve $\gamma := s_\infty \times \{o\}$ on Y. Now we shall perform an elementary transformation to Y along γ. Let $\pi\colon X := \mathrm{Bl}_\gamma(Y) \to Y$ be the blowing-up of Y along γ. Note that $g^{-1}(\gamma) = \gamma$ by construction. Hence by Lemma 3.3, $g\colon Y \to Y$ can be lifted to a non-isomorphic étale endomorphism $f\colon X \to X$. Since $N_{\gamma/Y} \simeq \mathcal{O}_\gamma \oplus \mathcal{O}_\gamma$, the exceptional divisor $\mathrm{Exc}(\pi)$ is isomorphic to the product $\gamma \times \mathbb{P}^1$. Let $\mathbb{S}'$ be the proper transform of the surface $\mathbb{S} \times \{o\}$ by π. Then there exists a divisorial contraction $\pi'\colon X \to Y'$ which contracts the divisor $\mathbb{S}'$ to an elliptic curve γ'. Since $f^{-1}(\mathbb{S}') = \mathbb{S}'$, there exists a non-isomorphic étale endomorphism $g'\colon Y' \to Y'$ such that $\pi' \circ f = g' \circ \pi'$. By the natural projection $p'\colon Y' \to C$, Y' can be regarded as a smooth fiber space over C and put $Y'_t := p'^{-1}(t)$ for $t \in C$.

Then

$$Y'_t \simeq \begin{cases} \mathbb{S}, & t \neq o, \\ \mathbb{P}^1 \times E, & t = o \end{cases},$$

and thus a jumping phenomenon occurs. Both $Y_\bullet := (Y, g)$ and $Y'_\bullet := (Y', g')$ are constant FESPs constructed from of the original constant ESP $X_\bullet := (X, f)$ by blowing-downs of an ESP.

In the case where $f\colon X \to X$ is a non-isomorphic étale endomorphism of a snooth projective 3-fold with $\kappa(X) \geq 0$, we can say more about a type of extremal rays of $\overline{\mathrm{NE}}(X)$.

Proposition 8.4. (*cf.* [9, Proposition 4.6 and Theorem 4.8]) *Let $f: X \to X$ be a non-isomorphic surjective endomorphism of a smooth projective 3-fold X with $\kappa(X) \geq 0$. If K_X is not nef, then the following assertions hold:*

(1) *The set of extremal rays R is a finite set and f induces a permutation of them.*
(2) *The contraction morphism* $\mathrm{Cont}_R: X \to X'$ *associated to any extremal ray R of* $\overline{\mathrm{NE}}(X)$ *is a divisorial contraction which is (the inverse of) the blowing-up along an elliptic curve C on X'.*
(3) *Let E be the exceptional divisor of* Cont_R. *Then $E \cong \mathbb{P}_C(\mathcal{O}_C \oplus \ell)$ for some torsion line bundle $\ell \in \mathrm{Pic}^0(C)$.*

Proof. The assertions (1) and (2) have already been proved in [9, Proposition 4.6 and Theorem 4.8]. Thus it is enough to prove only the assertion (3). By Remark 3.2, E is isomorphic to either the Atiyah surface $\mathbb{S}$ or $\mathbb{P}_C(\mathcal{O}_C \oplus \ell)$ for an elliptic curve C and a line bundle $\ell \in \mathrm{Pic}^0(C)$. If we replace f by its suitable power f^k ($k > 0$), we may assume that $f^{-1}(E) = E$. Hence, the restriction of f to E gives a non-isomorphic étale endomorphism $f|_E: E \to E$. Then, in the first case, $\ell \in \mathrm{Pic}^0(C)$ is of finite order by Proposition 4.8.

Next, we show that the case where $E \simeq \mathbb{S}$ cannot occur. Applying [13, Main Theorem], we infer that there exists a finite étale Galois covering $\rho: \widetilde{X} \to X$ such that there exists on $\widetilde{X}$ an abelian scheme structure $\varphi: \widetilde{X} \to T$ over a variety T with $\dim T \leq 2$. Then Lemma 3.12 shows the existence of an extremal ray $\widetilde{R}(\subset \overline{\mathrm{NE}}(\widetilde{X}))$ of divisorial type such that $\rho_* \widetilde{R} = R$. Let $\widetilde{E}$ be the exceptional divisor of the contraction morphism $\mathrm{Cont}_{\widetilde{R}}$. Then $\rho^{-1}(E) = \widetilde{E}$ and $\widetilde{E}$ is isomorphic to the Atiyah surface $p: \mathbb{S} \to \widetilde{C}$ over an elliptic curve $\widetilde{C}$ by Lemma 4.12.

First, suppose that $\dim T = 0$. Then $\widetilde{X}$ is an abelian 3-fold and contains no rational curves. Thus a contradiction is derived. Next, suppose that $\dim T = 2$. Assume that some fiber of $p: \widetilde{E} \to \widetilde{C}$ is mapped to a point by $\varphi|_{\widetilde{E}}: \widetilde{E} \to T$. Then by the rigidity lemma, there exists a morphism $u: \widetilde{C} \to T$ such that $u \circ p = \varphi|_{\widetilde{E}}$. This contradicts the assumption that p is a $\mathbb{P}^1$-bundle and φ is an elliptic bundle. Hence any fiber of p is mapped to a curve by $\varphi|_{\widetilde{E}}$ and $\dim \varphi(\widetilde{E}) \geq 1$. Suppose that $\dim \varphi(\widetilde{E}) = 1$ and let F be any fiber of the Stein factorization of $\varphi|_{\widetilde{E}}: \widetilde{E} \to T$. Then $p(F) = \widetilde{C}$ and $F^2 = 0$. Thus applying Proposition 5.2, we infer that F equals the canonical section s_∞ and hence $\widetilde{E} = s_\infty$, which derives a contradiction. Suppose that $\varphi(\widetilde{E}) = T$. Then $\kappa(T) = -\infty$, which contradicts the fact that $\kappa(T) = \kappa(\widetilde{X}) = \kappa(X) \geq 0$.

Finally, suppose that $\dim T = 1$. Then applying the same argument as above, each fiber of p is mapped onto T by φ. Hence $T \simeq \mathbb{P}^1$, and $\kappa(\widetilde{X}) = \kappa(T) = -\infty$, which again contradicts the fact that $\kappa(\widetilde{X}) = \kappa(X) \geq 0$. □

Here we insert some remark related with a new phenomenon in the case of $\kappa(X) = -\infty$.

Remark 8.5. Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. We encounter a new phenomenon which does not appear in the case of $\kappa(X) \geq 0$. In Remark 8.3, we have constructed such an example of a 3-fold X which enjoys the following properties;

- There exists an extremal ray R of divisorial type.
- The exceptional divisor E of the contraction morphism $\mathrm{Cont}_R: X \to X'$ is the Atiyah surface $\mathbb{S}$.

There exists no Seifert elliptic fiber space structure on X, as Proposition 6.2 shows.

Next, using ideas of the proof of Theorem 3.10, we can show the finiteness of extremal rays which are of *fiber type*.

Theorem 8.6. *Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Then there exist at most finitely extremal rays which are of fiber type on $\overline{\mathrm{NE}}(X)$.*

Proof. Choose an extremal ray $R(\subset \overline{\mathrm{NE}}(X))$ of fiber type arbitrarily and fix it. Then for any extremal ray $R'(\neq R)$ of fiber type, both FESPs (X, R) and (X, R') are of type (D) or of type (C) by Proposition 7.12.

First, we show that the case where both FESPs (X, R) and (X, R') are of type (D) cannot occur. Suppose the contrary. Then by Proposition 7.3, the contraction morphism $\pi := \mathrm{Cont}_R\colon X \to C$ (resp. $\pi' := \mathrm{Cont}_{R'}\colon X \to C'$) is a smooth del Pezzo fiber space over an elliptic curve C (resp. C'). Hence any fiber of π (resp. π') is covered by rational curves and contracted to a point by π' (resp. π). Thus by the rigidity lemma, there exists an isomorphism $v\colon C \simeq C'$ such that $\pi' = v \circ \pi$. Hence $R = R'$, which is a contradiction.

Hence, we may consider the latter case. We first show that both FESPs (X, R) and (X, R') are of type ($\mathrm{C}_{-\infty}$). Suppose the contrary that the FESP (X, R) is of type (C_0) or type (C_1). Let $\pi := \mathrm{Cont}_R\colon X \to S$ (resp. $\pi' := \mathrm{Cont}_{R'}\colon X \to S'$) be the contraction morphism associated to R (resp. R'). Then by Proposition 7.3, the surface S contains no rational curves. Hence any fiber of π' is mapped to a point by π. Thus by the rigidity lemma, there exists a morphism $u\colon S' \to S$ with $\pi = u \circ \pi'$. Since any fiber of π is connected, u is of degree one and hence an isomorpism by Zariski's main theorem. Thus $R = R'$, which derives a contradiction. Hence by Proposition 7.3, both S and S' are elliptic ruled surfaces. In particular, $\rho(S) = \rho(S') = 2$. Now, we follow the arguments of [15]. Let $\{H_1, H_2\}$ be a set of ample divisors of S such that $\{\mathrm{cl}(H_1), \mathrm{cl}(H_2)\}$ is a basis of $N^1(S)$. Since $R \neq R'$, we infer that $(\pi^* H_i) \cdot R' > 0$ for all $i = 1, 2$. Then there exists a positive rational number α such that $\pi^*(H_1 - \alpha H_2) \cdot R' = 0$. Hence if we set $D := H_1 - \alpha H_2$, then $\pi^*(D) \sim \pi'^*(D')$ for some Cartier divisor D' on S'.

We show that $D^2 = 0$. The proof is by contradiction. Suppose the contrary. Then the product $\pi^*(D)^2$ in $N_1(X)$ is numerically equivalent to δZ for a non-zero effective 1-cycle Z on X and for a non-zero rational number $\delta := D^2 \neq 0$. Hence $\pi^*(L)Z = \delta^{-1}\pi^*(LD^2) = 0$ and $\pi'^*(L')Z = \delta^{-1}\pi'^*(L'd'^2) = 0$ for any Cartier divisor L on S and any Cartier divisor L' on S'. Thus the numerical equivalence class $\mathrm{cl}(Z)$ is contained in $R \cap R'$. Hence $R = R'$, which contradicts the assumption.

We consider the algebraic equation $(H_1 - z H_2)^2 = 0$ of degree two on z, which depends only on H_1 and H_2. It has at most two roots, one of which equals α. For such α, consider the real 2-dimensional vector space $F := \pi^*(\mathrm{cl}(H_1 - \alpha H_2))^{\perp} \subset N_1(X)$. By costruction, we see that $R' \subset F$. Since there exists at most two extremal rays contained in the real 2-dimensional vector subspace F, there exist at most 4 extremal rays R' which is of fiber type and different from R. □

Remark 8.7. (1) Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold with $\kappa(X) = -\infty$. Then, with the aid of Theorem 8.6, we infer that the number of extremal rays of X is finite if and only if the number of extremal rays of X of divisorial type is finite.

(2) Let E be an elliptic curve and $\mu\colon E \to E$ multiplication by $n\,(> 1)$. Then the product

$X := E \times \mathbb{P}^1 \times \mathbb{P}^1$ admits a non-isomorphic étale endomorphism $f\colon X \to X$ of X defined by $f := \mu_n \times \mathrm{id}_{\mathbb{P}^1} \times \mathrm{id}_{\mathbb{P}^1}$. There exist exactly two extremal rays on X. The projections $p_{1,2}\colon X \to E \times \mathbb{P}^1$ and $p_{1,3}\colon X \to E \times \mathbb{P}^1$ are all extremal contractions.

9. Some reductions of a conic bundle to a $\mathbb{P}^1$-bundle

Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C) constructed from X by a sequence of blowing-downs of an ESP. In this section, we show that $f\colon X \to X$ can be lifted to a non-isomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$, where $\widetilde{X}$ is a suitable finite étale covering of X and is obtained from a $\mathbb{P}^1$-bundle by successive blowing-ups along elliptic curves. We follow some arguments by Mori and Mukai (cf. [30]) used in the classification of Fano 3-folds.

Proposition 9.1. *Let $f\colon Y \to Y$ be a finite étale covering of a smooth projective 3-fold Y with $\kappa(Y) = -\infty$. Suppose that;*

(1) *There exists a conic bundle $\varphi\colon Y \to S$ over a product surface $S := C \times E$ of a smooth curve C and an elliptic curve E.*
(2) *φ is not smooth and the discriminant locus Δ_φ of φ is contained in fibers of the first projection $p_1\colon S \to C$.*
(3) *There exists a Lie group homomorphism $\alpha\colon E \to E$ such that $g \circ \varphi = \varphi \circ f$ for an étale endomorphism $g := \mathrm{id}_C \times \alpha\colon S \to S$.*

Then, by replacing f by a suitable power f^k ($k > 0$), there exist a finite étale Galois covering $\psi\colon \widetilde{Y} \to Y$ and a commutative diagram

$$\begin{array}{ccc}
\widetilde{Y} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
M & \xrightarrow{\overline{f}} & M \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
\widetilde{S} & \xrightarrow{\widetilde{g}} & \widetilde{S} \\
\downarrow & & \downarrow \\
C & = & C
\end{array}$$

satisfying the following conditions:

(1) *$\widetilde{f}\colon \widetilde{Y} \to \widetilde{Y}$ is an étale endomorphism such that $f \circ \psi = \psi \circ \widetilde{f}$.*
(2) *$\pi\colon \widetilde{Y} \to M$ is a succession of blowing-ups of a smooth projective 3-fold M along elliptic curves.*
(3) *$\overline{f}\colon M \to M$ is an étale endomorphism.*
(4) *$q\colon M \to \widetilde{S}$ is a $\mathbb{P}^1$-bundle over a surface $\widetilde{S}$ which is a copy of S.*
(5) *$\widetilde{g}\colon \widetilde{S} \to \widetilde{S}$ is an étale endomorphism of $\widetilde{S} \simeq S$ such that $\widetilde{g} := \mathrm{id}_C \times \alpha$.*

Proof. By assumption, the discriminant loci Δ_φ consist of a disjoint union of elliptic curves $\Delta_1, \cdots, \Delta_r$ which are fibers of p_1 such that $g^{-1}(\Delta_i) = \Delta_i$ for any i.

Step 1. Suppose that φ is not an extremal contraction. Then, by [30, Proposition 4.8], $\varphi^{-1}(\Delta_i)$ is reducible for some Δ_i. Let $\varphi^{-1}(\Delta_i) = E_i^{(1)} + E_i^{(2)}$ be an irreducible decomposition. Then, by [30, Proposition 4.9], $\varphi|_{E_i^{(j)}}\colon E_i^{(j)} \to \Delta_i$ is a $\mathbb{P}^1$-bundle over Δ_i with fiber $\ell_i^{(j)}$ for each $j = 1, 2$. Replacing f by f^2, we may assume that $f^{-1}(E_i^{(j)}) = E_i^{(j)}$ for any $j = 1, 2$. Then, $-K_Y - E_i^{(j)}$ is φ-nef for each j and

$$\overline{\mathrm{NE}}(Y/S) \cap (-K_Y - E_i^{(j)})^{\perp} = \mathbb{R}_{\geq 0}\,[\ell_i^{(j')}]$$

for $j' \in \{1, 2\} \setminus j$. Hence, each $[\ell_i^{(j)}]$ spans an extremal ray $R_i^{(j)}$. Let $\pi_i^{(j)} := \mathrm{Cont}_{R_i^{(j)}}\colon Y \to Y_i^{(j)}$ be the contraction morphism associated to $R_i^{(j)}$. Then, there exists a conic bundle $\varphi_i^{(j)}\colon Y_i^{(j)} \to S$ such that $\varphi = \varphi_i^{(j)} \circ \pi_i^{(j)}$. Since $f_*[\ell_i^{(j)}] = [\ell_i^{(j)}]$ for any j, there exists an étale endomorphism $f_i^{(j)}\colon Y_i^{(j)} \to Y_i^{(j)}$ such that $f_i^{(j)} \circ \pi_i^{(j)} = \pi_i^{(j)} \circ f$. Thus we have obtained the following commutative diagram which we are looking for:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\pi_i^{(j)}"'] & Y \arrow[d, "\pi_i^{(j)}"] \\
Y_i^{(j)} \arrow[r, "f_i^{(j)}"] \arrow[d, "\varphi_i^{(j)}"'] & Y_i^{(j)} \arrow[d, "\varphi_i^{(j)}"] \\
S \arrow[r, "g"] \arrow[d, "p_1"'] & S \arrow[d, "p_1"] \\
C \arrow[r, equal] & C
\end{tikzcd}$$

Step 2. Suppose that φ is an extremal contraction. Then, by [30], each $\varphi^{-1}(\Delta_i)$ is irreducible. Let $e \in E$ be the zero element of the elliptic curve E and put $o_i := (\{e\} \times C) \cap \Delta_i$. Let $\mathcal{S}$ be the set of two irreducible components of $\varphi^{-1}(o_i)$. Then, an analytic continuation along a loop $\gamma \in \pi_1(\Delta_i, o_i)$ induces a permutation of $\mathcal{S}$. Hence, there exists a monodromy representation $\chi_i\colon \pi_1(\Delta_i, o_i) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$, which is surjective, since $\varphi^{-1}(\Delta_i)$ is irreducible. Let $\widetilde{\Delta}_i \to \Delta_i$ be an étale double covering associated to $\mathrm{Ker}\,\chi_i$. By the composition of a suitable isogeny $\Delta_i \to \widetilde{\Delta}_i$, we may assume that $\Delta_i \to \Delta_i$ is multiplication by m_i. Let

$$\rho_i := \mathrm{id}_C \times \mu_{m_i}\colon S_i = C \times \Delta_i \to S = C \times E$$

be a finite étale covering. Since μ_{m_i} commutes with $\alpha\colon \Delta_i \to \Delta_i$, g can be lifted to an étale endomorphism $\widetilde{g} := \mathrm{id}_C \times \alpha\colon S_i \to S_i$. Furthermore, $\widetilde{Y}_i := Y_i \times_S S_i \to Y$ is also a finite étale covering and there exists a conic bundle $\widetilde{\varphi}_i\colon \widetilde{Y}_i \to S$. Since $\varphi \circ f = g \circ \varphi$ and $\widetilde{g}$ is a lift of g, there is induced an étale endomorphism $\widetilde{f}_i\colon \widetilde{Y}_i \to \widetilde{Y}_i$ which is a lift of $f\colon Y \to Y$. By construction, $\widetilde{\Delta}_i := \rho_i^{-1}(\Delta_i)$ is reducible. Hence, by Step 1, there exists a blowing-up $\pi_i\colon \widetilde{Y}_i \to \widetilde{Y}_i'$ along an elliptic curve on a conic bundle $\widetilde{\varphi}_i\colon \widetilde{Y}_i' \to S_i$. Moreover, if we replace f by f^2, $\widetilde{f}_i$ descends to an étale endomorphism $\widetilde{f}_i'\colon \widetilde{Y}_i' \to \widetilde{Y}_i'$.

In the general case, suppose that $\varphi^{-1}(\Delta_i)$ is reducible for all $1 \leq i \leq k$, $(k \leq r)$ and irreducible otherwise. Let $\mu\colon E \to E$ be a multiplication mapping which factors through $\mu m_i\colon \Delta_i \to \Delta_i$ for all $i > k$. If we take the base change $\widetilde{Y} = Y \underset{E,\mu}{\times} E$ and perform the same procedure as above, then the proof of proposition is finished. □

Remark 9.2. Proposition 9.1 holds even if f is an isomorphism.

In the case where there exists an FESP of type (C_1), we encounter the situation produced in Proposition 9.1 with the aid of Proposition 7.3 and Corollary 8.1.

Theorem 9.3. *Let $f: X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_1) constructed from X by a sequence of blowing-downs of an ESP. Then, by relplacing f by a suitable power f^k ($k > 0$), there exist a finite étale Galois covering $\rho: \widetilde{X} \to X$ and a Cartesian morphism of constant ESPs*

$$(\widetilde{X}, \widetilde{f}) \xrightarrow{\pi'} (M, f') \xrightarrow{q} (\widetilde{S}, \widetilde{h}),$$

satisfying the following conditions:

(1) *$\pi': \widetilde{X} \to M$ is a succession of blowing-ups of a smooth projective 3-fold M along elliptic curves and $f': M \to M$ is a non-isomorphic étale endomorphism.*
(2) *$q: M \to \widetilde{S}$ is a $\mathbb{P}^1$-bundle over the product surface $\widetilde{S} := \widetilde{C} \times E$ of a smooth curve $\widetilde{C}$ and an elliptic curve E.*
(3) *$\widetilde{h} := \mathrm{id}_{\widetilde{C}} \times \alpha: \widetilde{S} \to \widetilde{S}$ for a Lie group homomorphism $\alpha: E \to E$.*
(4) *By q, the centers of the blowing-up π' are mapped onto the fibers of the first projection $p: \widetilde{S} \to \widetilde{C}$.*

Proof. By Corollary 8.1, replacing f by a suitable power f^k ($k > 0$), we may assume that the conclusions of Corollary 8.1 are satisfied. Since S is a minimal algebraic surface with $\kappa(S) = 1$ and $e(S) = 0$, the Iitaka fibration $\psi: S \to C$ gives a Seifert elliptic surface structure. By [40], an étale endomorphism $h: S \to S$ induces a finite automorphism of the base curve C. Hence, replacing f by a suitable power f^ℓ ($\ell > 0$), we may assume that $\psi \circ h = \psi$. Thus we have the following Cartesian morphism of constant ESP:

$$(X, f) \xrightarrow{\pi} (Y, g) \xrightarrow{\varphi} (S, h),$$

where π is a sequence of equivariant blowing-downs and φ is a conic bundle. By [13, Theorem 2.24], the irreducible component of the fixed point locus $\mathrm{Fix}(h)$ of h is a finite étale covering of C. Hence, by [13, Lemma 2.25], we can take a finite étale covering $C' \to C$ such that the normalization S' of the fiber product $S \times_C C'$ satisfies the following;

(i) S' is isomorphic over C' to the direct product $C' \times E$ of a smooth curve C' of genus $p_a(C') \geq 2$ and an elliptic curve E.
(ii) There exists a non-isomorphic étale endomorphism $\widetilde{h}: S' \to S'$ which is a lift of $h: S \to S$ such that $\widetilde{h} = \mathrm{id}_{C'} \times \mu_\alpha$ for a non-isomorphic Lie group homomorphism $\mu_\alpha: E \to E$.

Then, if we put $\overline{X} := X \times_C C'$ and $\overline{Y} = Y \times_C C'$, the following conditions are satisfied;

(a) $\overline{X}$ and $\overline{Y}$ are both non-singular and there exists a birational morphism $\overline{\pi}: \overline{X} \to \overline{Y}$ which is a succession of blowing-ups along elliptic curves.
(b) There exist isomorphisms $\overline{X} \cong X \times_S S'$ and $\overline{Y} \cong Y \times_S S'$.
(c) $\widetilde{\varphi}: \overline{Y} \to S'$ is a conic bundle.

(d) There exists a non-isomorphic étale endomorphism $\overline{f}\colon \overline{X} \to \overline{X}$ (resp. $\overline{g}\colon \overline{Y} \to \overline{Y}$) which is a lift of $f\colon X \to X$ (resp. $g\colon Y \to Y$.)

By Proposition 7.3, the discriminant locus $\mathrm{disc}(\overline{\varphi})$ is a disjoint union of elliptic curves supported in fibers of the first projection $S' \cong C' \times E \to C'$. Hence, by Proposition 9.1, if we replace f by a suitable power f^k $(k > 0)$, the following conditions are satisfied.

(A) There exists a finite étale Galois covering $\widetilde{Y} \to \overline{Y}$ and a non-isomorphic étale endomorphism $\widetilde{g}\colon \widetilde{Y} \to \widetilde{Y}$ which is a lift of $\overline{g}\colon \overline{Y} \to \overline{Y}$.

(B) There exists a succession of blowing-ups $\tau\colon \widetilde{Y} \to M$ of a smooth projective 3-fold M along elliptic curves and there induced a non-isomorphic étale endomorphism $f'\colon M \to M$.

(C) M is a $\mathbb{P}^1$-bundle over a surface $\widetilde{S}$ which is a copy of S'.

(D) By a non-isomorphic étale endomorphism $f'\colon M \to M$, there induced a non-isomorphic étale endomorphism $\widetilde{h} = \mathrm{id}_C \times \alpha\colon \widetilde{S} \to \widetilde{S}$.

Thus we are done. □

REMARK 9.4. We give such an example to illustrate our idea. Let $\widetilde{C} \to C$ be an étale double covering of a smooth curve C with $g(C) \geq 2$ and $i\colon \widetilde{C} \simeq \widetilde{C}$ an involution such that $\widetilde{C}/\langle i\rangle \simeq C$. Take a line bundle ℓ of order two on an elliptic curve E and set $p\colon S := \mathbb{P}_E(\mathcal{O}_E \oplus \ell) \to E$. Let s_k $(k = 0, 1)$ be two disjoint sections of p corresponding to a surjection $\mathcal{O}_E \oplus \ell \twoheadrightarrow \mathcal{O}_E$ (resp. $\mathcal{O}_E \oplus \ell \twoheadrightarrow \ell$). Then by [22, Theorem 2.(4)], there exists a relative automorphism $u \in \mathrm{Aut}(S/E)$ of order two such that $u(s_k) = s_{1-k}$ for $k = 0, 1$. Then $j := i \times u$ defines a free involutive automorphism of $\widetilde{C} \times S$ and let $Y := \widetilde{C} \times S/\langle j\rangle$ be its quotient. For an odd integer $n > 1$, let $\mu_n\colon E \to E$ be multiplication by n. Since $\mu_n^*\ell \sim \ell$ by [32, p75, (ii)], there is induced a non-isomorphic étale endomorphism $g\colon S \simeq S \times_{E,\mu_n} E \to S$. We define a finite étale endomorphism ψ of $\widetilde{C} \times S$ by $\psi := \mathrm{id}_{\widetilde{C}} \times g$. Since $g \circ j = j \circ g$, ψ commutes with the involution j of $\widetilde{C} \times S$ and descends to a non-isomorphic étale endomorphism $\varphi\colon Y \to Y$ over C. By the natural projection $Y \to T := C \times E \to \widetilde{C}/\langle i\rangle \simeq C$, Y is a $\mathbb{P}^1$-bundle over T and is also a fiber bundle over C whose fiber is isomorphic to S.

The same method applies to other situation. We consider the case where there exists an FESP of type (C_0) constructed from X by a sequence of blowing-downs of an ESP.

Theorem 9.5. *Let $f\colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type (C_0) constructed from X by a sequence of blowing-downs of an ESP. Then, by replacing f by a suitable power f^k $(k > 0)$, there exists a finite étale Galois covering $\rho\colon \widetilde{X} \to X$ and the Cartesian morphism of constant ESPs*

$$(\widetilde{X}, \widetilde{f}) \xrightarrow{\pi'} (M, f') \xrightarrow{q} (A, h'),$$

which satisfies the following properties;

(1) *There exists a succession of blowing-ups $\pi'\colon \widetilde{X} \to M$ of a smooth projective 3-fold M along elliptic curves such that $\pi' \circ \widetilde{f} = f' \circ \pi'$ for some non-isomorphic étale endomorphism $f'\colon M \to M$.*

(2) *$q\colon M \to A$ is a $\mathbb{P}^1$-bundle over an abelian surface A.*

(3) *If π' is not an isomorphism, then A can be chosen to be a direct product $E \times E'$ of elliptic curves E and E' such that $h' = \mathrm{id}_E \times \mu_\alpha$ for some Lie group homomorphism $\mu_\alpha \colon E' \to E'$.*

(4) *The centers of the blowing-up π' are mapped onto the fibers of the first projection $A := E \times E' \to E$.*

Proof. By Corollary 8.1 and Propositions 7.3, 3.8, there exists a constant FESP $Y_\bullet := (Y, g)$. Then there exist Cartesian morphisms of constant ESPs

$$X_\bullet := (X, f) \xrightarrow{\pi} Y_\bullet := (Y, g) \xrightarrow{\varphi} S_\bullet := (S, h)$$

such that the following conditions are satisfied;

(1) π is a succession of blowing-ups along elliptic curves on Y.

(2) $\varphi \colon Y \to S$ is a conic bundle.

(3) S is isomorphic to an abelian surface or a hyperelliptic surface.

Step 1. First we show that the proof can be reduced to the case where S is an abelian surface. Suppose that S is a hyperelliptic surface. Then some finite étale Galois cover $p \colon \widetilde{S} \to S$ of S is isomorphic to an abelian surface A. By [40], there exists a non-isomorphic étale endomorphism $\widetilde{h} \colon A \to A$ which is a lift of $h \colon S \to S$. Let $\widetilde{X} := X \times_S A$ (resp. $\widetilde{Y} := Y \times_S A$) be the pull-back of $\varphi \circ \pi \colon X \to S$ (resp. $\varphi \colon Y \to S$) by the étale morphism $p \colon A \to S$. Then the natural projection $\widetilde{X} \to X$ (resp. $\widetilde{Y} \to Y$) is a finite étale Galois covering. Thus there exist Cartesian morphisms of constant ESP

$$\widetilde{X}_\bullet := (\widetilde{X}, \widetilde{f}) \xrightarrow{\widetilde{\pi}} \widetilde{Y}_\bullet := (\widetilde{Y}, \widetilde{g}) \xrightarrow{\widetilde{\varphi}} A_\bullet := (A, \widetilde{h})$$

such that the following conditions are satisfied:

(1) $\widetilde{\pi}$ is a succession of blowing-ups along elliptic curves on a smooth 3-fold $\widetilde{Y}$.

(2) $\widetilde{f}$ (resp. $\widetilde{g}$) and is a lift of f (resp. g).

Thus we may assume that $S = A$ is an abelian surface.

Step 2. Then by Propositions 7.3 and 7.8, the set $M := \varphi(\mathrm{Exc}(\pi)) \cup \mathrm{disc}(\varphi)$ is a disjoint union of finitely many elliptic curves on S. Moreover, there is induced a permutation of the finite set M by the étale endomorphism $h \colon A \to A$. Hence if we replace an endomorphism f by its suitable power f^k ($k > 0$), we may assume that $h^{-1}(\Delta_i) = \Delta_i$ for any connected component Δ_i of M. Thus the abelian surface A is an elliptic bundle $p \colon A \to E$ over an elliptic curve E so that $p \circ h = p$. We have the following Cartesian morphisms of constant ESP;

$$X_\bullet := (X, f) \xrightarrow{\pi} Y_\bullet := (Y, g) \xrightarrow{\varphi} A_\bullet := (A, h).$$

Applying Proposition 9.1, we can finish the proof in the same way as that of Theorem 9.3. □

Suppose that there exists an FESP of type ($\mathrm{C}_{-\infty}$) constructed from X by a sequence of blowing-downs of an ESP. Note that we cannot necessarily obtain another *constant* FESP, since there may exist infinitely many extremal rays of $\overline{\mathrm{NE}}(X)$.

Theorem 9.6. *Let $f \colon X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Suppose that there exists an FESP $(Y_\bullet, R_\bullet)$ of type*

$(C_{-\infty})$ constructed from X by a sequence of blowing-downs of an ESP. Then, there exists a finite étale Galois covering $\rho\colon \widetilde{X} \to X$ which satisfies the following properties:

(1) *There exists a non-isomorphic étale endomorphism $\widetilde{X} \to \widetilde{X}$ such that $\rho \circ \widetilde{f} = f \circ \rho$.*

(2) *$Z_\bullet = (g_n\colon Z_n \to Z_{n+1})_n$ is an FESP constructed from $\widetilde{X}_\bullet = (\widetilde{X}, \widetilde{f})$ by a sequence of blowing-downs $\pi_\bullet = (\pi_n)_n\colon \widetilde{X}_\bullet \to Z_\bullet$ of an ESP.*

(3) *Z_n is a $\mathbb{P}^1$-bundle $\varphi_n\colon Z_n \to S_n$ over a smooth algebraic surface S_n.*

(4) *There exists an ESP $S_\bullet = (h_n\colon S_n \to S_{n+1})_n$ such that $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$.*

(5) *S_n is a $\mathbb{P}^1$-bundle $\alpha_n\colon S_n \to C$ over the Albanese elliptic curve C of X.*

(6) *The composite map $\alpha_n \circ \varphi_n \circ \pi_n\colon X \to C$ is independent of n and coincides with the Albanese map $\mathrm{Alb}_X\colon X \to C$ of X.*

(7) *There exists a non-isomorphic Lie group homomorphism $u\colon C \to C$ such that $u \circ \alpha_n = \alpha_{n+1} \circ h_n$.*

(8) *$S := S_0$ is isomorphic to either the Atiyah surface $\mathbb{S}$ or $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_0)$ for some $\ell_0 \in \mathrm{Pic}^0(C)$. Moreover, in the former case, we have $S_n \simeq \mathbb{S}$ for any n. In the latter case, any S_n is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for some $\ell_n \in \mathrm{Pic}^0(C)$.*

Proof. Set $Y_\bullet = (g_n\colon Y_n \to Y_{n+1})_n$ and $R_\bullet = (R_n)_n$, where $(g_n)_* R_n = R_{n+1}$ for any n. Let $\pi_\bullet = (\pi_n)_n\colon X_\bullet \to Y_\bullet$ be the Cartesian blowing-up. By assumption, each $\varphi_n\colon Y_n \to S_n$ is a conic bundle over a smooth surface S_n and there exists an ESP $S_\bullet = (h_n\colon S_n \to S_{n+1})_n$ such that $\varphi_{n+1} \circ g_n = h_n \circ \varphi_n$. Moreover, by the Albanese map $\alpha_n\colon S_n \to C_n$, S_n is a $\mathbb{P}^1$-bundle over an elliptic curve C_n and there exists a finite étale morphism $u_n\colon C_n \to C_{n+1}$ such that $u_n \circ \alpha_n = \alpha_{n+1} \circ h_n$. For any n, the composite map $\psi_n := \alpha_n \circ \varphi_n \circ \pi_n\colon X \to C_n$ is isomorphic to the Albanese map Alb_X of X and also gives the MRC fibration of X. Hence any C_n is isomorphic to the Albanese elliptic curve C of X and there exists an isomorphism $v_n\colon C_n \simeq C_0$ such that $\psi_0 = v_n \circ \psi_n$ for any n. Thus, if we replace α_n (resp. ψ_n) by the composite $v_n \circ \alpha_n$ (resp. $v_n \circ \psi_n$), we may assume that $\psi_n\colon X \to C$ is independent of n (i.e., $\psi_n \equiv \psi$) and coincides with the Albanese map Alb_X of X. By the universality of the Albanese map of X, the map $u_n\colon C \to C$ is independent of n, and $u := u_n$ satisfies $\psi \circ f = u \circ \psi$ and $\alpha_{n+1} \circ h_n = u \circ \alpha_n$. Since $\deg u = \deg h_n = \deg g_n = \deg f > 1$, $u\colon C \to C$ has a fixed point P. Hence C is endowed with an abelian group structure with P as the unit element and u is a group homomorphism. By Propositions 4.1 and 4.4, there exists a multiplication map $\mu_k\colon C \to C$ by some integer k such that the pull-back $\widetilde{S_0} := S_0 \times_{C,\mu_k} C$ is isomorphic to either the Atiyah surface $\mathbb{S}$ or $\mathbb{P}_C(\mathcal{O}_C \oplus \ell'_0)$ for some $\ell'_0 \in \mathrm{Pic}^0(C)$. We set $\widetilde{X} := X \times_{C,\mu_k} C$. Since $\mu_k \circ h = h \circ \mu_k$, there exists a non-isomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of $f\colon X \to X$. Hence if we replace X by $\widetilde{X}$ (resp. f by $\widetilde{f}$), we may assume from the beginning that either $S_0 = \mathbb{S}$ or $S_0 \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell'_0)$ for some $\ell'_0 \in \mathrm{Pic}^0(C)$. By Proposition 7.3, the discriminant locus $\Delta_{(\psi_0)}$ of ψ_0 is a disjoint union of elliptic curves, i.e., $\Delta_{(\psi_0)} = \bigsqcup_i \Delta_i$ such that $\alpha_0(\Delta_i) = C$ for all i. If $S_0 \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell')$ (resp. $\mathbb{S}$), then any elliptic curve Δ_i is a multi-section of α_0 (resp. the canonical section s_∞). Hence we can apply the same method as in the proof of Proposition 9.1. Then there exists some multiplication mapping $\mu_m\colon C \to C$ for some integer m such that the following conditions are satisfied.

- If we set $\widetilde{X} := X \times_{C,\mu_m} C$, then there exists a non-isomorphic étale endomorphism $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ which is a lift of $f\colon X \to X$.
- $\pi_0\colon \widetilde{X} \to Z_0$ is a succession of blowing-ups of a smooth projective 3-fold Z_0 along elliptic curves.

- $\varphi_0 : Z_0 \to S_0$ is a $\mathbb{P}^1$-bundle over an elliptic ruled surface S_0.
- S_0 is isomorphic to either $\mathbb{S}$ or $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_0)$ for some $\ell_0 \in \mathrm{Pic}^0(C)$.

By Corollary 1.2, we can construct the FESP $Z_\bullet := (g_n : Z_n \to Z_{n+1})_n$ of $f : X \to X$. Then applying Proposition 7.3, we infer that $Z_\bullet$ is an FESP of type (C). There exists a $\mathbb{P}^1$-bundle structure $\varphi_n : Z_n \to S_n$ over an elliptic ruled surface S_n for any n, since g_n is étale and $\varphi_0 : Z_0 \to S_0$ is a $\mathbb{P}^1$-bundle. The assertion (8) follows from Proposition 5.10. Thus all the properties in the statement are satisfied. □

10. Possibilities of the base surface of type ($\mathrm{C}_{-\infty}$)

In this section, as an illustration of how an *étaleness assumption* of a given endomorphism $f : X \to X$ is used, we consider the case where there exists an FESP $(Y_\bullet, R_\bullet)$ of type ($\mathrm{C}_{-\infty}$) constructed from (X, f) by a sequence of blowing-downs of an ESP, i.e., for $Y_\bullet = (g_n : Y_n \to Y_{n+1})_n$, each $\varphi_n : Y_n \to S_n$ is a conic bundle over an elliptic ruled surface S_n. Combining Propositions 4.8, 5.15 and 5.16, we show that the base space S_n is of very limited type. Theorem 10.1 below provides a key to many of the results in our subsequent articles. It is also related to the following basic fact concerning the existence of endomorphisms of elliptic ruled surfaces (cf. Corollary 4.2, Remark 4.3, Proposition 4.8):

- *Any* $\mathbb{P}^1$-bundle S over an elliptic curve C admits a non-isomorphic surjective endomorphism.
- Let $\mathcal{L}$ be a line bundle on an elliptic curve C. Then the elliptic ruled surface $S := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$ admits an ESP $S_\bullet = (g_n : S_n \to S_{n+1})_n$ of elliptic ruled surfaces over C such that $S_0 = S$ if and only if $\deg \mathcal{L} = 0$.
- S admits a non-isomorphic étale endomorphism if and only if $\mathcal{L} \in \mathrm{Pic}(C)$ is torsion.

One of the applications of Theorem 1.5 is to give classifications of 3-folds admitting an FESP of type (D), which will be considered in our subsequent article; Part III. Now, we shall start the proof of the following which is a variant of Theorem 1.5.

Theorem 10.1. *Let $f : X \to X$ be a non-isomorphic étale endomorphism of a smooth projective 3-fold X with $\kappa(X) = -\infty$. Let $Y_\bullet = (g_n : Y_n \to Y_{n+1})_n$ be an FESP and $\pi_\bullet = (\pi_n)_n : X_\bullet := (X, f) \to Y_\bullet$ a succession of Cartesian blowing-ups along elliptic curves. Suppose that there exists an ESP $S_\bullet = (u_n : S_n \to S_{n+1})_n$ of smooth algebraic surfaces S_n and a Cartesian morphism $\varphi_\bullet = (\varphi_n)_n : Y_\bullet \to S_\bullet$ such that the following hold:*

(1) *$\varphi_n : Y_n \to S_n$ is a $\mathbb{P}^1$-bundle for any n.*
(2) *Any S_n is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}_C(\mathcal{O}_C \oplus \ell_n)$ for a line bundle ℓ_n of degree 0 on the Albanese elliptic curve C of X.*

Then $\ell_n \in \mathrm{Pic}^0(C)$ is of finite order for any n.

Proof. By definition, we have a Cartesian morphism of ESPs

$$(X, f) \xrightarrow{\pi_\bullet} Y_\bullet \xrightarrow{\varphi_\bullet} S_\bullet \xrightarrow{\alpha_\bullet = (\alpha_n)} (C, h)$$

in which the following are satisfied for any n.

- π_n is a succession of Cartesian blowing-ups along elliptic curves,
- φ_n is a $\mathbb{P}^1$-bundle,
- the Albanese map $\alpha_n : S_n \to C$ gives S_n a $\mathbb{P}^1$-bundle structure over C,

- h is a non-isomorphic group homomorphism of an elliptic curve C, and
- the composite map $\alpha_n \circ \varphi_n \circ \pi_n \colon X \to C$ is independent of n and coincides with the Albanese map $\alpha_X \colon X \to C$ of X.

We may assume that $n = 0$ without loss of generality. The proof is by contradiction. Suppose the contrary that $S_0 \simeq \mathbb{P}_C(\mathcal{O}_C \oplus \ell_0)$ for $\ell_0 \in \mathrm{Pic}^0(C)$ which is of infinite order. Then, by Proposition 5.10, we see that $\ell_n \in \mathrm{Pic}^0(C)$ is of infinite order for *any* n.

Step 1: First, we show that there exists some extremal ray of divisorial type on X. Suppose the contrary. Then $\varphi := \varphi_0 \colon X \simeq Y_0 \to S := S_0$ is an extremal contraction giving a $\mathbb{P}^1$-bundle structure. Hence, applying Theorem 3.10 and Proposition 7.3 (2), there is induced a non-isomorphic étale endomorphism $u \colon S \to S$ such that $\varphi \circ f^k = u \circ \varphi$ for some $k > 0$. Thus by Proposition 4.8, $\ell_0 \in \mathrm{Pic}^0(C)$ is of finite order, which contradicts the assumption.

Step 2: Let $D_0^{(n)}$ (resp. $D_\infty^{(n)}$) be the section of $\alpha_n \colon S_n \to C$ corresponding to the first projection $\mathcal{O}_C \oplus \ell_n \twoheadrightarrow \ell_n$ (resp. the second projection $\mathcal{O}_C \oplus \ell_n \twoheadrightarrow \mathcal{O}_C$). By replacing ℓ_n by $\ell_n^{\otimes -1}$ if necessary, we may assume that $u_n^{-1}(D_0^{(n+1)}) = D_0^{(n)}$ and $u_n^{-1}(D_\infty^{(n+1)}) = D_\infty^{(n)}$ for all n. Then by Corollary 7.9, the $\mathbb{P}^1$-fiber space $\psi_n := \varphi_n \circ \pi_n \colon X \to S_n$ over S_n is a $\mathbb{P}^1$-bundle outside $D_0^{(n)} \cup D_\infty^{(n)}$. Proposition 7.10 yields that all the irreducible components of Δ_i of $\psi_0^{-1}(D_0 \sqcup D_\infty)$ are elliptic ruled surfaces and cross normally with each other.

We show that there exists an elliptic ruled surface $\Delta (\subset X)$ such that $\psi_0(\Delta)$ coincides with either of the two disjoint sections of α_0, and $\psi_n(\Delta) = S_n$ for $n \gg 0$.

Lemma 10.2. *Let $D_0 \sqcup D_\infty$ be a union of two disjoint sections of the $\mathbb{P}^1$-bundle $\alpha_0 \colon S_0 := \mathbb{P}_C(\mathcal{O}_C \oplus \ell_0) \to C$ defined as above. Then, there exists an irreducible component $\Delta (\subset \psi_0^{-1}(D_0 \sqcup D_\infty))$ which is isomorphic to an elliptic ruled surface of the form $\mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L})$, where $\mathcal{L} \in \mathrm{Pic}^0(C')$ of infinite order on an elliptic curve C'.*

Proof. By Proposition 7.10, all the irreducible components Δ_i of $\psi_0^{-1}(D_0 \sqcup D_\infty)$ are elliptic ruled surfaces which cross normally. Furthermore, they are either the proper transform of $\varphi_0^{-1}(D_0 \sqcup D_\infty)$ or the π_0-exceptional divisors. By Proposition 1.1, $f^{-n}(\Delta_i)$ is irreducible for any $n \geq 0$ and there exists an ESP $\Delta_{i,\bullet} = (f \colon \Delta_{i,n} \to \Delta_{i,n+1})_n$ of elliptic ruled surfaces $\Delta_{i,n} := f^n(\Delta_i)$ ($n \in \mathbb{Z}$). Then by Proposition 4.1, we see that any Δ_i isomorphic over C' to an elliptic ruled surface of the form $\mathbb{P}_{C'}(\mathcal{E})$ for a semi-stable vector bundle $\mathcal{E}$ such that $\mathcal{E}$ is either indecomposable of rank 2 or isomorphic to $\mathcal{O}_{C'} \oplus \mathcal{L}_i$ for a line bundle $\mathcal{L}_i$ of degree 0 on C'.

Suppose that for *any* i, Δ_i is isomorphic to $\mathbb{P}_{C'}(\mathcal{E})$ for an *indecomposable* vector bundle $\mathcal{E}$, or $\Delta_i \simeq \mathbb{P}_{C'}(\mathcal{O}_{C'} \oplus \mathcal{L}_i)$ for some *torsion* line bundle $\mathcal{L}_i \in \mathrm{Pic}^0(C')$ and we shall derive a contradiction. Since $f \colon X \to X$ is a finite étale covering, $\Delta_{i,n}$ is also an elliptic ruled surface of the same type as Δ_i for any n by Proposition 5.10. Hence, by Proposition 5.6 and Corollary 5.7, $\psi_k(\Delta_{i,\ell})$ is an irreducible curve on S_k for any $k, \ell \in \mathbb{Z}$. For any $m \in \mathbb{Z}$, let $\Delta_{i,\bullet}[m]$ be the ESP which is a shift of $\Delta_{i,\bullet}$ by m, i.e., $\Delta_{i,k}[m] = \Delta_{i,k+m}$ for any k and $\Delta_{i,\bullet}[m] = (f|_{\Delta_{i,k+m}} \colon \Delta_{i,k+m} \to \Delta_{i,k+m+1})_k$. Since $f \colon X \to X$ is an *endomorphism*, there exist the following Cartesian morphisms of ESPs;

$$\begin{tikzcd} \Delta_{i,\bullet}[m] \arrow[r, "i_\bullet"] & X_\bullet \arrow[d, "\psi_\bullet = \varphi_\bullet \circ \pi_\bullet"] \\ & S_\bullet \end{tikzcd}$$

Applied Lemma 2.6 to this Cartesian diagram, there is induced an ESP $\psi_\bullet(\Delta_{i,\bullet}[m])$ of elliptic curves such that the inclusion $\psi_\bullet(\Delta_{i,\bullet}[m]) \hookrightarrow S_\bullet$ is Cartesian. Set $\Gamma_{i,m} := \psi_0(\Delta_{i,m})$. Then by Proposition 6.9 (4.3), we have $\Gamma_{i,m} = D_\infty$ or D_0 for any m. Thus for any m and i, the surface $\Delta_{i,m} := f^m(\Delta_i)$ coincides with some of the irreducible components of $\psi_0^{-1}(D_0 \sqcup D_\infty)$, whose number is finite. Hence for any $m > 0$, the m-th power $f^m \colon X \to X$ gives rise to a permutation of the finite set M consisting of all the irreducible components of $\psi_0^{-1}(D_0 \sqcup D_\infty)$. Hence, a suitable power $f^k \colon X \to X$ of f gives rise to the identity permutation of the set M. In particular, all the irreducible components of π_0-exceptional divisors are preserved by f^k. Thus the MMP works compatibly with étale endomorphisms and there exists a constant ESP $Y_\bullet := (Y, g)$ of $X_\bullet := (X, f^k)$. Then by Theorem 3.10 and Proposition 7.3, there exists a non-isomorphic étale endomorphism $u_0 \colon S_0 \to S_0$ such that $\psi_0 \circ f^{kr} = u_0 \circ \psi_0$ for some $r > 0$. Hence by Proposition 4.8, $\ell_0 \in \mathrm{Pic}^0(C)$ is of finite order. Thus a contradiction is derived and we are done. □

Lemma 10.3. *Let Δ be as in Lemma 10.2. Then,*

(1) $\psi_n(\Delta) = S_n$ *for a sufficiently large positive integer n.*

(2) $\psi_0(f^{-n}(\Delta)) = S_0$ *for a sufficiently large positive integer n.*

Proof. We use the same notation as in the proof of Lemma 10.2. First we shall prove the assertion (2). The proof is by contradiction. Suppose that for infinitely many positive integer n, $\gamma_n := \psi_0(f^{-n}(\Delta))$ is an irreducible curve. Then $\gamma_n = D_0$ or D_∞ by the same argument as in the proof of Lemma 10.2. Hence, the irreducible surface $f^{-n}(\Delta)$ is contained in $\psi_0^{-1}(D_0 \sqcup D_\infty)$ for infinitely many n. Thus, there exists positive integers $p < q$ such that $f^{-p}(\Delta) = f^{-q}(\Delta)$. Since $T = f^{q-p}(T)$ for the elliptic ruled surface $T := f^{-q}(\Delta)$ and $f^{-1}(f^{n+1}(\Delta)) = f^n(\Delta)$ for n, $f^{q-p}|_T \colon T \to T$ defines a non-isomorphic étale endomorphism of T. Hence by Proposition 4.8, $T \simeq \mathbb{P}_{C''}(\mathcal{O}_{C''} \oplus \ell'')$ for some line bundle $\ell'' \in \mathrm{Pic}^0(C'')$ of finite order on an elliptic curve C''. This contradicts Proposition 5.10. Thus the assertion (2) has been proved. Since there exists a Cartesian morphism of ESPs $\psi_\bullet \colon X_\bullet \to S_\bullet$, the assertions (1) and (2) are equivalent. Thus we are done. □

Step 3: We are at the final stage of our proof. Using Lemma 10.3 and Proposition 5.16, we describe the structure of the surjective morphism $\psi_n|_\Delta \colon \Delta \to S_n$ ($n \gg 0$) in terms of line bundles of degree zero on elliptic curves. Let $o \in C$ be a zero element. Then we show that the intersection $\Delta \cap \alpha_X^{-1}(o)$ has an infinite number of connected components, which derives a contradiction.

Now let us continue the proof. By Lemma 10.3, there exists a positive integer n_0 such that $\psi_n(\Delta) = S_n$ for all $n \geq n_0$, where $\Delta = \mathbb{P}_{C'}(\mathcal{O} \oplus \mathcal{L})$ and $S_n = \mathbb{P}_C(\mathcal{O} \oplus \ell_n)$. Let $\alpha_\Delta \colon \Delta \to C'$ be the Albanese map of Δ. Then there exists a finite surjective morphism $v_n \colon C' \to C$ such that $\alpha_n \circ \psi_n|_\Delta = v_n \circ \alpha_\Delta$. Since $\alpha_n \circ \psi_n|_\Delta = \alpha_X|_\Delta$, the composite $\Delta \xrightarrow{\alpha_\Delta} C' \xrightarrow{v_n} C$ is the Stein factorization of the morphism $\alpha_X|_\Delta \colon \Delta \to C$ and v_n is independent of n. Hereafter, we denote v_n by v. By Lemma 5.12, there exists a non-zero integer a_n such that $v^*\ell_n \simeq \mathcal{L}^{\otimes a_n}$ for all $n \geq n_0$.

Lemma 10.4. *The sequence $\{a_n\}_{n \geq n_0}$ is unbounded. i.e.,*

$$\sup_{n \geq n_0} |a_n| = \infty .$$

Proof. Assume the contrary. Then there exists an integer b such that $a_n = b$ for infinitely many n's. In particular, there exist integers $p < q$ such that $v^*\ell_p \simeq \mathcal{L}^{\otimes b} \simeq v^*\ell_q$. Since $\ell_n \simeq h^*\ell_{n+1}$ for any n, we infer that $\ell_p \simeq (h^{q-p})^*\ell_q$. Thus we have $(h^{q-p} \circ v)^*\ell_q \simeq v^*\ell_q$. Since $\ell_q \in \mathrm{Pic}^0(C)$ is of infinite order, Lemma 5.11 yields that $h^{q-p} \circ v = v$ up to translation under the group law of the elliptic curve C. Thus a contradiction is derived, since $\deg h > 1$. □

Let $\alpha_\Delta \colon \Delta \to C'$ be the Albanese map of Δ. By replacing ℓ_n by $\ell_n^{\otimes -1}$ if necessary, we may assume that $\ell_n \simeq h^*\ell_{n+1}$ for any n. By the universality of the Albanese map, there is induced the following commutative diagram:

$$\begin{array}{ccccccccc}
\Delta & \xrightarrow{\psi_{n_0}} & S_{n_0} & \xrightarrow{u_{n_0}} & S_{n_0+1} & \xrightarrow{u_{n_0+1}} & \cdots & \xrightarrow{u_{n-1}} & S_n \\
{\scriptstyle \alpha_\Delta}\downarrow & & {\scriptstyle \alpha_{n_0}}\downarrow & & {\scriptstyle \alpha_{n_0+1}}\downarrow & & & & {\scriptstyle \alpha_n}\downarrow \\
C' & \xrightarrow[v]{} & C & \xrightarrow[h]{} & C & \xrightarrow[h]{} & \cdots & \xrightarrow[h]{} & C.
\end{array}$$

Then we infer that

$$\mathcal{L}^{\otimes a_{n_0}} \simeq v^*\ell_{n_0} \simeq (h^{n-n_0} \circ v)^*\ell_n ,$$

for any $n \geq n_0$. Note that by Lemma 5.12, the integer a_{n_0} is uniquely determined by the finite surjective morphism $u_{n-1} \circ \cdots \circ u_{n_0} \circ \psi_{n_0}|_\Delta \colon \Delta \to S_n$ between elliptic ruled surfaces.

For each integer $n \geq n_0$, define a set M_n by

$$M_n := \{(k_n, \beta_n) \in \mathbb{Z} \times \mathrm{Hom}_{\mathrm{group}}(C', C) \mid \beta_n^*\ell_n \simeq \mathcal{L}^{\otimes k_n}\} .$$

Then by Proposition 5.15, the set M_n can be endowed with the structure of an abelian group such that the first projection $p_n \colon M_n \to \mathbb{Z}$ is an injective group homomorphism. Thus $p_n(M_n)$ is an additive subgroup of $\mathbb{Z}$ generated by a unique positive integer ν_n.

Then we have $(a_{n_0}, h^{n-n_0} \circ v) \in M_n$ and the integer a_{n_0} is divisible by the integer ν_n for each $n \geq n_0$. Hence we see that the set $B := \{\nu_n \mid n \geq n_0\}$ is a finite set and $\operatorname{Sup} B < \infty$. Furthermore, by construction, $\psi_n(\Delta) = S_n$ and there exists a finite étale morphism $v \colon C' \to C$ such that $v \circ \alpha_\Delta = \alpha_n \circ \psi_n|_\Delta$ and $v^*\ell_n \simeq \mathcal{L}^{\otimes a_n}$ for any $n \geq n_0$. Hence we infer that $(a_n, v) \in M_n$ and the integer a_n is a multiple of ν_n. If we put $b_n := a_n/\nu_n$, then we have $\operatorname{Sup}\{b_n \mid n \geq n_0\} = \infty$ by Lemma 10.4. Let F_n be the fiber of the Albanese map $\alpha_n \colon S_n \to C$ over the zero element $o \in C$. Since $\alpha_X = \alpha_n \circ \psi_n$ and $\psi_n = \varphi_n \circ \pi_n$ for any n, we have $\alpha_X^{-1}(o) = \psi_n^{-1}(F_n)$. Hence

$$\Delta \cap \alpha_X^{-1}(o) = (\psi_n|_\Delta)^{-1}(F_n) \tag{2}$$

and the number of the connected components of the left hand side of (2) is independent of n. On the other hand, by Propositions 5.15 and 5.16, the number of the connected components of the right hand side of (2) is more than or equal to b_n^2, which is unbounded as n varies. Thus a contradiction is derived and the proof of Theorem 10.1 has been done. □

As a deeper application of Theorem 10.1, we shall prove Theorem 1.5.

Proof of Theorem 1.5. It follows from Lemma 4.6 that the property whether $\ell_n \in \mathrm{Pic}^0(C)$ is of finite order is invariant under a finite étale base change $\widetilde{C} \to C$. Hence applying Theorem 9.6, we can reduce to the case where $\varphi_n \colon Y_n \to S_n$ is a $\mathbb{P}^1$-bundle for any n. Thus

Theorem 10.1 yields the claim. □

Remark 10.5. Theorem 10.1 does not hold without the assumption that the endomorphism $f\colon X \to X$ is *étale*. We shall give such an example. Let C be an elliptic curve and $\mathcal{L} \in \mathrm{Pic}^0(C)$ a line bundle of infinite order. Let $S := \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$ be an elliptic ruled surface and Δ_1, Δ_2 mutually disjoint sections of $\alpha_S : S \to C$ corresponding to the first projection $\mathcal{O}_C \oplus \mathcal{L} \twoheadrightarrow \mathcal{O}_C$ and the second projection $\mathcal{O}_C \oplus \mathcal{L} \twoheadrightarrow \mathcal{L}$ respectively. Then by [38, Proposition 5], S admits a non-isomorphic endomorphism $\psi\colon S \to S$ which is ramified only along $\Delta_1 \cup \Delta_2$, and $\varphi^{-1}(\Delta_i) = \Delta_i$ for each i. Let $Y := S \times \mathbb{P}^1$ be the product variety. Then Y admits a non-isomorphic endomorphism $g := \psi \times \mathrm{id}_{\mathbb{P}^1} : Y \to Y$. Let E_i be the elliptic curve on Y defined by $E_i := \Delta \times \{0\}$ and $X_i := \mathrm{Bl}_{E_i}(Y)$ the blowing-up of Y along E_i. Then $g^{-1}(E_i) = E_i$ for each $i = 1, 2$. Hence by the universality of the blowing-up (cf. the proof of Lemma 3.3), $g\colon Y \to Y$ lifts to a non-isomorphic *ramified* endomorphism $f_i\colon X_i \to X_i$ of X_i. In other words, the endomorphism $f\colon Y \to Y$ is constructed from $f_i\colon X_i \to X_i$ by an equivariant blowing-down and $Y \to S$ is a trivial $\mathbb{P}^1$-bundle over S. Since $\mathcal{L} \in \mathrm{Pic}^0(C)$ is a *non-torsion* line bundle, the conclusion as in Theorem 1.5 does not hold.

As an immediate application of Theorem 10.1, we see that if there exists an FESP of type (C) constructed from the given variety by a sequence of blowing-downs of an ESP, then up to finite étale covering we can reduce the FESP to a $\mathbb{P}^1$-bundle over an elliptic ruled surface of very limited type.

Corollary 10.6. *Under the same assumption as in Theorem 9.6, up to finite étale covering, there exists an FESP $Y_\bullet$ of $X_\bullet := (X, f)$ such that in the assertion* (8), $\ell_n \in \mathrm{Pic}(C)$ *is a torsion line bundle for any n, and in particular $S_0 \simeq C \times \mathbb{P}^1$.*

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