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Author(s)	Saïdi, Mohamed; Williams, Nicholas
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Saïdi, M. and Williams, N.
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GALOIS COVERS OF TYPE $(p, \cdots, p)$, VANISHING CYCLES FORMULA, AND THE EXISTENCE OF TORSOR STRUCTURES

Mohamed SAÏDI and Nicholas WILLIAMS

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Abstract

In this article we prove a local Riemman-Hurwitz formula which compares the dimensions of the spaces of vanishing cycles in a finite Galois cover of type $(p, p, \cdots, p)$ between formal germs of p-adic curves and which generalises the formula proven in [6] in the case of Galois covers of degree p. We also investigate the problem of the existence of a torsor structure for a finite Galois cover of type $(p, p, \cdots, p)$ between p-adic schemes.

0. Introduction

Let K be a *complete discrete valuation ring of mixed characteristic*, R its valuation ring, and $k := R/\pi R$ the residue field of characteristic $p > 0$ which we assume to be *algebraically closed*. We suppose that K contains a primitive p-th root of 1. In [6] the first author proved a local Riemman-Hurwitz formula which compares the dimensions of the spaces of vanishing cycles in a finite Galois cover of degree p between formal germs of R-curves. This formula is quite explicit and involves the (usual) *"generic"different*, which measures the ramification at the level of generic fibres, and a certain *"special"different* which involves certain *"conductors"* attached to the induced covers between the formal boundaries of the formal germs (cf. loc. cit. Theorem 3.4).

In this paper we generalise this formula to the setting of Galois covers of type $(p, p, \cdots, p)$, i.e., with Galois group $\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$. In principle one can apply the formula in the Galois degree p case obtained in [6] iteratively to derive such a formula. However, the difficulty here lies in computing the *conductors involved in the special different at the various degree p intermediate covers*; the possibility of having generically purely inseparable extensions at the level of special fibres doesn't allow the use of the standard ramification theory as in [10] in order to compute these conductors. In this paper we are able to compute in §1 these conductors at the various degree p intermediate levels via direct, rather tedious, computations (cf. Theorem 1.1). Although our main result computing these conductors is stated only in the case of Galois covers of type (p, p) (cf. loc. cit.), it is quite straightforward to deduce from this result the relevant value conductors as well as the corresponding Riemman-Hurwitz formula in the case of general Galois covers of type $(p, p, \cdots, p)$ (cf. Example 1.7 for an illustration). In §2 we derive an explicit local Riemman-Hurwitz formula which compares the dimensions of the spaces of vanishing cycles in a finite Galois covers

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of type (p, p) between formal germs of R-curves, which can be easily iterated to deduce a similar local Riemman-Hurwitz formula in the general case of finite Galois covers of type $(p, p, \cdots, p)$.

In §3 we investigate the problem of the existence of a torsor structure for a finite Galois cover of type $(p, p, \cdots, p)$ between R-(formal) schemes where we allow R to be of equal characteristic $p > 0$. Let X be a *normal flat and geometrically connected* R-(formal) scheme with an *integral special fibre* $X_k := X \times_R k$, $\{f_i : Y_i \to X\}_{i=1}^n$ torsors under finite and flat R-group scheme G_i which are generically pairwise disjoint, $1 \leq i \leq n$, and $f : Y \to X$ the morphism of normalisation of X in (the fibre product over $X_K := X \times_R K$) $\prod_{i=1}^n Y_{i,K}$, where $Y_{i,K} := Y_i \times_R K$. Assume the special fibre $Y_k := Y \times_R k$ is *reduced*. Our main result Theorem 3.4 gives necessary and sufficient conditions for f to have the structure of a torsor under a finite and flat R-group scheme (necessarily isomorphic to $G_1 \times_R \cdots \times_R G_n$). In the case where X is a *relative curve* these conditions are equivalent to the condition that at least $n - 1$ of the group schemes G_i are étale (cf. Theorem 3.5). This latter fact is false in relative dimension > 1 (cf. 3.8).

Notations

In this paper $p \geq 2$ is a *prime integer*, K is (unless we specify otherwise) a *complete discrete valuation ring*, $char(K) = 0$, R its valuation ring, π a uniformising parameter, v_K will denote the valuation of K which is normalised by $v_K(\pi) = 1$, and $k := R/\pi R$ the residue field of characteristic $p > 0$ which we assume to be *algebraically closed*. We suppose R contains a primitive p-th root of 1.

For an R-scheme X we will denote by $X_K := X \times_R K$ (resp. $X_k := X \times_R k$) the *generic* (resp. *special*) fibre of X. If $X = \mathrm{Spf} A$ is a formal affine R-scheme we will denote $X_K := \mathrm{Spec}(A \otimes_R K)$ and $X_k := \mathrm{Spec}(A/\pi)$ the special fibre of X.

A *formal (resp. algebraic) R-curve* is an R-formal scheme of finite type (resp. scheme of finite type) flat, separated, and whose special fibre is equidimensional of dimension 1.

We will refer to a (generically separable) cover $Y \to X$ between normal connected (formal R-)schemes which is Galois with Galois group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ as a Galois cover of type (p, p).

Let X be a *proper, normal, (formal) R-curve* with X_k *geometrically reduced*. For $x \in X$ a *closed point* let $F_x = \mathrm{Spf}(\hat{O}_{X,x})$ be the *formal completion* of X at x, which we will refer to as the *formal germ* of X at x. Thus, $\hat{O}_{X,x}$ is the completion of the local ring of (the algebraisation) of X at x. Let $\{P_i\}_{i=1}^n$ be the *minimal prime ideals of $\hat{O}_{X,x}$ which contain* π; they correspond to the *branches* $\{\eta_i\}_{i=1}^n$ of the completion of X_k at x (i.e., closed points of the normalisation of X_k above x), and $X_i = X_{x,i} := \mathrm{Spf}(\hat{O}_{x,P_i})$ the formal completion of the localisation of F_x at P_i. The local ring $\hat{O}_{x,P_i}$ is a *complete discrete valuation ring* with uniformiser π. We refer to $\{X_i\}_{i=1}^n$ as the set of *boundaries of the formal germ* F_x. We have a canonical morphism $X_i \to F_x$ of formal schemes, $1 \leq i \leq n$.

With the same notations as above, let $x \in X$ be a *closed* point and $\widetilde{X_k}$ the *normalisation* of X_k. There is a one-to-one correspondence between the set of points of $\widetilde{X_k}$ above x and the set of boundaries of the formal germ at the point x. Let x_i be the point of $\widetilde{X_k}$ above x which corresponds to the boundary X_i, $1 \leq i \leq n$. Then the completion of $\widetilde{X_k}$ at x_i is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ over k, where t_i is a *local parameter* at x_i. The complete local ring $\hat{O}_{x,P_i}$ is a discrete valuation ring with uniformiser π, and residue field isomorphic to $k((t_i))$. Fix an isomorphism $k((t_i)) \simeq \hat{O}_{x,P_i}/\pi$. Let $T_i \in \hat{O}_{x,P_i}$

be an element which lifts (the image in $\hat{O}_{x,P_i}/\pi$ under the above isomorphism of) t_i; we shall refer to such an element T_i as a *parameter* of $\hat{O}_{x,P_i}$, or of the boundary X_i. Then there exists an isomorphism $R[[T_i]]\{T_i^{-1}\} \simeq \hat{O}_{x,P_i}$, where

$$R[[T]]\{T^{-1}\} := \Big\{ \sum_{i=-\infty}^{\infty} a_i T^i,\ \lim_{i\to-\infty} |a_i| = 0 \Big\}$$

and $|\ |$ is a normalised absolute value of R (cf. [1], §2, 5).

Given a power series $g \in k((z))$ where z is an indeterminate we write

$$g(z) = \sum_{i\in I\subset\mathbb{Z}} a_i z^i + \text{higher order terms},$$

meaning all remaining monomial terms in z are of the form cz^t where $c \in k$ and $t > i$ for at least one $0 \neq i \in I$. Also given a power series $H(Z) \in R[[T]]\{T^{-1}\}$ we write

$$H(Z) = (F(Z))^p + \sum_{i\in I\subset\mathbb{Z}} c_i Z^i + \text{higher order terms},$$

meaning all remaining monomial terms in Z are of the form dZ^t where either $v_K(d) > v_K(c_i)$ for all $i \in I$ or there exists at least one $i \in I$ such that $v_K(d) = v_K(c_i)$ and $t > i$.

Background

In this section we collect/improve some background material form [6] that will be used in this paper. Let $A := R[[T]]\{T^{-1}\}$ and $f : \mathrm{Spf}\,(B) \to \mathrm{Spf}\,(A)$ a non-trivial Galois cover of degree p. We assume that π (which is a uniformiser of A) is a unifomiser of B (this condition is satisfied after possibly base changing to a finite extension of R, cf. [2]). Proposition 2.3 in [6] shows that f has the structure of a torsor under one of the three group schemes, μ_p, $\mathcal{H}_n$ where $0 < n < v_K(\lambda)$, or $\mathcal{H}_{v_K(\lambda)}$ (cf. loc. cit. 2.1 for the definition of these group schemes and the local explicit description of torsors under these group schemes). To the torsor f are associated some data: the acting group scheme as above, the **degree of different** δ, the **conductor variable** m, and $c = -m$ the **conductor** (cf. loc. cit. definition 2.4. The notation c is introduced in this paper, only the conductor variable m was considered in loc. cit.). Adapting slightly the proof of Proposition 2.3 in [6] provides the following details in the three occurring cases:

(a) For the **group scheme $\boldsymbol{\mu_p}$** where $\delta = v_K(p)$, the torsor equation is of the form

$$Z^p = u$$

where $u = \sum_{i\in\mathbb{Z}} a_i T^i \in A^\times$ is a unit such that its image $\bar{u}$ modulo π is not a p-power. On the level of special fibres the induced μ_p-torsor is given by an equation $z^p = \bar{u}$ where $\bar{u} = \sum_{i\geq l} \bar{a}_i t^i \in k((t))$ for some integer l with $\bar{a}_l \neq 0$ (here t equals T modulo π). There are two cases to consider:

(a1) $\gcd(l, p) = 1$. We have $\bar{u} = t^l(\sum_{i\geq l} \bar{a}_i t^{i-l})$ and $\bar{v} := \sum_{i\geq l} \bar{a}_i t^{i-l} \in k[[t]]$ is a unit. Further, we can write $u = T^l v$ where $v := \sum_{i\geq l} a_i T^{i-l} \in R[[T]]\{T^{-1}\}$ is a unit whose reduction modulo π equals $\bar{v}$. After possibly multiplying u by a p-power we can assume $0 \leq l < p$. The unit $v \in A^\times$ admits an l-th root $s \in A$ since l is coprime to p and k is algebraically closed. Thus, $s^l = v$ in A and after replacing the parameter T by $T' := T.s$, which is also a parameter of A, our μ_p-torsor $f : \mathrm{Spf}\,(B) \to \mathrm{Spf}\,(A)$ is defined by the equation $Z^p = (T')^l$.

(a2) $\gcd(l,p)>1$, in which case l is divisible by p and $\bar{u}=\sum_{i\geq l}\bar{a}_it^i$. After multiplying u by T^{-l} (which is a p-power) we can assume $\bar{u}=\sum_{i\geq l}\bar{a}_it^{i-l}=\sum_{j\geq 0}\bar{a}_jt^j\in k[[t]]$. Let $m:=\min\{i\mid v_K(a_i)=0,\gcd(i-l,p)=1\}=\min\{j\mid \gcd(j,p)=1\}$. We can write $\bar{u}=\bar{a}_0+\bar{a}_1t^p+\cdots+\bar{a}_{[m/p]}t^{[m/p]p}+\bar{a}_mt^m+$higher order terms, and $u=a_0+a_1T^p+\cdots+a_{[m/p]}T^{[m/p]p}+\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i$. If $a\in A$ is a unit we can write $a=b^p+c$ with $b\in A$ a unit and $v_K(c)>0$. Thus, we can assume without loss of generality that $u=a_0^p+a_1^pT^p+\cdots+a_{[m/p]}^pT^{[m/p]p}+\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i$. Now $a_0^p+a_1^pT^p+\cdots+a_{[m/p]}^pT^{[m/p]p}=(a_0+a_1T+\cdots+a_{[m/p]}T^{[m/p]})^p-p(a_0+a_1T+\cdots+a_{[m/p]}T^{[m/p]})+$higher order terms, and after replacing u by $u(a_0+a_1T+\cdots+a_{[m/p]}T^{[m/p]})^{-p}$ we can assume without loss of generality that the torsor equation is: $Z^p=1+a_mT^m+\sum_{\substack{v_K(a_i)=0\\ i>m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i$. Further, $a_mT^m+\sum_{\substack{v_K(a_i)=0\\ i>m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i=T^mv$ where $v=a_m+\sum_{\substack{v_K(a_i)=0\\ i>m}}a_iT^{i-m}+\sum_{v_K(a_i)>0}a_iT^{i-m}\in A$ is a unit which admits an m-th root $u\in A$. Thus, $u^m=v$ in A and after replacing the parameter T by $T':=T.u$, which is also a parameter of A, our μ_p-torsor $f:\mathrm{Spf}\,(B)\to\mathrm{Spf}\,(A)$ is defined by the equation $Z^p=1+(T')^m$.

Simplified form: After a possible change of the parameter T of A, the torsor equation $Z^p=u$ can be reduced to either the form

$$\textbf{(a1)}\quad Z^p=T^h \text{ where } h\in\mathbb{F}_p^\times,$$

or of the form

$$\textbf{(a2)}\quad Z^p=1+T^m$$

where m is as defined above for these two cases. The conductor is given in both cases by (**a1**) $\mathbf{c}=\mathbf{0}$, and (**a2**) $\mathbf{c}=-\mathbf{m}$.

(b) For the **group scheme $\boldsymbol{\mathcal{H}_n}$** where $0<n<v_K(\lambda)$ and $\delta=v_K(p)-n(p-1)$, the torsor equation is of the form

$$(1+\pi^nZ)^p=1+\pi^{np}u$$

where $u=\sum_{i\in\mathbb{Z}}a_iT^i\in A^\times$ is a unit such that modulo π it is not a p-power. Reducing modulo π, on the special fibre the acting group scheme is α_p and the torsor is given by an equation $z^p=\bar{u}$ where $\bar{u}=\sum_{i\geq l}\bar{a}_it^i\in k((t))$ for some integer l with $\bar{a}_l\neq 0$ and which is defined up to addition of a p-power. We define $m:=\min\{i|v_K(a_i)=0,\gcd(i,p)=1\}\in\mathbb{Z}$. Then $\bar{u}=\bar{a}_{l/p}t^{p.l/p}+\cdots+\bar{a}_{[m/p]}t^{[m/p]p}+\bar{a}_mt^m+$ higher order terms, and $u=a_{l/p}T^{p.l/p}+\cdots+a_{[m/p]}T^{[m/p]p}+\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i$. If $a\in A$ is a unit we can write $a=b^p+c$ with $b\in A$ a unit and $v_K(c)>0$. Thus, we can assume without loss of generality that $u=a_{l/p}^pT^{p.l/p}+\cdots+a_{[m/p]}^pT^{[m/p]p}+\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i$. Now $1+\pi^{np}(a_{l/p}^pT^{p.l/p}+\cdots+a_{[m/p]}^pT^{[m/p]p})=(1+\pi^n(a_{l/p}T^{l/p}+\cdots+a_{[m/p]}T^{[m/p]}))^p-p\pi^n(a_{l/p}T^{l/p}+\cdots+a_{[m/p]}T^{[m/p]})+$ higher order terms. Thus, since $np<v_K(p)+n$, the torsor equation can be written $(1+\pi^nZ)^p=(1+\pi^n(a_{l/p}T^{l/p}+\cdots+a_{[m/p]}T^{[m/p]}))^p+\pi^{np}(\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i)$ and after multiplying by $(1+\pi^n(a_{l/p}T^{l/p}+\cdots+a_{[m/p]}T^{[m/p]}))^{-p}=1-p\pi^n(a_lT^{l/p}+\cdots+a_{[m/p]}T^{[m/p]})+$ higher order terms, we get an equation $(1+\pi^nZ)^p=1+\pi^{np}(\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i)$ and can assume $u=\sum_{\substack{v_K(a_i)=0\\ i\geq m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i$. Further, $a_mT^m+\sum_{\substack{v_K(a_i)=0\\ i>m}}a_iT^i+\sum_{v_K(a_i)>0}a_iT^i=T^mv$ where $v\in A$ is a unit which admits an m-th root $h\in A$. Thus, $h^m=v$ in A and after

replacing the parameter T by $T' := T.h$, which is also a parameter of A, the $\mathcal{H}_n$-torsor $f : \mathrm{Spf}\,(B) \to \mathrm{Spf}\,(A)$ is defined by the equation $Z^p = 1 + \pi^{np}(T')^m$.

Simplified form: After a change of the parameter T, the torsor equation can be reduced to the form

$$Z^p = 1 + \pi^{np}T^m$$

where m is as defined above. The conductor is given by $c = -m$.

(c) For the **group scheme $\boldsymbol{\mathcal{H}_{v_K(\lambda)}}$** where $\delta = 0$, the torsor equation is of the form

$$(1 + \lambda Z)^p = 1 + \lambda^p u$$

where $u = \sum_{i\in\mathbb{Z}} a_i T^i \in A^\times$ is a unit. On the special fibre the acting group scheme is $\mathbb{Z}/p\mathbb{Z}$ and the torsor is given by an equation $z^p - z = \bar{u}$ where $\bar{u} = \sum_{i\geq l} \bar{a}_i t^i$ for some integer l with $\bar{a}_l \neq 0$ and which is defined up to addition of an Artin-Schreier element of the form $b^p - b$. In fact, after such an Artin-Schreier transformation, $\bar{u}$ can be represented as: $\bar{u} = \bar{a}_m t^m + \bar{a}_{m+1}t^{m+1} + ... + \bar{a}_{-1}t^{-1} = \sum_{i=m}^{-1} \bar{a}_i t^i$ where $\bar{a}_m \neq 0$ and $m < 0$ is the conductor variable such that $\gcd(m, p) = 1$. Indeed, for $f(t) = \sum_{i\geq 0} a_i t^i \in k[[t]]$ we have $f(t) = (f(t) + f(t)^p + f(t)^{p^2} + \cdots) - (f(t) + f(t)^p + f(t)^{p^2} + \cdots)^p$. Moreover, $\bar{u} = \bar{a}_m t^m + \bar{a}_{m+1}t^{m+1} + ... + \bar{a}_{-1}t^{-1} = t^m\bar{v}$ where $\bar{v} = \bar{a}_m + \bar{a}_{m+1}t + ... + \bar{a}_{-1}t^{-m-1} \in k[[t]]$ is a unit. Let $v = a_m + a_{m+1}T + ... + a_{-1}T^{-m-1} \in R[[T]]$ be an element which lifts $\bar{v}$ and h an m-th root of v in $R[[T]]$. Then after replacing the parameter T by $T' := T.h$, which is also a parameter of A, our $\mathcal{H}_{v_K(\lambda)}$-torsor $f : \mathrm{Spf}\,(B) \to \mathrm{Spf}\,(A)$ is defined by the equation $Z^p = 1 + \lambda^p(T')^m$.

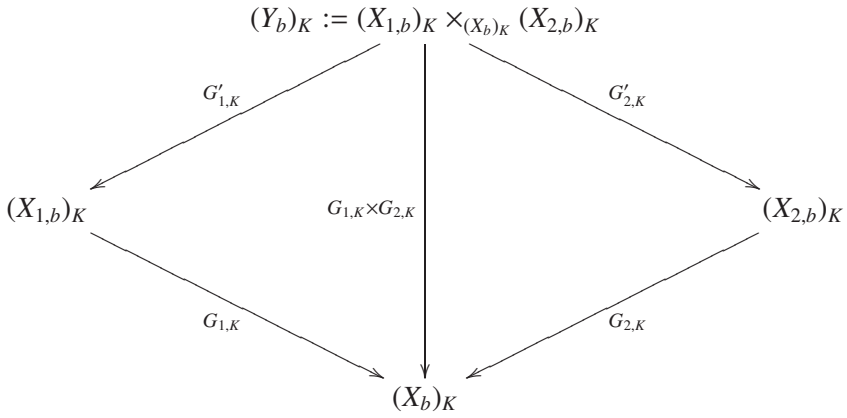
Simplified form: After a change of the parameter T, the torsor equation over R can be simplified to the form

$$Z^p = 1 + \lambda^p T^m$$

where m is as defined above. The conductor is given by $c = -m$.

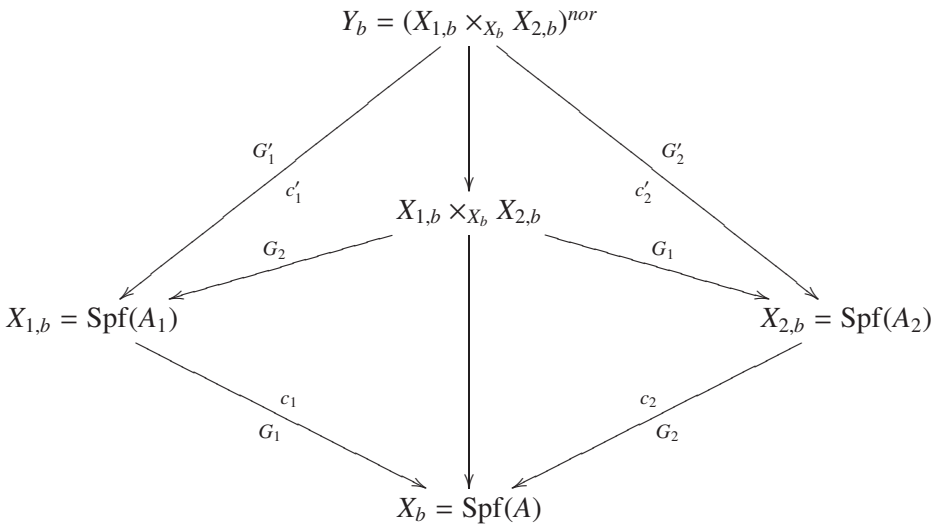
1. The type (p, p) case

In this section $A := R[[T]]\{T^{-1}\}$ and $X_b := \mathrm{Spf}\,(A)$. Let $f_{i,K} : (X_{i,b})_K \to (X_b)_K$ be two (generically) disjoint non-trivial degree p Galois covers. We have the following diagram:



where $G_{i,K}$ and $G'_{i,K}$ are the acting group schemes on the various covers and as $\mathrm{char}(K) = 0$,

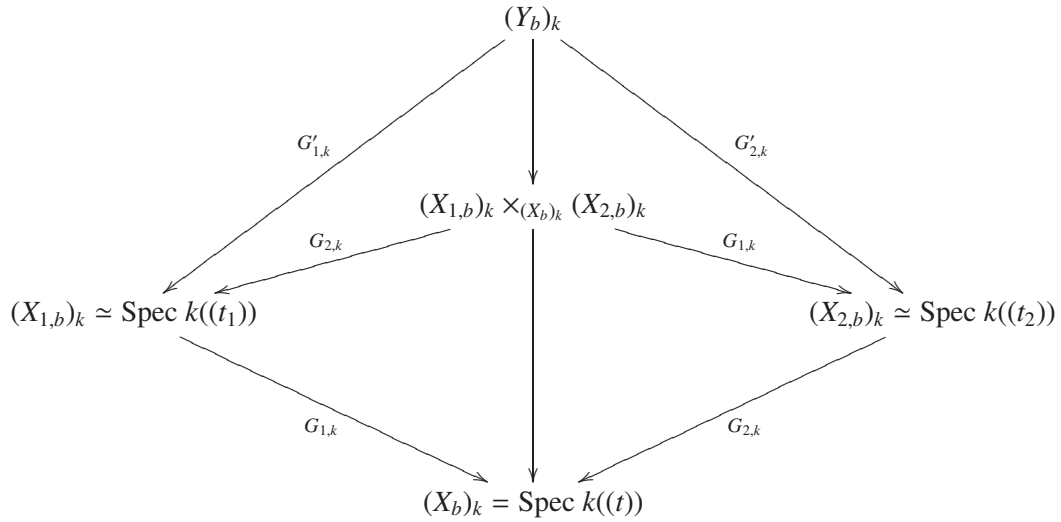
we have $G_{i,K} = G'_{i,K} \simeq \mathbb{Z}/p\mathbb{Z} \simeq \mu_p$ is étale for $i = 1, 2$. For $i = 1, 2$, let $f_i : X_{i,b} \to X_b$ be the Galois covers of degree p where $X_{i,b}$ is the *normalisation* of X_b in $(X_{i,b})_K$. Similarly, let Y_b be the *normalisation* of X_b in $(Y_b)_K$ so that $f : Y_b \to X_b$ is a non-trivial Galois cover of type (p, p). We assume that $(Y_b)_k$ is *reduced*. Note that in this case $X_{i,b}$ is isomorphic to $\mathrm{Spf}\left(R[[T_i]]\{T_i^{-1}\}\right)$ for $1 \leq i \leq 2$ (cf. [1], §2, 5]). We have the diagram:



where:

- Y_b and $X_{i,b}$ are normal for $i = 1, 2$, and Y_b is the normalisation $(X_{1,b} \times_{X_b} X_{2,b})^{nor}$ of the fibre product $(X_{1,b} \times_{X_b} X_{2,b})$.
- c_i (respectively c'_i) denotes the conductor of the torsor $X_{i,b} \to X_b$ (respectively $Y_b \to X_{i,b}$). The conductor c_i (respectively c'_i) is dependent on the conductor variable m_i (respectively m'_i) (cf. Background).
- G_i (respectively G'_i) denotes the finite and flat (commutative) R-group scheme of the torsor $X_{i,b} \to X_b$ (respectively $Y_b \to X_{i,b}$). We know G_i, G'_i are among the R-group schemes $\mathcal{H}_{v_K(\lambda)}$, μ_p, or $\mathcal{H}_n$ for $0 < n < v_K(\lambda)$ (cf. loc. cit.).

On the level of special fibres over k we have a diagram:



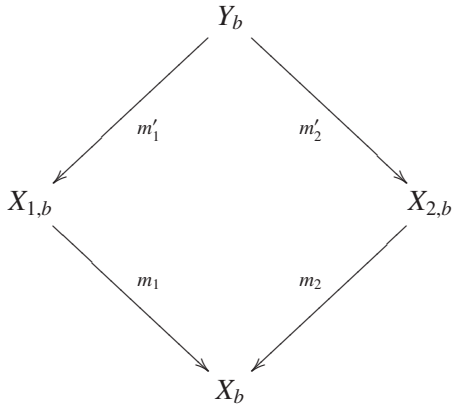
where:

- $(Y_b)_k$ and $(X_{i,b})_k$ are reduced for $i = 1, 2$.
- $k((t)) = A/(\pi)$ (respectively $A_1/\pi \simeq k((t_1))$, $A_2/\pi \simeq k((t_2))$ where t (respectively t_1 and t_2) is the reduction modulo π of T (respectively T_1, T_2, where T_i is some suitable parameter of $X_{i,b}$ for $i = 1, 2$).
- $G_{i,k} = G_i \times_R k$ and $G'_{i,k} = G'_i \times_R k$ are the acting group schemes over k, a field with characteristic p, so that these group schemes are necessarily isomorphic to either $\mathbb{Z}/p\mathbb{Z}$, μ_p or α_p.

We aim to express the **conductor** c'_1 in terms of c_1 and c_2 for the various torsor combinations and likewise for c'_2. To achieve this, we express the **conductor variables** m'_1 and m'_2 in terms of m_1 and m_2. We have six cases to consider by taking all possible pairs of the group schemes $\mathcal{H}_{v_K(\lambda)}$, μ_p and $\mathcal{H}_n$ over R acting on $X_{1,b}$, $X_{2,b}$. The following is one of our main results.

Theorem 1.1. *Let $X_b = \mathrm{Spf}\,(A)$ and suppose we have two (generically) disjoint non-trivial degree p Galois covers $f_{i,K} : (X_{i,b})_K \to (X_b)_K$, for $i = 1, 2$. Let $(Y_b)_K$ be the compositum of these covers.*

For $i = 1, 2$, let $f_i : X_{i,b} \to X_b$ be the Galois covers of degree p where $X_{i,b}$ is the normalisation of X_b in $(X_{i,b})_K$. Set Y_b as the normalisation of X_b in $(Y_b)_K$ so that $f : Y_b \to X_b$ is a non-trivial Galois cover of type (p, p). We assume that the ramification index of the corresponding extension of DVR's equals 1 and that the special fibre of Y_b is reduced. Thus, f_i is a non-trivial torsor under a finite flat R-group scheme G_i of rank p with conductor variable m_i for $i = 1, 2$. Let m'_i denote the conductor variable of the torsor $Y_b \to X_{i,b}$. Then, for all possible pairs of G_1 and G_2, we can express the conductors m'_i in terms of the m_i conductor variables for $i = 1, 2$ as follows:



1. *For $G_1 = G_2 = \mathcal{H}_{v_K(\lambda)}$ we have that $m'_1 = m_2$ and $m'_2 = m_1 p - m_2(p-1)$ when $m_1 \le m_2$, and $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$ when $m_1 > m_2$.*
2. *For $G_1 = \mathcal{H}_{v_K(\lambda)}$ and $G_2 = \mu_p$ we have that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$.*
3. *For $G_1 = \mathcal{H}_{v_K(\lambda)}$ and $G_2 = \mathcal{H}_n$ we have that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$.*
4. *For $G_1 = \mathcal{H}_n$ and $G_2 = \mu_p$ we have that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$.*
5. *For $G_1 = G_2 = \mu_p$ we have that $m'_1 = m_2$ and $m'_2 = m_1 p - m_2(p-1)$ when $m_1 \le m_2$, and $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$ when $m_1 > m_2$. In this case these results are*

only valid when at least one of m_1 and m_2 is non-zero (*cf. Proof and* Remark 1.8).

6. *For $G_1 = \mathcal{H}_{n_1}$ and $G_2 = \mathcal{H}_{n_2}$ we have that $m'_1 = m_2$ and $m'_2 = m_1 p - m_2(p-1)$ when $n_1 < n_2$, that $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$ when $n_1 > n_2$, that $m'_1 = m_2$ and $m'_2 = m_1 p - m_2(p-1)$ when both $n_1 = n_2$ and $m_1 < m_2$, and $m'_1 = m_2 p - m_1(p-1)$ and $m'_2 = m_1$ when both $n_1 = n_2$ and $m_1 \geq m_2$, where $0 < n, n_1, n_2 < v_K(\lambda)$.*

1.1. Proof of Theorem 1.1. Proof. We treat each of the six occurring cases individually. However, there is an important distinction between the first three cases and the remaining cases.

In the first three cases, that is when at least one of the acting group schemes is the étale group scheme $\mathcal{H}_{v_K(\lambda)}$, one can work modulo π at the level of special fibres for, in this case, $Y_b = X_{1,b} \times_{X_b} X_{2,b}$ which implies $(Y_b)_k = (X_{1,b})_k \times_{(X_b)_k} (X_{2,b})_k$. Indeed, suppose $G_1 = \mathcal{H}_{v_K(\lambda)}$ so that the torsor $X_{1,b} \to X_b$ is étale. Then, by base change, the torsor $X_{1,b} \times X_{2,b} \to X_{2,b}$ is automatically étale. The special fibre of $X_{2,b}$ is reduced (because it is dominated by Y_b whose special fibre is reduced) but as $X_{1,b} \times X_{2,b} \to X_{2,b}$ is étale, this implies the special fibre of $X_{1,b} \times X_{2,b}$ is also reduced. Then, by Theorem 3.4 in this paper, $X_{1,b} \times_{X_b} X_{2,b}$ is normal and equal to Y_b, as required.

In the last three cases, we do not have this situation, which means one must work above X_b over R without being permitted to reduce to the special fibre. However, we still proceed in a similar fashion, even if the computations are more involved. In particular, we start with the equation of $X_{i,b} \to X_b$, base change it to $X_{j,b}$ for $j \neq i$ and make appropriate (Kummer) transformations in order to find the torsor equations of $Y_b \to X_{j,b}$ and read off the conductors m'_j for $j = 1, 2$. Note that in each case we can perform a change of the parameter T of $A = R[[T]]\{T^{-1}\}$ so that one of the two torsor equations above $X_b = \mathrm{Spf}\,(A)$ is in its simplified form but we must assume the other equation remains in its original full power series form (cf. Background).

1. $(\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)})$. Here $m_1, m_2 < 0$. The $\mathcal{H}_{v_K(\lambda)}$ torsor equation $X_{i,b} \to X_b$ is given by $(1 + \lambda Z_i)^p = 1 + \lambda^p u_i$ where $u_i \in A^\times$. Modulo π, these torsor equations reduce to $z_i^p - z_i = \bar{u}_i$ on the special fibre, $i = 1, 2$. We start by computing m'_1. We can choose the parameter T so that $u_1 = T^{m_1}$, $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ is a power series, accordingly, $\bar{u}_1 = t^{m_1}$ and $\bar{u}_2 = \sum_{i=m_2}^{-1} \bar{a}_i t^i$ (cf. loc. cit.) where $\bar{a}_{m_2} \neq 0$. We can write t in terms of z_1 in $(X_{1,b})_k$: $z_1^p - z_1 = t^{m_1} \Leftrightarrow z_1^p \left(1 - z_1^{1-p}\right) = t^{m_1} \Leftrightarrow t = \left(z_1^{1/m_1}\right)^p \left(1 - z_1^{1-p}\right)^{1/m_1}$. Thus a parameter of $(X_{1,b})_k$ is z_1^{1/m_1} and so by letting $z := z_1^{1/m_1}$ we can write $t = z^p \left(1 - z^{-m_1(p-1)}\right)^{1/m_1}$. We can now proceed to base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to $(X_{1,b})_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$\begin{aligned} z_2^p - z_2 &= \sum_{i=m_2}^{-1} \bar{a}_i t^i = \sum_{i=m_2}^{-1} \bar{a}_i z^{ip} \left(1 - z^{-m_1(p-1)}\right)^{i/m_1} = \sum_{i=m_2}^{-1} \bar{a}_i z^{ip} \left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + \ldots\right) \\ &= \bar{a}_{m_2} z^{m_2 p} \left(1 - \frac{m_2}{m_1} z^{-m_1(p-1)} + \ldots\right) + \bar{a}_{m_2+1} z^{(m_2+1)p} \left(1 - \frac{m_2+1}{m_1} z^{-m_1(p-1)} + \ldots\right) + \ldots \\ &= \bar{a}_{m_2} z^{m_2 p} - \frac{m_2 \bar{a}_{m_2}}{m_1} z^{m_2 p - m_1(p-1)} + \text{higher order terms.} \end{aligned}$$

Expressing z^{ip} as $z^{ip} - z^i + z^i$ gives rise (after an Artin-Schreier transformation) to an equation of the form: $z_2^p - z_2 = \bar{a}_{m_2} z^{m_2} - \frac{m_2 \bar{a}_{m_2}}{m_1} z^{m_2 p - m_1(p-1)} +$ higher order terms. The conductor

variable m_1' is the smallest power of z in the above expression which is coprime to p. The expression above indicates there are two candidates, namely m_2 and $m_2p - m_1(p-1)$. Note that $m_2p - m_1(p-1) \leq m_2$ is equivalent to $m_1 \geq m_2$. Therefore, when $m_1 \geq m_2$ we have $m_1' = m_2p - m_1(p-1)$ and when $m_1 < m_2$ we have $m_1' = m_2$. The formula for m_2' is obtained in a similar way as a consequence of the symmetry occurring in this case.

2. $(\mathcal{H}_{v_K(\lambda)}, \mu_p)$. Here $m_1 < 0$ while $m_2 \geq 0$ hence $m_1 \leq m_2$. The torsor equation for $\mathcal{H}_{v_K(\lambda)}$ is given by $(1+\lambda Z_1)^p = 1 + \lambda^p u_1$ and for μ_p by $Z_2^p = u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p - z_1 = \bar{u}_1$ and $z_2^p = \bar{u}_2$ with acting group schemes $\mathbb{Z}/p\mathbb{Z}$ and μ_p respectively on the special fibre.

We start by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i\in\mathbb{Z}} a_i T^i$ remains as a power series and therefore, accordingly, $\bar{u}_1 = t^{m_1}$ and $\bar{u}_2 = \sum_{i\geq l} \bar{a}_i t^i$ for some integer l where $\bar{a}_l \neq 0$. As in case 1 of this proof, we have that the parameter of $(X_{1,b})_k$ is $z := z_1^{1/m_1}$ and we can write $t = z^p\left(1 - z^{-m_1(p-1)}\right)^{1/m_1}$. We now have two cases to treat, namely $(a1)$ and $(a2)$, depending on whether or not l is coprime to p.

(a1) In this case, $\gcd(l,p) = 1 \Rightarrow m_2 = 0$. We base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to $(X_{1,b})_K$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$z_2^p = \sum_{i\geq l} \bar{a}_i t^i = \sum_{i\geq l} \bar{a}_i z^{ip}(1 - z^{-m_1(p-1)})^{i/m_1} = \sum_{i\geq l} \bar{a}_i z^{ip}\left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + \ldots\right)$$

$$= \bar{a}_l z^{lp}\left(1 - \frac{l}{m_1} z^{-m_1(p-1)} + \ldots\right) + \bar{a}_{l+1} z^{(l+1)p}\left(1 - \frac{l+1}{m_1} z^{-m_1(p-1)} + \ldots\right) + \ldots.$$

As this is a μ_p-torsor equation, the factor $\bar{a}_l z^{lp}$ can be eliminated by multiplication by a suitable p-power to obtain an equation: $z_2^p = 1 - \frac{l}{m_1} z^{-m_1(p-1)} +$ higher order terms. So the conductor variable is $m_1' = -m_1(p-1)$, as this is the smallest power of z in the above expression which is coprime to p.

(a2) In this case $\gcd(l,p) \neq 1$. By the details outlined at the start of this paper (cf. Background), we know that the torsor equation of $(X_{2,b})_k \to (X_b)_k$ can be expressed as follows: $z_2^p = 1 + \bar{a}_{m_2} t^{m_2} + \sum_{i>m_2} \bar{a}_i t^i = 1 + \sum_{i\geq m_2} \bar{a}_i t^i$. By a suitable change of variables, we can express this μ_p-torsor as: $z_2^p = 1 + \sum_{i\geq m_2} \bar{a}_i t^i \Leftrightarrow (z_2 - 1)^p = \sum_{i\geq m_2} \bar{a}_i t^i \Rightarrow z_2^p = \sum_{i\geq m_2} \bar{a}_i t^i$. We can now proceed to base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to $(X_{1,b})_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$z_2^p = \sum_{i\geq m_2} \bar{a}_i z^{pi}\left(1 - z^{-m_1(p-1)}\right)^{i/m_1} = \sum_{i\geq m_2} \bar{a}_i z^{pi}\left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + \ldots\right)$$

$$= \bar{a}_{m_2} z^{m_2 p}\left(1 - \frac{m_2}{m_1} z^{-m_1(p-1)} + \ldots\right) + \bar{a}_{m_2+1} z^{(m_2+1)p}\left(1 - \frac{m_2+1}{m_1} z^{-m_1(p-1)} + \ldots\right) + \ldots$$

$$= \bar{a}_{m_2} z^{m_2 p} - \frac{m_2 \bar{a}_{m_2}}{m_1} z^{m_2 p - m_1(p-1)} + \text{higher order terms.}$$

So $m_1' = m_2p - m_1(p-1)$ as this is the smallest power of z in the above expression which is not divisible by p.

We now determine m_2'. We choose T so that $u_1 = \sum_{i\in\mathbb{Z}} a_i T^i$ and u_2 is given by T^h in the case (a1) and by $1 + T^{m_2}$ in the case (a2). After reducing these equations modulo π, we have

$z_1^p - z_1 = \sum_{i=m_1}^{-1} \bar{a}_i t^i$ and (a1) $z_2^p = t^h$ or (a2) $z_2^p = 1 + t^{m_2}$ on the special fibre.

(a1) We can write t in terms of z_2 in $(X_{2,b})_k$ since $z_2^p = t^h \Leftrightarrow t = \left(z_2^{1/h}\right)^p$. This implies that $z := z_2^{1/h}$ is a parameter of $(X_{2,b})_k$ and we have that $t = z^p$. We base change the equation of $(X_{1,b})_k \to (X_b)_k$ to $(X_{2,b})_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$: $z_1^p - z_1 = \sum_{i=m_1}^{-1} \bar{a}_i t^i = \sum_{i=m_1}^{-1} \bar{a}_i z^{ip} = \bar{a}_{m_1} z^{m_1 p}$ + higher order terms. The leading term $z^{m_1 p}$ (as well as all the other terms z^{ip}) is a multiple of p but, as in case 1 of this proof, after an Artin-Schreier transformation we obtain: $z_1^p - z_1 = \bar{a}_{m_1} z^{m_1}$ + higher order terms. Therefore, the conductor variable $m_2' = m_1$.

(a2) As above, we write t in terms of z_2 in $(X_{2,b})_k$: $z_2^p = 1 + t^{m_2} \Leftrightarrow z_2^p - 1 = t^{m_2} \Leftrightarrow (z_2 - 1)^p = t^{m_2} \Leftrightarrow t = \left((z_2 - 1)^{1/m_2}\right)^p$. This means that the parameter of $(X_{2,b})_k$ is $z := (z_2 - 1)^{1/m_2}$ and so, from the above, we obtain $t = z^p$. Now, we base change the equation of $(X_{1,b})_k \to (X_b)_k$ to $(X_{2,b})_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$: $z_1^p - z_1 = \sum_{i=m_1}^{-1} \bar{a}_i t^i = \sum_{i=m_1}^{-1} \bar{a}_i z^{ip} = \bar{a}_{m_1} z^{m_1 p}$ + higher order terms. After an Artin-Schreier transformation we obtain: $z_1^p - z_1 = \bar{a}_{m_1} z^{m_1}$ + higher order terms. Therefore, as in the (a1) case, the conductor variable $m_2' = m_1$.

3. $(\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_n)$. Here $m_1 < 0$ while $m_2 \in \mathbb{Z}$. The torsor equation for $\mathcal{H}_{v_K(\lambda)}$ is given by $(1 + \lambda Z_1)^p = 1 + \lambda^p u_1$ and for $\mathcal{H}_n$ by $(1 + \pi^n Z_2)^p = 1 + \pi^{np} u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p - z_1 = \bar{u}_1$ and $z_2^p = \bar{u}_2$ with acting group schemes $\mathbb{Z}/p\mathbb{Z}$ and α_p respectively on the special fibre.

We start by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ remains as a power series and therefore, accordingly, $\bar{u}_1 = t^{m_1}$ and $\bar{u}_2 = \sum_{i \geq l} \bar{a}_i t^i$ for some integer l where $\bar{a}_l \neq 0$. Recall that $l = m_2$ here. As in case 1 of this proof, we have that the parameter of $(X_{1,b})_k$ is $z := z_1^{1/m_1}$ and we can write $t = z^p \left(1 - z^{-m_1(p-1)}\right)^{1/m_1}$. We base change the torsor equation of $(X_{2,b})_k \to (X_b)_k$ to $(X_{1,b})_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{1,b})_k$:

$$\begin{aligned} z_2^p = \sum_{i \geq l} \bar{a}_i t^i &= \sum_{i \geq l} \bar{a}_i z^{pi} \left(1 - z^{-m_1(p-1)}\right)^{i/m_1} = \sum_{i \geq l} \bar{a}_i z^{pi} \left(1 - \frac{i}{m_1} z^{-m_1(p-1)} + \ldots\right) \\ &= \bar{a}_l z^{pl} \left(1 - \frac{l}{m_1} z^{-m_1(p-1)} + \ldots\right) + \bar{a}_{l+1} z^{p(l+1)} \left(1 - \frac{l+1}{m_1} z^{-m_1(p-1)} + \ldots\right) + \ldots \\ &= \bar{a}_l z^{pl} - \frac{l \bar{a}_l}{m_1} z^{lp - m_1(p-1)} + \text{higher order terms.} \end{aligned}$$

As this is an α_p-torsor equation, the term $\bar{a}_l z^{pl}$ can be removed and we can ignore the terms involving i's which are divisible by p. So the conductor variable $m_1' = m_2 p - m_1(p-1)$, as this would be the smallest power of z which is not divisible by p.

It remains to compute m_2' in this case. This time we choose the parameter T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ is the power series and $u_2 = T^{m_2}$. After reducing modulo π, we have $\bar{u}_1 = \sum_{i=m_1}^{-1} \bar{a}_i t^i$ and $\bar{u}_2 = t^{m_2}$ on the special fibre. We can write t in terms of z_2 in $(X_{2,b})_k$ since $z_2^p = t^{m_2} \Leftrightarrow t = \left(z_2^{1/m_2}\right)^p$. This implies that $z := z_2^{1/m_2}$ is the parameter of $(X_{2,b})_k$ and we have that $t = z^p$. We base change the equation of $(X_{1,b})_k \to (X_b)_k$ to $(X_{2,b})_k$ to obtain the torsor equation for $(Y_b)_k \to (X_{2,b})_k$: $z_1^p - z_1 = \sum_{i=m_1}^{-1} \bar{a}_i t^i = \sum_{i=m_1}^{-1} \bar{a}_i z^{ip} = \bar{a}_{m_1} z^{m_1 p}$ + higher order terms. After an Artin-Schreier transformation we obtain: $z_1^p - z_1 = \bar{a}_{m_1} z^{m_1}$ + higher order terms. Therefore, the conductor variable $m_2' = m_1$.

We remind the reader that in the remaining three cases, we cannot reduce modulo π and work at the level of special fibres. Thus, the computations here are slightly more involved. It will be useful to recall in advance here the following equality given by the Binomial Theorem

$$1 + (\pi^n bZ)^p = (1 + \pi^n bZ)^p - \sum_{k=1}^{p-1} \binom{p}{k} (\pi^n bZ)^k \quad (*)$$

which can be generalised to the Multimonomial Theorem (or Identity)

$$1 + \sum_i \left(\pi^n b_i Z^i\right)^p = \left(1 + \sum_i \pi^n b_i Z^i\right)^p - p \sum_i \pi^n b_i Z^i + \text{higher order terms} \quad (**).$$

We also mention here that to circumvent our inability to take, say, p-th roots of coefficients belonging to the ring R, we can adopt the following technique for a given element a_i where $v_K(a_i) = 0$; namely, it can be expressed as $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ such that $v_K(b_i) = 0$ and $v_K(c_i) > 0$.

4. $(\mathcal{H}_n, \mu_p)$. Here $m_1 \in \mathbb{Z}$ while $m_2 \geq 0$. The torsor equation for $\mathcal{H}_n$ is given by $(1 + \pi^n Z_1)^p = 1 + \pi^{np} u_1$ and for μ_p by $Z_2^p = u_2$ where $u_1, u_2 \in A^\times$. We start by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ remains as a power series. We can express T in terms of Z_1 in order to read off the parameter for $X_{1,b}$:

$$\begin{aligned}
(1 + \pi^n Z_1)^p = 1 + \pi^{np} T^{m_1} &\Leftrightarrow \pi^{np} Z_1^p + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{nk} Z_1^k + 1 = 1 + \pi^{np} T^{m_1} \\
&\Leftrightarrow \pi^{np} Z_1^p + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{nk} Z_1^k = \pi^{np} T^{m_1} \\
&\Leftrightarrow Z_1^p \left(\pi^{np} + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{nk} Z_1^{k-p}\right) = \pi^{np} T^{m_1} \\
&\Leftrightarrow Z_1^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{n(k-p)} Z_1^{k-p}\right) = T^{m_1} \\
&\Leftrightarrow T = \left(Z_1^{1/m_1}\right)^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{n(k-p)} Z_1^{k-p}\right)^{1/m_1}.
\end{aligned}$$

We know from the proof of case 3 that $Z := Z_1^{1/m_1}$ is the parameter of $X_{1,b}$ modulo π (hence it is a parameter of $X_{1,b}$) and so we can write:

$$T = Z^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n(p-k)} Z^{-m_1(p-k)}\right)^{1/m_1}.$$

For convenience, set $B = \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n(p-k)} Z^{-m_1(p-k)}$ so that $T = Z^p (1 + B)^{1/m_1}$. We now have two cases to treat, namely $(a1)$ and $(a2)$, depending on whether or not $l = \min\{i | v_K(a_i) = 0\}$ is coprime to p.

(a1) In this case, $\gcd(l, p) = 1 \Rightarrow m_2 = 0$. We base change the torsor equation of $X_{2,b} \to X_b$ to $X_{1,b}$ to obtain:

$$\begin{aligned} Z_2^p &= \sum_{i\in\mathbb{Z}} a_i T^i = \sum_{i\in\mathbb{Z}} a_i Z^{ip}(1+B)^{i/m_1} = \sum_{i\in\mathbb{Z}} a_i Z^{ip}\left(1+\frac{i}{m_1}B+...\right) \\ &= \sum_{i\in\mathbb{Z}} a_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip}B + ... \\ &= \sum_{v_K(a_i)=0} a_i Z^{ip} + \sum_{v_K(a_i)>0} a_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip}B + ... \end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$ to obtain:

$$\begin{aligned} Z_2^p &= \sum_{v_K(b_i)=0} b_i^p Z^{ip} + \sum_{v_K(c_i)>0} c_i Z^{ip} + \sum_{v_K(a_i)>0} a_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip}B + ... \\ &= \sum_{v_K(b_i)=0} \left(b_i Z^i\right)^p + \sum_{v_K(d_i)>0} d_i Z^{ip} + \sum_{i\in\mathbb{Z}} \frac{ia_i}{m_1} Z^{ip}B + ..., \end{aligned}$$

where we set $d_i = c_i$ if $v_K(a_i) = 0$ and $d_i = a_i$ if $v_K(a_i) > 0$. Now, as this is generically a μ_p-torsor we can take the p-power term $\left(b_l Z^l\right)^p$ in the first summation into factor, so that we get a new equation:

$$Z_2^p = 1 + \sum_{\substack{v_K(b_i)=0 \\ i\neq l}} \left(b_l^{-1} b_i Z^{i-l}\right)^p + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} + \sum_{i\in\mathbb{Z}} \frac{ib_l^{-p}a_i}{m_1} Z^{p(i-l)}B + ...,$$

which can be rewritten using the identity $(**)$, and after multiplying by a suitable p-power, as:

$$Z_2^p = 1 - p\sum_{\substack{v_K(b_i)=0 \\ i\neq l}} b_l^{-1} b_i Z^{i-l} + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} + \sum_{i\in\mathbb{Z}} \frac{ib_l^{-p}a_i}{m_1} Z^{p(i-l)}B + \text{higher order terms.}$$

The summation $\sum_{v_K(d_i)>0} d_i b_l^{-p} Z^{p(i-l)}$ does not contribute to the conductor variable since the powers of Z involved are p-powers so we can safely exclude it. Indeed, if the coefficient with smallest K-valuation in the right hand side of the above equation occurs in the above summation say in the term $d_i b_l^{-p} Z^{p(i-l)}$ then $v_K(d_i)$ is necessarily divisible by p (since $(Y_b)_k$ is reduced), and we can assume without loss of generality that this summation is of the form $\pi^{pt}(f(Z)^p + \pi g(Z))$, where $f(Z) \in A_1$ is a unit and $g(Z) \in A_1$. Writing $1 + \pi^{pt} f(Z)^p = (1 + \pi^t f(Z))^p - \sum_{k=1}^{p-1} \binom{p}{k} (\pi^t f(Z))^k$ and multiplying the above equation by $(1 + \pi^t f(Z))^{-p}$ we obtain an equation:

$$Z_2^p = 1 - p\sum_{\substack{v_K(b_i)=0 \\ i\neq l}} b_l^{-1} b_i Z^{i-l} - \sum_{k=1}^{p-1} \binom{p}{k} \left(\pi^t f(Z)\right)^k + \sum_{i\in\mathbb{Z}} \frac{ib_l^{-p}a_i}{m_1} Z^{p(i-l)}B + \text{higher order terms.}$$

Furthermore the summation $\sum_{k=1}^{p-1} \binom{p}{k} (\pi^t f(Z))^k$ doesn't contribute anymore towards the coefficient with smallest possible valuation in the right hand side of the above equation. For the rest of this proof we will automatically operate in this way and ignore such summations. Then, up to multiplying the coefficients by units, we have:

$$Z_2^p = 1 - \pi^{v_K(p)} \sum_{\substack{v_K(b_i)=0 \\ i \neq l}} b_l^{-1} b_i Z^{i-l} + \pi^{v_K(p)-n(p-1)} \sum_{i \in \mathbb{Z}} \frac{i b_l^{-p} a_i}{m_1} Z^{p(i-l)-m_1(p-1)} + \text{higher order terms.}$$

Clearly the smallest power of π is $v_K(p) - n(p-1)$ and so we look to the summation with that coefficient for the conductor variable. For zero valuation coefficients, the index of the summation will start at the integer l, the index corresponding to the lowest zero valuation coefficient. As m_1' is the smallest exponent appearing in the relevant summation which is coprime to p, we have that $m_1' = -m_1(p-1)$.

(a2) In this case $\gcd(l, p) \neq 1$. Again, we take $T = Z^p (1+B)^{1/m_1}$ where B is as defined previously. We then base change the μ_p-torsor equation of $X_{2,b} \to X_b$ to $X_{1,b}$ to obtain:

$$\begin{aligned} Z_2^p &= \sum_{i \in \mathbb{Z}} a_i T^i = 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i \\ &= 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(a_i)>0} a_i Z^{ip} (1+B)^{i/m_1}. \end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$ so that:

$$\begin{aligned} Z_2^p &= 1 + \sum_{v_K(b_i)=0} b_i^p Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(c_i)>0} c_i Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(a_i)>0} a_i Z^{ip} (1+B)^{i/m_1} \\ &= 1 + \sum_{v_K(b_i)=0} b_i^p Z^{ip} (1+B)^{i/m_1} + \sum_{v_K(d_i)>0} d_i Z^{ip} (1+B)^{i/m_1}, \end{aligned}$$

where we again set $d_i = c_i$ if $v_K(a_i) = 0$ and $d_i = a_i$ if $v_K(a_i) > 0$. Now we continue by expansion of the binomial terms:

$$\begin{aligned} Z_2^p &= 1 + \sum_{v_K(b_i)=0} b_i^p Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots\right) + \sum_{v_K(d_i)>0} d_i Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots\right) \\ &= 1 + \sum_{v_K(b_i)=0} \left(b_i Z^i\right)^p + \sum_{v_K(b_i)=0} \frac{i}{m_1} b_i^p Z^{ip} B + \sum_{v_K(d_i)>0} d_i Z^{ip} \\ &\quad + \sum_{v_K(d_i)>0} \frac{i}{m_1} d_i Z^{ip} B + \text{higher order terms.} \end{aligned}$$

The summation $\sum_{v_K(d_i)>0} d_i Z^{ip}$ does not contribute to the conductor variable since the powers of Z involved are p-powers so we can safely exclude it (cf. Proof in case (a1)). Now using the identity $(**)$, and after multiplying by a suitable p-power, we obtain an equation:

$$\begin{aligned} Z_2^p &= 1 - p \sum_{v_K(b_i)=0} b_i Z^i + \sum_{v_K(b_i)=0} \frac{i}{m_1} b_i^p Z^{ip} B + \text{higher order terms} \\ &= 1 - p \sum_{i \geq m_2} b_i Z^i + \sum_{i \geq m_2} \frac{i}{m_1} b_i^p Z^{ip} B + \text{higher order terms,} \end{aligned}$$

which equals:

$$1 - p\sum_{i\geq m_2} b_i Z^i + \sum_{i\geq m_2}\frac{i}{m_1} b_i^p Z^{ip}\left(\sum_{k=1}^{p-1}\binom{p}{k}\pi^{-n(p-k)}Z^{-m_1(p-k)}\right) + \text{higher order terms}$$

$$= 1 - \pi^{v_K(p)}\sum_{i\geq m_2} b_i Z^i + \pi^{v_K(p)-n(p-1)}\sum_{i\geq m_2}\frac{ib_i^p}{m_1}Z^{ip-m_1(p-1)} + \text{higher order terms;}$$

up to multiplying the coefficients by units. The second summation has the smallest π valuation and so the conductor variable is $m_1' = m_2 p - m_1(p-1)$.

We now determine m_2'. We choose the parameter T so that $u_1 = \sum_{i\in\mathbb{Z}} a_i T^i$ and $u_2 = T^h$ in the case (a1) while $u_2 = 1 + T^{m_2}$ in the case (a2).

(a1) In this case, $m_2 = 0$. The parameter of $X_{2,b}$ is $Z := Z_2^{1/h}$ where $T = Z^p$ is obtained from the torsor equation $Z_2^p = T^h \Leftrightarrow \left(Z_2^{1/h}\right)^p = T$. We base change the torsor equation of $X_{1,b} \to X_b$ to $X_{2,b}$ to obtain:

$$(1+\pi^n Z_1)^p = 1 + \pi^{np}\sum_{i\in\mathbb{Z}} a_i T^i = 1 + \pi^{np}\sum_{i\in\mathbb{Z}} a_i Z^{pi} = 1 + \pi^{np}\sum_{v_K(a_i)=0} a_i Z^{pi} + \pi^{np}\sum_{v_K(a_i)>0} a_i Z^{pi}.$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$. Hence

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1 + \pi^{np}\sum_{v_K(b_i)=0} b_i^p Z^{pi} + \pi^{np}\sum_{v_K(c_i)>0} c_i Z^{pi} + \pi^{np}\sum_{v_K(a_i)>0} a_i Z^{pi}\\ &= 1 + \sum_{v_K(b_i)=0}\left(\pi^n b_i Z^i\right)^p + \sum_{v_K(c_i)>0}\pi^{np} c_i Z^{pi} + \sum_{v_K(a_i)>0}\pi^{np} a_i Z^{pi}\\ &= 1 + \sum_{v_K(b_i)=0}\left(\pi^n b_i Z^i\right)^p + \sum_{v_K(d_i)>0}\pi^{np} d_i Z^{pi},\end{aligned}$$

where we again set $d_i = c_i$ if $v_K(a_i) = 0$ and $d_i = a_i$ if $v_K(a_i) > 0$. Using the identity $(**)$, and after multiplying by a suitable p-power, we get: $(1+\pi^n Z_1)^p = 1 - p\sum_{v_K(b_i)=0}\pi^n b_i Z^i + \sum_{v_K(d_i)>0}\pi^{np} d_i Z^{pi}$ + higher order terms $= 1 - \pi^{v_K(p)+n}\sum_{i\geq m_1} b_i Z^i$ + higher order terms; up to multiplying the coefficients by units. Now, m_2' is the smallest exponent appearing in this leading summation which is coprime to p and so $m_2' = m_1$.

(a2) In this case $\gcd(l, p) \neq 1$. From the torsor equation $Z_2^p = 1 + T^{m_2}$ we can deduce:

$$\begin{aligned}Z_2^p = 1 + T^{m_2} \Leftrightarrow Z_2^p - 1 = T^{m_2} &\Leftrightarrow (Z_2-1)^p - \sum_{k=1}^{p-1}\binom{p}{k}(-1)^k Z_2^k = T^{m_2}\\ &\Leftrightarrow (Z_2-1)^p\left(1 - (Z_2-1)^{-p}\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k Z_2^k\right) = T^{m_2}\end{aligned}$$

which implies:

$$T = \left((Z_2-1)^{\frac{1}{m_2}}\right)^p\left(1 - (Z_2-1)^{-p}\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k Z_2^k\right)^{\frac{1}{m_2}},$$

and so we take $Z := (Z_2-1)^{1/m_2}$, which we already know to be the parameter of $X_{2,b}$ by case 2 of this proof, in order to write:

$$T = Z^p\left(1 - Z^{-m_2 p}\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k(1+Z^{m_2})^k\right)^{\frac{1}{m_2}}.$$

For simplicity, let us denote $\sum_{k=1}^{p-1} \binom{p}{k}(-1)^k(1+Z^{m_2})^k$ by B so that we can write $T = Z^p(1-Z^{-m_2 p}B)^{\frac{1}{m_2}}$. We can now base change the torsor equation of $X_{1,b} \to X_b$ to $X_{2,b}$ to obtain:

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1+\pi^{np}\sum_{i\in\mathbb{Z}} a_i T^i = 1+\pi^{np}\sum_{i\in\mathbb{Z}} a_i Z^{pi}(1-Z^{-m_2 p}B)^{\frac{i}{m_2}}\\ &= 1+\pi^{np}\sum_{i\in\mathbb{Z}} a_i Z^{pi}\left(1-\frac{i}{m_2}Z^{-m_2 p}B+\ldots\right)\\ &= 1+\pi^{np}\sum_{i\in\mathbb{Z}} a_i Z^{pi}-\pi^{np}\sum_{i\in\mathbb{Z}}\frac{ia_i}{m_2}Z^{p(i-m_2)}B+\ldots.\end{aligned}$$

Partitioning any summation above over the index i into the terms where $v_K(a_i)=0$ and the terms where $v_K(a_i)>0$ gives:

$$\begin{aligned}(1+\pi^n Z_1)^p = 1&+\pi^{np}\sum_{v_K(a_i)=0} a_i Z^{pi}-\pi^{np}\sum_{v_K(a_i)=0}\frac{ia_i}{m_2}Z^{p(i-m_2)}B\\ &+\pi^{np}\sum_{v_K(a_i)>0} a_i Z^{pi}-\pi^{np}\sum_{v_K(a_i)>0}\frac{ia_i}{m_2}Z^{p(i-m_2)}B+\ldots.\end{aligned}$$

Again, for the terms with $v_K(a_i)=0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i)=0$ but $v_K(c_i)>0$ and so we have:

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1+\pi^{np}\sum_{v_K(b_i)=0} b_i^p Z^{pi}\\ &\quad+\pi^{np}\sum_{v_K(c_i)>0} c_i Z^{pi}-\pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i^p}{m_2}Z^{p(i-m_2)}B-\pi^{np}\sum_{v_K(c_i)>0}\frac{ic_i}{m_2}Z^{p(i-m_2)}B\\ &\quad+\pi^{np}\sum_{v_K(a_i)>0} a_i Z^{pi}-\pi^{np}\sum_{v_K(a_i)>0}\frac{ia_i}{m_2}Z^{p(i-m_2)}B+\text{higher order terms}\\ &= 1+\sum_{v_K(b_i)=0}\left(\pi^n b_i Z^i\right)^p-\pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i^p}{m_2}Z^{p(i-m_2)}B+\text{higher order terms;}\end{aligned}$$

excluding summations whose coefficients have positive valuation. Using the identity $(**)$, and after multiplying by a suitable p-power, yields:

$$\begin{aligned}(1+\pi^n Z_1)^p &= 1-p\sum_{v_K(b_i)=0}\pi^n b_i Z^i-\pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i^p}{m_2}Z^{p(i-m_2)}B+\text{higher order terms}\\ &= 1-\pi^{v_K(p)+n}\sum_{i\geq l} b_i Z^i-\pi^{np}\sum_{v_K(b_i)=0}\frac{ib_i}{m_2}Z^{p(i-m_2)}\left(\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k(1+Z^{m_2})^k\right)\\ &\quad+\text{higher order terms}\\ &= 1-\pi^{v_K(p)+n}\sum_{i\geq l} b_i Z^i+\pi^{v_K(p)+np}\sum_{v_K(b_i)=0}\frac{ib_i}{m_2}Z^{p(i-m_2)}(1+Z^{m_2})\\ &\quad+\text{higher order terms;}\end{aligned}$$

up to multiplying the coefficients by units. Since $v_K(p)+n$ is the smallest exponent of π and

$l = \min\{i | v_K(a_i) = 0, \gcd(i,p) = 1\} = m_1$ is the starting index, we have that $m_2' = m_1$ is the conductor variable.

5. (μ_p, μ_p). Here $m_1, m_2 \geq 0$. The first μ_p-torsor is given by the equation $Z_1^p = u_1$ and the second μ_p-torsor by $Z_2^p = u_2$ where $u_1, u_2 \in A^\times$. Modulo π, these torsor equations reduce to $z_1^p = \bar{u}_1$ and $z_2^p = \bar{u}_2$ with acting group schemes μ_p on the special fibre. On the special fibre the reduced power series are of the form $\bar{u} = \sum_{i \geq l} \bar{a}_i t^i$ for some integer l. Depending on whether or not these l are coprime to p or not, there are three cases to consider. In particular, the pairs $(a1, a1)$, $(a1, a2)$ and $(a2, a2)$. We only treat the cases $(a1, a2)$ and $(a2, a2)$, the case $(a2, a1)$ is treated in a similar way to the case $(a1, a2)$. For the case $(a1, a1)$ see Remark 1.8.

(a1,a2) Here $m_1 = 0$ and $m_2 > 0$. We start by computing m_1'. We can choose the parameter T so that $u_1 = T^h$ but $u_2 = 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i$ remains as a power series. From the torsor equation $Z_1^p = T^h$, we can write $T = Z^p$ where $Z = Z_1^{1/h}$ is the parameter of $X_{1,b}$. Then we can base change the torsor equation of $X_{2,b} \to X_b$ to $X_{1,b}$ and obtain:

$$Z_2^p = 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i = 1 + \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi}.$$

For the terms with $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$ and so we have:

$$\begin{aligned} Z_2^p &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{pi} + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} \\ &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} \left(b_i Z^i\right)^p + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi}. \end{aligned}$$

Using the identity $(**)$, and after multiplying by a suitable p-power, we get:

$$\begin{aligned} Z_2^p &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} + \text{higher order terms} \\ &= 1 - \pi^{v_K(p)} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i + \sum_{\substack{v_K(c_i)>0 \\ i \geq m_2}} c_i Z^{pi} + \sum_{v_K(a_i)>0} a_i Z^{pi} + \text{higher order terms,} \end{aligned}$$

up to multiplying the coefficients by units. Therefore, $m_1' = m_2$. Here we ignored the last two summands in the above equality (cf. proof of case 4 computing m_1').

Now we want to determine m_2'. We can choose the parameter T so that $u_1 = \sum_{i \in \mathbb{Z}} a_i T^i$ is a power series and $u_2 = 1 + T^{m_2}$ is in simplified form. We know from case 4 that the torsor equation $Z_2^p = 1 + T^{m_2}$ gives rise to: $T = Z^p \left(1 - Z^{-m_2 p} \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_2})^k\right)^{\frac{1}{m_2}}$ where $Z := (Z_2 - 1)^{1/m_2}$ is the parameter of $X_{2,b}$. We have that $Z_1^p = \sum_{i \in \mathbb{Z}} a_i T^i = \sum_{\substack{v_K(a_i)=0 \\ i \geq l}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i$ where l is such that $\gcd(l, p) = 1$. We can write $T = Z^p (1 - Z^{-m_2 p} B)^{\frac{1}{m_2}}$ by letting $B = \sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_2})^k$. Then we base change the torsor equation of $X_{1,b} \to X_b$ to $X_{2,b}$ and obtain:

$$Z_1^p = \sum_{\substack{v_K(a_i)=0 \\ i\geq l}} a_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} + \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}}.$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$ and so we have:

$$\begin{aligned} Z_1^p &= \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_i^p Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} + \sum_{\substack{v_K(c_i)>0 \\ i\geq l}} c_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} \\ &\quad + \sum_{v_K(a_i)>0} a_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} \\ &= \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_i^p Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}} + \sum_{v_K(d_i)>0} d_i Z^{ip} \left(1 - Z^{-m_2 p} B\right)^{\frac{i}{m_2}}, \end{aligned}$$

where $d_i = c_i$ if $v_K(a_i) = 0$ and $d_i = a_i$ for $v_K(a_i) > 0$. Hence:

$$\begin{aligned} Z_1^p &= \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_i^p Z^{ip} \left(1 - \frac{i}{m_2} Z^{-m_2 p} B + \ldots\right) + \sum_{v_K(d_i)>0} d_i Z^{ip} \left(1 - \frac{i}{m_2} Z^{-m_2 p} B + \ldots\right) \\ &= \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_i^p Z^{ip} - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_i^p Z^{ip} \frac{i}{m_2} Z^{-m_2 p} B + \sum_{v_K(d_i)>0} d_i Z^{ip} \\ &\quad - \sum_{v_K(d_i)>0} d_i Z^{ip} \frac{i}{m_2} Z^{-m_2 p} B + \text{higher order terms}. \end{aligned}$$

Taking into factor the p-power $b_l^p Z^{lp} = \left(b_l Z^l\right)^p$ we obtain:

$$\begin{aligned} Z_1^p &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-p} b_i^p Z^{p(i-l)} - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \\ &\quad - \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \text{higher order terms} \\ &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i>l}} \left(b_l^{-1} b_i Z^{i-l}\right)^p - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \\ &\quad - \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \text{higher order terms}. \end{aligned}$$

Using the identity $(**)$, and after multiplying by a suitable p-power, we get

$$\begin{aligned} Z_1^p &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-1} b_i Z^{i-l} - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \\ &\quad - \sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} B + \text{higher order terms} \\ &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-1} b_i Z^{i-l} - \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p Z^{p(i-l)} \frac{i}{m_2} Z^{-m_2 p} \left(\sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_2})^k\right) \end{aligned}$$

$$+ \text{ higher order terms}$$

$$= 1 - \pi^{v_k(p)} \sum_{\substack{v_K(b_i)=0 \\ i>l}} b_l^{-1} b_i Z^{i-l} + \pi^{v_k(p)} \sum_{\substack{v_K(b_i)=0 \\ i\geq l}} b_l^{-p} b_i^p \frac{i}{m_2} Z^{p(i-l)-m_2 p}(1+Z^{m_2})$$

$$+ \text{ higher order terms,}$$

up to multiplying the coefficients by units. Then, clearly, $m_2' = -m_2(p-1)$ is the conductor variable. Here we ignored the term $\sum_{v_K(d_i)>0} b_l^{-p} d_i Z^{p(i-l)}$ in the above summation (cf. proof of case 4 computing m_1').

(a2,a2) Here both $m_1, m_2 > 0$ and we start by computing m_1'. The parameter T can be chosen so that $u_1 = 1 + T^{m_1}$ but $u_2 = \sum_{i\in\mathbb{Z}} a_i T^i$ remains as a power series. From the torsor equation $Z_1^p = 1 + T^{m_1}$, we know from case 4 that we can write T as: $T = Z^p\left(1 - Z^{-m_1 p}\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k(1+Z^{m_1})^k\right)^{\frac{1}{m_1}}$ where $Z := (Z_1-1)^{1/m_1}$ is the parameter of $X_{1,b}$. As before, for purposes of convenience, we write $T = Z^p(1 - Z^{-m_1 p}B)^{\frac{1}{m_1}}$ where $B = \sum_{k=1}^{p-1}\binom{p}{k}(-1)^k(1+Z^{m_1})^k$. Then we can base change the torsor equation for $X_{2,b} \to X_b$ to $X_{1,b}$ and obtain:

$$\begin{aligned}
Z_2^p &= 1 + \sum_{\substack{v_K(a_i)=0 \\ i\geq m_2}} a_i T^i + \sum_{v_K(a_i)>0} a_i T^i \\
&= 1 + \sum_{\substack{v_K(a_i)=0 \\ i\geq m_2}} a_i Z^{ip}\left(1 - Z^{-m_1 p}B\right)^{\frac{i}{m_1}} + \sum_{v_K(a_i)>0} a_i Z^{ip}\left(1 - Z^{-m_1 p}B\right)^{\frac{i}{m_1}} \\
&= 1 + \sum_{\substack{v_K(a_i)=0 \\ i\geq m_2}} a_i Z^{ip}\left(1 - \frac{i}{m_1}Z^{-m_1 p}B + \ldots\right) + \sum_{v_K(a_i)>0} a_i Z^{ip}\left(1 - \frac{i}{m_1}Z^{-m_1 p}B + \ldots\right) \\
&= 1 + \sum_{\substack{v_K(a_i)=0 \\ i\geq m_2}} a_i Z^{ip} - \sum_{\substack{v_K(a_i)=0 \\ i\geq m_2}} a_i Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \sum_{v_K(a_i)>0} a_i Z^{ip} \\
&\quad - \sum_{v_K(a_i)>0} a_i Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \text{higher order terms.}
\end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$ to obtain:

$$\begin{aligned}
Z_2^p &= 1 + \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} - \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \sum_{v_K(c_i)>0} c_i Z^{ip} - \sum_{v_K(c_i)>0} c_i Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B \\
&\quad + \sum_{v_K(a_i)>0} a_i Z^{ip} - \sum_{v_K(a_i)>0} a_i Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \text{higher order terms} \\
&= 1 + \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} \left(b_i Z^i\right)^p - \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \sum_{v_K(d_i)>0} d_i Z^{ip} \\
&\quad - \sum_{v_K(d_i)>0} d_i Z^{ip}\frac{i}{m_1}Z^{-m_1 p}B + \text{higher order terms,}
\end{aligned}$$

where we take $d_i = c_i$ if $v_K(a_i) = 0$ and $d_i = a_i$ with $v_K(a_i) > 0$. Using the identity $(**)$, and

after multiplying by a suitable p-power, we get:

$$\begin{aligned} Z_2^p &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i - \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} Z^{-m_1 p} B + \text{higher order terms} \\ &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i - \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} Z^{-m_1 p} \left(\sum_{k=1}^{p-1} \binom{p}{k} (-1)^k (1 + Z^{m_1})^k \right) \\ &\quad + \text{higher order terms.} \end{aligned}$$

Then up to multiplying the coefficients by units we have:

$$Z_2^p = 1 - \pi^{v_K(p)} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i Z^i + \pi^{v_K(p)} \sum_{\substack{v_K(b_i)=0 \\ i \geq m_2}} b_i^p \frac{i}{m_1} Z^{(i-m_1)p} (1 + Z^{m_1}) + \text{higher order terms.}$$

In order to determine which is the smallest power of Z and hence the conductor variable m_1', we need to compare m_2 with $m_2 p - m_1(p-1)$ since both summations have coefficients with the same π valuation. Note that $m_2 \leq m_2 p - m_1(p-1)$ is equivalent to $m_1 \leq m_2$. Assuming $m_1 \leq m_2$, we have $m_1' = m_2$ and, by symmetry, $m_2' = m_1 p - m_2(p-1)$. The case $m_1 \leq m_2$ is entirely similar.

6. $(\mathcal{H}_{n_1}, \mathcal{H}_{n_2})$. Here $m_1, m_2 \in \mathbb{Z}$ and both are coprime to p. The $\mathcal{H}_{n_1}$-torsor equation is given by $(\pi^{n_1} Z_1 + 1)^p = 1 + \pi^{pn_1} u_1$ and the $\mathcal{H}_{n_2}$-torsor equation by $(\pi^{n_2} Z_2 + 1)^p = 1 + \pi^{pn_2} u_2$ where $u_1, u_2 \in A^\times$. The two torsors have associated conductor variables m_1, m_2 respectively.

We begin by computing m_1'. We can choose the parameter T so that $u_1 = T^{m_1}$ but $u_2 = \sum_{i \in \mathbb{Z}} a_i T^i$ remains as a power series. By case 4, we can express T in terms of $Z := Z_1^{1/m_1}$, the parameter for $X_{1,b}$, as: $T = Z^p \left(1 + \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n_1(p-k)} Z^{-m_1(p-k)}\right)^{\frac{1}{m_1}}$ and, for convenience, we set $B = \sum_{k=1}^{p-1} \binom{p}{k} \pi^{-n_1(p-k)} Z^{-m_1(p-k)}$ so that we can write $T = Z^p (1+B)^{\frac{1}{m_1}}$. Then we base change the torsor equation for $X_{2,b} \to X_b$ to $X_{1,b}$ to obtain:

$$\begin{aligned} Z_2^p &= 1 + \pi^{pn_2} \sum_{i \in \mathbb{Z}} a_i T^i = 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i T^i + \pi^{pn_2} \sum_{v_K(a_i) > 0} a_i T^i \\ &= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} (1+B)^{\frac{i}{m_1}} + \pi^{pn_2} \sum_{v_K(a_i) > 0} a_i Z^{ip} (1+B)^{\frac{i}{m_1}} \\ &= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots\right) + \pi^{pn_2} \sum_{v_K(a_i) > 0} a_i Z^{ip} \left(1 + \frac{i}{m_1} B + \ldots\right) \\ &= 1 + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} + \pi^{pn_2} \sum_{\substack{v_K(a_i)=0 \\ i \geq m_2}} a_i Z^{ip} \frac{i}{m_1} B + \pi^{pn_2} \sum_{v_K(a_i) > 0} a_i Z^{ip} \\ &\quad + \pi^{pn_2} \sum_{v_K(a_i) > 0} a_i Z^{ip} \frac{i}{m_1} B + \text{higher order terms.} \end{aligned}$$

For the terms where $v_K(a_i) = 0$, we can express $a_i = b_i^p + c_i$ for some $b_i, c_i \in R$ with $v_K(b_i) = 0$ and $v_K(c_i) > 0$ to obtain:

$$\begin{aligned}
Z_2^p &= 1 + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} B + \pi^{pn_2} \sum_{v_K(c_i)>0} c_i Z^{ip} \\
&\quad + \pi^{pn_2} \sum_{v_K(c_i)>0} c_i Z^{ip} \frac{i}{m_1} B + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} \\
&\quad + \pi^{pn_2} \sum_{v_K(a_i)>0} a_i Z^{ip} \frac{i}{m_1} B + \text{higher order terms} \\
&= 1 + \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} \left(\pi^{n_2} b_i Z^i\right)^p + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} B + \pi^{pn_2} \sum_{v_K(d_i)>0} d_i Z^{ip} \\
&\quad + \pi^{pn_2} \sum_{v_K(d_i)>0} d_i Z^{ip} \frac{i}{m_1} B + \text{higher order terms,}
\end{aligned}$$

where $d_i = c_i$ if $v_K(a_i) = 0$ and $d_i = a_i$ if $v_K(a_i) > 0$. Using the identity $(**)$, and after multiplying by a suitable p-power, we get:

$$\begin{aligned}
(\pi^{n_2} Z_2 + 1)^p &= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} \pi^{n_2} b_i Z^i + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} B + \text{higher order terms} \\
&= 1 - p \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} \pi^{n_2} b_i Z^i \\
&\quad + \pi^{pn_2} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} \left(\sum_{k=1}^{p-1} \binom{p}{k} \pi^{n_1(k-p)} Z^{-m_1(p-k)} \right) + \text{higher order terms.}
\end{aligned}$$

Then up to multiplying the coefficients by units we have:

$$\begin{aligned}
Z_2^p &= 1 - \pi^{v_K(p)+n_2} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i Z^i \\
&\quad + \pi^{v_K(p)+pn_2-n_1(p-1)} \sum_{\substack{v_K(b_i)=0 \\ i\geq m_2}} b_i^p Z^{ip} \frac{i}{m_1} Z^{-m_1(p-1)} + \text{higher order terms.}
\end{aligned}$$

We have to compare $v_K(p) + n_2$ with $v_K(p) + pn_2 - n_1(p-1)$ in order to determine the smallest power of π. Note that $v_K(p) + n_2 < v_K(p) + pn_2 - n_1(p-1)$ is equivalent to $n_1 < n_2$ and so when this happens, $m_1' = m_2$ and when $n_1 > n_2$, we have $m_1' = m_2 p - m_1(p-1)$. We also need to consider the case where $n_1 = n_2$. Comparing m_2 with $m_2 p - m_1(p-1)$ we have that:

$$m_1' = \min\{m_2, m_2 p - m_1(p-1)\} = \begin{cases} m_2 & \text{if } m_1 < m_2 \\ m_2 p - m_1(p-1) & \text{if } m_1 \geq m_2 \end{cases}$$

Now we want to determine m_2' but by the symmetry present in this case this is entirely similar to the above consideration. In particular, if $n_1 < n_2$ then $m_2' = m_1 p - m_2(p-1)$, if $n_1 > n_2$ then $m_2' = m_1$ and, finally, if $n_1 = n_2$ then:

$$m_2' = \min\{m_2, m_1 p - m_2(p-1)\} = \begin{cases} m_1 p - m_2(p-1) & \text{if } m_1 < m_2 \\ m_1 & \text{if } m_1 \geq m_2 \end{cases}$$

All six possible cases have now been treated. □

We are also able to state when the Galois cover $Y_b \to X_b$ has a torsor structure by taking into account when base changing in the above proof without additional modification resulted in the equation of the normalisation (see also Theorem 3.4).

Theorem 1.2. *Let $f_i : X_{i,b} \to X_b$ be non-trivial Galois covers of degree p above the formal boundary X_b which are generically disjoint for $i = 1, 2$. Let G_i be the corresponding group schemes for $i = 1, 2$ and let Y_b be as defined in* Theorem 1.1. *Then $Y_b = X_{1,b} \times_{X_b} X_{2,b}$, in which case $Y_b \to X_b$ is a torsor under $G_1 \times_{\mathrm{Spec} R} G_2$, if and only if at least one of the two group schemes G_i is the étale group scheme $\mathcal{H}_{v_K(\lambda)}$.*

Proof. See Theorem 3.5. □

Definition 1.3. For the extension B/A of DVR's where $X_b = \mathrm{Spf}\,(A)$ and $Y_b = \mathrm{Spf}\,(B)$ are as in Theorem 1.1, we define the **special different(s)** by

$$d_{s_1} = (c_1 - 1)p(p-1) + (c_1' - 1)(p-1)$$

$$d_{s_2} = (c_2 - 1)p(p-1) + (c_2' - 1)(p-1)$$

Lemma 1.4. *The above two special differents are in fact equal: $d_{s_1} = d_{s_2}$.*

Proof. Follows immediately by substituting the possible values for c_1' and c_2' under each of the six cases given in Corollary 1.2. □

This d_{s_i}, $i = 1, 2$, coincides in fact with the term $\varphi(s)$ which appears in Kato's vanishing cycles formula in the case of a Galois cover of type (p, p) (Theorem 6.7 in [3]). We will also see this variable makes an appearance in our genus formula in Theorem 2.2.1 in the next section.

Corollary 1.5. *We have the following relationship between conductors:*

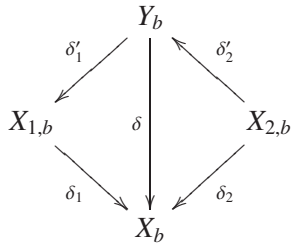
$$c_2' - c_1' = (c_1 - c_2)p.$$

Proof. Follows from rearranging the relationship between conductors given by $d_{s_1} = d_{s_2}$. □

Had Corollary 1.5 been given, we would have only needed to perform half of the computations in the proof of Theorem 1.1. In particular, with m_1' (or equivalently c_1') obtained, m_2' (or equivalently c_2') would be determined by this relationship. In fact, this formula could have been derived independently using the theory of higher ramification groups as per [10] but only for the $(\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)})$ case, the first of the six cases in Theorem 1.1. This is because this ramification theory only holds when the residue field extension is separable and so both group schemes G_i, $i = 1, 2$, must be étale.

From the calculations in the proof of Theorem 1.1 it is also possible to compute the degree

of the different δ in the extension B/A. Since the ramification index of this extension $e = 1$, we see easily that $\delta = \delta_1 + \delta'_1 = \delta_2 + \delta'_2$ (cf. Theorem 1.6 for notations).



Theorem 1.6. *With the situation described in* Theorem 1.1*, let δ'_i (resp. δ_i) denote the degree of the different corresponding to the extension $Y_b \to X_{i,b}$ (resp. $X_{i,b} \to X_b$), $i = 1, 2$. Then, for all possible pairs (G_1, G_2), we can explicitly state the values for δ'_i as follows:*

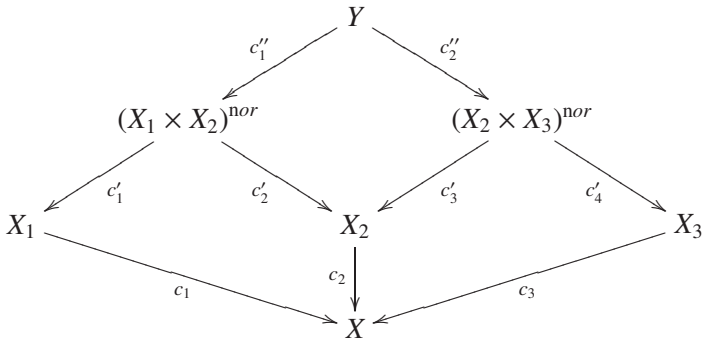
Table 1. Degree of the differents δ'_1, δ'_2

(G_1, G_2)	δ'_1	δ'_2
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_{v_K(\lambda)}$	0	0
$\mathcal{H}_{v_K(\lambda)}, \mu_p$	$v_K(p)$	0
$\mathcal{H}_{v_K(\lambda)}, \mathcal{H}_n$	$v_K(p) - n(p-1)$	0
$\mathcal{H}_n, \mu_p$	$v_K(p) - \frac{v_K(p)-n(p-1)}{p}(p-1)$	$v_K(p) - \frac{v_K(p)+n}{p}(p-1)$
μ_p, μ_p	$v_K(p) - \frac{v_K(p)}{p}(p-1)$	$v_K(p) - \frac{v_K(p)}{p}(p-1)$
$\mathcal{H}_{n_1}, \mathcal{H}_{n_2}$	*If $n_1 \le n_2$ then δ'_1 equals:* $v_K(p) - \frac{v_K(p)+n_2}{p}(p-1)$ *If $n_1 > n_2$ then δ'_1 equals:* $v_K(p) - \frac{v_K(p)+n_2 p-n_1(p-1)}{p}(p-1)$	*If $n_1 \le n_2$ then δ'_2 equals:* $v_K(p) - \frac{v_K(p)+n_1 p-n_2(p-1)}{p}(p-1)$ *If $n_1 > n_2$ then δ'_2 equals:* $v_K(p) - \frac{v_K(p)+n_1}{p}(p-1)$

Proof. For an arbitrary rank p torsor with conductor variable m and torsor equation $Z^p = 1 + \pi^{np}T^m +$ higher terms, where $0 \le n \le v_K(\lambda)$, the degree of the different is given by $\delta = v_K(p) - n(p-1)$. This means that computing the degree of the different δ reduces to obtaining n from the exponent of π in the coefficient of the term corresponding to the conductor variable m. From our calculations obtained in the proof of Theorem 1.1, we can simply read off the n value in each of the cases and substitute into the formula $v_K(p)-n(p-1)$ to obtain the degree of the different at that particular stage. Strictly speaking, we can only

rely on this approach for the last three cases because in the first three cases we worked modulo π on the special fibre. In the first three cases to compute δ we use the fact that the degree of the different is preserved by étale base change. □

EXAMPLE 1.7. With the type (p, p) case established it is possible to manually perform the same calculations in the type $(p, ..., p)$ setting. We illustrate this with a type (p, p, p) example, using the (p, p)-type results in Theorem 1.1 and Corollary 1.2 iteratively at each stage to determine the conductors in terms of the base level conductors.



Suppose $X_i \to X$ are torsors under the R-group scheme G_i of rank p for $i = 1, 2, 3$ which are pairwise generically disjoint, i.e X_i is generically disjoint from X_j for $i \neq j$. Write $(X_1 \times X_2)^{nor}$ and $(X_2 \times X_3)^{nor}$ for the normalisation of X in $X_{1,K} \times_{X_K} X_{2,K}$ and $X_{2,K} \times_{X_K} X_{3,K}$, respectively, and Y for the normalisation of X in $X_{1,K} \times_{X_K} X_{2,K} \times_{X_K} X_{3,K}$. For the purposes of an example, let $G_i = \mathcal{H}_{v_K(\lambda)}$ for all i and assume $c_1 \leq c_2 \leq c_3$ and $c_2' \leq c_3'$. Then, by applying the type (p, p) formula iteratively we can compute the conductor c_1'' as follows (similarly one can compute c_2''): $c_1'' = c_3' p - c_2'(p-1) = (c_3 p - c_2(p-1))\, p - c_1(p-1) = c_3 p^2 - c_2 p(p-1) - c_1(p-1)$.

REMARK 1.8. We discuss an example which illustrates the case (**a1**, **a1**) occuring in the proof of Theorem 1.1, the case (μ_p, μ_p). Here $m_1 = m_2 = 0$. In this case one can show that the group schemes acting on the torsors $Y_b \to X_{i,b}$ with conductor variables m_i' are $\mathcal{H}_{n_i'}$ with $0 < n_i' < v_K(\lambda)$ for $i = 1, 2$. Moreover, one can show $n_1' = n_2'$ and $m_1' = m_2'$. Suppose $u_1 = T^h$ and $u_2 = T^l(v(T))$ where $v(T) = \sum_{i \geq 0} a_i T^i$ such that $a_0 \in R^\times$ is a unit. The conductor variables m_1', m_2' are in fact encoded in $v(T)$. The proof is complicated to present in general. Instead, we treat an instructive example to illustrate the computations involved.

Suppose $p \neq 2$, $u_1 = T$ and $u_2 = T + T^3 = T(1 + T^2)$. Then the μ_p-torsors above X_b generically defined by $Z_1^p = T$ and $Z_2^p = T(1 + T^2)$ are linearly disjoint. We begin by computing m_1'. We can write $T = Z^p$ where $Z = Z_1$ is the parameter of $X_{1,b}$. Then we can base change the torsor equation for $X_{2,b} \to X_b$ to $X_{1,b}$:

$$Z_2^p = T(1 + T^2) = Z^p(1 + Z^{2p}).$$

Removing the multiplicative factor Z^p (which is a p-power) gives rise to an equation of the form:

$$Z_2^p = 1 + Z^{2p} = \left(1 + Z^2\right)^p - \sum_{k=1}^{p-1} \binom{p}{k} Z^{2k}.$$

Multiplying this equation by the p-power $\left(1+Z^2\right)^{-p} = 1 - pZ^2 + ...$, results in an equation of the form:

$$Z_2^p = 1 - \sum_{k=1}^{p-1} \binom{p}{k} Z^{2k} + \text{higher order terms.}$$

The smallest power of Z which is coprime to p is obtained when $k = 1$. Therefore, $m_1' = 2$.

Now we want to determine m_2'. We have that $Z_2^p = T(1+T^2)$ which we can write $Z_2^p = T' \Leftrightarrow Z^p = T'$ where the parameter of $X_{2,b}$ is $Z = Z_2$, and the relation $T' = T(1+T^2)$ implies: $T = T'\left(1+T^2\right)^{-1} = T'\left(1 - T^2 + T^4 - T^6 + ...\right) = T' + \left(-T'T^2 + T'T^4 - T'T^6 + ...\right)$. From this we deduce that T can be expressed as $T'+$ higher powers of T'. In particular, $T = T' - T'^3 + T'^5 - T'^7 + ...$ We can now proceed to base change the torsor equation of $X_{1,b} \to X_b$ to $X_{2,b}$:

$$\begin{aligned} Z_1^p = T &= T' - T'^3 + T'^5 - T'^7 + \text{higher order terms} \\ &= Z^p - Z^{3p} + Z^{5p} - Z^{7p} + \text{higher order terms} \\ &= Z^p\left(1 - Z^{2p} + Z^{4p} - Z^{6p} + \text{higher order terms}\right). \end{aligned}$$

Removing the multiplicative factor Z^p, gives rise to an equation of the form:

$$\begin{aligned} Z_1^p &= 1 - Z^{2p} + Z^{4p} - Z^{6p} + \text{higher order terms} \\ &= \left(1 - Z^2 + Z^4 ...\right)^p - p\sum(Z^2 + Z^4 + ...) + \text{higher order terms;} \end{aligned}$$

by using the formula $(**)$. Multiplying this equation by the inverse p-power $(1 - Z^2 + Z^4...)^{-p}$, results in an equation of the form:

$$\begin{aligned} Z_1^p &= 1 - p\sum(Z^2 + Z^4 + ...) + \text{higher order terms} \\ &= 1 - \pi^{v_K(p)}\sum(Z^2 + Z^4 + ...) + \text{higher order terms;} \end{aligned}$$

up to multiplying the coefficients by units. Therefore, the conductor variable is $m_2' = 2 = m_1'$ and $n_2' = \frac{v_K(p)}{p} = n_1'$.

2. Computation of vanishing cycles

2.1. Computation of vanishing cycles in Galois cover of degree p. In this section we recall some results from [6].

Definition 2.1.1. For an R-curve X and a closed point $x \in X$, we let $\mathcal{X} := \mathrm{Spf}\hat{\mathcal{O}}_{X,x}$ denote the formal spectrum of the completion of the local ring of X at x. Assume X_k is reduced. Then the **genus of the point** x is given by:

$$g_x := \delta_x - r_x + 1$$

where

- $\delta_x = \dim_k\left(\tilde{\mathcal{O}}_x/\mathcal{O}_x\right)$,
- r_x is the number of maximal ideals in $\tilde{\mathcal{O}}_x$.

Here $\mathcal{O}_x := \hat{\mathcal{O}}_{X,x}/\pi$ denotes the stalk of the special fibre $\mathcal{X}_k$ at x and $\tilde{\mathcal{O}}_x$ denotes its normali-

sation in its total ring of fractions.

If $g_x = 0$, the point x is either a smooth or an ordinary multiple point (where $\delta_x = r_x - 1$). Here is a result which provides an explicit formula—a local Riemann-Hurwitz formula—comparing the (above) genus in a Galois cover of degree p.

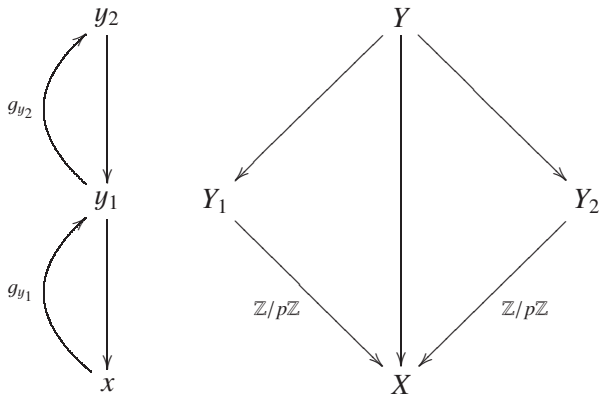
Theorem 2.1.2. (*cf.* Theorem 3.4 *in* [6]) *Let* $X := \mathrm{Spf}\left(\hat{\mathcal{O}}_x\right)$ *be the formal germ of an R-curve at a closed point* x *with* X_k *reduced* (*cf.* Notations). *Let* $f : Y \to X$ *be a Galois cover of degree* p *with* Y *normal and local. Assume that the special fibre* Y_k *of* Y *is reduced. Let* $\{\wp_i\}_{i\in I}$ *be the minimal prime ideals of* $\hat{\mathcal{O}}_x$ *which contain* π, *and let* $X_{b_i} := \mathrm{Spf}\left(\hat{\mathcal{O}}_{\wp_i}\right)$ *be the formal completion of the localisation of* X *at* $\wp_i$. *For each* $i \in I$ *the above cover* f *induces a torsor* $f_{b_i} : Y_{b_i} \to X_{b_i}$ *under a finite and flat R-group scheme* G_i *of rank* p *above the boundary* X_{b_i} *with conductor* c_i, *we write* $c_i = 1$ *in case this torsor is trivial. If* $y \in Y$ *is the closed point of* Y, *then:*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

where g_y *(resp.* g_x*) denotes the genus of* y *(resp.* x*),* d_η *is the degree of the divisor of ramification in the morphism* $f_K : Y_K \to X_K$ *induced by* f *on the generic fibre, and* $d_s = \sum_{i\in I}(c_i - 1)(p - 1)$.

We will refer to this formula simply as the genus formula. The following corollary is immediate from Theorem 2.1.2.

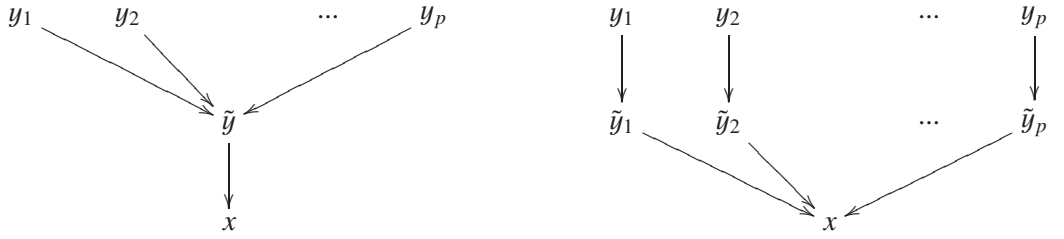
2.2. Computation of vanishing cycles in Galois covers of type (p, p)**.** In this section we prove that the degree p genus formula in Theorem 2.1.2 can be extended to the case of Galois covers of type (p, p). Let $f : Y \to X$ be a Galois cover of type (p, p) where X is a formal germ of an R-curve, Y is local and normal, and Y_k is reduced. We can express f as the compositum of two, generically disjoint, degree p Galois covers $Y_i \to X$ with Y_i normal and local for $i = 1, 2$ as follows:



Let $\{x_i\}_i \subset X_K$ be the (finite) set of branched points in the cover $f_K : Y_K \to X_K$ between generic fibres and $\{y_{ij}\}_{i,j} \subset Y_K$ the set of ramified points in f_K with $r = \mathrm{Card}\{\{y_{ij}\}_{i,j}\}$. Thus, for fixed i the $\{y_{ij}\}_{i,j}$ are the points of Y_K above x_i.

We assume there are r_1 points ($\subset Y_1$) ramified in $Y_{1,K} \to X_K$ and r_2 ramified points ($\subset Y$) in $Y_K \to Y_{1,K}$. Because the Galois group G is of type (p, p) an inertia subgroup of G has at most cardinality p since $\mathrm{char}(K) = 0$ and so the inertia subgroups must be cyclic. Therefore,

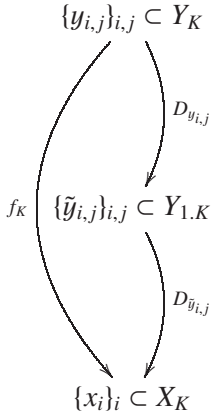
two cases occur: at the first stage we have one point above a branched point then at the second stage there must be p points which sit above it *or* at the first stage we have p points above a branched point then at the second stage there is 1 point above each of these p points. In summary, for a branched point $x = x_i$, we have one of the two situations occurring:



The diagram on the left depicts ramification occurring at the first stage while the diagram on the right depicts ramification occurring at the second stage. Since these two cases are disjoint, this gives us that $r = r_1 p + r_2$ where:

- $r_1 = \text{Card}\{$ ramified points in $Y_1 \to X\}$
- $r_2 = \text{Card}\{$ ramified points in $Y \to Y_1\}$

For the branched points $\{x_i\}_i \subset X_K$ in the cover $f_K : Y_K \to X_K$, we can visualise the general picture, including decomposition groups, as follows:



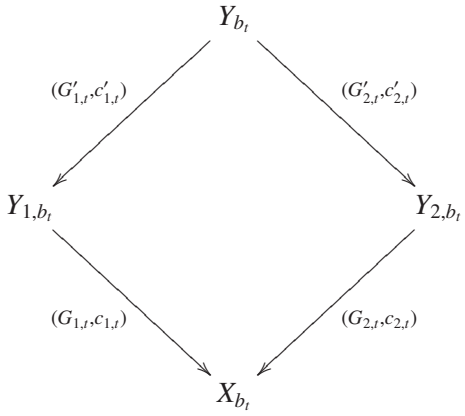
where $D_{\tilde{y}_{i,j}} \leq \mathbb{Z}/p\mathbb{Z}$ and $D_{y_{i,j}} \leq \mathbb{Z}/p\mathbb{Z}$ denote the decomposition groups of the point $\tilde{y}_{i,j}$ at the first stage and the point $y_{i,j}$ above $\tilde{y}_{i,j}$ at the second stage respectively. If only one point sits above x_i in Y_1 then the order of the decomposition group $D_{\tilde{y}_{i,j}}$ (resp. $D_{y_{i,j}}$) will equal p (resp. 1) and, otherwise, the opposite is true. This means we have a natural test for ramification in the first and second step as follows:

$$p = |D_{\tilde{y}_{i,j}}| \Leftrightarrow p \neq |D_{y_{i,j}}| = 1 \Leftrightarrow x_i \text{ ramifies at 1st stage}$$

$$p \neq |D_{\tilde{y}_{i,j}}| = 1 \Leftrightarrow p = |D_{y_{i,j}}| \Leftrightarrow x_i \text{ ramifies at 2nd stage}$$

Now we turn to address the decomposition above the boundaries. Let $\{X_{b_t}\}_t$ denote the boundaries of X. For each t, the Galois cover $f : Y \to X$ induces a Galois cover $f_t : Y_{b_t} \to X_{b_t}$ above the boundary X_{b_t} (note that Y_{b_t} is not necessarily connected). Unlike the degree p case, the cover $Y_{b_t} \to X_{b_t}$ is not a torsor under $G_{1,t} \times G_{2,t}$ unless at least one of the group schemes $G_{1,t}$ and $G_{2,t}$ is étale (see Theorem 1.2 and Theorem 3.5). However, at

each intermediate degree p stage, the corresponding cover is indeed a torsor where $c_{1,t}$ and $c_{2,t}$ (respectively $c'_{1,t}$ and $c'_{2,t}$) are the conductors associated to the torsor under the finite flat R-group schemes $G_{1,t}$ and $G_{2,t}$ respectively (respectively $G'_{1,t}$ and $G'_{2,t}$). Below is the picture when Y_{b_t} is connected; the case we refer to as being *unibranched throughout*.



Our main Theorem in this section compares the genus in a Galois cover of type (p, p).

Theorem 2.2.1. *Let $X := \mathrm{Spf}\left(\hat{\mathcal{O}}_x\right)$ be the formal germ of an R-curve at a closed point x with X_k reduced. Let $f : Y \to X$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}$—that is, of type (p, p)—where Y is normal and local and the special fibre Y_k of Y is reduced.*

Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two generically disjoint degree p Galois covers such that Y is the compositum of Y_1 and Y_2. Let $\{X_{b_t}\}_{t\in I}$ denote the boundaries of X. The Galois cover f_1 induces a torsor $Y_{1,b_t} \to X_{b_t}$ under a finite and flat R-group scheme of type p with conductor $c_{1,t}$ for each t. Similarly, $c'_{1,t}$ denotes the conductor associated to the torsor $Y_{b_t} \to Y_{1,b_t}$. In cases these torsors are trivial we write $c_{1,t} = 1$ and $c'_{1,t} = 1$, respectively. We let r_1 (resp. r_2) denote the number of ramified points in $Y_{1,K} \to X_K$ (resp. $Y_K \to Y_{1,K}$).

If $y \in Y$ is the closed point of Y, then:

$$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$

where g_y (resp. g_x) denotes the genus of y (resp. x), $d_\eta := (r_1 + r_2)p(p-1)$ is the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ induced by f on the generic fibre and

$$d_s = \sum_{\substack{\text{boundary unibranched}\\ \text{throughout}}} \left[(c'_{1,t} - 1)(p-1) + (c_{1,t} - 1)p(p-1)\right] + \sum_{\substack{\text{boundary unibranched}\\ \text{then } p\text{-branched}}} (c_{1,t} - 1)p(p-1) + \sum_{\substack{\text{boundary } p\text{-branched}\\ \text{then unibranched}}} (c'_{1,t} - 1)(p-1).$$

Proof. By Theorem 2.1.2 we have the following genus formula for the degree p Galois cover $Y_1 \to X$ expressing g_{y_1} in terms of g_x where y_1 is the point of Y_1 above x: $2g_{y_1} - 2 = p(2g_x - 2) + r_1(p-1) - \sum_{t\in I}(c_{1,t} - 1)(p-1)$. Each boundary X_{b_t} is either unibranched or p-branched in Y_1 and so we can break up the d_s summation as follows:

$$2g_{y_1} - 2 = p(2g_x - 2) + r_1(p-1) - \sum_{\substack{t\in I\\ X_{b_t}\text{ unibranched}}} (c_{1,t} - 1)(p-1) - \sum_{\substack{t\in I\\ X_{b_t}\ p\text{-branched}}} (c_{1,t} - 1)(p-1).$$

Also by Theorem 2.1.2 we have the following genus formula for the degree p Galois cover $Y \to Y_1$ expressing g_y in terms of g_{y_1}: $2g_y-2 = p\left(2g_{y_1}-2\right)+r_2p(p-1)-\sum_{t\in I}(c'_{1,t}-1)(p-1)$. Again, we rewrite the d_s summation into unibranched or p-branched cases:

$$2g_y-2 = p(2g_{y_1}-2)+r_2p(p-1)-\sum_{\substack{t\in I\\ Y_{1,b_t}\text{ unibranched}}}(c'_{1,t}-1)(p-1)-\sum_{\substack{t\in I\\ Y_{1,b_t}\ p\text{-branched}}}(c'_{1,t}-1)(p-1).$$

Tracing a boundary X_{b_t} through the entire type (p,p) Galois cover $f : Y \to X$, keeping in mind that under the cover $Y_1 \to X$ the boundary can be p-branched or unibranched and, likewise, under the cover $Y \to Y_1$, we have four possible cases which can arise. In particular, the boundary is unibranched throughout, unibranched and then p-branched p-branched and then unibranched or finally p-branched throughout. Now, substituting, our first genus formula expressing g_{y_1} in terms of g_x into the second genus formula expressing g_y in terms of g_{y_1}, will give us a genus formula expressing g_y in terms of g_x, as required:

$$\begin{aligned}
2g_y-2 &= p(2g_{y_1}-2)+r_2p(p-1)-\sum_{\substack{t\in I\\ Y_{1,b_t}\text{ uni.}}}(c'_{1,t}-1)(p-1)-\sum_{\substack{t\in I\\ Y_{1,b_t}\ p\text{-b.}}}(c'_{1,t}-1)(p-1)\\
&= p\left(p(2g_x-2)+r_1(p-1)-\sum_{\substack{t\in I\\ X_{b_t}\text{ uni.}}}(c_{1,t}-1)(p-1)-\sum_{\substack{t\in I\\ X_{b_t}\ p\text{-b.}}}(c_{1,t}-1)(p-1)\right)\\
&\quad +r_2p(p-1)-\sum_{\substack{t\in I\\ Y_{1,b_t}\text{ uni.}}}(c'_{1,t}-1)(p-1)-\sum_{\substack{t\in I\\ Y_{1,b_t}\ p\text{-b.}}}(c'_{1,t}-1)(p-1)\\
&= p^2(2g_x-2)+(r_1+r_2)\,p(p-1)-\sum_{\substack{t\in I\\ \text{uni., uni.}}}\left[(c_{1,t}-1)p(p-1)+(c'_{1,t}-1)(p-1)\right]\\
&\quad -\sum_{\substack{t\in I\\ \text{uni., }p\text{-b.}}}(c_{1,t}-1)p(p-1)-\sum_{\substack{t\in I\\ p\text{-b., uni.}}}(c'_{1,t}-1)(p-1)-\sum_{\substack{t\in I\\ p\text{-b., }p\text{-b.}}}0.
\end{aligned}$$

So, we obtain a genus formula in the form $2g_y-2 = p^2(2g_x-2)+d_\eta-d_s$ where d_η and d_s are as expressed in the statement of the Theorem. □

For illustration purposes, we explain this picture on the boundary in the case of a unibranched throughout cover above an open disc and, so in what follows $X = \mathrm{Spf}R[[T]]$ and $X_{b_t} = \mathrm{Spf}R[[T]]\{\frac{1}{T}\}$.

Case: unibranched throughout. Write, as above, c_1 for the conductor in the first stage and c'_1 for the conductor in the second stage above X_{b_t}. By Theorem 2.1.2 we have that $g_{y_1} = \frac{(r_1-c_1-1)(p-1)}{2}$ and by the genus formula given in Theorem 2.1.2 we can write $2g_y-2 = p(2g_{y_1}-2)+r_2p(p-1)-(c'_1-1)(p-1)$ and so substituting the first equation into the second results in

$$\begin{aligned}
2g_y-2 &= p(2g_{y_1}-2)+r_2p(p-1)-(c'_1-1)(p-1)\\
&= p\left((r_1-c_1-1)(p-1)-2\right)+r_2p(p-1)-(c'_1-1)(p-1)\\
&= \underbrace{-2p^2+2p^2}_{0}-2p+(r_1+r_2)p(p-1)-(p-1)\left(c'_1-1+p(c_1+1)\right)
\end{aligned}$$

$$= p^2(0-2) + (r_1+r_2)p(p-1) - (p-1)\left(c_1' - 1 + p(c_1+1) - 2p\right)$$
$$= p^2(0-2) + \underbrace{(r_1+r_2)p(p-1)}_{d_\eta} - \underbrace{(p-1)\left((c_1'-1) + p(c_1-1)\right)}_{d_s}.$$

The results from the above discussion are summarised in the table below.

Table 2. Values for d_η and d_s in the (p, p) setting above one boundary

1st step	2nd step	d_η	d_s
uni	uni	$(r_1+r_2)p(p-1)$	$(c_1'-1)(p-1) + (c_1-1)p(p-1)$
uni	p	$(r_1+r_2)p(p-1)$	$p(p-1)(c-1)$
p	uni	$(r_1+r_2)p(p-1)$	$(p-1)(c-1)$
p	p	$(r_1+r_2)p(p-1)$	0

From this we can deduce that the general form for the genus formula for (p, p)-type covers in case $X = \mathrm{Spf}(\hat{O}_x)$ has a *unique* boundary (or equivalently X_k is unibranch) is given by

$$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$

where $d_\eta = (r_1+r_2)p(p-1)$ and where

$$d_s = \begin{cases} (c_1'-1)(p-1) + (c_1-1)p(p-1) & \text{boundary unibranched throughout} \\ (c-1)p(p-1) & \text{boundary unibranched, then } p\text{-branched} \\ (c-1)(p-1) & \text{boundary } p\text{-branched, then unibranched} \\ 0 & \text{boundary } p\text{-branched throughout} \end{cases}$$

We can derive from the above formula some interesting results:

Proposition 2.2.2. *Let $X = \mathrm{Spf}\,(R[[T]])$ be the formal germ of an R-curve at a smooth point x and let $f : Y \to X$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Assume Y is normal and local and that the special fibre Y_k of Y is reduced. Let $X_b = \mathrm{Spf}(R[[T]]\{T^{-1}\})$ be the boundary of X and $f_b : Y_b \to X_b$ the induced Galois cover on the boundaries. Let y be the unique closed point of Y_k and $d_\eta := (r_1+r_2)p(p-1)$ be the degree of the divisor of ramification in the morphism $f_K : Y_K \to X_K$ induced by f on the generic fibre and c_1 and c_1' are as in cases 2 and 3 below and where c is the only acting conductor at the relevant unibranched stage. Then:*

1. *If Y_k is unibranched above x then $g_y = \frac{(p(r_1+r_2-c_1-1)-c_1'-1)(p-1)}{2}$.*
2. *The morphism $Y_k \to X_k$ is unibranched and then p-branched above x then $g_y = \frac{(p(r_1+r_2-c-1)-2)(p-1)}{2}$.*
3. *The morphism $Y_k \to X_k$ is p-branched and then unibranched above x then $g_y = \frac{(p(r_1+r_2-2)-c-1)(p-1)}{2}$.*
4. *The morphism Y_k is p^2-branched above x then $g_y = \frac{(p(r_1+r_2-2)-2)(p-1)}{2}$.*

Proof. Follows directly from rearranging the type (p, p) vanishing cycles formula with $g_x = 0$. □

In this situation, we have the following test for whether y is a smooth point or not.

Corollary 2.2.3. *With the same assumptions as in* Proposition 2.2.2*, y is a smooth point if and only if we are in the case 1 of loc. cit. and $p(r_1 + r_2 - 1) = 1 + c'_1 + c_1 p$ holds.*

Proof. ($\Rightarrow$) Suppose y is a smooth point. Then $\delta_y = \dim_k(\tilde{\mathcal{O}}_y/\mathcal{O}_y) = 0$ and $r_y = 1$ since there is one branch and so $g_y = \delta_y - r_y + 1 = 0 - 1 + 1 = 0$. If $g_y = 0$ in the unibranched case then, by the previous proposition, $p(r_1 + r_2 - c_1 - 1) - c'_1 - 1 = 0$ which rearranges to $p(r_1 + r_2 - 1) = 1 + c'_1 + c_1 p$.

($\Leftarrow$) Suppose that we are in case 1 and $p(r_1 + r_2 - 1) = 1 + c'_1 + c_1 p$, then $g_y = 0$. As there is one branch $r_y = 1$ and so we have that $\delta_y = g_y + r_y - 1 = 0 + 1 - 1$ which in turn implies y is a smooth point. □

3. On the existence of a torsor structure

In this section we discuss the question of the existence of a torsor structure for a Galois cover of type (p, p) between formal normal R-schemes. In addition to the notations set at the beginning of this paper, in this section we allow R to be a *complete discrete valuation ring of equal characteristic $p > 0$* with *algebraically closed residue field k*. Let X be a (*formal*) R-scheme of finite type which is *normal*, geometrically connected, and flat over R. We further assume that the special fibre X_k of X is *integral*. Let $f_K : Y_K \to X_K$ be an *étale torsor* under a finite étale K-group scheme $\widetilde{G}$ of rank p^t ($t \geq 1$), with Y_K *geometrically connected*, and $f : Y \to X$ the corresponding morphism of *normalisation*. (Thus, Y is the normalisation of X in Y_K.) We are interested in the following question.

Question 3.1. When is $f : Y \to X$ a *torsor* under a finite and flat R-group scheme G which extends $\widetilde{G}$, i.e., with $G_K = \widetilde{G}$?

The following is well known.

Theorem 3.2 (Proposition 2.4 in [7]; Theorem 5.1 in [11])**.** *If* $\mathrm{char}(K) = 0$ *we assume that X is* locally factorial*. Let η be the generic point of X_k and $\mathcal{O}_\eta$ the local ring of X at η, which is a discrete valuation ring with fraction field $K(X)$: the function field of X. Let $f_K : Y_K \to X_K$ be an étale torsor under a finite étale K-group scheme $\widetilde{G}$ of* **rank p***, with Y_K connected, and let $K(X) \to L$ be the corresponding extension of function fields. Assume that the ramification index above $\mathcal{O}_\eta$ in the field extension $K(X) \to L$ equals 1. Then $f : Y \to X$ is a torsor under a finite and flat R-group scheme G of rank p which extends $\widetilde{G}$ (i.e., with $G_K = \widetilde{G}$).*

Strictly speaking the above references treat the case where $\mathrm{char}(K) = 0$. For the equal characteristic $p > 0$ case see [8], Theorem 2.2.1. Theorem 3.2 also holds when X is the formal spectrum of a complete discrete valuation ring (cf. [6], Proposition 2.3, and the references therein in the unequal characteristic case, as well as Proposition 2.3.1 in [9] in the equal characteristic $p > 0$ case). It is well known that the analog of Theorem 3.2 is *false* in general. There are counterexamples to the statement in Theorem 3.2 where $\widetilde{G}$ is cyclic of rank p^2, see [11], Example 6.2.12, for instance.

Next, we describe the setting in this section. Let $n \geq 1$, and for $i \in \{1, \cdots, n\}$ let

$$f_{i,K} : X_{i,K} \to X_K$$

be an étale torsor under an étale finite *commutative* K-group scheme $\widetilde{G}_i$, with $X_{i,K}$ *geometri-*

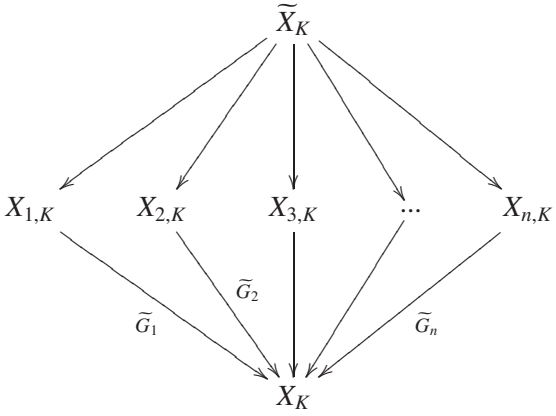
cally connected, such that the $\{f_{i,K}\}_{i=1}^n$ are *generically pairwise disjoint*, i.e. $f_{i,K}$ and $f_{j,K}$ are generically disjoint for $i \neq j$. Assume that $f_{i,K} : X_{i,K} \to X_K$ extends to a torsor

$$f_i : X_i \to X$$

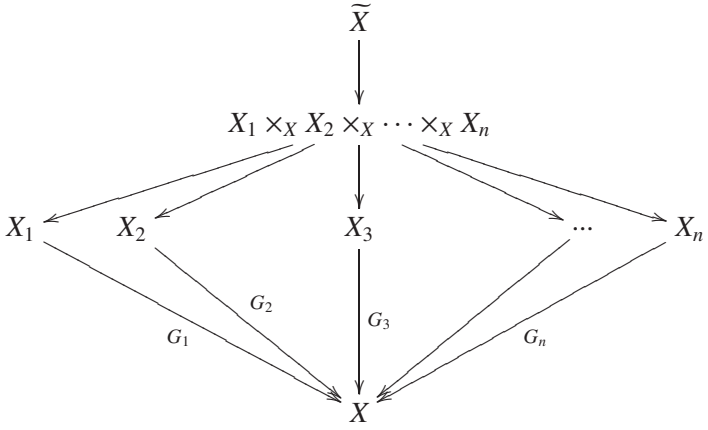
under a finite and flat (necessarily commutative) R-group scheme G_i with $(G_i)_K = \widetilde{G}_i$, and with X_i *normal*, $\forall i \in \{1, \cdots, n\}$. (Thus, X_i is the normalisation of X in $X_{i,K}$.) Let

$$\widetilde{X}_K := X_{1,K} \times_{X_K} X_{2,K} \times_{X_K} \cdots \times_{X_K} X_{n,K},$$

and $\widetilde{X}$ the *normalisation* of X in $\widetilde{X}_K$. Thus, $\widetilde{X}_K$ is the generic fibre of $\widetilde{X}$ and we have the following commutative diagrams



and



where $X_1 \times_X X_2 \times_X \cdots \times_X X_n$ denotes the fibre product of the $\{X_i\}_{i=1}^n$ over X, the morphism $\widetilde{X} \to X_1 \times_X X_2 \times_X \cdots \times_X X_n$ is birational and is induced by the natural *finite* morphisms $\widetilde{X} \to X_i$, $\forall i \in \{1, \cdots, n\}$. Note that $f_K : \widetilde{X}_K \to X_K$ (resp. $\tilde{f} : X_1 \times_X X_2 \times_X \cdots \times_X X_n \to X$) is a torsor under the étale finite commutative K-group scheme $\widetilde{G} := \widetilde{G}_1 \times_{\mathrm{Spec}K} \widetilde{G}_2 \times_{\mathrm{Spec}K} \cdots \times_{\mathrm{Spec}K} \widetilde{G}_n$ (resp. a torsor under the finite and flat commutative R-group scheme $G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$), as follows easily from the various definitions. Note that $\left(G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n\right)_K = \widetilde{G}$.

In this setup Question 1 reads as follows.

Question 3.3. When is $f : \widetilde{X} \to X$ a torsor under a finite and flat (necessarily commutative) R-group scheme G which extends $\widetilde{G}$, i.e., with $G_K = \widetilde{G}$?

Our main result in this paper is the following.

Theorem 3.4. *We use the same notations as above. Assume that $\widetilde{X}_k$ is* **reduced**. *Then the following three statements are equivalent.*

1. *$f : \widetilde{X} \to X$ is a torsor under a finite and flat commutative R-group scheme G, in which case $G = G_1 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$ necessarily.*
2. *$\widetilde{X} = X_1 \times_X X_2 \times_X \cdots \times_X X_n$, in other words $X_1 \times_X X_2 \times_X \cdots \times_X X_n$ is normal.*
3. *$(X_1 \times_X X_2 \times_X \cdots \times_X X_n)_k$ is reduced.*

Note that the above condition in Theorem 3.4 that $\widetilde{X}_k$ is reduced is always satisfied after possibly passing to a finite extension R'/R of R (cf. [2]). It implies that the $(X_i)_k$ are reduced, $\forall i \in \{1, \cdots, n\}$. Moreover, Theorem 3.2 and Theorem 3.4 provide a "complete" answer to Question 1 in the case of Galois covers of type $(p, \cdots, p)$, i.e., the case where $\mathrm{rank}(G_i) = p,\ \forall i \in \{1, \cdots, n\}$.

In the case of (relative) *smooth curves* one can prove the following more precise result when $\mathrm{rank}(G_i) = p,\ \forall i \in \{1, \cdots, n\}$.

Theorem 3.5. *We use the same notations and assumptions as in* Theorem 3.4. *Assume further that X is a (relative)* ***smooth R-curve****, $n \geq 2$, and* $\mathrm{rank}(G_i) = p$ *for $1 \leq i \leq n$. Then the three (equivalent) conditions in* Theorem 3.4 *are equivalent to the following.*

4. ***At least n-1*** *of the finite flat R-group schemes G_i acting on $f_i : X_i \to X$ are* ***étale****, for $i \in \{1, \cdots, n\}$.*

Remarks 3.6. **1)** Theorem 3.4 holds true if X is the formal spectrum of a complete discrete valuation ring (cf. the details of the proof of Theorem 3.4 below which applies as it is in this case).
2) In 3.8 we provide examples showing that Theorem 3.5 *doesn't hold* in relative dimension > 1.

Proof of Theorem 3.4

Next, we prove Theorem 3.4. We start by the following.

Proposition 3.7. *Let G be a finite and flat commutative R-group scheme whose generic fibre is a product of group schemes of the form $G_K = \widetilde{G}_1 \times_{\mathrm{Spec}K} \widetilde{G}_2 \times_{\mathrm{Spec}K} \cdots \times_{\mathrm{Spec}K} \widetilde{G}_n$, where the $\{\widetilde{G}_i\}_{i=1}^n$ are finite and flat commutative K-group schemes. Then G is a product of finite and flat commutative R-group schemes $\{G_i\}_{i=1}^n$, i.e., $G = G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$, with $(G_i)_K = \widetilde{G}_i$.*

Proof. First, we treat the case $n = 2$. Thus, we have $G_K = \widetilde{G}_1 \times_{\mathrm{Spec}K} \widetilde{G}_2$ and need to show $G = G_1 \times_{\mathrm{Spec}R} G_2$ where $(G_i)_K = \widetilde{G}_i$, for $i = 1, 2$. Let G_i be the *schematic closure* of $\widetilde{G}_i$ in G, for $i = 1, 2$ (cf. [5], 2.1). Therefore, G_1 and G_2 are closed subgroup schemes of G which are finite and flat over $\mathrm{Spec}R$ (cf. loc. cit.). We have a short exact sequence

$$1 \to G_1 \to G \to G/G_1 \to 1,$$

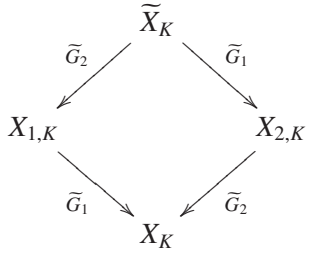
and likewise

$$1 \to G_2 \to G \to G/G_2 \to 1,$$

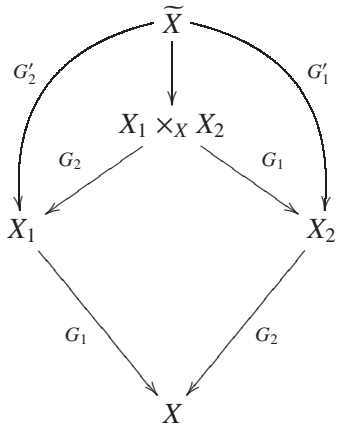
of finite and flat commutative R-group schemes (cf. loc. cit.). It remains for the proof to show that the composite homomorphism $G_2 \to G \to G/G_1$ is an isomorphism. The morphism $G \to G/G_1$ is finite. The morphism $G_2 \to G$ is a closed immersion, hence finite. The composite $G_2 \to G/G_1$ of the above morphisms is then finite. We will show it is an isomorphism. The morphism $G_2 \to G/G_1$ is a closed immersion since its kernel is trivial. Indeed, on the generic fibre the kernel is trivial: $(G_1 \cap G_2)_K = \widetilde{G_1} \cap \widetilde{G_2} = \{1\}$. The map $G_2 \to G/G_1$ is then an isomorphism as both group schemes have the same rank. Similarly, the morphism $G_1 \to G/G_2$ is an isomorphism. Therefore, $G = G_1 \times_{\mathrm{Spec}R} G_2$ as required. Now an easy devissage argument along the above lines of thought, using induction on n, reduces immediately to the above case $n = 2$. □

Proof of Theorem 3.4. $(1 \Rightarrow 2)$ Assume that $f : \widetilde{X} \to X$ is a torsor under a finite and flat R-group scheme G. In particular, $G_K = \widetilde{G}$ and G is necessarily commutative. We will show that $\widetilde{X} = X_1 \times_X X_2 \times_X \cdots \times_X X_n$, i.e., show that $X_1 \times_X X_2 \times_X \cdots \times_X X_n$ is normal (this will imply that $G = G_1 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$ necessarily, as $G_1 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$ is the group scheme of the torsor $\tilde{f} : X_1 \times_X X_2 \times_X \cdots \times_X X_n \to X$). One reduces easily by a devissage argument to the case $n = 2$ which we will treat below.

Assume $n = 2$. We have the following commutative diagrams of torsors



and



where $\widetilde{X} \to X_i$ is a torsor under a finite and flat R-group scheme G_j', for $j \neq i$. Moreover,

$G'_1 = \left(\widetilde{G}_1\right)^{\text{schematic closure}}$, and $G'_2 = \left(\widetilde{G}_2\right)^{\text{schematic closure}}$ (where the schematic closure is taken inside G) holds necessarily, so that $G = G'_1 \times_{\text{Spec}R} G'_2$ (cf. Proposition 3.7). Note that $\widetilde{X}/G'_1 = X_2$ must hold as the quotient $\widetilde{X}/G'_1$ is normal: since $\left(\widetilde{X}/G'_1\right)_k$ is reduced (as $\widetilde{X}_k$ is reduced and $\widetilde{X}$ dominates $\widetilde{X}/G'_1$), and $\left(\widetilde{X}/G'_1\right)_K = X_{2,K}$ is normal (cf. [4], 4.1.18). Similarly $\widetilde{X}/G'_2 = X_1$ holds. We want to show that $\widetilde{X} = X_1 \times_X X_2$, and we claim that this reduces to showing that the natural morphism $G \to G_1 \times_{\text{Spec}R} G_2$ (cf. the map ϕ below) is an isomorphism. Indeed, if one has two torsors, in this case $\widetilde{X} \to X$ and $X_1 \times_X X_2 \to X$ above the same X, under isomorphic group schemes, which are isomorphic on the generic fibres, and if we have a morphism $\widetilde{X} \to X_1 \times_X X_2$ which is compatible with the torsor structure and the given identification of group schemes (cf. above diagrams and the definition of ϕ below), then this morphism must be an isomorphism. (This is a consequence of Lemma 4.1.2 in [11]. In [11] $\text{char}(K) = 0$ is assumed, the same proof however applies if $\text{char}(K) = p$.) We have two short exact sequences of finite and flat commutative R-group schemes (cf. above diagrams and discussion for the equalities $G_1 = G/G'_2$ and $G_2 = G/G'_1$)

$$1 \to G'_2 \to G \to G_1 = G/G'_2 \to 1,$$

and

$$1 \to G'_1 \to G \to G_2 = G/G'_1 \to 1.$$

The morphisms $G \to G_1$, and $G \to G_2$, are finite. Consider the following exact sequence

$$1 \to \text{Ker}(\phi) \to G \to G_1 \times_{\text{Spec}R} G_2,$$

where $\phi : G \to G_1 \times_{\text{Spec}R} G_2$ is the morphism induced by the above morphisms. We want to show that the map $\phi : G \to G_1 \times_{\text{Spec}R} G_2$ is an isomorphism. We have $\text{Ker}(\phi) = G'_1 \cap G'_2$ by construction. However, $G'_1 \cap G'_2 = \{1\}$ since $G = G'_1 \times_{\text{Spec}R} G'_2$ by Proposition 3.7, and therefore $\text{Ker}(\phi) = \{1\}$ which means $\phi : G \to G_1 \times_{\text{Spec}R} G_2$ is a closed immersion. Finally, G and $G_1 \times_{\text{Spec}R} G_2$ have the same rank as group schemes which implies ϕ is an isomorphism, as required.

(2 $\Rightarrow$ 3) Clear.

(3 $\Rightarrow$ 1) By assumption $(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced. Moreover, we have $(X_1 \times_X X_2 \times_X ... \times_X X_n)_K = \widetilde{X}_K$ is normal. Hence $X_1 \times_X X_2 \times_X ... \times_X X_n$ is normal (cf. [4], 4.1.18), and $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$. We know that $\tilde{f} : X_1 \times_X X_2 \times_X ... \times_X X_n \to X$ is a torsor under the group scheme $G_1 \times_{\text{Spec}R} G_2 \times_{\text{Spec}R} .... \times_{\text{Spec}R} G_n$, so $f : \widetilde{X} \to X$ is a torsor under the same group scheme. □

Proof of Theorem 3.5

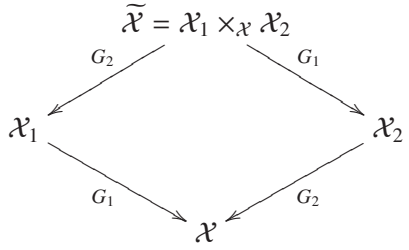
Next, we prove Theorem 3.5.

Proof. (1 $\Rightarrow$ 4) Suppose that $f : \widetilde{X} \to X$ is a torsor under a finite and flat R-group scheme G; in which case $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$ and $G = G_1 \times_{\text{Spec}R} \cdots \times_{\text{Spec}R} G_n$ (cf. Theorem 3.4). We will show that *at least* $n-1$ of the finite flat R-group schemes G_i (acting on $f_i : X_i \to X$) are étale, for $i \in \{1, \cdots, n\}$. We argue by induction on the rank of G.

Base case: The base case pertains to $\text{rank}(G) = p^2$ and $n = 2$. Thus, $\text{rank}(G_1) = \text{rank}(G_2) = p$. We assume $\widetilde{X} = X_1 \times_X X_2$ and prove that at least one of the two group

schemes G_1 or G_2 is étale. We assume that X is a scheme, and not a formal scheme, in which case the argument of proof is the same.

Let $x \in X$ be a *closed* point of X and $\mathcal{X}$ the *boundary of the formal germ* of X at x, so $\mathcal{X}$ is isomorphic to $\mathrm{Spec}\left(R[[T]]\{T^{-1}\}\right)$ (cf. Background). We have a natural morphism $\mathcal{X} \to X$ of schemes. Write $\mathcal{X}_1 := \mathcal{X} \times_X X_1$, $\mathcal{X}_2 := \mathcal{X} \times_X X_2$, and $\widetilde{\mathcal{X}} := \mathcal{X} \times_X \widetilde{X}$. Thus, by base change, $\widetilde{\mathcal{X}} \to \mathcal{X}$ (resp. $\mathcal{X}_1 \to \mathcal{X}$, and $\mathcal{X}_2 \to \mathcal{X}$) is a torsor under the group scheme G (resp. under G_1, and G_2) and we have the following commutative diagram:



Note that $\widetilde{\mathcal{X}}$ is normal as $(\widetilde{\mathcal{X}})_k$ is reduced (recall $(\widetilde{X})_k$ is reduced) and $(\widetilde{\mathcal{X}})_K$ is normal (cf. [4], 4.1.18), hence $\widetilde{\mathcal{X}} = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ holds (cf. Theorem 3.4 and Remarks 3.6, 1).

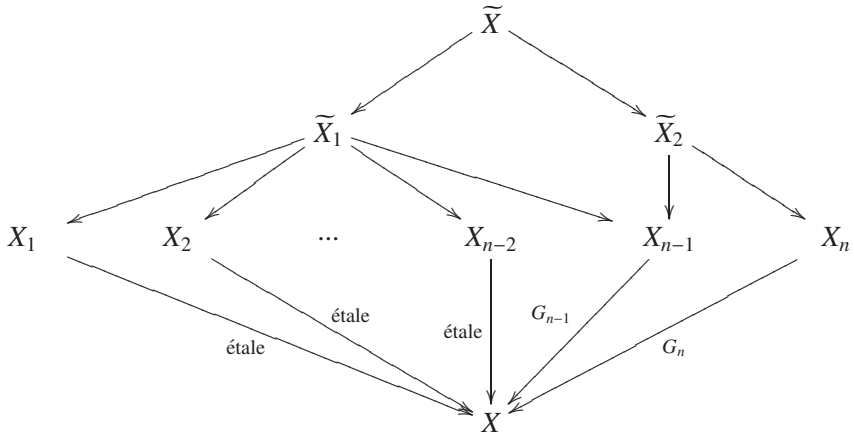
Assume now that G_1 and G_2 are both *non-étale* R-group schemes. Then we prove that $\widetilde{\mathcal{X}} \to \mathcal{X}$ can not have the structure of a torsor under a finite and flat R-group scheme which would then be a contradiction. More precisely, we will prove that $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ can not be normal in this case, hence the above conclusion (cf. Theorem 3.4).

We will assume for simplicity that $\mathrm{char}(K) = 0$ and K contains a primitive p-th root of 1. A similar argument as the one used below holds in equal characteristic $p > 0$. First, $\widetilde{\mathcal{X}}$ is connected as $\widetilde{\mathcal{X}}_k$ is *unibranch* (the finite morphism $\widetilde{X}_k \to X_k$ is radicial). As the group schemes G_1 and G_2 are non étale, their special fibres $(G_1)_k$ and $(G_2)_k$ are radicial isomorphic to either μ_p or α_p. We treat the case $(G_1)_k$ is isomorphic to $\mu_p := \mu_{p,k}$ and $(G_2)_k$ is isomorphic to $\alpha_p := \alpha_{p,k}$; the remaining cases are treated similarly. Recall $\mathcal{X}$ is isomorphic to $\mathrm{Spec}\left(R[[T]]\{T^{-1}\}\right)$. For a suitable choice of the parameter T the torsor $\mathcal{X}_2 \to \mathcal{X}$ is given by an equation $Z_2^p = 1 + \pi^{np} T^m$ where n is a positive integer (satisfying a certain condition) and $m \in \mathbb{Z}$ (cf. Background. Also see Proposition 2.3.1 in [9] for the equal characteristic case), and the torsor $\mathcal{X}_1 \to \mathcal{X}$ is given by an equation $Z_1^p = f(T)$ where $f(T) \in R[[T]]\{T^{-1}\}$ is a unit whose reduction $\overline{f(T)}$ modulo π is not a p-power (cf. loc. cit.). We claim that $\widetilde{\mathcal{X}} = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ can not hold. Indeed, by base change $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2 \to \mathcal{X}_2$ is a G_1-torsor which is generically given by an equation $Z^p = f(T)$, where $f(T)$ is viewed as a function on $\mathcal{X}_2$. But in $\mathcal{X}_2$ the function T becomes a p-power modulo π as one easily deduces from the equation $Z_2^p = 1 + \pi^{np} T^m$ defining the torsor $\mathcal{X}_2 \to \mathcal{X}$. Indeed, after a change of variables we can write the above equation as $(1 + \pi^n Z_2')^p = 1 + \pi^{np} T^m$ which reduces, after an easy computation, to an equation $z_2'^p = t^m$ hence $((z_2')^{\frac{1}{m}})^p = t$. In particular, the reduction $\overline{f(T)}$ modulo π of $f(T)$, viewed as a function on $(\mathcal{X}_2)_k$, is a p-power. This means that $(\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2)_k$ is not reduced and $\widetilde{\mathcal{X}} \to \mathcal{X}_2$ can not be a $G_1 \simeq \mu_{p,R}$-torsor (cf. the proof of Proposition 2.3 in [6]), and a fortiori $\widetilde{\mathcal{X}} \neq \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$.

Inductive hypothesis: Given G, we assume that the $(1 \Rightarrow 4)$ part in Theorem 3.5 holds true for smaller values of $n \geq 2$. Then $\widetilde{X}_1 := X_1 \times_X X_2 \times_X \ldots \times_X X_{n-1}$ is normal (since

its special fibre is reduced (as it is dominated by $\widetilde{X}$ whose special fibre is reduced) and its generic fibre is normal (cf. [4], 4.1.18)), hence at least $n-2$ of the corresponding G_i's, for $i \in \{1, \cdots, n-1\}$, are étale by the induction hypothesis. We will assume, without loss of generality, that G_i is étale for $1 \leq i \leq n-2$.

Inductive step: We have the following picture for our inductive step (the case for n):



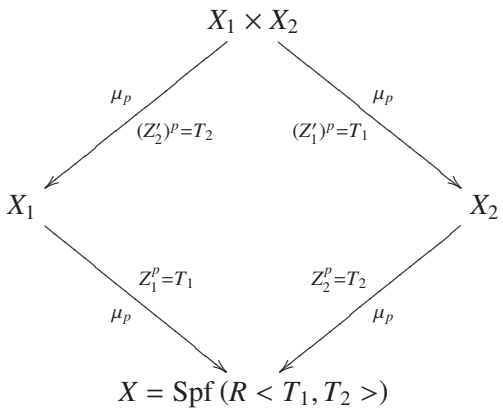
We argue by contradiction. Suppose that neither G_{n-1} nor G_n is étale. This would mean that $\widetilde{X}_2 \to X$, where $\widetilde{X}_2$ is the normalisation of X in $(X_{n-1})_K \times_{X_K} (X_n)_K$, does not have the structure of a torsor (as this would contradict the induction hypothesis). This implies that $\widetilde{X} \to X$ does not have the structure of a torsor since it factorises $\widetilde{X} \to \widetilde{X}_2 \to X$, for otherwise $\widetilde{X}_2 \to X$ being a quotient of $\widetilde{X} \to X$ would be a torsor. Of course, $\widetilde{X} \to X$ is a torsor to start with by assumption and so this is a contradiction. Therefore, at least one of G_{n-1} and G_n is étale, as required.

$(1 \Leftarrow 4)$ Suppose that at least $n-1$ of the G_i are étale, say: $G_1, G_2, \cdots, G_{n-1}$ are étale. Write $\widetilde{X}_1 := X_1 \times_X X_2 \times_X ... \times_X X_{n-1}$. Then $\widetilde{X}_1 \to X$ is a torsor under the finite *étale* R-group scheme $G'_1 := G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_{n-1}$. Moreover, $X_1 \times_X X_2 \times_X ... \times_X X_n = \widetilde{X}_1 \times_X X_n$, and $X_1 \times_X X_2 \times_X ... \times_X X_n \to X_n$ is an étale torsor under the group scheme G'_1 (by base change). In particular, $(X_1 \times_X X_2 \times_X ... \times_X X_n)_k$ is reduced as $(X_n)_k$ is reduced. Indeed, $\widetilde{X}$ dominates X_n and $\widetilde{X}_k$ is reduced. Hence $\widetilde{X} = X_1 \times_X X_2 \times_X ... \times_X X_n$ (cf. Theorem 3.4) and $\widetilde{X} \to X$ is a torsor under the group scheme $G := G_1 \times_{\mathrm{Spec}R} G_2 \times_{\mathrm{Spec}R} \cdots \times_{\mathrm{Spec}R} G_n$. □

3.8. Counterexample to Theorem 3.5 in higher dimensions. Theorem 3.5 is not valid (under similar assumptions) for (formal) smooth R-schemes of relative dimension ≥ 2. Here is a counterexample. Assume $\mathrm{char}(K) = 0$ and K contains a primitive p-th root of 1. Let $X = \mathrm{Spf}(A)$ where $A := R < T_1, T_2 >$ is the free R-Tate algebra in the two variables T_1 and T_2. Let $G_1 = G_2 = \mu_p := \mu_{p,R}$, neither being an étale R-group scheme. For $i = 1, 2$, consider the G_i-torsor $X_i \to X$ which is generically defined by the equation

$$Z_i^p = T_i.$$

We have the following commutative diagram:



The torsor $X_1 \times_X X_2 \to X_2$ is a $G_1 = \mu_p$-torsor defined generically by the equation

$$(Z_1')^p = T_1$$

where T_1 is viewed as a function on X_2. This function is not a p-power modulo π as follows easily from the fact that the torsor $X_2 \to X$ is defined generically by the equation $Z_2^p = T_2$. In particular, $X_1 \times_X X_2 \to X_2$ is a non trivial μ_p-torsor, and $(X_1 \times_X X_2)_k \to (X_2)_k$ is a non trivial $\mu_{p,k}$-torsor. Hence $(X_1 \times_X X_2)_k$ is necessarily reduced (as $(X_2)_k$ is reduced since $(X_2)_k \to (X_1)_k$ is a non trivial $\mu_{p,k}$-torsor). Thus, $X_1 \times_X X_2$ is normal (cf. Theorem 3.4) and $X_1 \times_X X_2 = \widetilde{X}$, where $\widetilde{X}$ is the normalisation of X in $(X_1 \times_X X_2)_K$, which contradicts the statement of Theorem 3.5 in this case.

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Mohamed Saïdi
College of Engineering, Mathematics, and Physical Sciences
University of Exeter
Harrison Building
North Park Road
EXETER EX4 4QF
United Kingdom
e-mail: M.Saidi@exeter.ac.uk

Nicholas Williams
London
United Kingdom
e-mail: Nicholas.Williams09@alumni.imperial.ac.uk