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BIHARMONIC SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD

Norihito KOISO and Hajime URAKAWA

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Abstract

In this paper, we solve affirmatively B.-Y. Chen's conjecture for hypersurfaces in the Euclidean space, under a generic condition. More precisely, every biharmonic hypersurface of the Euclidean space must be minimal if their principal curvatures are simple, and the associated frame field is irreducible.

1. Introduction

In this paper, we solve B.-Y.Chen's conjecture for hypersurfaces of the Euclidean space in the case that every principal curvature is simple and some generic condition is satisfied.

A map $\varphi : (M, g) \to (N, h)$ is called *harmonic* if it is a critical point of the energy functional $E(\varphi) = \dfrac{1}{2}\displaystyle\int_M |d\varphi|^2 \, v_g$. Its Euler-Lagrange equation is that the tension field $\tau = \tau(\varphi)$ vanishes. Recall that an isometric immersion $\varphi : (M, g) \to (N, h)$ is minimal if and only if it is harmonic.

In 1983, Eells and Lemaire [8] introduced the notion of k-energy. A map $\varphi : (M, g) \to (N, h)$ is called *biharmonic* if it is a critical point of the *bienergy* $E_2(\varphi) = \dfrac{1}{2}\displaystyle\int_M |\tau(\varphi)|^2 \, v_g$.

A minimal isometric immersion is always biharmonic, and many researchers have asked whether the converse is true, namely under which conditions, a biharmonic isometric immersion is minimal. In this connection, Chen [4] proposed the conjecture

B.-Y. Chen's conjecture: *Every biharmonic submanifold of the Euclidean space must be minimal.*

Caddeo, Montald and Oniciuc [2] raised the generalized Chen's conjecture:

The generalized B.-Y. Chen's conjecture: *Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be minimal.*

In 2010, Ou [15] gave a counter example of the generalized conjecture.

On the other hand, Hasanis and Vlachosin [9], and Defever [6] showed the Chen's conjecture is true for hypersurfaces in the 4-dimensional Euclidean space. Recently, Akutagawa and Maeta [1] showed that any complete, proper (i.e., the preimage of each compact subset is compact) biharmonic submanifold of the Euclidean space is minimal.

The main theorem in this paper is as follows:

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Main theorem. *Let M be an n-dimensional biharmonic hypersurface of the $(n+1)$-dimensional Euclidean space E^{n+1}. Assume that all the principal curvatures are simple and that $g(\nabla_{v_i} v_j, v_k) \neq 0$ for all distinct triplets $\{v_i, v_j, v_k\}$ of unit principal curvature vectors in the kernel of $d\tau$. Then, M is minimal.*

For more precise statement of this theorem, see Theorem 8.2 and Definition 6.1 of irreducibility.

We emphasize that we need not completeness assumption to M in Theorem 8.2.

The outline of this paper is as follows. In Section 2, we prepare several materials on biharmonic submanifolds M in a Riemannian manifold $\overline{M}$, and show that every n-dimensional Riemannian manifold M can be embedded as a biharmonic, but not minimal hypersurface in some $(n+1)$-dimensional Riemannian manifold $\overline{M}$ (Theorem 2.3).

In Section 3, we treat n-dimensional non-minimal biharmonic submanifold M^n of the $(n+m)$-dimensional space form $\overline{M}^{n+m}(K)$ with constant sectional curvature K. We show that if $K \leq 0$, then $|\tau|^2$ does not attain a local maximum.

In Section 4, we treat non-minimal biharmonic hypersurfaces M^n of the space form $\overline{M}^{n+1}(K)$ with $K \leq 0$. In this case, $-\tau/2$ becomes a simple principal curvature of M. Let $\{\lambda_i\}_{i \leq n}$ be the principal curvatures of M, where $\lambda_n = -\tau/2$, and $\{v_i\}$ be the corresponding orthonormal principal curvature vectors. Since τ is not constant, $F = \{x \in M \mid \tau(x) = (\text{constant c})\}$ is a hypersurface of M around every generic point of M, and every v_i is tangent to F except v_n. We call F a *characteristic hypersurface* of M. Every v_i becomes a principal curvature vector of F in M except v_n. Let μ_i be the principal curvature of F in M for the direction v_i. We show that $\{\lambda_i, \mu_i\}_{i<n}$ satisfies an over-determined ODE along the v_n-curves (Proposition 4.7).

In Section 5, we analyze the over-determined ODE, and show that the set of all initial values of the ODE is an algebraic manifold in $\mathbf{R}^{2(n-1)}$.

In Section 6, we introduce the notion of the *irreducibility* of the frame field $\{v_i\}_{i<n}$ of F (Definition 6.1). We show that $\{\lambda_i\}$ and $\{\mu_i\}$ are *linearly related* under irreducibility assumption.

In Section 7, we show that $\{\lambda_i\}$ and $\{\mu_i\}$ are constant along F.

Finally, in Section 8, we give a proof to our main theorem, Theorem 8.2.

2. Biharmonic submanifolds in a Riemannian manifold

A smooth map between Riemannian manifolds $\varphi : (M, g) \to (\overline{M}, \overline{g})$ is said to be *biharmonic* if it is a critical point of the *bienergy* $E_2(\varphi) = \dfrac{1}{2}\displaystyle\int_M |\tau|^2 v_g$. The Euler-Lagrange equation is given by

$$\overline{\Delta}\tau - g^{ij}\overline{R}(\tau, \partial_i)\partial_j = 0, \tag{2.1}$$

where ∂_i is the partial differentiation with respect to the local coordinates of M, $R, \overline{R}$ are the curvature tensors of M and $\overline{M}$, respectively, and

$$\alpha(\partial_i, \partial_j) := \overline{\nabla}_{\partial_i}(\varphi_*\partial_j) - \varphi_*(\nabla_{\partial_i}\partial_j), \quad \tau := \operatorname{tr}\alpha, \quad \Delta := -g^{ij}\nabla_i\nabla_j, \quad \overline{\Delta} := -g^{ij}\overline{\nabla}_i\overline{\nabla}_j. \tag{2.2}$$

In the following, we assume that M is a submanifold of $(\overline{M}, \overline{g})$ and the inclusion map ι is a biharmonic map with respect to the induced metric $g = \iota^*\overline{g}$. Such a submanifold M is called

a *biharmonic submanifold* of $\overline{M}$. Then, $\iota^*T\overline{M}$ is decomposed into $TM \oplus TM^\perp$ and α, τ have their values in $TM^\perp$. We also denote the normal connection of $TM^\perp$ by ∇.

We decompose the equation (2.1) into the tangential direction and the normal direction. For $\overline{\Delta}\tau$, we have

$$
\begin{aligned}
&g((\overline{\Delta}\tau)^\top, \partial_i) = \overline{g}((\nabla^j\alpha)(\partial_j, \partial_i), \tau) + 2\overline{g}(\alpha(\partial_j, \partial_i), \nabla^j\tau), \\
&(\overline{\Delta}\tau)^\perp = \Delta\tau + g^{ij}g^{k\ell}\overline{g}(\alpha(\partial_j, \partial_k), \tau)\alpha(\partial_i, \partial_\ell).
\end{aligned} \tag{2.3}
$$

For $g^{jk}\overline{R}(\tau, \partial_j), \partial_k)^\top$, we have

$$
\begin{aligned}
g((g^{jk}\overline{R}(\tau, \partial_j), \partial_k))^\top, \partial_i) &= g^{jk}g(\overline{R}(\partial_i, \partial_k), \partial_j), \tau) \\
&= g^{jk}\overline{g}((\nabla_{\partial_i}\alpha)(\partial_k, \partial_j) - (\nabla_{\partial_k}\alpha)(\partial_i, \partial_j), \tau) \quad \text{(by Codazzi eq.)} \\
&= \overline{g}(\nabla_{\partial_i}\tau, \tau) - \overline{g}((\nabla^j\alpha)(\partial_j, \partial_i), \tau) \\
&= \frac{1}{2}\nabla_i|\tau|^2 - \overline{g}((\nabla^j\alpha)(\partial_j, \partial_i), \tau).
\end{aligned} \tag{2.4}
$$

Now we introduce the following notions:

$$
\begin{aligned}
&(\delta\alpha)(\partial_i) := -(\nabla^j\alpha)(\partial_j, \partial_i), \quad (\alpha^2)(V) := g^{ij}g^{k\ell}\overline{g}(\alpha(\partial_j, \partial_k), V)\alpha(\partial_i, \partial_\ell), \\
&(\overline{R}^\perp_V)(\partial_i, \partial_j) := (\overline{R}(V, \partial_i)\partial_j)^\perp.
\end{aligned} \tag{2.5}
$$

By (2.1) $\sim$ (2.5), the equations of biharmonic submanifolds can be written as follows:

Lemma 2.1. *A submanifold M of $(\overline{M}, \overline{g})$ is a biharmonic submanifold if and only if the following two equations hold:*

$$\Delta\tau + \alpha^2(\tau) - \operatorname{tr}\overline{R}^\perp_\tau = 0, \tag{2.6a}$$

$$-2\overline{g}((\delta\alpha)(\partial_i), \tau) + 2\overline{g}(\alpha(\partial_j, \partial_i), \nabla^j\tau) - \frac{1}{2}\nabla_i|\tau|^2 = 0. \tag{2.6b}$$

By using these equations, we first give examples of biharmonic hypersurfaces which are not minimal submanifolds. In the following, we regard α, τ to be real values in terms of the unit normal vector field N. We need first the following lemma.

Lemma 2.2. *In a Riemannian manifold ($\overline{M} = M \times \mathbf{R}$, $\overline{g} = g(t) + dt^2$), the second fundamental form α_{ij} and the symmetric bilinear form $\beta_{ij} := \overline{g}(\overline{R}^\perp_N(\partial_i, \partial_j), N)$ on M are given by*

$$\alpha_{ij} = -\frac{1}{2}g'_{ij}, \quad \beta_{ij} = -\frac{1}{2}g''_{ij} + \frac{1}{4}g^{k\ell}g'_{ik}g'_{j\ell}, \tag{2.7}$$

where, $*'$ means the differentiation with respect to t.

As a consequence, for every pair of symmetric bilinear forms α and β on M, there exists a Riemannian metric $\overline{g} = g(t) + dt^2$ such that α coincides with the second fundamental form at $t = 0$ and β coincides with $\overline{g}(\overline{R}^\perp_N(\partial_i, \partial_j), N)$ at $t = 0$, respectively.

Proof. We add t to the coordinates $\{x^i\}$ of M to become the coordinates of $\overline{M}$. We denote the differentiation with respect to x^i by ∂_i, and the differentiation with respect to t by ∂_t. For the second fundamental form α, it is well known that

$$\alpha_{ij} = \overline{g}(\overline{\nabla}_{\partial_i}\partial_j, \partial_t) = -\frac{1}{2}g'(\partial_i, \partial_j) = -\frac{1}{2}g'_{ij}. \tag{2.8}$$

Furthermore, since $\overline{g}(\overline{\nabla}_{\partial_i}\partial_t, \partial_t) = 0$ and $\overline{g}(\overline{\nabla}_{\partial_i}\partial_t, \partial_j) = -\overline{g}(\partial_t, \overline{\nabla}_{\partial_i}\partial_j) = \frac{1}{2}g'_{ij}$, we have $\overline{\nabla}_{\partial_i}\partial_t = \frac{1}{2}g^{k\ell}g'_{ik}\partial_\ell$. Thus, since $\overline{g}(\overline{\nabla}_{\partial_t}\partial_t, \partial_i) = -\overline{g}(\partial_t, \overline{\nabla}_{\partial_t}\partial_i) = 0$ we have $\overline{\nabla}_{\partial_t}\partial_t = 0$. Therefore, we obtain

(2.9)

$$\begin{aligned}
\overline{g}(\overline{R}(\partial_t, \partial_i)\partial_j, \partial_t) &= \overline{g}(\overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_i}\partial_j - \overline{\nabla}_{\partial_i}\overline{\nabla}_{\partial_t}\partial_j, \partial_t) \\
&= \partial_t\{\overline{g}(\overline{\nabla}_{\partial_i}\partial_j, \partial_t)\} - \overline{g}(\overline{\nabla}_{\partial_i}\partial_j, \overline{\nabla}_{\partial_t}\partial_t) - \partial_i\{\overline{g}(\overline{\nabla}_{\partial_t}\partial_j, \partial_t)\} + \overline{g}(\overline{\nabla}_{\partial_t}\partial_j, \overline{\nabla}_{\partial_i}\partial_t) \\
&= -\frac{1}{2}g''_{ij} - 0 - 0 + \frac{1}{4}\overline{g}(g^{pq}g'_{jp}\partial_q, g^{k\ell}g'_{ki}\partial_\ell) \\
&= -\frac{1}{2}g''_{ij} + \frac{1}{4}g^{k\ell}g'_{ki}g'_{\ell j}.
\end{aligned}$$

Since for any α and β, we can solve (2.7) as a system of equations for g' and g'', the latter half statement holds. □

By using Lemma 2.2, we have the following.

Theorem 2.3. *Every n dimensional Riemannian manifold M can be embedded into an $(n+1)$-dimensional Riemannian manifold $\overline{M}$ as a biharmonic hypersurface, but not minimal.*

Proof. For every $c \neq 0$, we apply Lemma 2.2 to $\alpha = cg$, $\beta = (c^2/n)g$, and we construct a Riemannian metric $\overline{g}$. Then, it holds that $\tau = nc$, which implies (2.6b), and that $c^2 \cdot nc - nc^2/n \cdot nc = 0$, which implies (2.6a), respectively. □

Note that our Riemannian metric $\overline{g}$ satisfies that the sectional curvature $(g_{ii})^{-1}\overline{g}(\overline{R}(N, \partial_i)\partial_i, N) = (g_{ii})^{-1}\beta_{ii} = c^2/n > 0$.

3. Biharmonic submanifolds M^n in a space form $\overline{M}^{n+m}$

For submanifolds M^n in a space form $\overline{M}^{n+m}$ of sectional curvature K, Codazzi equation holds: $\nabla_i\alpha_{jk} = \nabla_j\alpha_{ik}$ ([13], Corollary 4.4). Therefore, we have

$$\begin{aligned}
&(\overline{R}^\perp_\tau)(\partial_i, \partial_j) = Kg(\partial_i, \partial_j)\tau, \quad \operatorname{tr}\overline{R}^\perp_\tau = nK\tau, \\
&(\delta\alpha)(\partial_i) = -(\nabla^j\alpha)(\partial_j, \partial_i) = -(\nabla_i\alpha)(\partial_j, \partial^j) = -\nabla_i\tau, \\
&-2\overline{g}((\delta\alpha)(\partial_i), \tau) = 2\overline{g}(\nabla_i\tau, \tau) = \nabla_i|\tau|^2.
\end{aligned} \tag{3.1}$$

Thus, (2.6) is written as

$$\Delta\tau + \alpha^2(\tau) - nK\tau = 0, \tag{3.2a}$$

$$\frac{1}{2}\nabla_i|\tau|^2 + 2\overline{g}(\alpha(\partial_j, \partial_i), \nabla^j\tau) = 0. \tag{3.2b}$$

By taking the inner product of (3.2a) and τ, we have

$$\begin{aligned}
&\overline{g}(\Delta\tau, \tau) = -\nabla^i(\overline{g}(\nabla_i\tau, \tau)) + \overline{g}(\nabla_i\tau, \nabla^i\tau) = \frac{1}{2}\Delta|\tau|^2 + |\nabla\tau|^2, \\
&\overline{g}(\alpha^2(\tau), \tau) = |\alpha_\tau|^2,
\end{aligned} \tag{3.3}$$

where we put $(\alpha_\tau)_{ij} := \overline{g}(\alpha(\partial_i, \partial_j), \tau)$. Therefore, $|\tau|^2$ satisfies

$$\frac{1}{2}\Delta|\tau|^2 + |\nabla\tau|^2 + |\alpha_\tau|^2 - nK|\tau|^2 = 0. \tag{3.4}$$

From this elliptic equation for $|\tau|^2$, we have the following

Proposition 3.1 ([14] Proposition 2.4)**.** *Assume that $\overline{M}^{n+m}$ is a space form of constant sectional curvature K less than or equal to 0, and that M is a biharmonic submanifold of $\overline{M}$. If $|\tau|^2$ admits a local maximum at some point, then M^n is minimal. In particular, if $|\tau|^2$ is constant, then M is minimal.*

Proof. If $|\tau|^2$ admits a local maximum at some point, $\Delta|\tau|^2 \geq 0$ holds at the point. Therefore, we have $|\alpha_\tau|^2 = 0$, which implies that $\tau = 0$. Thus, the local maximum must be 0, so it hold that $\tau \equiv 0$, locally. By real analyticity of biharmonic submanifolds, τ must be 0. □

4. An over-determined system of ODE

In this section, we assume that the ambient space $\overline{M}^{n+1}$ is a space form of sectional curvature K, and M^n is a biharmonic hypersurface in $\overline{M}$.

We can study more precisely biharmonic hypersurfaces since we can diagonalize the second fundamental form. Indeed, by taking an orthonormal frame consisting of unit principal curvature vectors $\{v_i\}$, we can diagonalize the second fundamental form as $\alpha(v_i, v_j) = \lambda_i\delta_{ij}$. Then, by (3.2b), we have

$$\Delta\tau + |\alpha|^2\tau - nK\tau = 0, \tag{4.1a}$$

$$\tau\nabla_i\tau + 2\alpha_{i\ell}\nabla^\ell\tau = (\tau + 2\lambda_i)\nabla_i\tau = 0 \quad (\text{for all } 1 \leq i \leq n). \tag{4.1b}$$

Therefore, in the case of $K \leq 0$, if we assume that there is no principal curvature satisfying that $\tau + 2\lambda_i = 0$, then τ must be constant, and then M^n is minimal. Thus, we have

Proposition 4.1. *If M^n is a non-minimal biharmonic submanifold of the space form $\overline{M}^{n+1}$ of constant curvature K with $K \leq 0$, $(-1/2)\tau$ is a principal curvature.*

Proposition 4.1 is essentially important to continue our arguments below to obtain main theorem.

From now on, we assume that M is a biharmonic hypersurface, but not minimal.

Thus, in the case that $K \leq 0$, the mean curvature is not constant. In the case that $K \geq 0$, we assume that the mean curvature is not constant. We always assume that $n \geq 2$.

Let $\{\lambda_i\}$ be the principal curvatures, and let us denote their unit principal curvature vectors by $\{v_i\}$, and put $\tau := \sum\lambda_i$. In the following, all the subscripts of the tensor fields mean the ones with respect to not the local coordinates, but $\{v_i\}$. For examples, $\alpha_{ij} = \alpha(v_i, v_j) = \delta_{ij}\lambda_i$. And we denote the differentiation with respect to v_i by $v_i[*]$. Furthermore, $g(\nabla_{v_i}v_j, v_k) = v_i[g(v_j, v_k)] - g(v_j, \nabla_{v_i}v_k) = -g(v_j, \nabla_{v_i}v_k)$, in particular, $g(\nabla_{v_i}v_j, v_j) = 0$, which we will use frequently.

Note that v_i is not uniquely determined when λ_i has multiplicity. To select them suitably, we need the following formula.

Lemma 4.2. *For all $i, j, k \leq n$, we have*

$$\nabla_i\alpha_{jk} = \delta_{jk}v_i[\lambda_j] + (\lambda_j - \lambda_k)g(\nabla_{v_i}v_j, v_k), \tag{4.2a}$$

$$\delta_{jk}v_i[\lambda_j] - \delta_{ik}v_j[\lambda_i] = -(\lambda_j - \lambda_k)g(\nabla_{v_i}v_j, v_k) + (\lambda_i - \lambda_k)g(\nabla_{v_j}v_i, v_k). \tag{4.2b}$$

Proof. Using $\alpha_{ij} = \delta_{ij}\lambda_i$,

$$\begin{aligned}\nabla_i\alpha_{jk} &= v_i[\alpha_{jk}] - \alpha(\nabla_{v_i}v_j, v_k) - \alpha(v_j, \nabla_{v_i}v_k)\\ &= v_i[\delta_{jk}\lambda_j] - \lambda_k g(\nabla_{v_i}v_j, v_k) - \lambda_j g(v_j, \nabla_{v_i}v_k)\\ &= \delta_{jk}v_i[\lambda_j] - \lambda_k g(\nabla_{v_i}v_j, v_k) + \lambda_j g(\nabla_{v_i}v_j, v_k).\end{aligned} \tag{4.3}$$

Hence the first equation holds. The second equation is derived from Codazzi equation: $\nabla_i\alpha_{jk} = \nabla_j\alpha_{ik}$. □

In case $k = j \neq i$ and $\lambda_j = \lambda_i$ in (4.2b), we have $v_i[\lambda_j] = 0$. It means that if λ_i has multiplicity > 1, then $v_i[\lambda_i] = 0$. We renumber the indices $\{i\}$ so that $\tau + 2\lambda_n = 0$. From (4.1b), if $\lambda_i \neq \lambda_n$, then $v_i[\lambda_n] = 0$. Therefore, if λ_n has multiplicity > 1, then $v_i[\lambda_n] = 0$ for all $i \leq n$, τ is constant on M, and M is minimal. Since M is not minimal, we conclude that λ_n is simple.

For other λ_i with multiplicity > 1, we reselect v_i as follows. For an index i_0, let E be the tangent sub-bundle on M generated by $\{v_i \mid \lambda_i = \lambda_{i_0}\}$. Since λ_n is simple, E has trivial normal connection $\nabla^{E\perp}$ along each integral v_n curve. Therefore, we can choose orthonormal bases $\{v_i\}$ of E so that $\nabla^{E\perp}_{v_n}v_i = 0$, i.e., $g(\nabla_{v_n}v_i, v_j) = 0$ for all i and j satisfying $\lambda_i = \lambda_j = \lambda_{i_0}$. Moreover, we have $v_i[\tau] = -2v_i[\lambda_n] = 0$ for $i < n$.

We summarize the above selection of $\{v_i\}$ as follows, and, from now on, we assume the frame field $\{v_i\}$ satisfies the property.

Lemma 4.3. *The principal curvature $\lambda_n = -\tau/2$ is simple, and it holds that $v_i[\tau] = 0$ for any $i < n$. Moreover, we can choose the frame field $\{v_i\}$ so that $g(\nabla_{v_n}v_i, v_j) = 0$ if $\lambda_i = \lambda_j$.*

Moreover, the chosen vector fields v_i have the following good property.

Lemma 4.4. *For any $i \leq n$, it holds that*

$$\nabla_{v_n}v_i = 0. \tag{4.4}$$

Proof. We consider the covariant differentiation of (4.1b).

$$\begin{aligned}0 &= \nabla_j(\tau\nabla_i\tau + 2\alpha_{i\ell}\nabla^\ell\tau)\\ &= \nabla_j\tau\nabla_i\tau + \tau\nabla_j\nabla_i\tau + 2\nabla_j\alpha_{i\ell}\nabla^\ell\tau + 2\alpha_{i\ell}\nabla_j\nabla^\ell\tau\\ &= \nabla_j\tau\nabla_i\tau + \tau\nabla_j\nabla_i\tau + 2\nabla_j\alpha_{in}\nabla_n\tau + 2\lambda_i\nabla_j\nabla_i\tau\\ &= (\tau + 2\lambda_i)\nabla_i\nabla_j\tau + 2\nabla_n\tau\nabla_n\alpha_{ij} + \nabla_i\tau\nabla_j\tau.\end{aligned} \tag{4.5}$$

Exchanging i and j in (4.5), and taking the difference between them, we have $(\lambda_i - \lambda_j)\nabla_i\nabla_j\tau = 0$. Assume that $\lambda_i \neq \lambda_j$. Then it holds that $\nabla_i\nabla_j\tau = 0$. Substituting it into (4.5), we have $\nabla_n\alpha_{ij} = 0$, because $\nabla_i\tau = 0$ or $\nabla_j\tau = 0$ and $\nabla_n\tau \neq 0$. Substitute it into (4.2a) replaced i, j, k by n, i, j. Then, we have $0 = (\lambda_i - \lambda_j)g(\nabla_{v_n}v_i, v_j)$. Therefore, $g(\nabla_{v_n}v_i, v_j) = 0$ if $\lambda_i \neq \lambda_j$.

On the other hand, $g(\nabla_{v_n}v_i, v_j) = 0$ if $\lambda_i = \lambda_j$ by Lemma 4.3. Thus, we have $g(\nabla_{v_n}v_i, v_j) = 0$ for any $i, j \leq n$. We also have

$$\nabla_n\alpha_{ij} = 0 \tag{4.6}$$

for $i, j \le n$ and $i \ne j$, by (4.2a). □

Since τ is not constant, on a neighborhood of a point satisfying that $d\tau \ne 0$, the set $\tau =$ (a constant) is a hypersurface of M.

DEFINITION 4.5. We call each F defined by $\tau =$ (a constant) *a characteristic hypersurface* of M.

For $i < n$, (4.1b) implies that $v_i[\tau] = 0$, because $\lambda_i \ne \lambda_n = -\tau/2$. Therefore, the set $\{v_i \mid 1 \le i < n\}$ is a locally defined orthonormal frame field of the tangent bundle of F.

Moreover, every v_i is a principal curvature vector filed of F as follows.

Lemma 4.6. *Every vector v_i $(i < n)$ is a principal curvature vector of hypersurface F in M. We denote by β the second fundamental form, and by μ_i the principal curvature for the direction v_i. Then, it holds that, for $i, j < n$,*

$$\beta(v_i, v_j) = g(\nabla_{v_i} v_j, v_n) = \delta_{ij}\mu_i, \quad \nabla_{v_i} v_n = -\mu_i v_i. \tag{4.7}$$

Proof. We consider (4.2b) with $i = n$, $j, k < n$ and $j \ne k$. We have

$$0 = -(\lambda_j - \lambda_k) g(\nabla_{v_n} v_j, v_k) + (\lambda_n - \lambda_k) g(\nabla_{v_j} v_n, v_k). \tag{4.8}$$

Since $\nabla_{v_n} v_j = 0$ by Lemma 4.4 and $\lambda_k \ne \lambda_n$, we have

$$0 = g(\nabla_{v_j} v_n, v_k) = -g(v_n, \nabla_{v_j} v_k) = -\beta(v_j, v_k). \tag{4.9}$$

It also implies that $\nabla_{v_j} v_n$ is parallel to v_j. Therefore, $\nabla_{v_j} v_n = g(\nabla_{v_j} v_n, v_j) v_j = -\mu_j v_j$. □

Now, we derive an over-determined ODE.

Proposition 4.7. *Let M^n be a biharmonic hypersurface of the space form $\overline{M}^{n+1}$ of sectional curvature $K \le 0$. Then, λ_i and μ_i satisfy the following ordinary differential equations:*

$$-\tau'' + \tau' \textstyle\sum_{i<n} \mu_i + \tau\left(\frac{1}{4}\tau^2 - nK + \sum_{i<n} \lambda_i^2\right) = 0, \tag{4.10a}$$

$$(\lambda_i)' = \left(\frac{1}{2}\tau + \lambda_i\right)\mu_i, \tag{4.10b}$$

$$(\mu_i)' = \mu_i^2 - \frac{1}{2}\tau\lambda_i + K. \tag{4.10c}$$

Here, τ is a function of $\{\lambda_i\}$ defined by $\tau = (2/3)\sum_{i<n} \lambda_i$, and $*'$ is the differentiation $v_n[*]$.

Proof. For (4.10a), we calculate $\nabla_i \nabla_j \tau$ for $i, j \le n$.

$$\begin{aligned} \nabla_i \nabla_j \tau &= (\nabla_{v_i}(\nabla\tau))(v_j) = v_i[v_j[\tau]] - \nabla_{\nabla_{v_i} v_j} \tau = v_i[v_j[\tau]] - \textstyle\sum_{k \le n} g(\nabla_{v_i} v_j, v_k) \nabla_{v_k} \tau \\ &= v_i[v_j[\tau]] - g(\nabla_{v_i} v_j, v_n) \nabla_n \tau. \end{aligned} \tag{4.11}$$

Therefore, $\nabla_i \nabla_i \tau = -\mu_i \tau'$ for all $i < n$, and $\nabla_n \nabla_n \tau = \tau''$. Substituting it into (4.1a), we have an expression of $\Delta\tau$:

$$\Delta\tau = -\nabla_n \nabla_n \tau - \textstyle\sum_{i<n} \nabla_i \nabla_i \tau = -\tau'' + \tau' \sum_{i<n} \mu_i. \tag{4.12}$$

Thus, we have

$$0 = \Delta\tau + |\alpha|^2\tau - nK\tau \tag{4.13}$$
$$= -\tau'' + \tau'\textstyle\sum_{i<n}\mu_i + \tau(\frac{1}{4}\tau^2 - nK + \sum_{i<n}{\lambda_i}^2),$$

which is (4.10a).

For $v_n[\lambda_i]$, we use (4.2b) with $j = n$, $k = i < n$. We have, using (4.6),

$$-v_n[\lambda_i] = -(\lambda_n - \lambda_i)g(\nabla_{v_i}v_n, v_i) = (-\frac{1}{2}\tau - \lambda_i)\mu_i, \tag{4.14}$$

which is (4.10b)

For $v_n[\mu_i]$, differentiating the equation of the definition of μ_i, we have

$$\begin{aligned} v_n[\mu_i] = v_n[g(\nabla_{v_i}v_i, v_n)] &= g(\nabla_{v_n}\nabla_{v_i}v_i, v_n) + g(\nabla_{v_i}v_i, \nabla_{v_n}v_n) \\ &= g(R(v_n, v_i)v_i + \nabla_{v_i}\nabla_{v_n}v_i + \nabla_{[v_n,v_i]}v_i, v_n) \text{ (by } \nabla_{v_n}v_n = 0) \\ &= K + \lambda_n\lambda_i + g(\nabla_{\nabla_{v_n}v_i - \nabla_{v_i}v_n}v_i, v_n) \text{ (by Gauss, } \nabla_{v_n}v_i = 0) \\ &= K - \frac{1}{2}\tau\lambda_i + \mu_i g(\nabla_{v_i}v_i, v_n) \text{ (by } \nabla_{v_n}v_i = 0, \nabla_{v_i}v_n = -\mu_i v_i) \\ &= K + \mu_i^2 - \frac{1}{2}\tau\lambda_i, \end{aligned} \tag{4.15}$$

which is (4.10c). □

In the case $K \leq 0$, note here that we can conclude non-existence of non-minimal biharmonic hypersurfaces if the overdetermined differential system (4.10) has only solutions satisfying $\tau \equiv$ const.

In the case $n = 2$, we can prove by a different manner the B.-Y. Chen's theorem:

Corollary 4.8 (Chen [4], Jiang [12])**.** *Every biharmonic submanifold in the* 3 *dimensional space form of non-positive sectional curvature is minimal.*

Proof. Substituting $n = 2$, (4.10b), (4.10c) and $\tau' = (2/3)\lambda_1'$ into (4.10a), we have

$$(2/27)\lambda_1\{14{\lambda_1}^2 - 16{\mu_1}^2 - 9nK - 12K\} = 0. \tag{4.16}$$

Since $\lambda_1 = (3/2)\tau$ is not constant,

$$14{\lambda_1}^2 - 16{\mu_1}^2 - 9nK - 12K = 0. \tag{4.17}$$

Substituting (4.10b) and (4.10c) into the equation $\{14{\lambda_1}^2 - 16{\mu_1}^2 - 9nK - 12K\}' = 0$, we have

$$-16\mu_1\{3{\lambda_1}^2 - 2{\mu_1}^2 - 2K\} = 0. \tag{4.18}$$

If $\mu_1 \equiv 0$, by (4.10b), $\tau = (2/3)\lambda_1$ must be constant. Thus, the submanifold is minimal. If $\mu_1 \neq 0$ at a point, we have that $3{\lambda_1}^2 - 2{\mu_1}^2 - 2K = 0$. Then, together with (4.17), λ_1 is constant. We have done. □

5. Solutions to the over-determined system of ODE

In this section, we assume that M^n is a biharmonic hypersurface in the space form $\overline{M}^{n+1}$ of sectional curvature K.

We will analyze the algebraic ordinary differential system (4.10). In the following we only treat the case $i \leq n-1$, and put $n_1 := n-1$. And for the summations which will be treated in this section in i, we always assume that i run over the set $1 \leq i \leq n_1$. In this section, we will treat a more general setting that it would occur that $\lambda_i = \lambda_j$ for different i and j.

We will denote by $\lambda = (\lambda_1, \cdots, \lambda_{n_1}) \in \mathbf{R}^{n_1}$, and $\mu = (\mu_1, \cdots, \mu_{n_1}) \in \mathbf{R}^{n_1}$. The solutions to the ordinary differential system can be regarded as real analytic maps on a neighborhood of the origin of $\mathbf{R}$ into $\mathbf{R}^{2n_1} = \{(\lambda, \mu)\}$, and our τ is regarded as a function $\tau = (2/3)\sum_{i \leq n_1} \lambda_i$ on $\mathbf{R}^{2n_1}$.

Let $S \subset \mathbf{R}^{2n_1}$ be the set of all the (λ, μ) such that the equation (4.10) has a solution with initial value (λ, μ).

Lemma 5.1. *Each $(\lambda, \mu) \in S$ is a zero point of the following polynomial.*

$$P_0 := -\frac{4}{3}\sum \lambda_i {\mu_i}^2 - \frac{2}{3}\tau \sum {\mu_i}^2 + \frac{4}{9}\sum \mu_i \sum \lambda_i \mu_i + \frac{4}{3}\tau \sum {\lambda_i}^2 + \frac{2}{9}\tau \Big(\sum \mu_i\Big)^2 + \frac{1}{2}\tau^3 - \frac{2}{3}(2n+1)K\tau. \tag{5.1}$$

Proof. Assume that $(\lambda, \mu) = (\lambda(t), \mu(t))$ is a solution to (4.10). Then,

$$\tau' = \frac{2}{3}\sum \lambda_i' = \frac{2}{3}\sum \Big(\frac{1}{2}\tau \mu_i + \lambda_i \mu_i\Big) = \frac{1}{3}\tau \sum \mu_i + \frac{2}{3}\sum \lambda_i \mu_i, \tag{5.2}$$

$$\tau'' = \frac{1}{3}\tau' \sum \mu_i + \frac{1}{3}\tau \sum \mu_i' + \frac{2}{3}\sum \lambda_i' \mu_i + \frac{2}{3}\sum \lambda_i \mu_i'.$$

Substituting this, (4.10b) and (4.10c) into (4.10a), we obtain the desired polynomial. □

Starting at P_0, we determine the polynomial P_k inductively as follows: We substitute the solution $(\lambda(t), \mu(t))$ to the ordinary differential equation of normal form, (4.10b), (4.10c) in the polynomial P_k, and differentiate it with respect to t, and substitute (4.10b), (4.10c) into $\lambda_i'(t), \mu_i'(t)$. Then, we obtain a polynomial in $\{\lambda_i(t), \mu_i(t)\}$. We define P_{k+1}, this polynomial.

Proposition 5.2. *The set S of all initial values of* (4.10) *coincides with the algebraic manifold $\cap_{k=0}^{\infty}(P_k)^{-1}(0)$.*

Proof. We proved that $S \subset \cap_{k=0}^{\infty}(P_k)^{-1}(0)$. Conversely, let (λ, μ) be a point of $\cap_{k=0}^{\infty}(P_k)^{-1}(0)$, and $(\lambda(t), \mu(t))$ be the solution to the partial system (4.10b), (4.10c) with initial value (λ, μ). Then, the k-th derivative of the left hand side of (4.10a) vanishes for any $k \geq 0$. Since the solution $(\lambda(t), \mu(t))$ is real analytic, it means that (4.10a) is satisfied. □

For the ODE (4.10), we can classify all solutions giving minimal hypersurfaces.

Proposition 5.3. *All the solutions to* (4.10) *satisfying that $\tau \equiv 0$ are classified as follows: In the case where $K = 0$:*

$$\mu_i = -\frac{1}{t + c_i},\ \lambda_i = \frac{a_i}{t + c_i}, \quad \text{or} \quad \mu_i = 0,\ \lambda_i = a_i. \tag{5.3}$$

In the case where $K = -1$:

$$\mu_i = -\tanh(t + c_i),\ \lambda_i = \frac{a_i}{\cosh(t + c_i)}, \quad \text{or} \quad \mu_i = \pm 1,\ \lambda_i = a_i e^{\pm t}. \tag{5.4}$$

In the case where $K = 1$:

$$\mu_i = \tan(t + c_i),\ \lambda_i = \frac{a_i}{\cos(t + c_i)}. \tag{5.5}$$

Here, the sum of all the a_i corresponding to the same c_i must be zero: $\sum_{\{i \mid c_i = c_k\}} a_i = 0$ for every k. (regarding $c_i = \infty$ for the singular solution in the case $K = 0$, and $c_i = \pm\infty$ in the case that $K = -1$). For example, if all the c_i are different each other, it must be that $a_i = 0$ for all i. If all c_i are same, our condition is only that $\sum_i a_i = 0$.

Proof. Substituting $\tau = 0$ in system (4.10), we see that equation (4.10a) is automatically satisfied, and we have

$$\lambda_i' = \lambda_i \mu_i, \quad \mu_i' = {\mu_i}^2 + K. \tag{5.6}$$

We solve the second equation, and substitute the solution into the first equation. Then we get (5.3), (5.4), or (5.5). Each solution satisfies $\sum \lambda_i = (3/2)\tau = 0$ if and only if the last condition for a_i is satisfied. □

Proposition 5.4. *There exist constant solutions to* (4.10) *which satisfy $\tau \neq 0$ only in the case $n = 4$, $K > 0$, and it holds that $\lambda_i = \pm\sqrt{K}$, $\mu_i = 0$, $\tau = \pm 2\sqrt{K}$ in this case.*

Proof. Substituting $\lambda_i' = 0$, $\mu_i' = 0$, $\tau' = \tau'' = 0$ into (4.10), we have

$$\left(\frac{1}{2}\tau + \lambda_i\right)\mu_i = 0, \quad {\mu_i}^2 - \frac{1}{2}\tau\lambda_i + K = 0, \tag{5.7}$$
$$\frac{1}{4}\tau^2 - nK + \sum_i {\lambda_i}^2 = 0.$$

By the third one of (5.7), we have that $K > 0$, because $\tau \neq 0$. In the case where $\lambda_i = -\tau/2$ in the first and second ones of (5.7), we have ${\mu_i}^2 + (1/4)\tau^2 + K = 0$ which does not occur. Thus, we have $\mu_i = 0$. Then, we have by the second one, $\lambda_i = 2K/\tau$. By using together with $\tau = (2/3)n_1\lambda_i$, we have $\tau = \pm(2/\sqrt{3})\sqrt{n_1 K}$, $\lambda_i = \pm\sqrt{3K}/\sqrt{n_1}$. Substituting this into the third one of (5.7), we obtain that $n = 4$ and the other claims. □

Corollary 5.5. *In the case of the space form of constant curvature, the biharmonic hypersurface all of whose principal curvatures are constant and different each other must be minimal.*

Remark 5.6. There exist examples of biharmonic hypersurfaces having principal curvatures with multiplicities. They are isoparametric hypersurfaces in S^{n+1} (Ichiyama, Inoguchi and Urakawa, [10], [11]). In the case where $K \leq 0$, there are no such biharmonic hypersurfaces which are not minimal by Proposition 3.1.

Lemma 5.7. *Let $\{\lambda_i, \mu_i\}$ be a solution to* (4.10) *with $\tau \neq 0$. If $\lambda_i \equiv \lambda_j$, then $\mu_i \equiv \mu_j$ or $\lambda_i \equiv -(1/2)\tau$. Conversely, if $\mu_i \equiv \mu_j$, then $\lambda_i \equiv \lambda_j$.*

Proof. If $\lambda_i \equiv \lambda_j$, then $(\tau/2 + \lambda_i)(\mu_i - \mu_j) \equiv 0$, hence $\tau/2 + \lambda_i \equiv 0$ or $\mu_i \equiv \mu_j$. Conversely, if $\mu_i \equiv \mu_j$, then $\tau(\lambda_i - \lambda_j) \equiv 0$. Since $\tau \neq 0$, we have $\lambda_i \equiv \lambda_j$. □

Note that the case $\lambda_i \equiv -\tau/2$ does not occur when the solution comes from a non-minimal

biharmonic hypersurface, because $\lambda_n = -\tau/2$ is simple. Next we consider solutions with same λ_i, under $K = 0$.

Lemma 5.8. *The solution to* (4.10) *with* $K = 0$ *satisfying that all the* λ_i *are the same, must satisfy* $\tau \equiv 0$.

Proof. We assume $\tau \neq 0$. We may write as $\lambda_i = \lambda$. Since $\tau = (2/3)n_1\lambda$, we get $\lambda \neq -(1/2)\tau$. Thus, by Lemma 5.7, all the μ_i are equal to each other, we may write as $\mu_i = \mu$. Substituting these into (4.10), we have

$$\lambda' = \frac{n_1+3}{3}\lambda\mu, \quad \mu' = \mu^2 - \frac{n_1}{3}\lambda^2, \quad \frac{n_1(n_1+9)}{9}\lambda^3 + n_1\mu\lambda' - \lambda'' = 0. \tag{5.8}$$

Differentiating the first equation, and substituting into the third one into which the first and second equations are substituted, we obtain

$$\lambda\{n_1(n_1+6)\lambda^2 + ({n_1}^2 - 9)\mu^2\} = 0. \tag{5.9}$$

Thus, $\tau = 0$ if $n_1 \geq 3$.

In the case that $n_1 \leq 2$, we have $\mu = c\,\lambda$. Substituting this into λ', μ', and eliminating λ', we have $(1 + c^2)\lambda^2 = 0$ which implies that $\tau = 0$. □

We can also solve (4.10b), (4.10c) with $K = 0$ if we know the function τ.

Proposition 5.9. *The ordinary differential system* (4.10b), (4.10c) *with* $K = 0$ *can be solved as follows if we regard* τ *as a known function. If we put* $\lambda_i = r_i \sin\theta_i$, $\mu_i = r_i \cos\theta_i$,

$$\theta_i = \frac{1}{2}\int \tau\, dt, \quad r_i = \frac{-1}{\int \cos\theta_i\, dt}. \tag{5.10}$$

Proof. Let us rewrite (4.10b), (4.10c) in terms of r_i, θ_i,

$$\begin{aligned} r_i' \sin\theta_i + \theta_i' r_i \cos\theta_i &= (\frac{1}{2}\tau + r_i \sin\theta_i) r_i \cos\theta_i, \\ r_i' \cos\theta_i - \theta_i' r_i \sin\theta_i &= {r_i}^2 \cos^2\theta_i - \frac{1}{2}\tau\, r_i \sin\theta_i. \end{aligned} \tag{5.11}$$

By (5.11), we have $r_i' = {r_i}^2 \cos\theta_i$, and $\theta_i' r_i = (1/2)\tau r_i$ which solve θ_i, and r_i. □

6. Irreducibility of principal curvature vector fields

In this section, we assume that M^n is a non-minimal biharmonic hypersurface in the Euclidean space $\overline{M} = E^{n+1}$, and all the principal curvatures $\{\lambda_i\}$ are simple.

We denote the covariant differentiation on the characteristic hypersurface F by $\widetilde{\nabla}$, and the second fundamental form of F in E^{n+1} by $\widetilde{\alpha}$. The second fundamental form of F in M is denoted by β.

Note that the unit normal frame fields N and v_n on M are parallel with respect to the normal connection on F in E^{n+1}. In fact, since $\overline{g}(\overline{\nabla}_{v_i} N, v_n) = -\overline{g}(N, \overline{\nabla}_{v_i} v_n) = -\alpha_{in} = 0$ for every $i < n$, we have $\overline{g}(\widetilde{\nabla}_{v_i} N, v_n) = 0$, and $\widetilde{\nabla}_{v_i} N = 0$, $\widetilde{\nabla}_{v_i} v_n = 0$.

Since the N-component of $\widetilde{\alpha}$ coincides with the restriction of α to the tangent space TF of F because of $\overline{g}(\overline{\nabla}_{v_i} v_j, N) = \alpha(v_i, v_j)$, we use the same notation α for it. The v_n-component β of $\widetilde{\alpha}$ is $\beta_{ij} = \delta_{ij}\mu_i$ by (4.7). Thus, $\widetilde{\alpha}$ can be diagonalized by the frames $\{N, v_n\}$ whose

eigenvalues are λ_i and μ_i.

We calculate the covariant differentiation of $\widetilde{\alpha}$ as follows:

$$
\begin{aligned}
(6.1)\quad \widetilde{\nabla}_i\widetilde{\alpha}_{jk} &= (\widetilde{\nabla}_{v_i}\widetilde{\alpha})(v_j, v_k)\\
&= \widetilde{\nabla}_{v_i}(\widetilde{\alpha}(v_j, v_k)) - \widetilde{\alpha}(\widetilde{\nabla}_{v_i}v_j, v_k) - \widetilde{\alpha}(v_j, \widetilde{\nabla}_{v_i}v_k)\\
&= \widetilde{\nabla}_{v_i}(\delta_{jk}(\lambda_k N + \mu_k v_n)) - g(\widetilde{\nabla}_{v_i}v_j, v_k)(\lambda_k N + \mu_k v_n) - g(v_j, \widetilde{\nabla}_{v_i}v_k)(\lambda_j N + \mu_j v_n)\\
&= \delta_{jk}(v_i[\lambda_k]N + v_i[\mu_k]v_n) - g(\widetilde{\nabla}_{v_i}v_j, v_k)(\lambda_k N + \mu_k v_n) + g(\widetilde{\nabla}_{v_i}v_j, v_k)(\lambda_j N + \mu_j v_n)\\
&= \{\delta_{jk}v_i[\lambda_k] + (\lambda_j - \lambda_k)g(\widetilde{\nabla}_{v_i}v_j, v_k)\}N + \{\delta_{jk}v_i[\mu_k] + (\mu_j - \mu_k)g(\widetilde{\nabla}_{v_i}v_j, v_k)\}v_n.
\end{aligned}
$$

Since $\widetilde{\nabla}_i\widetilde{\alpha}_{jk} = \widetilde{\nabla}_j\widetilde{\alpha}_{ik}$, the N-component of $\widetilde{\alpha}$ coincides with

$$
(6.2)\qquad \delta_{jk}v_i[\lambda_k] + (\lambda_j - \lambda_k)g(\widetilde{\nabla}_{v_i}v_j, v_k) = \delta_{ik}v_j[\lambda_k] + (\lambda_i - \lambda_k)g(\widetilde{\nabla}_{v_j}v_i, v_k).
$$

For $i \neq j = k$, it holds that

$$
(6.3)\qquad v_i[\lambda_j] = (\lambda_i - \lambda_j)g(\widetilde{\nabla}_{v_j}v_i, v_j).
$$

In the case that all the i, j and k are different each other,

$$
(6.4)\qquad \widetilde{\nabla}_i\alpha_{jk} = (\lambda_j - \lambda_k)g(\widetilde{\nabla}_{v_i}v_j, v_k) = (\lambda_i - \lambda_k)g(\widetilde{\nabla}_{v_j}v_i, v_k).
$$

We conclude that the quantities s^{λ}_{ijk} defined by

$$
(6.5)\qquad s^{\lambda}_{ijk} := (\lambda_j - \lambda_k)g(\widetilde{\nabla}_{v_i}v_j, v_k)
$$

are symmetric for all distinct triplets $\{i, j, k\}$.

By the same way, we obtain the relations of μ, by considering the v_n-component: For all the i, j and k which are different each other, we can conclude that

$$
\begin{aligned}
(6.6)\qquad & v_i[\mu_j] = (\mu_i - \mu_j)g(\widetilde{\nabla}_{v_j}v_i, v_j),\\
& s^{\mu}_{ijk} := (\mu_j - \mu_k)g(\widetilde{\nabla}_{v_i}v_j, v_k) \text{ are symmetric for all the } i, j, k.
\end{aligned}
$$

Assume that all the principal curvatures are simple. If $g(\widetilde{\nabla}_{v_i}v_j, v_k) \neq 0$, it holds that $s^{\lambda}_{ijk} \neq 0$, which implies that $g(\widetilde{\nabla}_{v_j}v_k, v_i)$, $g(\widetilde{\nabla}_{v_k}v_i, v_j) \neq 0$. Thus, we obtain the relations that

$$
(6.7)\qquad \frac{\mu_i - \mu_j}{\lambda_i - \lambda_j} = \frac{\mu_j - \mu_k}{\lambda_j - \lambda_k} = \frac{\mu_k - \mu_i}{\lambda_k - \lambda_i} = \frac{s^{\mu}_{ijk}}{s^{\lambda}_{ijk}}.
$$

Therefore, if $g(\widetilde{\nabla}_{v_i}v_j, v_k) \neq 0$ for every distinct triplet $\{i, j, k\}$, then all the $\dfrac{\mu_i - \mu_j}{\lambda_i - \lambda_j}$ coincide each other for every distinct pair $\{i, j\}$. Thus, if we denote the common quantity by φ, then all $\mu_i - \varphi\lambda_i$ have the same value. If we denote it by ψ, then it holds that $\mu_i - \varphi\lambda_i = \psi$ for all i.

Conversely, if there exist φ and ψ satisfying that $\mu_i = \varphi\lambda_i + \psi$ for all i, and there exist at least two different λ_i, φ and ψ are uniquely determined. Really, we assume the following weaker conditions:

DEFINITION 6.1. Put $J = \{\{i, j\} \mid 1 \leq i, j \leq n_1,\ i \neq j\}$. If a distinct triplet $\{i, j, k\}$ satisfies $g(\nabla_{v_i}v_j, v_k) \neq 0$, then we define $\{i, j\} \sim \{j, k\} \sim \{i, k\}$. Let $\sim_J$ be the equivalence relation on J generated by $\sim$. If all $\{i, j\} \in J$ are equivalent under $\sim_J$, the frame field $\{v_i\}$ is *irreducible*.

Otherwise, the frame field is *reducible*.

DEFINITION 6.2. If there exist functions φ, ψ satisfying $\mu_i = \varphi\lambda_i + \psi$ for all $1 \le i \le n_1$, we say that $\{\lambda_i\}$ and $\{\mu_i\}$ are *linearly related*.

As we saw, we have

Lemma 6.3. *If the frame field $\{v_i\}$ is irreducible, then $\{\lambda_i\}$ and $\{\mu_i\}$ are linearly related.*

When $n_1 = 3$, the frame field $\{v_i\}$ is reducible if and only if all $g(\nabla_{v_1} v_2, v_3) = g(\nabla_{v_2} v_3, v_1) = g(\nabla_{v_3} v_1, v_2) = 0$.

Proposition 6.4. *If $\{\lambda_i\}$ and $\{\mu_i\}$ have linear relation $\mu_i = \varphi\lambda_i + \psi$, and if there exist at least three distinct λ_i, then φ and ψ are constant on F.*

Proof. For $i \neq j$, we have (6.3)

$$v_i[\lambda_j] = (\lambda_i - \lambda_j) g(\widetilde{\nabla}_{v_j} v_i, v_j),$$

and (6.6)

$$v_i[\mu_j] = (\mu_i - \mu_j) g(\widetilde{\nabla}_{v_j} v_i, v_j).$$

Since $\mu_i = \varphi\lambda_i + \psi$, we obtain that

$$v_i[\varphi]\,\lambda_j + \varphi\, v_i[\lambda_j] + v_i[\psi] = \varphi\,(\lambda_i - \lambda_j)\, g(\widetilde{\nabla}_{v_j} v_i, v_j) = \varphi\, v_i[\lambda_j]. \tag{6.8}$$

Thus, we have $v_i[\varphi]\lambda_j + v_i[\psi] = 0$ $(i \neq j)$. Since we assume the existence of three different λ_j, we can conclude that $v_i[\varphi] = v_i[\psi] = 0$. Therefore, φ and ψ are constant on F. □

7. Constantness of principal curvatures

In this section, we assume that M^n is a non-minimal biharmonic hypersurface in the Euclidean space E^{n+1}.

In the following, we assume that $\{\lambda_i(t)\}$ and $\{\mu_i(t)\}$ have linear relation $\mu_i(t) = \varphi(t)\lambda_i(t) + \psi(t)$. This assumption holds, under the case $n_1 = 2$ or the condition that $\{v_i\}$ is irreducible in the case $n_1 \ge 3$.

In this section, we do not assume that all the λ_i are simple. However, by Lemma 5.8, for the solutions other than the one satisfying that $\tau \equiv 0$, there exist at least two λ_i, so φ and ψ are uniquely determined.

In the following, we will assume that $\tau \neq 0$, and treat the solutions having λ_i different each other.

Lemma 7.1. *The functions φ, ψ must satisfy the following two ordinary differential equations:*

$$\varphi' = -\frac{1}{2}\tau(\varphi^2 + 1) + \varphi\psi, \quad \psi' = (\psi - \frac{1}{2}\tau\varphi)\psi. \tag{7.1}$$

Proof. By substituting $\mu_i = \varphi\lambda_i + \psi$ and (4.10c) into (4.10b), we have

$$\left\{\psi' - \psi^2 + \frac{1}{2}\tau\varphi\psi\right\} + \left\{\varphi' + \frac{1}{2}\tau(\varphi^2 + 1) - \psi\varphi\right\}\lambda_i = 0. \tag{7.2}$$

Due to Lemma 5.8, there exist λ_i which are different each other, and we have (7.1) for φ', ψ'. □

Later on, we will proceed calculations dividing by φ, we first have to show in the following Lemma, which enables us to assume that $\varphi \neq 0$.

Lemma 7.2. *The function φ is not* 0 *at a generic point.*

Proof. Assume that $\varphi \equiv 0$. By substituting this into the first equation of Lemma 7.1, we obtain $\tau = 0$. We get Lemma 7.2. □

By Lemma 7.2, we will always assume that $\varphi \neq 0$. Furthermore, we will have several lemmas for later uses. We define the function $\Lambda_k := \sum_{i \leq n_1} {\lambda_i}^k$ $(k = 0, 1, \dots)$. Note that $\Lambda_0 = n_1$ by definition.

Lemma 7.3. *The differentiations λ_i' and τ' can be expressed in terms of φ, ψ, and λ_i as follows:*

$$\lambda_i' = \frac{1}{2}\tau\psi + (\psi + \frac{1}{2}\tau\varphi)\lambda_i + \varphi(\lambda_i)^2, \quad \tau' = \frac{1}{3}(n_1 + 3)\tau\psi + \frac{1}{2}\tau^2\varphi + \frac{2}{3}\varphi\Lambda_2. \tag{7.3}$$

Proof. The first equation of (7.3) can be obtained by substituting $\mu_i = \varphi\lambda_i + \psi$ simply into (4.10b), and we get the second one by summing it up. □

Lemma 7.4. *The functions Λ_k satisfy the following ordinary differential equations.*

$$\Lambda_k' = k\left\{\frac{1}{2}\tau\psi\Lambda_{k-1} + (\frac{1}{2}\tau\varphi + \psi)\Lambda_k + \varphi\Lambda_{k+1}\right\}. \tag{7.4}$$

Proof. Lemma 7.3 implies that

$$\begin{aligned} \Lambda_k' &= \sum((\lambda_i)^k)' = k\sum {\lambda_i}^{k-1}\lambda_i' \\ &= k\sum {\lambda_i}^{k-1}\left\{\frac{1}{2}\tau\psi + (\psi + \frac{1}{2}\tau\varphi)\lambda_i + \varphi(\lambda_i)^2\right\} \\ &= k\sum\left\{\frac{1}{2}\tau\psi{\lambda_i}^{k-1} + (\frac{1}{2}\tau\varphi + \psi){\lambda_i}^k + \varphi{\lambda_i}^{k+1}\right\}, \end{aligned} \tag{7.5}$$

from which we obtain immediately (7.4). □

Lemma 7.5. *The function τ satisfies the following ordinary differential equation:*

$$\tau'' - \left\{n_1\psi + \frac{3\tau(\varphi^2 + 1)}{2\varphi}\right\}\tau' + \frac{\tau^2(\tau\varphi + (n_1 + 3)\psi)}{2\varphi} = 0. \tag{7.6}$$

Proof. We differentiate $\Lambda_1 = (3/2)\tau$ in t, and apply Lemma 7.4, and express Λ_1 in terms of τ. Then we have

$$\Lambda_2 = \frac{6\tau' - 3\tau^2\varphi - 2(n_1 + 3)\tau\psi}{4\varphi}, \tag{7.7}$$

and $\sum\mu_i = \varphi\Lambda_1 + n_1\psi$. Substituting these into $\sum\mu_i$, $\sum{\lambda_i}^2$ of (4.10a), we obtain (7.6). □

Lemma 7.6. *The function φ is not constant.*

Proof. By Lemma 7.2, $\varphi \neq 0$. Substituting $\varphi = c$ $(\neq 0)$ (7.1) in Lemma 7.1, we have $2c\psi - (1+c^2)\tau = 0$, $\psi' = \psi^2 - (1/2)c\tau\psi$, thus we obtain $\tau' = (2c)^{-1}\tau^2$. Substituting it and its differentiation into (7.6) in Lemma 7.5, we have $(1+c^2)\tau^3 = 0$, which completes the proof. □

Next, we will show that ψ is not constant if $\tau \neq 0$. To do it, we first show that $\psi \neq 0$. Assume that $\psi(t_1) = 0$ at some point t_1 for the solution $(\lambda_i, \mu_i, \varphi, \psi)$. Then, Lemma 7.1 and the uniqueness of solution to the ordinary differential equation imply that $\psi(t) \equiv 0$. Then, it holds that $\mu_i = \varphi\,\lambda_i$ for all i. All the angles $\theta_i = \arctan(\lambda_i/\mu_i) = \arctan(1/\varphi)$ in Proposition 5.9 are equal to each other, so we may write them as θ. Then, if we write by p, one of indefinite integrals $\int \cos\theta\, dt$, we have that $\varphi = \cot\theta$, $p' = \cos\theta$, and

$$\theta = \frac{1}{2}\int \tau\, dt, \quad r_i = \frac{-1}{p+b_i}, \tag{7.8}$$
$$\mu_i = \frac{-\cos\theta}{p+b_i}, \quad \lambda_i = \frac{-\sin\theta}{p+b_i}, \quad \tau = -\frac{2}{3}\sin\theta\cdot\textstyle\sum_{i\leq n_1}\dfrac{1}{p+b_i}.$$

We define shortly $s_k := \sum_i (p+b_i)^{-k}$. Since $\tau \neq 0$, p is not constant, and we can define s_k for generic t. Then, substituting these into (5.1) in Lemma 5.1, we have

$$\cos^2\theta = \frac{{s_1}^3 + 6s_1 s_2}{3(2s_1 s_2 + 3s_3)}. \tag{7.9}$$

Differentiating this and substitute $p' = \cos\theta$ and $\theta' = \tau/2$, we have

$$\begin{aligned} 2s_1\sin^2\theta = -(2s_1 s_2 + 3s_3)^{-2}\{&4{s_1}^3{s_2}^2 - 4{s_1}^4 s_3 + 9{s_1}^2 s_2 s_3 \\ &+ 18{s_2}^2 s_3 + 36 s_1 {s_3}^2 - 9{s_1}^3 s_4 - 54 s_1 s_2 s_4\}. \end{aligned} \tag{7.10}$$

By eliminating $\cos\theta$ and $\sin\theta$ from (7.9) and (7.10), we have

$$\begin{aligned} Q(p) := &-4{s_1}^5 s_2 + 12{s_1}^3{s_2}^2 - 18{s_1}^4 s_3 + 63{s_1}^2 s_2 s_3 \\ &+ 54{s_2}^2 s_3 + 162 s_1 {s_3}^2 - 27{s_1}^3 s_4 - 162 s_1 s_2 s_4 \\ = &\ 0. \end{aligned} \tag{7.11}$$

Since $\tau \neq 0$, p is not constant. Thus, $Q(p)$ must vanish identically as a rational function in p. Thus, we obtain in particular, $\lim_{p\to\infty} p^7 Q(p) = 0$. On the other hand, since $\lim_{p\to\infty} p^k s_k = n_1$ for each s_k, we obtain that $0 = -2{n_1}^3(n_1-3)(n_1+3)(2n_1+3)$. Thus we have that $n_1 = 3$.

Next, denote by m, the number of b_i which are equal to b_1. Let us consider the coefficients of $(p+b_1)^{-7}$ in the partial fraction decomposition of $Q(p)$. Then, since the coefficients are equal to the one exchanging each s_k into m in $Q(p)$, the coefficient is equal to $-2m^3(m-3)(m+3)(2m+3)$, and it must vanish. Thus, we obtain that $m = 3$.

It means that all the b_i are equal to each other. But, in this case, all the λ_i must be equal to each other, and due to Lemma 5.8, we obtain that $\tau \equiv 0$.

Therefore, we obtain the following lemma.

Lemma 7.7. *The function ψ does not attain* 0.

Finally, we can show that ψ is not constant by Lemma 7.7.

Lemma 7.8. *The function ψ is not constant.*

Proof. Assume that $\psi = c \neq 0$. Then, by substituting it into (7.1) in Lemma 7.1, we have $\varphi' = (1/2)(2c\varphi - \tau(1+\varphi^2))$, and $c(2c - \tau\varphi) = 0$. Then, we have $\varphi = 2c/\tau$, $\tau' = \tau^3/(4c)$. Together it and its differentiation, and Lemma 7.5, we obtain $\tau^3 = 0$. Thus, we have $\tau \equiv 0$. □

Proposition 7.9. *Assume that $n_1 \geq 3$, all the principal curvatures λ_i are simple, and $\tau \neq 0$. Then, all the τ, φ, ψ, λ_i, and μ_i must be constant on F.*

Proof. In the following, we will show inductively that Λ_k are constant along each F. $\Lambda_0 = n_1$, and $\Lambda_1 = (3/2)\tau$ are constant along each F. Assume that all the Λ_ℓ with $\ell \leq k$, are constant along each F. Then, their differentiations with respect to t are also constant along each F. Therefore, by Lemma 7.4,

$$\frac{1}{2}\tau\psi\Lambda_{k-1} + (\frac{1}{2}\tau\varphi + \psi)\Lambda_k + \varphi\Lambda_{k+1} \tag{7.12}$$

are also constant along each F. Since φ and ψ are constant along each F by Proposition 6.4, and $\varphi \neq 0$, Λ_{k+1} is also constant along each F.

Therefore, all the elementary symmetric polynomials in λ_i are constant, and every λ_i is also constant along each F. □

8. Proof of Main theorem

In this section, we assume that M^n is a non-minimal biharmonic hypersurface in the Euclidean space E^{n+1}, all the principal curvatures of M are simple, and the frame field is irreducible.

For every distinct triplet $\{i, j, k\}$, (6.3) and (6.5) hold, i.e.,

(8.1)

$$v_i[\lambda_j] = (\lambda_i - \lambda_j)g(\widetilde{\nabla}_{v_j}v_i, v_j), \text{ and } s^\lambda_{ijk} := (\lambda_j - \lambda_k)g(\widetilde{\nabla}_{v_i}v_j, v_k) \text{ are symmetric in } i, j \text{ and } k.$$

By Proposition 7.9, λ_j are constant along each F, which imply that $v_i[\lambda_j] = 0$. Thus, since λ_i are simple, it holds that $g(\widetilde{\nabla}_{v_j}v_i, v_j) = 0$. Therefore, $g(\widetilde{\nabla}_{v_j}v_j, v_i) = -g(\widetilde{\nabla}_{v_j}v_i, v_j) = 0$. Combining with $g(\widetilde{\nabla}_{v_j}v_j, v_j) = 0$, we get $\widetilde{\nabla}_{v_j}v_j = 0$.

By the definition of the curvature tensor field and $g(\widetilde{\nabla}_{v_i}v_j, v_k) = s^\lambda_{ijk}/(\lambda_j - \lambda_k)$, we obtain

(8.2)

$$\begin{aligned}
&g(\widetilde{R}(v_i, v_j)v_j, v_i) = g(\widetilde{\nabla}_{v_i}\widetilde{\nabla}_{v_j}v_j - \widetilde{\nabla}_{v_j}\widetilde{\nabla}_{v_i}v_j - \widetilde{\nabla}_{[v_i,v_j]}v_j, v_i)\\
&= -v_j[g(\widetilde{\nabla}_{v_i}v_j, v_i)] + g(\widetilde{\nabla}_{v_i}v_j, \widetilde{\nabla}_{v_j}v_i) - g(\widetilde{\nabla}_{\widetilde{\nabla}_{v_i}v_j - \widetilde{\nabla}_{v_j}v_i}v_j, v_i) \quad (\text{by } \widetilde{\nabla}_{v_j}v_j = 0)\\
&= \textstyle\sum_k g(\widetilde{\nabla}_{v_i}v_j, v_k)g(\widetilde{\nabla}_{v_j}v_i, v_k) - \sum_k \Big\{g(\widetilde{\nabla}_{v_i}v_j, v_k) - g(\widetilde{\nabla}_{v_j}v_i, v_k)\Big\}g(\widetilde{\nabla}_{v_k}v_j, v_i) \quad (\text{by } g(\widetilde{\nabla}_{v_i}v_j, v_i) = 0)\\
&= \sum_{k\neq i,j} \frac{s^\lambda_{ijk}}{\lambda_j - \lambda_k}\cdot\frac{s^\lambda_{jik}}{\lambda_i - \lambda_k} - \sum_{k\neq i,j}\left(\frac{s^\lambda_{ijk}}{\lambda_j - \lambda_k} - \frac{s^\lambda_{jik}}{\lambda_i - \lambda_k}\right)\cdot\frac{s^\lambda_{kji}}{\lambda_j - \lambda_i}\\
&= \sum_{k\neq i,j} \frac{2(s^\lambda_{ijk})^2}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}.
\end{aligned}$$

Therefore, the scalar curvature $\widetilde{s}$ of F is expressed as

$$\widetilde{s} = \sum_{i\neq j} g(\widetilde{R}(v_i, v_j)v_j, v_i) = 2\sum_{i<j,\, k\neq i,j} \frac{2(s^{\lambda}_{ijk})^2}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}. \tag{8.3}$$

Note that the terms of the right hand side of (8.3) having the same $s^{\lambda}_{i_1 i_2 i_3}$ $(i_1 < i_2 < i_3)$ are $(i, j, k) = (i_1, i_2, i_3), (i_1, i_3, i_2), (i_2, i_3, i_1)$, and by a simple computation, their sum vanishes as follows:

$$\frac{2(s^{\lambda}_{i_1 i_2 i_3})^2}{(\lambda_{i_1} - \lambda_{i_3})(\lambda_{i_2} - \lambda_{i_3})} + \frac{2(s^{\lambda}_{i_1 i_2 i_3})^2}{(\lambda_{i_1} - \lambda_{i_2})(\lambda_{i_3} - \lambda_{i_2})} + \frac{2(s^{\lambda}_{i_1 i_2 i_3})^2}{(\lambda_{i_2} - \lambda_{i_1})(\lambda_{i_3} - \lambda_{i_1})} = 0. \tag{8.4}$$

Thus, we have the following

Lemma 8.1. *Every F has zero scalar curvature.*

On the other hand, applying Gauss equation to F regarding as a submanifold of E^{n_1+2}, we have, for every $i \neq j$,

$$\begin{aligned} g(\widetilde{R}(v_i, v_j)v_j, v_i) &= \overline{g}(\widetilde{\alpha}(v_i, v_i), \widetilde{\alpha}(v_j, v_j)) - \overline{g}(\widetilde{\alpha}(v_i, v_j), \widetilde{\alpha}(v_i, v_j)) \\ &= \lambda_i\lambda_j + \mu_i\mu_j. \end{aligned} \tag{8.5}$$

Thus, we have

$$\begin{aligned} 0 &= \textstyle\sum_{i\neq j} g(\widetilde{R}(v_i, v_j)v_j, v_i) = \sum_{i<j} (2\lambda_i\lambda_j + 2\mu_i\mu_j) \\ &= (\textstyle\sum_i \lambda_i)^2 - \sum_i {\lambda_i}^2 + (\sum_i \mu_i)^2 - \sum_i {\mu_i}^2 \\ &= (n_1 - 1)\psi(n_1\psi + 3\tau\varphi) + \frac{1}{4}(1 + \varphi^2)(9\tau^2 - 4\Lambda_2). \end{aligned} \tag{8.6}$$

We rewrite (8.6) as follows.

$$4(n_1 - 1)\psi(n_1\psi + 3\tau\varphi) + (1 + \varphi^2)(9\tau^2 - 4\Lambda_2) = 0. \tag{8.7}$$

By substituting $\mu_i = \varphi\lambda_i + \psi$ into (5.1) of Lemma 5.1, we have

$$\begin{aligned} &4({n_1}^2 - 9)\tau\psi^2 + 2\varphi\psi\{(6n_1 - 9)\tau^2 + 4(n_1 - 6)\Lambda_2\} \\ &\qquad + 3\{3\tau^3(1 + \varphi^2) + 8\tau\Lambda_2 - 8\varphi^2\Lambda_3\} \\ &= 0 \end{aligned} \tag{8.8}$$

Together with (8.7) and (8.8), and differentiating twice (8.7) and (8.8), and eliminating Λ_3, Λ_2, and τ, we obtain our main theorem.

Theorem 8.2. *Every biharmonic hypersurface M in the Euclidean space is minimal if we assume that all the principal curvatures are simple, and the frame $\{v_i\}$ of a characteristic submanifold $F = \{\tau = c\}$ for a constant c, is irreducible.*

Proof. In the following, we will proceed to eliminate Λ_3, Λ_2, and τ exactly. By differentiating (8.7) in t, we have

$$9(n_1-1)\tau\varphi\psi^2+2n_1(n_1-1)\psi^3+\frac{1}{2}\psi\{9\tau^2(1+2\varphi^2)+4\Lambda_2(-1+(n_1-3)\varphi^2)\} \tag{8.9}$$
$$+\varphi(1+\varphi^2)(3\tau\Lambda_2-2\Lambda_3)=0.$$

Eliminating Λ_3 from (8.8) and (8.9), we have

$$-9\tau^3(1+\varphi^2)^2-6\tau^2\varphi\psi\{2(n_1-6)+(2n_1-21)\varphi^2\} \tag{8.10}$$
$$-4\tau\psi^2\{{n_1}^2-9+({n_1}^2-27n_1+18)\varphi^2\}$$
$$+24n_1(n_1-1)\varphi\psi^3$$
$$+4\Lambda_2\{3\tau(-2+\varphi^2+3\varphi^4)+2\varphi(-n_1+3+(2n_1-3)\varphi^2)\psi\}=0.$$

Eliminating Λ_2 from (8.7) and (8.10), we have

$$9\tau^3(1+\varphi^2)^2(-7+8\varphi^2) \tag{8.11}$$
$$+6\tau^2\varphi\psi(1+\varphi^2)(-17n_1+33+2(11n_1-3)\varphi^2)$$
$$+4\tau\psi^2\{-7{n_1}^2+6n_1+9-(n_1-9)(5n_1-3)\varphi^2+4n_1(5n_1-3)\varphi^4\}$$
$$+8n_1(n_1-1)\varphi\psi^3(-n_1+6+2n_1\varphi^2)=0.$$

To eliminate τ, differentiate (8.11) in t, and apply Lemma 7.3 to τ', and eliminate Λ_2 by using (8.7). Then, we obtain the following four equations:

By differentiating the first term of (8.11), we have

$$\{9\tau^3(1+\varphi^2)^2(-7+8\varphi^2)\}' \tag{8.12}$$
$$=9\tau^2(1+\varphi^2)(-7+8\varphi^2)$$
$$\times\{6\tau^2\varphi(1+\varphi^2)+\tau\psi(n_1+3+(7n_1-3)\varphi^2)+2n_1(n_1-1)\varphi\psi^2\}$$
$$-54\tau^3\varphi(1+\varphi^2)(-1+4\varphi^2)(\tau(1+\varphi^2)-2\varphi\psi).$$

By differentiating the second term of (8.11), we have

$$\{6\tau^2\varphi\psi(1+\varphi^2)(-17n_1+33+2(11n_1-3)\varphi^2)\}' \tag{8.13}$$
$$=\tau\varphi\psi\{-17n_1+33+2(11n_1-3)\varphi^2\}$$
$$\times\{21\tau^2\varphi(1+\varphi^2)+2\tau\psi(2n_1+9+(14n_1-3)\varphi^2)+8n_1(n_1-1)\varphi\psi^2\}$$
$$-3\tau^2\psi(\tau(1+\varphi^2)-2\varphi\psi)\{-17n_1+33+3(5n_1+27)\varphi^2+10(11n_1-3)\varphi^4\}.$$

By differentiating the third term of (8.11), we have

(8.14)
$$(4\tau\psi^2p)'=4\tau^2\varphi\psi^2p+\frac{8n_1(n_1-1)\varphi\psi^4p}{3(1+\varphi^2)}+4\tau p\left\{2\psi^3+\frac{1}{3}(n_1+3)\psi^3+\frac{2(n_1-1)\varphi^2\psi^3}{1+\varphi^2}\right\}$$
$$-4(5n_1-3)\tau\varphi\psi^2(-n_1+9+8n_1\varphi^2)(\tau(1+\varphi^2)-2\varphi\psi),$$

where $p=-7{n_1}^2+6n_1+9-(n_1-9)(5n_1-3)\varphi^2+4n_1(5n_1-3)\varphi^4$.

And finally differentiating the fourth term of (8.11), we have

$$\{8n_1(n_1-1)\varphi\psi^3(-n_1+6+2n_1\varphi^2)\}' \tag{8.15}$$
$$=-4n_1(n_1-1)\psi^3(-n_1+6+2n_1\varphi^2)(\tau(1+4\varphi^2)-8\varphi\psi)$$

$$- 16{n_1}^2(n_1 - 1)\varphi^2\psi^3(\tau(1+\varphi^2) - 2\varphi\psi).$$

Together (8.12), (8.13), (8.14) and (8.15), by multiplying the denominator, and setting all the terms of the resulting equation in order of τ, we obtain the following equation:

$$\begin{aligned}(8.16)\quad & 324\tau^4\varphi(1+\varphi^2)^3(-3+2\varphi^2)\\
&+ 9\tau^3\psi(1+\varphi^2)^2\{-4(n_1+24) - (257n_1 - 249)\varphi^2 + 4(53n_1+15)\varphi^4\}\\
&+ 6\tau^2\varphi\psi^2(1+\varphi^2)\{-101{n_1}^2 - 159n_1 + 468\\
&\qquad - (265{n_1}^2 - 783n_1 - 90)\varphi^2 + 4(85{n_1}^2 + 33n_1 - 18)\varphi^4\}\\
&+ 4\tau\psi^3\{-4{n_1}^3 - 78{n_1}^2 + 81n_1 + 81\\
&\qquad\qquad - 3(53{n_1}^3 - 69{n_1}^2 - 222n_1 + 126)\varphi^2\\
&\qquad\qquad - 9(3{n_1}^3 - 89{n_1}^2 + 11n_1 + 27)\varphi^4\\
&\qquad\qquad + 4n_1(59{n_1}^2 + 21n_1 - 36)\varphi^6\}\\
&+ 8n_1(n_1-1)\varphi\psi^4\{-(n_1-3)(7n_1+27) - (n_1-15)(5n_1+3)\varphi^2 + 4n_1(5n_1+6)\varphi^4\}\\
&= 0.\end{aligned}$$

The equation (8.11) is of third order in τ, and (8.16) is of fourth order in τ. Therefore, we use Euclid's algorithm to eliminate τ. Namely, dividing (8.16) by (8.11), the remainder, denoted by f_2, is of order two in τ. And dividing (8.11) by f_2, we denote the remainder by f_1, and finally dividing f_2 by f_1, we denote the remainder by f_0, then f_0 does not include τ. Note that the remainder is of the form of rational function r/q (where q, r are polynomials). But, if we multiply by q in advance, we may ignore q, and may use r in the next step. Finally the obtained numerator is of the form which is the multiplication of the following three polynomials:

$$(8.17)\qquad 72n_1(n_1-1)\varphi^3\psi^4(1+\varphi^2)^2,$$

$$\begin{aligned}(8.18)\quad & \{-105(3{n_1}^2 + n_1 + 12) + (1026{n_1}^2 + 875n_1 - 5901)\varphi^2\\
&\quad - 2(567{n_1}^2 + 1264n_1 - 351)\varphi^4 + 8(45{n_1}^2 + 34n_1 - 159)\varphi^6\}^2,\end{aligned}$$

$$\begin{aligned}(8.19)\quad & -7(17n_1 - 33)^2(5{n_1}^4 - 6{n_1}^3 - 13{n_1}^2 + 3n_1 + 27)\\
&\quad - \{100964{n_1}^6 - 269159{n_1}^5 - 362329{n_1}^4 + 1289439{n_1}^3\\
&\qquad + 839475{n_1}^2 - 4210164n_1 + 2755134\}\varphi^2\\
&\quad + 3\{167725{n_1}^6 - 789504{n_1}^5 + 904142{n_1}^4 + 651168{n_1}^3\\
&\qquad - 3766311{n_1}^2 + 5957928n_1 - 3211164\}\varphi^4\\
&\quad - \{825166{n_1}^6 - 5345493{n_1}^5 + 11702488{n_1}^4 + 1018458{n_1}^3\\
&\qquad - 21744558{n_1}^2 + 11059011n_1 + 2628288\}\varphi^6\\
&\quad + \{576422{n_1}^6 - 5693096{n_1}^5 + 13627127{n_1}^4 - 9103320{n_1}^3\\
&\qquad - 22777452{n_1}^2 + 44245224n_1 - 20731545\}\varphi^8\\
&\quad - 12\{13322{n_1}^6 - 199503{n_1}^5 + 758200{n_1}^4 - 367722{n_1}^3\end{aligned}$$

$$
\begin{aligned}
&\quad - 1262466{n_1}^2 + 1720305n_1 - 683640\}\varphi^{10} \\
&- 4\{1094{n_1}^6 + 56460{n_1}^5 - 446779{n_1}^4 + 834336{n_1}^3 \\
&\quad + 731808{n_1}^2 - 2056212n_1 + 843453\}\varphi^{12} \\
&+ 16(7n_1 - 9)^2(2{n_1}^2 - 9)(5{n_1}^2 - 26n_1 - 43)\varphi^{14}.
\end{aligned}
$$

The multiplication f_0 of three equations (8.17), (8.18) and (8.19) should be zero identically. This implies that one of the factors of f_0 should be zero identically. Note that each factor of f_0 is a polynomial only in φ or ψ, and its coefficient of the highest term is non-zero for every natural number n_1. Therefore, if f_0 vanishes identically, then φ or ψ must be a constant. Thus, by Lemmas 7.6 and 7.8, we obtain that $\tau \equiv 0$.

We have done. □

Since Euclid's algorithm for polynomials is a tedious calculation, we will give examples of calculation using a computer in Appendix.

9. Appendix: Euclid's algorithm using a computer

We give two examples of calculations using formula manipulation systems, mathematica[1] and Maple[2]. Both are commercial softwares, but free softwares probably have similar functions.

With mathematica, we calculate as follows.

```
f3 = ⟨the left hand side of (8.11)⟩;
f4 = ⟨the left hand side of (8.16)⟩;
f2 = Numerator[Factor[PolynomialRemainder[f4, f3, tau]]];
f1 = Numerator[Factor[PolynomialRemainder[f3, f2, tau]]];
f0 = Numerator[Factor[PolynomialRemainder[f2, f1, tau]]]
```

For Maple, we prepare a function which calculates remainder of multivariable polynomials.

```
with(PolynomialTools);
polynomialremainder := proc(poly1, poly2, var)
  local cf1, cf2, deg11, deg12, top2, ratio, i, j, poly;
  cf1 := CoefficientList(poly1, var);
  deg11 := numelems(cf1);
  cf2 := CoefficientList(poly2, var);
  deg12 := numelems(cf2);
  top2 := cf2[deg12];
  for j from deg11 by -1 to deg12 do
    ratio := cf1[j]/top2;
    for i from 0 to deg12-1 do
      cf1[j-i] := cf1[j-i]-cf2[deg12-i]*ratio
    end do
```

[1]Mathematica is a registered trademark of Wolfram Research Inc.

[2]Maple is a registered trademark of Waterloo Maple Inc.

```
  end do;
  poly := 0;
  for i to deg11 do
    poly := poly+cf1[i]*var^(i-1)
  end do;
  return poly
end proc;
```

Using this, we calculate as follows.

```
f3 := ⟨the left hand side of (8.11)⟩;
f4 := ⟨the left hand side of (8.16)⟩;
f2 := numer(factor(polynomialremainder(f4, f3, tau)));
f1 := numer(factor(polynomialremainder(f3, f2, tau)));
f0 := numer(factor(polynomialremainder(f2, f1, tau)));
```

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