COMPARISON THEOREMS IN PSEUDO-HERMITIAN GEOMETRY AND APPLICATIONS

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Abstract

In this paper, we study the theory of geodesics with respect to the Tanaka-Webster connection in a pseudo-Hermitian manifold, aiming to generalize some comparison results in Riemannian geometry to the case of pseudo-Hermitian geometry. Some Hopf-Rinow type, Cartan-Hadamard type and Bonnet-Myers type results are established.

Introduction

A CR structure on an $(2m + 1)$-dimensional manifold M^{2m+1} is an $2m$-dimensional distribution $H(M)$ endowed with a formally integrable complex structure J. The geometry of CR manifolds goes back to Poincaré and received a great attention in the works of Cartan, Tanaka, Chern-Moser, and others (cf. [10]). There have been, over the last twenty or thirty years, many studies in geometry and analysis on CR manifolds (cf. [9], [2], [8], [15], [4], [7]).

A pseudo-Hermitian manifold, which is an odd-dimensional analogue of Hermitian manifolds, is a CR manifold M endowed with a pseudo-Hermitian structure θ. The pseudo-Hermitian structure θ determines uniquely a global nowhere zero vector field ξ and it, combining with the complex structure J, induces a Riemannian metric g_θ on M too. It turns out that the Levi-Civita connection ∇^θ of g_θ is not convenient for investigating the pseudo-Hermitian manifold, because it is not compatible with the CR structure. From [14], [16], we know that there is a unique canonical linear connection ∇ (the Tanaka-Webster connection), which is compatible with both the metric g_θ and the CR structure (see Proposition 2.1). This connection always has nonvanishing torsion $T_\nabla(\cdot, \cdot)$, whose partial component $T_\nabla(\xi, \cdot)$ is an important pseudo-Hermitian invariant, called the pseudo-Hermitian torsion. A Sasakian manifold, which is an odd dimensional analogue of Kähler manifolds, is a pseudo-Hermitian manifold with vanishing pseudo-Hermitian torsion. Besides its similarity with Kähler geometry, interest in Sasakian manifolds has been from theoretical physics with AdS/CFT correspondence, which provides a duality between field theories and string theories.

In this paper, we investigate the theory of ∇-geodesics on the pseudo-Hermitian manifold aiming to generalize some comparison results in Riemannian geometry, such as Hopf-Rinow type, Cartan-Hadamard-type and Bonnet-Myers-type theorems, etc., to the case of pseudo-Hermitian geometry. For this purpose, we shall study the exponential maps, conjugate points and Jacobi fields with respect to the Tanaka-Webster connection ∇. The organization of this

paper is the following: In Section 1, we recall some basic notions and properties of pseudo-Hermitian manifolds. Section 2 is devoted to the exponential map $\exp^\nabla$ and Hopf-Rinow type theorem. We find that the Gauss lemma for ∇-geodesics is no longer true due to the torsion of ∇. As a result, a short ∇-geodesic is not necessarily a length-minimizing curve. This causes some trouble for establishing Hopf-Rinow type theorem. In order to study the metric properties of (M, ∇), a natural distance δ between any two points is introduced by taking infimum of the lengths of all broken ∇-geodesics joining the two points. In terms of the distance δ, we establish a partial Hopf-Rinow type result for pseudo-Hermitian manifolds, which states that if (M, δ) is complete, then (M, ∇) is complete. In Section 3, we investigate the Jacobi fields along a ∇-geodesic. As in the case for a Riemannian manifold, for any two vector $v, w \in T_pM$, $(d\exp^\nabla_p)_{tv}(tw)$ is a Jacobi field along the geodesic $\gamma(t) = exp^\nabla_p(tv)$. We compute the Taylor expansion of $\|(d\exp^\nabla_p)_{tv}(tw)\|^2$ to show that the behavior of ∇-geodesics $exp^\nabla_p((v + sw)t)$ is affected by both the curvature and the torsion of (M, ∇). Next, by generalizing a result in [1], the decomposition of a Jacobi field $V(t)$ along any geodesic $\gamma(t)$ is given with respect to $\gamma'(t)$ and its complementary space. Finally in this section, we give explicitly the Jacobi fields along geodesics in the Heisenberg group. In Section 4, we study Cartan-Hadamard type result for pseudo-Hermitian manifolds. The main result in this section asserts that if M^{2m+1} is a complete Sasakian manifold with non-positive horizontal curvature, then for any $p \in M$, $exp^\nabla_p : T_pM \to N$ is a covering map. Hence the covering space of M is diffeomorphic to R^{2m+1}. Finally, Section 5 is devoted to establishing a index comparison theorem. As applications, we get a Bonnet-Myers type result concerning conjugate points along geodesics in Sasakian manifolds with either positive horizontal sectional curvature or positive Ricci curvature. We should mention that various geodesics on pseudo-Hermitian manifolds have been investigated by several authors yet from somewhat different viewpoints. It is known that the general theory of sub-Riemannian geodesics was established in [13], which is a Hamiltonian description about geodesics in cotangent bundles. Since a pseudo-Hermitian manifold may be regarded as a special sub-Riemannian manifold, one may apply the sub-Riemannian geodesic theory to pseudo-Hermitian manifolds. Actually Barletta and Dragomir [1] re-expressed the sub-Riemannian geodesic equation via the Tanaka-Webster connection ∇ and studied the relationship between the sub-Riemannian geodesics and the ∇-geodesics on a pseudo-Hermitian manifold. Besides, they also established some Cartan-Hadamard type and Bonnet-Myers type results for conjugates along horizontal ∇-geodesics. In fact, the main results in [1] involve horizontal geodesics. By getting rid of the horizontal restriction for geodesics, we are able to generalize some results in [1] to ∇-geodesics with initial tangent vectors in any directions. We shall find that although the connection ∇ shares some common notions and properties with general linear connections, it displays some special features of a pseudo-Hermitian manifold too. Furthermore, some interesting geometric properties of pseudo-Hermitian manifolds are invisible from the Levi-Civita connection ∇^θ, but visible from the Tanaka-Webster connection ∇.

1. Preliminaries

In this section, we collect some facts and notations concerning pseudohermitian structures on CR manifolds (cf. [9], [2] for details).

DEFINITION 1.1. Let M^{2m+1} be a real $(2m + 1)$-dimensional orientable C^∞ manifold. A CR structure on M is a complex rank m subbundle $H^{1,0}M$ of $TM \otimes C$ satisfying
(i) $H^{1,0}M \cap H^{0,1}M = \{0\}$ ($H^{0,1}M = \overline{H^{1,0}M}$);
(ii) $[\Gamma(H^{1,0}M), \Gamma(H^{1,0}M)] \subseteq \Gamma(H^{1,0}M)$.
The pair $(M, H^{1,0}M)$ is called a CR manifold.

The complex subbundle $H^{1,0}M$ corresponds to a real subbundle of TM:

$$(1.1) \qquad H(M) = Re\{H^{1,0}M \oplus H^{0,1}M\}$$

which is endowed with a natural complex structure J as follows

$$(1.2) \qquad J(V + \overline{V}) = i(V - \overline{V})$$

for any $V \in H^{1,0}M$. Equivalently, the CR structure may be described by the pair $(H(M), J)$. A C^∞ map $f : (M, H(M), J) \to (N, H(N), \widetilde{J})$ between two CR manifolds is called a CR map if $df(H(M)) \subset H(N)$ and $(df \circ J)|_{H(M)} = (\widetilde{J} \circ df)|_{H(M)}$.

Let E be the conormal bundle of $H(M)$ in T^*M, whose fiber at each point $x \in M$ is given by

$$(1.3) \qquad E_x = \{\omega \in T_x^*M : \ker \omega \supseteq H_x(M)\}.$$

Since both TM and $H(M)$ are orientable, $E \simeq T(M)/H(M)$ is an orientable line bundle. It is known that any orientable real line bundle over a connected manifold is trivial. Therefore E admits globally defined nowhere vanishing sections.

DEFINITION 1.2. A globally defined nowhere vanishing section $\theta \in \Gamma(E)$ is called a pseudo-Hermitian structure on M. The Levi-form L_θ associated with a pseudo-Hermitian structure θ is defined by

$$(1.4) \qquad L_\theta(X, Y) = d\theta(X, JY)$$

for any $X, Y \in H(M)$. If L_θ is positive definite for some θ, then $(M, H(M), J)$ is said to be strictly pseudoconvex.

Henceforth we assume that $(M, H(M), J)$ is a strictly pseudoconvex CR manifold endowed with a pseudo-Hermitian structure θ such that L_θ is positive definite. The quadruple $(M, H(M), J, \theta)$ is called a pseudo-Hermitian manifold, which is sometimes denoted simply by (M, θ).

For a pseudo-Hermitian manifold $(M, H(M), J, \theta)$, there exists a unique globally defined nowhere zero vector field ξ such that (cf. [9])

$$(1.5) \qquad \theta(\xi) = 1, \quad d\theta(\xi, \cdot) = 0.$$

This vector field ξ is referred to as the Reeb vector field, which is transverse to $H(M)$. Consequently we have the following decomposition

$$(1.6) \qquad TM = L_\xi \oplus H(M),$$

where L_ξ is the trivial line bundle generated by ξ. The subbundles L_ξ and $H(M)$ will be called the vertical and horizontal distributions respectively. Correspondingly, a vector $V \in TM$ is

called vertical (resp. horizontal) if $V \in L_\xi$ (resp. $H(M)$). For convenience, we extend J to a $(1, 1)$-tensor field on M by requiring that

$$(1.7) \qquad\qquad J\xi = 0.$$

Let $\pi_H : TM \to H(M)$ be the natural projection morphism. Set $G_\theta = \pi_H^* L_\theta$, that is,

$$(1.8) \qquad\qquad G_\theta(X, Y) = L_\theta(\pi_H X, \pi_H Y)$$

for any $X, Y \in TM$. Then one may introduce a Riemannian metric, called the Webster metric, as follows

$$(1.9) \qquad\qquad g_\theta = \theta \otimes \theta + G_\theta$$

which is sometimes denoted by $\langle \cdot, \cdot \rangle$ for simplicity. Clearly

$$(1.10) \qquad\qquad \theta(X) = \langle \xi, X \rangle, \ d\theta(X, Y) = \langle JX, Y \rangle.$$

In terms of (1.9) and (1.10), we find that (1.6) is actually an orthogonal decomposition. In addition, $\theta \wedge (d\theta)^m$ is, up to a constant, the volume form of (M, g_θ).

On a pseudo-Hermitian manifold, we have the following canonical linear connection which preserves both the CR and the metric structures.

Proposition 1.1 ([14], [16]). *Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold. Then there exists a unique linear connection ∇ such that*
(i) $\nabla_X \Gamma(H(M)) \subset \Gamma(H(M))$ for any $X \in \Gamma(TM)$;
(ii) $\nabla g_\theta = 0$, $\nabla J = 0$ (hence $\nabla \xi = \nabla \theta = 0$);
(iii) The torsion T_∇ of ∇ is pure, that is, for any $X, Y \in H(M)$, $T_\nabla(X, Y) = d\theta(X, Y)\xi$ and $T_\nabla(\xi, JX) + J T_\nabla(\xi, X) = 0$.

The connection ∇ in Proposition 1.1 is called the Tanaka-Webster connection. Note that the torsion of the Tanaka-Webster connection is always non-zero. The pseudo-Hermitian torsion, denoted by τ, is the TM-valued 1-form defined by $\tau(X) = T_\nabla(\xi, X)$. The anti-symmetry of T_∇ implies that

$$(1.11) \qquad\qquad \tau(\xi) = 0.$$

Using (iii) of Proposition 1.1 and the definition of τ, the total torsion of the Tanaka-Webster connection may be expressed as

$$(1.12) \qquad\qquad T_\nabla(X, Y) = 2(\theta \wedge \tau)(X, Y) + 2d\theta(X, Y)\xi$$

for any $X, Y \in TM$. Set

$$(1.13) \qquad\qquad A(X, Y) = g_\theta(\tau X, Y)$$

for any $X, Y \in TM$. Then the properties of ∇ in Proposition 1.1 also imply that $\tau(H^{1,0}(M)) \subset H^{0,1}(M)$ and A is a trace-free symmetric tensor field.

Lemma 1.2 (cf. [9]). *The Levi-Civita connection ∇^θ of (M, g_θ) is related to the Tanaka-Webster connection by*

$$\nabla^\theta = \nabla - (d\theta + A)\xi + \tau \otimes \theta + 2\theta \odot J$$

where $(\theta \odot J)(X, Y) = \frac{1}{2}(\theta(X)JY + \theta(Y)JX)$ for any $X, Y \in TM$.

For a pseudo-Hermitian manifold (M, θ), the curvature tensor R with respect to its Tanaka-Webster connection is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

for any $X, Y, Z \in \Gamma(TM)$. Clearly R satisfies

$$(1.14) \qquad \langle R(X, Y)Z, W \rangle = -\langle R(Y, X)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$$

where the second equality is because of $\nabla g_\theta = 0$. However, the symmetric property $\langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$ is no longer true for a general pseudo-Hermitian manifold due to the failure of the first Bianchi identity.

For a horizontal 2-plane $P = span_R\{X, Y\} \subset H(M)$, the horizontal sectional curvature of P is defined by

$$(1.15) \qquad K^H(P) = \frac{\langle R(X \wedge Y), X \wedge Y \rangle}{\langle X \wedge Y, X, \wedge Y \rangle}.$$

We define the Ricci tensor of ∇ by

$$(1.16) \qquad \mathrm{Ric}(Y, Z) = \mathrm{trace}\{X \mapsto R(X, Z)Y\}$$

for any $Y, Z \in TM$.

DEFINITION 1.3. A pseudo-Hermitian manifold $(M, H(M), J, \theta)$ is called a Sasakian manifold if its pseudo-Hermitian torsion τ is zero.

From [9], we know that if (M, θ) is a Sasakian manifold, then

$$(1.17) \qquad \langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle.$$

Consequently, if one of X, Y, Z and W is vertical, then

$$(1.18) \qquad \langle R(X, Y)Z, W \rangle = 0.$$

We denote the length of a continuous piecewise smooth curve $c : [a, b] \to (M, g_\theta)$ by $L[c]$.

Lemma 1.3 (see also §7 in [1]). *Let $\alpha : [a, b] \times (-\varepsilon, \varepsilon) \to (M, \theta)$ be a smooth map. Set $T = d\alpha(\frac{\partial}{\partial t})$, $V = d\alpha(\frac{\partial}{\partial s})$. Write $\alpha(t, s)$ as $\alpha_s(t)$ ($a \le t \le b$, $-\varepsilon < s < \varepsilon$). Assume that $\alpha_0(t)$ is parameterized by arc length. Then*

$$\frac{d}{ds}L(\alpha_s)|_{s=0} = \langle V, T \rangle|_a^b - \int_a^b \{\langle V, \nabla_T T \rangle - \langle T_\nabla(V, T), T \rangle\}dt$$

Proof. The usual first variation formula of arc length gives (cf. [5])

$$(1.19) \qquad \frac{d}{ds}L(\alpha_s)|_{s=0} = \langle V, T \rangle|_a^b - \int_a^b \langle V, \nabla_T^\theta T \rangle dt.$$

The lemma follows immediately from Lemma 1.2, (1.12) and (1.19). □

2. Exponential map and Hopf-Rinow Type Results

Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold with the Tanaka-Webster connection ∇. A C^1 curve $\gamma : [0, l] \to M$ is called a ∇-geodesic if $\nabla_{\gamma'} \gamma' = 0$ on $[0, l]$. Since the linear connection ∇ is of class C^∞, a C^1-geodesic with respect to ∇ is automatically of class C^∞ (cf. [11]). A smooth curve $\gamma : [0, l] \to M$ is referred to as a slant curve if the angle between $\gamma'(t)$ and $\xi_{\gamma(t)}$ is constant along γ. In particular, if $\gamma'(t)$ is perpendicular (resp. parallel) to $\xi_{\gamma(t)}$ for each t, then γ is called a horizontal (resp. vertical) curve. Since $\nabla g_\theta = 0$ and $\nabla \xi = 0$, it is clear that any ∇-geodesic $\gamma(t)$ must be a slant curve. In particular, if the initial tangent vector of the ∇-geodesic is horizontal (resp. vertical), then γ should be horizontal (resp. vertical).

Given a point $p \in M$ and a vector $v \in T_p M$, the ODE theory implies that there exists a unique ∇-geodesic $\gamma_v(t)$ satisfying $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Since a parameterization which makes γ into a geodesic, if any, is determined up to an affine transformation of t, the parameter t is called an affine parameter. As usual, the exponential map $\exp_p^\nabla : T_p M \to M$ is defined by

$$\exp_p^\nabla(v) = \gamma_v(1)$$

for all $v \in T_p M$ such that 1 is in the domain of γ_v. Since ∇ preserves the metric g_θ, we find that $|\gamma_v'(t)| = |v|$ for each t and thus $L(\gamma_v; [0, 1]) = |v|$, where $L(\cdot)$ denotes the length of the curve. The linear connection ∇ of M is said to be complete if every ∇-geodesic can be extended to a geodesic $\gamma(t)$ defined for $-\infty < t < \infty$, where t is an affine parameter. Hence, if (M, ∇) is complete, then $\exp^\nabla$ is defined on all of TM and vice versa.

From Proposition 8.2 in [11], we know that there is a neighborhood D_p of each point p (more precisely, the zero vector at p) in $T_p M$ which is mapped diffeomorphically onto a neighborhood U_p of p in M by the exponential map. Choosing a linear frame $u = \{X_1, ..., X_n\}$ at p, the diffeomorphism $\exp_p^\nabla : D_p \to U_p$ defines a local coordinate system in U_p in a natural manner. This local coordinate system is called a normal coordinate system at p. The following result holds for any linear connection on a manifold.

Proposition 2.1 (cf. [11]). *Let $x^1, ..., x^n$ be a normal coordinate system with origin p. Let $U(p; \rho)$ be the neighborhood of p defined by $\sum (x^i)^2 < \rho^2$. Then there is a positive number a such that if $0 < \rho < a$, then*
(1) $U(p; \rho)$ is convex in the sense that any two points of $U(p; \rho)$ can be joined by a ∇-geodesic which lies in $U(p; \rho)$;
(2) Each point of $U(p; \rho)$ has a normal coordinate neighborhood containing $U(p; \rho)$.

For each $v \in T_p M$, the tangent space $T_v(T_p M)$ can be identified with $T_p M$ in a natural way. Due to the Webster metric, each tangent space $T_p M$ comes equipped with an inner product. Therefore $T_v(T_p M)$ inherits an inner product from $(g_\theta)_p(\cdot, \cdot)$.

Proposition 2.2. *Let $c(s)$ be a curve in $T_p M$ such that every point of c is at the same distance from the origin of $T_p M$. Let $\rho_s(t) : [0, 1] \to T_p M$ be the ray from 0 to $c(s)$ in $T_p M$. Set $\alpha(t, s) = \exp(\rho_s(t))$ and $\gamma(t) = \alpha(t, 0)$. Assume that $\exp^\nabla$ is defined through ρ_s. Then*

$$(2.1) \qquad \langle V(t), \gamma'(t) \rangle = \int_0^t [2\langle J\gamma', V \rangle \theta(\gamma') + \theta(\gamma')\langle \tau(\gamma'), V \rangle - \theta(V)\langle \tau(\gamma'), \gamma' \rangle]$$

where $V(t) = d\alpha(\frac{\partial}{\partial s})_{(t,0)}$ is the variation vector field of α along the ∇-geodesic γ.

Proof. From the definition of $\exp^\nabla$ and α, we know that the lengths of the curve $t \mapsto \alpha(t, s)$ are independent of s. Note that $V(0) = 0$. Then Lemma 1.3 and (1.12) yield

$$0 = \frac{dL[\exp(\rho_s)|_{[0,t]}]}{ds}|_{s=0}$$

$$= \langle V(t), \gamma'(t)\rangle - \int_0^t [2\theta(\gamma')\langle J\gamma', V\rangle + \theta(\gamma')\langle \tau(V), \gamma'\rangle - \theta(V)\langle \tau(\gamma'), \gamma'\rangle]dt.$$

Consequently

$$\langle V(t), \gamma'(t)\rangle = \int_0^t [2\langle J\gamma', V\rangle\theta(\gamma') + \theta(\gamma')\langle \tau(\gamma'), V\rangle - \theta(V)\langle \tau(\gamma'), \gamma'\rangle].$$

$\square$

REMARK 2.1. For $v \in T_pM$, we assume that $w \in T_v(T_pM)$ is perpendicular to v when w is also regarded as a vector in T_pM. Clearly there exists a curve $c(s)$ in T_pM such that $c(0) = v$, $c'(0) = w$ and such that every point of c is at the same distance from the origin of T_pM. In this circumstance, (2.1) becomes

$$(2.2) \qquad \langle (d\exp^\nabla)_{tv}(tw), (d\exp^\nabla)_{tv}(v)\rangle$$

$$= \int_0^t [2\langle J\gamma', V\rangle\theta(\gamma') + \theta(\gamma')\langle \tau(\gamma'), V\rangle - \theta(V)\langle \tau(\gamma'), \gamma'\rangle].$$

In particular, if M is Sasakian, then

$$(2.3) \qquad \langle (d\exp^\nabla)_{tv}(tw), (d\exp^\nabla)_{tv}(v)\rangle = 2\int_0^t \langle J\gamma', V\rangle\theta(\gamma').$$

From either (2.2) or (2.3), we see that the Gauss lemma is no longer true for $\exp^\nabla$ due to the torsion. However, it still holds for some special geodesics.

Corollary 2.3. *Let $\rho(t) = tv$ $(t \in [0, 1])$ be a ray through the origin in T_pM and let $w \in T_v(T_pM)$ be a vector perpendicular to v. Assume that $\exp^\nabla$ is defined along ρ. If either v is vertical or M is Sasakian and v is horizontal, then we have*

$$\langle (d\exp^\nabla)_{tv}(tw), (d\exp^\nabla)_{tv}(v)\rangle = 0$$

for each $t \in [0, 1]$.

Proof. First, assume that v is vertical. Then $\gamma'(t)$ is vertical for each t. In terms of (1.7) and (1.11), we deduce from (2.2) that $\langle (d\exp^\nabla)_{tv}(tw), (d\exp^\nabla)_{tv}(v)\rangle = 0$.

Next, assume that $v \in H_p(M)$. Then $\gamma(t) = \exp^\nabla \rho(t)$ is a horizontal geodesic. Furthermore, if M is Sasakian, we get from immediately (2.3) the required result. $\square$

In order to investigate the metric properties of the ∇-geodesics, we shall introduce a distance function determined by the connection ∇ and g_θ. A continuous curve $c : [a, b] \to M$ is called a broken ∇-geodesic if there exist $a = a_1 < a_2 < \cdots < a_n = b$ such that $c : [a_i, a_{i+1}] \to M$ is a ∇-geodesic for $i = 1, ..., n - 1$. Clearly a broken ∇-geodesic is piecewise C^∞-differentiable. For any $p, q \in M$, let $\Gamma(p, q)$ denote all broken ∇-geodesics

joining p and q. We define the distance between p and q, $\delta(p, q)$, by

$$(2.4) \qquad\qquad \delta(p, q) = \inf_{\gamma \in \Gamma(p,q)} L(\gamma).$$

For any continuous curve connecting p and q, it can be covered by a finite number of normal coordinate neighborhoods. Thus there always exist broken ∇-geodesics joining the two points, so the distance δ is finite. It is easy to verify that (M, δ) is a metric space. Let d denote the usual Riemannian distance function determined by g_θ. Clearly $d(p, q) \leq \delta(p, q)$ for any $p, q \in M$. Hence if (M, d) is complete, then (M, δ) is complete too.

Proposition 2.4. *The distance function δ defines the same topology as the manifold topology of M.*

Proof. It is known that the Riemannian distance function d defines the same topology as the original topology of M (cf. [11], [3]). Therefore we only need to verify that d and δ define the same metric space topology.

Suppose first that U is an open subset of the metric space topology defined by d. Since $d(p, q) \leq \delta(p, q)$ for any $p, q \in M$, we find that U must be an open subset of the metric space topology defined by δ.

Assume now that W is an open subset defined by δ, that is, for any point $p \in W$, there exits $\varepsilon > 0$ such that $B_\delta(p; \varepsilon) \subset W$, where $B_\delta(p; \varepsilon) = \{q : \delta(p, q) < \varepsilon\}$. Set

$$D_p(\varepsilon) = \{v \in T_p M : \|v\| < \varepsilon\}, \quad B^\nabla(p; \varepsilon) = \exp_p^\nabla(D_p(\varepsilon)).$$

Let $\exp^{\nabla^\theta}$ denote the Riemannian exponential map. For sufficiently small ε, we know that $\exp_p^{\nabla^\theta}(D_p(\varepsilon)) = B_d(p; \varepsilon)$ (cf. [11], [3]). Moreover, both $\exp_p^\nabla : D_p(\varepsilon) \to B^\nabla(p; \varepsilon)$ and $\exp_p^{\nabla^\theta} : D_p(\varepsilon) \to B_d(p; \varepsilon)$ are diffeomorphisms. Consequently $B^\nabla(p; \varepsilon)$ is an open subset in (M, d). Note that if γ is a ∇-geodesic joining p to a point q in the normal coordinate neighborhood $B^\nabla(p; \varepsilon)$, then $\delta(p, q) \leq L(\gamma) < \varepsilon$. This implies that $B^\nabla(p; \varepsilon) \subset B_\delta(p; \varepsilon)$, and thus $B^\nabla(p; \varepsilon) \subset W$. We then conclude that W is also an open subset defined by d. □

Theorem 2.5. *Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold. If (M, δ) is complete, then (M, ∇) is complete, that is, $\exp^\nabla$ is defined on all of TM.*

Proof. Given $p \in M$ and $0 \neq v \in T_p M$, we assume that γ is a ∇-geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Suppose $[0, t_0)$ is the largest open interval for which such a γ exists. Note that the parameter t must be proportional to arc length. Hence, if t_0 is finite and $t_i \nearrow t_0$, then $\delta(\gamma(t_i), \gamma(t_j)) \leq L(\gamma|_{[t_i, t_j]}) = c|t_j - t_i| \to 0$ as $i, j \to \infty$ $(i < j)$ for some positive constant c. Consequently $\{\gamma(t_i)\}$ is a Cauchy sequence with some limit q in (M, δ). Define $\gamma(t_0) = q$. Let $U(q; \rho)$ be a normal coordinate system as in Proposition 2.1. For sufficiently large i, $\gamma(t_i) \in U(q; \rho)$. Let $\sigma : [0, r_0) \to M$ be a ∇-geodesic with $\sigma(0) = \gamma(t_i)$ and $\sigma'(0) = \gamma'(t_i)$ and let $[0, r_0)$ be the largest open interval for which $\sigma(t)$ exists. According to Proposition 2.1, $\gamma(t_i)$ has a normal coordinate neighborhood containing $U(q; \rho)$. Thus $r_0 > t_0 - t_i$ and $\gamma(t_0) \in \sigma$. Therefore $\gamma \cup \sigma$ is a smooth ∇-geodesic, and γ extends past t_0, which is a contradiction. □

The Hopf-Rinow theorem in Riemannian geometry tells us that the completeness of (M, ∇^θ) is equivalent to the completeness of (M, d). Since the completeness of (M, d) yields

the completeness of (M, δ), we have

Corollary 2.6. *Let $(M, H(M), J, \theta)$ be a pseudo-Hermitian manifold. If (M, ∇^θ) is complete, then (M, ∇) is complete.*

3. Jacobi fields on pseudo-hermitian manifolds

From now on, we always assume that (M, ∇) is a complete pseudo-hermitian manifold of dimension $2m + 1$. Let us consider a 1-parameter family of ∇-geodesics given by a map $\alpha(t, s) : [a, b] \times (-\varepsilon, \varepsilon) \to M$ such that for each fixed s, $\alpha(t, s)$ is a ∇-geodesic. Set $T = d\alpha(\frac{\partial}{\partial t})$ and $V = d\alpha(\frac{\partial}{\partial s})$. Since $[\partial/\partial t, \partial/\partial s] = 0$, we have

$$(3.1) \qquad \nabla_T V - \nabla_V T = T_\nabla(T, V).$$

Therefore

$$(3.2) \qquad \nabla_T \nabla_T V = \nabla_T \nabla_V T + \nabla_T(T_\nabla(T, V)).$$

By the definition of curvature tensor and the geodesic equation $\nabla_T T = 0$, it follows from (3.2) that (see also Theorem I.4.1 in [3])

$$(3.3) \qquad \nabla_T \nabla_T V = R(T, V)T + \nabla_T(T_\nabla(T, V)).$$

The equation above is called the Jacobi equation. A vector field V satisfying the equation (3.3) is called a Jacobi field along the geodesic γ. For example, it is easy to verify that $(a + bt)\gamma'$ is a Jacobi field for any $a, b \in R$. However, a general solution of the Jacobi equation can not be given so explicitly and should depend on both the curvature and the torsion tensors. Since (3.3) is an ODE system of second order, a Jacobi field V is determined uniquely by $V(0)$ and $V'(0)$. Let J_γ denote the real linear space of all Jacobi fields along γ in (M, ∇). Then $\dim_R J_\gamma = 4m + 2$. We have shown that the variation field of a 1-parameter family of geodesics is a Jacobi field. Conversely, if V is a Jacobi field, then V comes from a variation of geodesics too. In fact, let $c(s)$ be a curve such that $c'(0) = V(0)$, and let $\gamma'(0)$, $V(0)$ and $T_\nabla(\gamma'(0), V(0))$ be extended respectively to parallel fields $\gamma'(0)_s$, $V'(0)_s$ and $T_\nabla(\gamma'(0), V(0))_s$ along $c(s)$. Then the variation field of

$$\alpha(t, s) := \exp_{c(s)}^{\nabla}\{t[\gamma'(0)_s + sV'(0)_s - sT_\nabla(\gamma'(0), V(0))_s]\}$$

is a Jacobi field $\widetilde{V}(t) = d\alpha(\frac{\partial}{\partial s})|_{s=0}$. Clearly $\widetilde{V}(0) = V(0)$. Using (3.1), we compute

$$\begin{aligned}
\widetilde{V}'(0) &= \{\nabla_{\frac{\partial}{\partial t}} d\alpha(\frac{\partial}{\partial s})|_{s=0}\}|_{t=0} \\
&= \nabla_{\frac{\partial}{\partial s}}\{[\gamma'(0)_s + sV'(0)_s - sT_\nabla(\gamma'(0), V(0))_s]\}|_{s=0} + T_\nabla(\gamma'(0), V(0)) \\
&= V'(0).
\end{aligned}$$

Since $\widetilde{V}(t)$ and $V(t)$ have the same initial conditions, we conclude that $\widetilde{V}(t) = V(t)$. Consequently a Jacobi field V with $V(0) = 0$ and $V'(0) = w$ may be given by

$$V(t) = \frac{\partial}{\partial s} \exp_p^{\nabla}(t(v + sw))|_{s=0} = (d\exp_p^{\nabla})_{tv}(tw).$$

If we put $v = w$, then $V(t) = t\gamma'(t)$, which implies

$$|(d \exp_p^\nabla)_v(v)| = |v|$$

or in other words, that $\exp_p^\nabla : T_pM \to M$ is an isometry in the radial direction.

We would like to obtain information on $\|(d \exp_p^\nabla)_{tv}(tw)\|^2$ by calculating its Taylor expansion, where $v, w \in T_pM$ are two unit vectors with $\langle v, w \rangle = 0$. Let $\gamma(t) = \exp_p^\nabla(tv)$ and $V(t) = (d \exp_p^\nabla)_{tv}(tw)$. From the above discussion, we know that V is a Jacobi field along γ satisfying $V(0) = 0$, $V'(0) = w$. Then

$$\langle V, V \rangle|_{t=0} = 0, \quad \langle V, V \rangle' = 2\langle V, V' \rangle|_{t=0} = 0, \quad \langle V, V \rangle''|_{t=0} = 2\langle V', V' \rangle|_{t=0} + 2\langle V'', V \rangle|_{t=0} = 2.$$

Note that $V''|_{t=0} = R(v, V(0))v + (T_\nabla(\gamma', V))'|_{t=0} = T_\nabla(v, w)$, so

$$\langle V, V \rangle'''|_{t=0} = 6\langle V'', V' \rangle|_{t=0} + 2\langle V''', V \rangle|_{t=0} = 6\langle T_\nabla(v, w), w \rangle.$$

Also

$$V'''|_{t=0} = (\nabla_{\gamma'}R)(\gamma', V)\gamma'|_{t=0} + R(\gamma', V')\gamma'|_{t=0} + [T_\nabla(\gamma', V)]''|_{t=0}$$
$$= R(v, w)v + [T_\nabla(\gamma', V)]''|_{t=0}.$$

Then

$$\langle V, V \rangle'''' = 8\langle V''', V' \rangle|_{t=0} + 6\langle V'', V'' \rangle|_{t=0} + 2\langle V'''', V \rangle|_{t=0}$$
$$= 8\langle R(v, w)v + [T_\nabla(\gamma', V)]''_{t=0}, w \rangle + 6\langle T_\nabla(v, w), T_\nabla(v, w) \rangle.$$

Therefore

(3.4) $\|d \exp^\nabla(tw)\|^2 = t^2 + \langle T_\nabla(v, w), w \rangle t^3$

$$+ \frac{1}{4!}\{8\langle R(v, w)v + [T_\nabla(\gamma', V)]''_{t=0}, w \rangle + 6\langle T_\nabla(v, w), T_\nabla(v, w) \rangle\}t^4 + O(t^5).$$

In particular, if (M, θ) is Sasakian, then (3.4) and (2.12) imply that

(3.5) $\|d \exp^\nabla(tw)\|^2 = t^2 + 2\langle Jv, w \rangle\langle \xi, w \rangle t^3 - \frac{1}{3}\langle R(w, v)v, w \rangle t^4$

$$+ \langle Jv, w \rangle^2 t^4 + O(t^5).$$

Set $\rho_s(t) = (v + sw)t$. We discover from either (3.4) or (3.5) that the behavior of geodesics $\exp_p(\rho_s)$ is affected by both the curvature and the torsion of M. Let us check some special cases of (3.5) on a Sasakian manifold. For example, if v is vertical, then $\exp^\nabla$ nearly preservers the "width" between the ray ρ_0 and ρ_s with the error term $O(t^5)$. Next, if v w are both horizontal and $\langle R(w, v)v, w \rangle$ is negative, then the geodesics locally diverge when compared to the rays ρ_s. Finally, if v, w are both horizontal and satisfy the additional condition $\langle Jv, w \rangle = 0$, then the expansion (3.5) is almost same as that for Riemannian exponential map (cf. [5]). In this circumstance, if $\langle R(w, v)v, w \rangle$ is positive, then the corresponding geodesics locally converge by comparison with the rays ρ_s.

Let V be a Jacobi field along a ∇-geodesic γ in a pseudo-Hermitian manifold M. Using (1.12) and the properties that $\nabla d\theta = 0$, $\nabla \theta = 0$, we compute

(3.6) $\nabla_{\gamma'}T_\nabla(\gamma', V) = \nabla_{\gamma'}\{2d\theta(\gamma', V)\xi + 2(\theta \wedge \tau)(\gamma', V)\}$

$$= 2d\theta(\gamma', \nabla_{\gamma'}V)\xi + \theta(\gamma')\nabla_{\gamma'}\tau(V) - \theta(\nabla_{\gamma'}V)\tau(\gamma') - \theta(V)\nabla_{\gamma'}\tau(\gamma').$$

Consequently V satisfies

$$(3.7) \qquad \nabla_{\gamma'}\nabla_{\gamma'}V = R(\gamma', V)\gamma' + 2d\theta(\gamma', \nabla_{\gamma'}V)\xi + \theta(\gamma')\nabla_{\gamma'}\tau(V)$$
$$- \theta(\nabla_{\gamma'}V)\tau(\gamma') - \theta(V)\nabla_{\gamma'}\tau(\gamma').$$

Note that $\langle \nabla_{\gamma'}\nabla_{\gamma'}V, \gamma' \rangle$ is not necessarily vanishing. Therefore, unlike the Riemannian case, the tangential component $\langle V, \gamma' \rangle$ is not linear in general. Although $(a + bt)\gamma'$ is a Jacobi field for any $a, b \in R$, the tangential component of a Jacobi field may contain some nonlinear part. We shall investigate this nonlinear tangential part of a Jacobi field on a Sasakian manifold.

Suppose now that M is a Sasakian manifold. Then (3.7) is simplified to

$$(3.8) \qquad \nabla_{\gamma'}\nabla_{\gamma'}V = R(\gamma', V)\gamma' + 2\langle J\gamma', \nabla_{\gamma'}V \rangle \xi.$$

First let us consider a vertical Jacobi field V. Writing $V(t) = f(t)\xi_{\gamma(t)}$ and substituting it into (3.8), we get

$$f'' = 0,$$

that is, $f(t) = a + bt$ for some $a, b \in R$. Hence we find that any vertical Jacobi field along γ is of the form $(a + bt)\xi_{\gamma(t)}$ for some $a, b \in R$.

Lemma 3.1. *Let M be a Sasakian manifold and let γ be a ∇-geodesic in M. Then any Jacobi field V along γ satisfies*

$$\frac{d}{dt}\langle V, \gamma' \rangle - 2\langle \xi, \gamma' \rangle \langle J\gamma', V \rangle = \text{const.}$$

Proof. Taking the inner product of (3.8) with γ', we get

$$(3.9) \qquad \frac{d^2}{dt^2}\langle V, \gamma' \rangle = 2\langle \xi, \gamma' \rangle \frac{d}{dt}\langle J\gamma', V \rangle.$$

Note that $\langle \xi, \gamma' \rangle$ is constant along γ. It follows from (3.9) that

$$\frac{d}{dt}\langle V, \gamma' \rangle = 2\langle \xi, \gamma' \rangle \langle J\gamma', V \rangle + \alpha$$

for some constant α. $\qquad\qquad\square$

Theorem 3.2. *Let M be a Sasakian manifold and let γ be a ∇-geodesic parameterized by arc length. Then every Jacobi field V along γ can be uniquely decomposed in the following form:*

$$V = a\gamma' + bt\gamma' + W$$

where $a, b \in R$ and W is a Jacobi field along γ such that

$$\langle W, \gamma' \rangle = 2\langle \xi, \gamma' \rangle \int_0^t \langle V, J\gamma' \rangle.$$

In particular, if either i) γ is horizontal, or ii) $V_{\gamma(t)} \perp J\gamma'(t)$ for every t, then W is perpendicular to γ.

Proof. Set

$$(3.12) \qquad a = \langle V, \gamma' \rangle_{\gamma(0)}, \quad b = \langle V', \gamma' \rangle_{\gamma(0)} - 2\langle \xi, \gamma' \rangle \langle J\gamma', V \rangle_{\gamma(0)}$$

and

(3.13) $W = V - a\gamma' - bt\gamma'$.

Since $(a + bt)\gamma'$ is a Jacobi field, W is a Jacobi field too. Then Lemma 3.1 implies that

(3.14) $\dfrac{d}{dt}\langle W, \gamma'\rangle = 2\langle \xi, \gamma'\rangle\langle J\gamma', W\rangle + \beta$

for some $\beta \in R$. In particular, by taking $t = 0$, then (3.12), (3.13) and (3.14) lead to

(3.15) $\beta = \langle W', \gamma'\rangle_{\gamma(0)} - 2\langle \xi, \gamma'\rangle\langle J\gamma', W\rangle_{\gamma(0)}$

$\qquad = \langle V', \gamma'\rangle_{\gamma(0)} - b - 2\langle \xi, \gamma'\rangle\langle J\gamma', V\rangle_{\gamma(0)} = 0$.

Note that $\langle W, \gamma'\rangle_{\gamma(0)} = \langle V, \gamma'\rangle_{\gamma(0)} - a = 0$. Then we integrate (3.14) from 0 to t and employ (3.15) to find

$$\langle W, \gamma'\rangle_{\gamma(t)} = 2\langle \xi, \gamma'\rangle \int_0^t \langle J\gamma', W\rangle dt.$$

This completes the proof. $\square$

REMARK 3.1. (i) If γ is a vertical geodesic, then $J\gamma' = 0$. As a result of Theorem 3.2, any Jacobi field V along the vertical geodesic can be written uniquely as

$$V = a\gamma' + bt\gamma' + W$$

with $\langle W, \gamma'\rangle \equiv 0$. On the other hand, for any ∇-geodesic in a Sasakian manifold, we have already shown that $\xi_{\gamma(t)}$ is a Jacobi field which obviously satisfies $\xi \perp J\gamma'$. Since $\langle \xi, \gamma'\rangle$ is constant, we may write $\xi_{\gamma(t)} = \langle \xi, \gamma'\rangle\gamma' + W$, where $W = \xi - \langle \xi, \gamma'\rangle\gamma'$ is clearly a Jacobi field with $\langle W, \gamma'\rangle \equiv 0$.
(ii) In [1], the authors established a similar decomposition for Jacobi fields along horizontal ∇-geodesics in pseudo-Hermitian manifolds. Here we give the decomposition for Jacobi fields along general ∇-geodesics in Sasakian manifolds.

EXAMPLE 3.1. Let us consider the Jacobi fields along a ∇-geodesic γ in the Heisenberg group H_m. Since H_m is a Sasakian manifold with zero curvature (cf. [9]), the Jacobi equation becomes

(3.16) $V'' = 2\langle J\gamma', V'\rangle\xi$.

First, we assume that γ' is not vertical. Set $\gamma'_H = \gamma' - \langle \gamma', \xi\rangle\xi$. Since $\langle \gamma', \xi\rangle =$const., we find that γ'_H and $J\gamma'_H$ are parallel along γ and

$$\|\gamma'_H\| = \|J\gamma'\| = \sqrt{1 - \langle \gamma', \xi\rangle^2} \neq 0.$$

Choose a basis $\{v_1, ..., v_{2m}\}$ in $H_{\gamma(0)}(H_m)$ such that

$$\{v_1 = J\gamma'_H(0)/\|J\gamma'_H(0)\|, v_2 = \gamma'_H(0)/\|\gamma'_H(0)\|, v_3, ..., v_{2m}\}$$

is an orthonormal basis of $HM_{\gamma(0)}$. Let $\{E_A(t)\}_{A=0}^{2m}$ be parallel vector fields along $\gamma(t)$ such that $E_0(t) = \xi_{\gamma(t)}$ and $E_i(0) = v_i$ ($i = 1, ..., 2m$). Suppose $V(t)$ is a Jacobi field along γ. Then we write

$$V(t) = \sum_{A=0}^{2m} f_A(t) E_A(t)$$

and substitute V into (3.16) to find

(3.17) $$f_0'' = 2\sqrt{1 - \langle \gamma', \xi \rangle^2} f_1', \quad f_i'' = 0 \quad (1 \le i \le 2m).$$

Consequently

(3.18) $$f_0(t) = a_1 \sqrt{1 - \langle \gamma', \xi \rangle^2} t^2 + a_0 t + b_0, \quad f_i(t) = a_i t + b_i \quad (1 \le i \le 2m),$$

where a_A, b_A are constants $(0 \le A \le 2m)$. Hence we deduce

(3.19) $$V = (a_1 \sqrt{1 - \langle \gamma', \xi \rangle^2} t^2 + a_0 t + b_0) \xi_{\gamma(t)} + \sum_{i=1}^{2m} (a_i t + b_i) E_i(t).$$

Next we assume that $\gamma'(t) = \xi_{\gamma(t)}$, and choose an orthonormal basis $\{v_1, ..., v_{2m}\}$ in $H_{\gamma(0)} M$. Let $\{E_A(t)\}_{A=0}^{2m}$ be parallel vector fields with $E_0 = \xi_{\gamma(t)}$ and $E_i(0) = v_i$ ($i = 1, ..., 2m$). Note that $J\gamma' = 0$ in this case. Similarly we may deduce from (3.16) the following general solution of the Jacobi equation

(3.20) $$V(t) = \sum_{A=0}^{2m} (a_A t + b_A) E_i(t).$$

4. Cartan-Hadamard Type Theorem

We say that a point q is conjugate to p if q is a singular value of $\exp_p^\nabla : T_p M \to M$. The conjugacy is said to be along γ_v if $d \exp_p^\nabla$ is singular at v. The following result is known for any linear connection.

Proposition 4.1 (cf. [12]). *Let $\gamma : [0, 1] \to M$ be a ∇-geodesic with $\gamma(0) = p$, $\gamma(1) = q$. Then q is conjugate to p along a ∇-geodesic γ if and only if there exists a non-zero Jacobi field V along γ such that $V(0) = V(1) = 0$. Hence q is conjugate to p if and only if p is conjugate to q.*

It follows immediately from Proposition 4.1 that

Corollary 4.2. *Let $\gamma : [0, l] \to M$ be a ∇-geodesic. If $\gamma(0)$ and $\gamma(l)$ are not conjugate, then a Jacobi field V along γ is determined by its values at $\gamma(0)$ and $\gamma(l)$.*

Let us come back to investigate the conjugate points of pseudo-Hermitian manifolds. First we consider the special case that the geodesic is vertical.

Proposition 4.3. *Let $(M^{2m+1}, H(M), J, \theta)$ be a Sasakian manifold. If $\gamma : [0, l] \to M$ is a vertical geodesic, then $\gamma|_{(0,l]}$ contains no point conjugate to $\gamma(0)$.*

Proof. Suppose γ is a vertical geodesic and V is a Jacobi field along γ with

(4.1) $$V_{\gamma(0)} = V_{\gamma(l)} = 0.$$

Since M is Sasakian and $\gamma'(t) = \xi_{\gamma(t)}$, we have $R(\gamma', V)\gamma' = 0$ and $J\gamma' = 0$. Therefore (3.8) becomes $V'' = 0$. By Corollary 4.2 and (4.1), we conclude that $V \equiv 0$. □

Sometimes it is convenient to consider the decomposition

(4.2) $$TM_{\gamma(t)} = \text{span}\{\xi\}_{\gamma(t)} \oplus HM_{\gamma(t)}$$

along the geodesic γ. Accordingly, we write a vector field V along γ as

(4.3) $$V = V_\xi + V_H$$

where $[\cdot]_H$ and $[\cdot]_\xi$ denote the horizontal and vertical components of the vector respectively. Since the Tanaka-Webster connection ∇ preserves the above decomposition, the Jacobi equation (3.8) may be decomposed as

(4.4) $$\nabla_{\gamma'}\nabla_{\gamma'}V_H = R(\gamma', V_H)\gamma', \quad \nabla_{\gamma'}\nabla_{\gamma'}V_\xi = 2\langle J\gamma', \nabla_{\gamma'}V_H\rangle\xi$$

by employing the curvature property (1.18) of Sasakian manifolds. To solve the first equation in (4.4), we may assume the initial conditions $V_H(0)$ and $V'_H(0)$. Whenever V_H is known, the second equation in (4.4) can be solved by assuming $V_\xi(0)$ and $V'_\xi(0)$.

The first equation of (4.4) yields

$$\langle V_H, \gamma'\rangle'' = \langle V''_H, \gamma'\rangle = \langle R(\gamma', V_H)\gamma', \gamma'\rangle = 0.$$

Therefore the horizontal component V_H may be written uniquely as

(4.5) $$V_H = (at + b)\gamma' + V_H^\perp$$

where $\langle V_H^\perp, \gamma'\rangle \equiv 0$. Consequently we see that the possible nonlinear part of $\langle V, \gamma'\rangle$ comes from $\langle V_\xi, \gamma'\rangle$ which is determined by the integral of $\langle J\gamma', V_H\rangle$ on $[0, t]$.

The discussion about the expansion $\|d\exp^\nabla(tw)\|^2$ in §3 provides some clue to the following result.

Theorem 4.4. *Let $(M^{2m+1}, H(M), J, \theta)$ be a Sasakian manifold with $K^H \leq 0$. Let $\gamma : [0, \beta] \to M$ be a ∇-geodesic. Then $\gamma((0, \beta])$ contains no point conjugate to $\gamma(0)$ along γ. Therefore, if (M, ∇) is complete with $K^H \leq 0$, then M has no conjugate points along any ∇-geodesic.*

Proof. Suppose $\gamma : [0, \beta] \to M$ is a ∇-geodesic and V is a Jacobi field along γ with $V(0) = V(\beta) = 0$. So $V_H(0) = V_H(\beta) = 0$ and $V_\xi(0) = V_\xi(\beta) = 0$. From (4.4), we have

(4.6) $$\langle \nabla_{\gamma'}\nabla_{\gamma'}V_H, V_H\rangle = \langle R(\gamma', V_H)\gamma', V_H\rangle \geq 0.$$

Consequently

(4.7) $$\frac{d}{dt}\langle \nabla_{\gamma'}V_H, V_H\rangle = \langle \nabla_{\gamma'}\nabla_{\gamma'}V_H, V_H\rangle + \langle \nabla_{\gamma'}V_H, \nabla_{\gamma'}V_H\rangle \geq 0,$$

that is, $\langle \nabla_{\gamma'}V_H, V_H\rangle(t)$ is non-decreasing. It follow from $V_H(0) = V_H(\beta) = 0$ that $\langle \nabla_{\gamma'}V_H, V_H\rangle \equiv 0$, which yields

(4.8) $$\frac{d}{dt}\langle V_H, V_H\rangle = 2\langle V'_H, V_H\rangle = 0.$$

From (4.8), we find that $|V_H| =$const., and thus $V_H \equiv 0$. From the second equation in (4.4), we get

$$\nabla_{\gamma'}\nabla_{\gamma'}V_\xi = 0,$$

which implies $V_\xi = (at + b)\xi$ for some $a, b \in R$. In view of $V_\xi(0) = V_\xi(\beta) = 0$, we find $V_\xi \equiv 0$. Therefore we conclude that $V \equiv 0$. $\qquad\qquad\square$

Remark 4.1. In [1], the authors proved that if M has nonpositive horizontal sectional curvature, then there is no horizontally conjugate point along any horizontal geodesic in a pseudo-Hermitian manifold.

As a consequence of Theorem 4.4, we shall now give a Cartan-Hadamard type result for Sasakian manifolds with non-positive horizontal sectional curvature. Let us first recall the following notion in [9].

Definition 4.1. Let $(N, \widetilde{\theta})$ and (M, θ) be two pseudo-Hermitian manifolds. We say that a CR map $f : N \to M$ is an isopseudo-Hermitian map if $f^*\theta = \widetilde{\theta}$.

Lemma 4.5. *Let $f : (N, \widetilde{\theta}) \to (M, \theta)$ be an isopseudo-Hermitian map. If $\dim N = \dim M = 2m + 1$, then $f : (N, g_{\widetilde{\theta}}) \to (M, g_\theta)$ is a local isometry with $df(\widetilde{\xi}) = \xi$.*

Proof. The assumption $f^*\theta = \widetilde{\theta}$ implies that $f^*d\theta = d\widetilde{\theta}$, and thus $f^*[\theta \wedge (d\theta)^m] = \widetilde{\theta} \wedge (d\widetilde{\theta})^m$. This yields that f is a local diffeomorphism. Furthermore, we get

$$(4.9) \qquad\qquad\qquad f^*G_\theta = \widetilde{G}_{\widetilde{\theta}},$$

since f is a CR map. At any point $p \in N$, we have

$$(4.10) \qquad\qquad\qquad \theta_q(df(\widetilde{\xi}_p)) = \widetilde{\theta}_p(\widetilde{\xi}_p) = 1$$

where $q = f(p) \in M$. For any vector $Y \in T_qM$, there exists a vector $X \in T_pN$ such that $df(X) = Y$, since $df : T_pN \to T_qM$ is a linear isomorphism. Therefore

$$(4.11) \qquad d\theta(df(\widetilde{\xi}_p), Y) = d\theta(df(\widetilde{\xi}_p), df(X)) = (df^*\theta)(\widetilde{\xi}_p, X) = d\widetilde{\theta}(\widetilde{\xi}_p, X) = 0.$$

Combining (4.10) and (4.11), we find that $df(\widetilde{\xi}_p) = \xi_q$. Consequently

$$f^*[\theta \otimes \theta + G_\theta] = \widetilde{\theta} \otimes \widetilde{\theta} + \widetilde{G}_{\widetilde{\theta}},$$

that is, $f^*g_\theta = g_{\widetilde{\theta}}$. $\qquad\qquad\square$

Next we need the following lemma, whose proof is a slight modification of that for Lemma 1.32 in [5].

Lemma 4.6. *Let $\varphi : (N, \widetilde{\theta}) \to (M, \theta)$ be an isopseudo-Hermitian map between two pseudo-Hermitian manifolds of same dimension. If $(N, \widetilde{\nabla})$ is complete, then φ is a covering map.*

Proof. By Lemma 4.5, we know that φ is a local isometry, and it preserves the CR structures. Clearly φ maps a $\widetilde{\nabla}$-geodesic to a ∇-geodesic. Fix $p \in M$ and let $\{p_\alpha\} = \varphi^{-1}(p)$. Let $D(r)$ and $D_\alpha(r)$ be the balls about zero of radius r in T_pM, $T_{p_\alpha}N$, respectively. Set $B^\nabla(p; r) = \exp_p^\nabla(D(r))$ and $B^{\widetilde{\nabla}}(p_\alpha; r) = \exp_{p_\alpha}^{\widetilde{\nabla}}(D_\alpha(r))$. Assume r is small enough so that $B^\nabla(p; r)$ is contained in a normal coordinate neighborhood around p. Write $U = B^\nabla(p; r)$ and $U_\alpha = B^{\widetilde{\nabla}}(p_\alpha; r)$ for simplicity. We will show that $\varphi^{-1}(U)$ is the disjoint union $\cup_\alpha U_\alpha$ and that $\varphi : U_\alpha \to U$ is a diffeomorphism for each α.

Since φ preserves locally the pseudo-Hermitian structures, we have $\varphi(\exp_{p_\alpha}^{\widetilde{\nabla}}) = \exp_p^\nabla d\varphi$, where $d\varphi : T_{p_\alpha}N \to T_pM$ is the differential map. Since $\exp_p^\nabla \circ d\varphi : D_\alpha(r) \to U$ is a diffeomorphism, so is $\varphi : U_\alpha \to U$. Clearly $\cup_\alpha U_\alpha \subset \varphi^{-1}(U)$. We shall show the opposite inclusion. Given $\widetilde{q} \in \varphi^{-1}(U)$, and set $q = \varphi(\widetilde{q})$. Let γ be the ∇-geodesic from q to p in the normal coordinate neighborhood U. Let $v = d\varphi^{-1}(\gamma'(q)) \in T_{\widetilde{q}}N$, and let $\widetilde{\gamma}$ be the ∇-geodesic from $\widetilde{q}$ in direction v. Since $(N, \widetilde{\nabla})$ is complete, $\widetilde{\gamma}$ may be extended arbitrary far. Let t_0 be the length of γ from q to p, and set $\widetilde{p} = \widetilde{\gamma}(t_0)$. Since $\varphi \circ \widetilde{\gamma} = \gamma$, $\varphi(\widetilde{p}) = p$. Hence $\widetilde{q} \in B^{\widetilde{\nabla}}(\widetilde{p}; r) \subset \cup_\alpha U_\alpha$.

It remains to show that $U_\alpha \cap U_\beta$ is empty if $\alpha \neq \beta$. Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. For any $\widetilde{q} \in U_\alpha \cap U_\beta$, let $\sigma(t)$ (resp. $\tau(t)$) be the geodesic from p_α to $\widetilde{q}$ in U_α (resp. p_μ to $\widetilde{q}$ in U_β). Then $\varphi(\sigma(t))$ and $\varphi(\tau(t))$ are two ∇-geodesics from p to $\varphi(\widetilde{q})$ in U. Since U is a normal coordinate neighborhood, $\varphi(\sigma(t)) = \varphi(\tau(t))$, which implies that $L(\sigma) = L(\tau)$. When $\widetilde{q}$ approaches to $(\partial U_\alpha) \cap U_\beta$, this is impossible. This completes the proof of the lemma.   □

Theorem 4.7. *Let $(M^{2m+1}, H(M), J, \theta)$ be a Sasakian manifold with $K^H \leq 0$. If (M, ∇) is complete, then for any $p \in M$, $\exp_p^\nabla : T_pM \to M$ is a covering map. Hence the universal covering space of M is diffeomorphic to R^{2m+1}.*

Proof. In terms of Theorem 4.4, we find that $\exp_p^\nabla : T_pM \to M$ has nonsingular differential. Set $\widehat{g} = (\exp_p^\nabla)^* g_\theta$, $\widehat{\theta} = (\exp_p^\nabla)^* \theta$, $\widehat{H} = \ker \widehat{\theta}$ and $\widehat{J} = (\exp_p^\nabla)_*^{-1} \circ J \circ (\exp_p^\nabla)_*$. Then $(T_pM, \widehat{H}, \widehat{J}, \widehat{\theta})$ is a pseudo-Hermitian manifold and $\exp_p^\nabla : (T_pM, \widehat{\theta}) \to (M, \theta)$ is an isopseudo-Hermitian map.

To simplify notations, we write $N = T_pM$ in what follows. Let $\widehat{\nabla}$ be the Tanaka-Webster connection of $(N, \widehat{H}(N), \widehat{\theta}, \widehat{J})$. The origin $o \in T_pM$ is a pole of $(N, \widehat{\nabla})$, since the ray $\rho(t) = tv$ $(0 \leq t < +\infty)$ in each direction $v \in T_pM$ is a $\widehat{\nabla}$-geodesic in N. First, we claim that $(N, \widehat{\nabla})$ is complete. Let $\widehat{\sigma} : [0, r_0) \to N$ be any $\widehat{\nabla}$-geodesic in N and let $[0, r_0)$ be the largest open interval for which $\widehat{\sigma}(t)$ exists. Suppose that $r_0 < +\infty$. Using the vector space structure of N, we may write $\widehat{\sigma}(t) = l(t)v(t)$ with $v(t) \in S^{2m}(1)$, where $l(t) = |\widehat{\sigma}(t)|_{g_\theta(p)}$ and $S^{2m}(1)$ is the unit sphere at 0. Set $\gamma_t(s) = \exp_p^\nabla(sv(t))$ $(0 \leq s < +\infty)$ and $\sigma(t) = \exp_p^\nabla(\widehat{\sigma}(t))$ $(0 \leq t < r_0)$. Then γ_t and σ are ∇-geodesics. Since (M, ∇) is complete, σ may be extended arbitrarily far. At least, $\sigma : [0, r_0] \to M$ is well-defined. Let $\{t_i\}$ be a sequence in $[0, r_0]$ such that $t_i \to r_0$. By compactness of $S^{2m}(1)$, we may pass to a sequence of $\{v(t_i)\}$, denoted still by $\{v(t_i)\}$ for simplicity, such that $v(t_i) \to v$ as $i \to +\infty$. Let $\gamma_v : [0, \infty) \to M$ be the ∇-geodesic such that $\gamma_v'(0) = v$. Then the theory of ordinary differential equations (continuous dependence of solutions on initial data) gives that $\gamma_{t_i}(s)$ converges uniformly to $\gamma_v(s)$ on any closed subinterval of $[0, \infty)$. Consequently there exists a finite number l_0 such $l(t_i) \to l_0$ and $\gamma_v(l_0) = \sigma(r_0)$. Set $\widehat{q} = l_0 v \in N$. Let $\widehat{U}$ be a normal coordinate neighborhood of $\widehat{q}$ as in Theorem 2.1. For i sufficiently large, $\widehat{\sigma}(t_i) \in \widehat{U}$. Let $\widehat{\tau} : (-\delta, \delta) \to N$ be the unique $\widehat{\nabla}$-geodesic such that $\widehat{\sigma}(t_i) \in \widehat{\tau}$ and $\widehat{\tau}(0) = \widehat{q}$. Then $\widehat{\sigma} \cup \widehat{\tau}$ is a smooth $\widehat{\nabla}$-geodesic which extends past r_0. This leads to a contradictions and thus proves the claim. In view of Lemma 4.6, we may conclude that $\exp_p^\nabla : T_pM \to M$ is a covering map.   □

Corollary 4.8. *Let $(M^{2m+1}, H(M), J, \theta)$ be a simply connected Sasakian manifold with $K^H \leq 0$. If (M, ∇) is complete, then M is diffeomorphic to R^{2m+1}.*

REMARK 4.2. (i) From Theorem 4.7, we know that any compact Sasakian manifolds with non-positive horizontal sectional curvature are aspherical. Therefore, by a standard fact from homotopy theory, their homotopy type is uniquely determined by their fundamental groups. In [6], the author gave some interesting results about the fundamental groups of compact Sasakian manifolds.

(ii) We should mention that the above Cartan-Hadamard type theorem is invisible from the Levi-Civita connection ∇^θ of any Sasakian manifold, because its curvature tensor with respect to ∇^θ always satisfies $R^\theta(\xi, X, \xi, X) = 1$ for any unit vector $X \in H(M)$.

5. Basic index lemma and Bonnet-Myers theorem

Let (M, θ) be a Sasakian manifold and $\gamma : [0, l] \to M$ be a ∇-geodesic, parametrized by arc-length. Given a piecewise differentiable vector field X along γ, we set

$$(5.1) \qquad I_0^l(X) = \int_0^l \{\langle \nabla_{\gamma'} X, \nabla_{\gamma'} X \rangle - \langle R(X, \gamma')\gamma', X \rangle\} dt.$$

The polarization of $I_0^l(X)$ or itself for simplicity will be called the index form at γ.

Theorem 5.1. *Let (M, θ) be a Sasakian manifold and let $\gamma : [0, l] \to M$ be a ∇-geodesic parametrized by arc-length, such that $\gamma(0)$ has no conjugate point along γ. Let Y be a horizontal Jacobi field along γ such that $Y_{\gamma(0)} = 0$ and let X be a piecewise differentiable vector field along γ such that $X_{\gamma(0)} = 0$. If $X_{\gamma(l)} = Y_{\gamma(l)}$ then*

$$I_0^l(X) \geq I_0^l(Y)$$

and the equality holds if and only if $X = Y$.

Proof. Let $J_{\gamma,0}$ be the space of all Jacobi fields $Z \in J_\gamma$ such that $Z_{\gamma(0)} = 0$. Clearly $\dim_R J_{\gamma,0} = 2m + 1$. Set $\widehat{\gamma}_{\gamma(t)} = t\gamma'$ and $\widehat{\xi}_{\gamma(t)} = t\xi$. From §3, we know that $\widehat{\gamma}, \widehat{\xi} \in J_{\gamma,0}$.

First we assume that the geodesic γ is not vertical. This implies that $\widehat{\gamma}$ and $\widehat{\xi}$ are linearly independent in $J_{\gamma,0}$. Let us complete $\widehat{\gamma}$ and $\widehat{\xi}$ to a linear basis $\{\widehat{\gamma}, \widehat{\xi}, V_2, ..., V_{2m}\}$ of $J_{\gamma,0}$. Clearly the vectors $\{\widehat{\gamma}(t), \widehat{\xi}(t), V_2(t), ..., V_{2m}(t)\}$ are linearly independent in $T_{\gamma(t)} M$ for each $0 < t \leq l$, because $\gamma(0)$ has no conjugate point along γ.

We set $V_j^H = V_j - \theta(V_j)\xi_{\gamma(t)}$, $2 \leq j \leq 2m$, and claim that

$$\{\widehat{\gamma}(t), \widehat{\xi}_{\gamma(t)}, V_{2,\gamma(t)}^H, ..., V_{2m,\gamma(t)}^H\}$$

are linearly independent for each $t \in (0, l]$ too. To prove this, suppose that there exist $\{c_A\}_{A=0}^{2m}$ such that

$$(5.2) \qquad 0 = c_0\widehat{\gamma}(t) + c_1\widehat{\xi}_{\gamma(t)} + \sum_{j=2}^{2m} c_i V_{j,\gamma(t)}^H = c_0\widehat{\gamma}(t) + \{c_1 - c_j t^{-1}\theta(V_j)_{\gamma(t)}\}\widehat{\xi}_{\gamma(t)} + \sum_{i=2}^{2m} c_j V_j$$

at a fixed $t \in (0, l]$. It follows from (5.2) that

$$c_0 = 0, \quad c_1 - c_j t^{-1}\theta(V_j)_{\gamma(t)} = 0, \quad c_j = 0 \quad (j = 2, ..., 2m),$$

because $\{\widehat{\gamma}(t), \widehat{\xi}(t), V_2(t), ..., V_{2m}(t)\}$ are linearly independent. Thus $c_A = 0$ $(A = 0, 1, ..., 2m)$.

To simplify the notations, let us set $Z_0 = \widehat{\gamma}, Z_1 = \widehat{\xi}$ and $Z_j = V_j^H$ $(2 \leq j \leq 2m)$. A simple observation shows that the vector fields $\{Z_A\}_{A=0}^{2m}$ satisfy the following equation

(5.3) $$\nabla_{\gamma'}\nabla_{\gamma'}Z = R(\gamma', Z)\gamma'.$$

Let $\widehat{J}_{\gamma,0}$ denote the space of all solutions Z of (5.3) such that $Z_{\gamma(0)} = 0$. Clearly $\dim_R \widehat{J}_{\gamma,0} = 2m + 1$. In fact, $\{Z_A(t)\}_{A=0}^{2m}$ is a linear basis in $\widehat{J}_{\gamma,0}$ and they are linearly independent for each t. Consequently there are piecewise differentiable functions $f^A(t)$ such that

(5.4) $$X_{\gamma(t)} = \sum_{A=0}^{2m} f^A(t)Z_{A,\gamma(t)}.$$

Now we compute

(5.5) $$|X'|^2 = \langle \sum_A (\frac{df^A}{dt}Z_A + f^A Z'_A), \sum_B (\frac{df^B}{dt}Z_B + f^B Z'_B) \rangle$$

$$= (|\sum_A \frac{df^A}{dt}Z_A|^2 + |\sum_A f^A Z'_A|^2) + 2\sum_{A,B}\langle \frac{df^A}{dt}Z_A, f^B Z'_B \rangle,$$

(5.6) $$-\langle R(X, \gamma')\gamma', X\rangle = -\sum_A f^A\langle R(Z_A, \gamma')\gamma', X\rangle = \sum_A f^A\langle Z''_A, X\rangle = \sum_{A,B}\langle f^A Z''_A, f^B Z_B\rangle$$

and

(5.7) $$\sum_{A,B}\langle \frac{df^A}{dt}Z_A, f^B Z'_B\rangle + |\sum_A f^A Z'_A|^2 + \sum_{A,B}\langle f^A Z''_A, f^B Z_B\rangle$$

$$= \frac{d}{dt}\sum_{A,B}\langle f^A Z_A, f^B Z'_B\rangle - \sum_{A,B}\langle f^A Z_A, \frac{df^B}{dt}Z'_B\rangle.$$

Note that

(5.8) $$\langle Z_A, Z'_B\rangle = \langle Z_B, Z'_A\rangle$$

because $[\langle Z_A, Z'_B\rangle - \langle Z_B, Z'_A\rangle]' = 0$ and $[\langle Z_A, Z'_B\rangle - \langle Z_B, Z'_A\rangle]_{\gamma(0)} = 0$. By employing (5.5), (5.6), (5.7) and (5.8), we deduce

(5.9) $|X'|^2 - \langle R(X, \gamma')\gamma', X\rangle$

$$= |\sum_A \frac{df^A}{dt}Z_A|^2 + \frac{d}{dt}\sum_{A,B}\langle f^A Z_A, f^B Z'_B\rangle - \sum_{A,B}\langle f^A Z_A, \frac{df^B}{dt}Z'_B\rangle + \sum_{A,B}\langle \frac{df^A}{dt}Z_A, f^B Z'_B\rangle$$

$$= |\sum_A \frac{df^A}{dt}Z_A|^2 + \frac{d}{dt}\sum_{A,B}\langle f^A Z_A, f^B Z'_B\rangle + \sum_{A,B}\frac{df^A}{dt}f^B[\langle Z_A, Z'_B\rangle - \langle Z_B, Z'_A\rangle]$$

$$= |\sum_A \frac{df^A}{dt}Z_A|^2 + \frac{d}{dt}\sum_{A,B}\langle f^A Z_A, f^B Z'_B\rangle.$$

Then integrating (5.9) gives

(5.10) $$I_0^l(X) = \sum_{A,B}\langle f^A Z_A, f^B Z'_B\rangle_{\gamma(l)} + \int_0^l |\sum_A \frac{df^A}{dt}Z_A|^2$$

Since Y is a horizontal Jacobi field with $Y_{\gamma(0)} = 0$, we know from (4.4) that Y automatically satisfies (5.3). Thus there are numbers a^A ($A = 0, 1, ..., 2m$) such that

$$(5.11) \qquad Y = \sum_{j=0}^{2m} a^A Z_A.$$

Applying (5.10) to Y, we get

$$(5.12) \qquad I_0^l(Y) = \sum_{A,B=0}^{2m} \langle a^A Z_A, a^B Z_B' \rangle_{\gamma(l)}.$$

The assumption $X_{\gamma(b)} = Y_{\gamma(b)}$ implies that $f^A(l) = a^A$ ($0 \le A \le 2m$). From (5.10) and (5.12), one may conclude that $I_0^l(X) \ge I_0^l(Y)$ and the equality holds if and only if $X = Y$.

Now we assume that $\gamma : [0, l] \to M$ is a vertical geodesic. In this case, $\widehat{\gamma} = \widehat{\xi} \in J_{\gamma,0}$. Let us complete $\widehat{\gamma}$ to a linear basis $\{\widehat{\gamma}, V_1, ..., V_{2m}\} \in J_{\gamma,0}$. Set $Z_0 = \widehat{\gamma}$ and $Z_j = V_j - \theta(V_j)\xi$ ($j = 1, ..., 2m$). Since $\gamma(0)$ has no conjugate point along the vertical geodesic, we may prove similarly that $\{Z_{0,\gamma(t)}, Z_{1,\gamma(t)}, ..., Z_{2m,\gamma(t)}\}$ are solutions of (5.3), and they are linearly independent in $T_{\gamma(t)}M$ for each $0 < t \le l$. Suppose X is a piecewise differentiable vector field along γ such that $X_{\gamma(a)} = 0$. Then

$$X_{\gamma(t)} = \sum_{A=0}^{2m} f^A(t) Z_{A,\gamma(t)}$$

for some piecewise differentiable functions f^A ($0 \le A \le 2m$). The remaining argument is similar to the first case. $\qquad \square$

REMARK 5.1. (i) The above proof actually yields the following result: Let Y be any solution of (5.3) along γ such that $Y_{\gamma(0)} = 0$. If X is any piecewise differentiable vector field along γ such that $X_{\gamma(0)} = 0$ and $X_{\gamma(l)} = Y_{\gamma(l)}$, then $I_0^l(X) \ge I_0^l(Y)$ and the equality holds if and only if $X = Y$. Note that Y is not necessarily to be horizontal if Y is already a solution of (5.3); (ii) A similar basic index result was established in [1] for a horizontal Jacobi field along a horizontal ∇-geodesic.

Taking $Y = 0$ in Theorem 5.1, we have

Corollary 5.2. *Let (M, θ) be a Sasakian manifold and let $\gamma : [0, l] \to M$ be a ∇-geodesic, parametrized by arc length and such that $\gamma(0)$ has no conjugate point along γ. If X is a piecewise differentiable vector field along γ such that $X_{\gamma(0)} = X_{\gamma(l)} = 0$, then $I_0^l(X) \ge 0$ and equality holds if and only if $X = 0$.*

As applications of Theorem 5.1, we now prove the Bonnet-Myers type result for Sasakian manifolds.

Theorem 5.3. *Let (M, θ) be a Sasakian manifold and let $\gamma : [a, b] \to M$ be a ∇-geodesic parametrized by arc length. Assume that the horizontal sectional curvature satisfies $K^H(\sigma) \ge k_0 > 0$ for any horizontal 2-plane $\sigma \subset T_x M$, $x \in M$. If $b - a \ge \pi/\sqrt{k_0(1 - \langle \gamma', \xi \rangle^2)}$, then $\gamma(a)$ has a conjugate point along $\gamma|_{(a,b]}$.*

Proof. Since there is no conjugate point along a vertical geodesic, we only need to consider the case that γ' is not parallel to $\xi_{\gamma(t)}$. Let $Y(t) \in H(M)_{\gamma(t)}$ be a unit vector field along γ such that $\nabla_{\gamma'} Y = 0$ and Y is perpendicular to γ. Thus $Y \perp \gamma'_H$. Set $f(t) = \sin[\pi(t - a)/(b - a)]$

and $X = f(t)Y$. If $\gamma(a)$ has no conjugate point in $\gamma|_{(a,b]}$, then Corollary 5.2 implies that

$$
\begin{aligned}
0 < I_a^b(X) &= \int_a^b \left\{ f'^2 |Y|^2 - f^2 \langle R(Y, \gamma') \gamma', Y \rangle \right\} dt \\
&= \int_a^b \left\{ f'^2 |Y|^2 - f^2 \langle R(Y, \gamma'_H) \gamma'_H, Y \rangle \right\} dt \\
&\leq \int_a^b \left\{ \frac{\pi^2}{(b-a)^2} \cos^2[\pi(t-a)/(b-a)] - k_0 (1 - \langle \gamma', \xi \rangle^2) \sin^2[\pi(t-a)/(b-a)] \right\} dt \\
&\leq \frac{b-a}{\pi} \int_0^\pi \left\{ \frac{\pi^2}{(b-a)^2} \cos^2 s - k_0 (1 - \langle \gamma', \xi \rangle^2) \sin^2 s \right\} ds \quad (s = \pi(t-a)/(b-a)) \\
&= \frac{b-a}{2} \left\{ \frac{\pi^2}{(b-a)^2} - k_0 (1 - \langle \gamma', \xi \rangle^2) \right\}
\end{aligned}
$$

because X is clearly non-zero. Consequently $b - a < \pi/\sqrt{k_0(1 - \langle \gamma', \xi \rangle^2)}$. By the assumption, we conclude that $\gamma(a)$ has a conjugate point in $\gamma|_{(a,b]}$. $\qquad\square$

The following result is more general than Theorem 5.3.

Theorem 5.4. *Let (M, θ) be a Sasakian manifold and let $\gamma : [a, b] \to M$ be a ∇-geodesic parametrized by arc length. Assume that the Ricci curvature satisfies $\mathrm{Ric}(X, X) \geq (2m-1)k_0 \langle X, X \rangle$, $X \in H(M)$, for some constant $k_0 > 0$. If $b - a \geq \pi/\sqrt{k_0(1 - \langle \gamma', \xi \rangle^2)}$, then $\gamma(a)$ has a conjugate point along $\gamma|_{(a,b]}$.*

Proof. Suppose that $\gamma : [a, b] \to M$ is a geodesic with no conjugate points along γ and γ' is not parallel to $\xi_{\gamma(t)}$. Let $\{Y_1, ..., Y_{2m-1}\}$ be parallel horizontal vector fields along γ such that $\{\gamma'_H/|\gamma'_H|, Y_1, ..., Y_{2m-1}\}$ is an orthonormal basis of $H(M)_{\gamma(t)}$ for every t. Set $f(t) = \sin[\pi(t-a)/b-a]$ and $X_i = f(t)Y_i$ $(i = 1, ..., 2m-1)$. Clearly each X_i is a no-zero vector field with $X_i(\gamma(a)) = X_i(\gamma(b)) = 0$. It follows from Corollary 5.2 that

$$
\begin{aligned}
0 < \sum_{j=1}^{2m-1} I_a^b(X_j) &= \sum_{j=1}^{2m-1} \int_a^b \left\{ f'(t)^2 |Y_j|^2 - f^2(t) \langle R(Y_j, \gamma') \gamma', Y_j \rangle \right\} \\
&= \int_a^b \left\{ (2m-1) f'(t)^2 - \mathrm{Ric}(\gamma'_H, \gamma'_H) f^2(t) \right\} dt \\
&\leq \frac{(2m-1)(b-a)}{\pi} \int_0^\pi \left\{ \frac{\pi^2}{(b-a)^2} \cos^2 s - k_0 (1 - \langle \gamma', \xi \rangle^2) \sin^2 s \right\} ds \\
&= \frac{(2m-1)(b-a)}{2} \left\{ \frac{\pi^2}{(b-a)^2} - k_0 (1 - \langle \gamma', \xi \rangle^2) \right\}
\end{aligned}
$$

This implies that $b - a < \pi/\sqrt{k_0(1 - \langle \gamma', \xi \rangle^2)}$. We conclude that under the assumption $b - a \geq \pi/\sqrt{k_0(1 - \langle \gamma', \xi \rangle^2)}$, $\gamma(a)$ should has a conjugate point in $\gamma|_{(a,b]}$. $\qquad\square$

REMARK 5.2. In view of Proposition 4.3, we see that the results for conjugate points in Theorems 5.3 and 5.4 are optimal, since the length $\pi/\sqrt{k_0(1 - \langle \gamma', \xi \rangle^2)} \to \infty$ as the constant $\langle \gamma', \xi \rangle \to 1$.

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