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## A REMARK ON CONDITIONS THAT A DIFFUSION IN THE NATURAL SCALE IS A MARTINGALE

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### Abstract

We consider a diffusion processes  $\{X_t\}$  on an interval in the natural scale. Some results are known under which  $\{X_t\}$  is a martingale, and we give simple and analytic proofs for them.

### 1. Introduction

Let  $-\infty \leq l_- < l_+ \leq \infty$  and let  $m$  be a Borel measure with  $\text{supp } m = (l_-, l_+)$ . We denote by  $\{\{X_t\}_{t \geq 0}, \{P_x\}_{x \in (l_-, l_+)}\}$  the minimal diffusion process on  $(l_-, l_+)$  with the speed measure  $m$  and the scale function  $s(x) = x$ . It is well known that a local martingale  $\{X_t\}$  is a martingale if and only if  $\{X_T : T \text{ is a stopping time with } T \leq t\}$  is uniformly integrable for any  $t \geq 0$ . Here our aim is to have more explicit condition for the one-dimensional diffusions in the natural scale. If  $|l_{\pm}| < \infty$ ,  $\{X_t\}$  is bounded so that it is a martingale. If  $l_- = -\infty$ ,  $l_+ < \infty$ , this can be reduced to the case of  $l_- < \infty$ ,  $l_+ = \infty$  by replacing  $X_t$  by  $-X_t$ . Hence it suffices to consider the following two cases.

Case I :  $-\infty < l_-, l_+ = +\infty$ ,    Case II :  $l_- = -\infty, l_+ = +\infty$ .

Let  $P(l_-, l_+)$  be the set of Borel measures on  $(l_-, l_+)$ , and for  $\mu \in P(l_-, l_+)$  let  $P_{\mu}(\cdot) := \int_{(l_-, l_+)} P_x(\cdot) \mu(dx)$ . According to Lemma 4.1 ([1], Lemma 2),  $\{X_t^{\tau}\}$  is a  $P_{\mu}$ -martingale for some  $\mu \in P(l_-, l_+)$  with  $\int_{(l_-, l_+)} |x| \mu(dx) < \infty$  if and only if  $\{X_t^{\tau}\}$  is  $P_x$ -martingale for any  $x \in (l_-, l_+)$ . We further set

$$\begin{aligned} \tau_a &:= \inf \{t \geq 0 \mid X_t = a\}, & \tau_{\pm} &:= \lim_{a \rightarrow l_{\pm}} \tau_a, & \tau &:= \tau_+ \wedge \tau_- \\ X_t^{\tau} &:= X_{t \wedge \tau}. \end{aligned}$$

Kotani [1] showed the following theorem.

**Theorem 1.1** ([1]).  *$\{X_t^{\tau}\}$  is a  $P_x$ -martingale for any  $x \in (l_-, l_+)$  if and only if*

Case I :

$$\int_{[r, l_+)} xm(dx) = \infty, \quad r \in (l_-, \infty)$$

Case II :

$$\int_{[r, l_+)} xm(dx) = \infty \text{ and } \int_{(l_-, r]} |x|m(dx) = \infty, \quad r \in (-\infty, \infty).$$

By Feller’s criterion,  $P_x(\tau_{\#} = \infty) = 1$  if  $|l_{\#}| = \infty, \# = \pm\infty$ . Thus Theorem 1.1 implies that  $\{X_t^\tau\}$  is a martingale if and only if the boundaries at infinity are natural. Hulley, Platen [2] derived another condition. Let

$$\mathcal{L}f := \frac{d^2}{dmdx}f$$

be the generator of  $\{X_t\}$  and for  $\lambda > 0$  let  $f_-$  (resp.  $f_+$ ) be the positive increasing (resp. positive decreasing) solution to the equation  $\mathcal{L}f = \lambda f$ , which are unique up to constants unless the boundary is regular.

**Theorem 1.2** ([2]).  $\{X_t^\tau\}$  is a  $P_x$ -martingale for any  $x \in (l_-, l_+)$  if and only if

Case I :

$$\lim_{z \rightarrow \infty} f'_-(z) = \infty$$

Case II :

$$\lim_{z \rightarrow \infty} f'_-(z) = \infty \text{ and } \lim_{z \rightarrow -\infty} f'_+(z) = -\infty.$$

Gushchin, Urusov, and Zervos [3] derived a condition that  $\{X_t^\tau\}$  is a submartingale or a supermartingale.

**Theorem 1.3** ([3]).  $\{X_t^\tau\}$  is a  $P_x$ -submartingale if and only if  $\int_r^\infty xm(dx) = \infty, r \in (l_-, l_+)$ .

By [2] Proposition 3.16, 3.17, this condition is equivalent to  $\lim_{t \rightarrow \infty} f'_-(t) = \infty$ . Together with Theorem 1.3 we thus have

**Theorem 1.4.**  $\{X_t^\tau\}$  is a  $P_x$ -submartingale if and only if  $\lim_{t \rightarrow \infty} f'_-(t) = \infty$ .

Moreover in [3], they further derived a condition in Case I such that  $\{X_t^\tau\}$  is a strict  $P_x$  supermartingale, that is,  $\{X_t^\tau\}$  is a  $P_x$ -supermartingale but is not a  $P_x$ -martingale.

**Theorem 1.5** ([3]). Let  $-\infty < l_-, l_+ = \infty$ . Then  $\{X_t^{\tau-}\}$  is a strict  $P_x$ -supermartingale if and only if

$$\lim_{t \rightarrow \infty} E_x[X_{t \wedge \tau_-}] = l_-$$

for any  $x \in (l_-, l_+)$ .

We believe that Theorem 1.5 is also true for  $l_- = -\infty$ . The goal of this paper is :

- (1) To give a simple analytic proof of Theorem 1.4 without using the results in [2]. We note that the proofs of Proposition 3.16, 3.17 in [2] is more or less probabilistic using Tanaka’s formula.
- (2) To give a simple analytic proof of Theorem 1.5 ; the original proof of that in [3] is done by embedding  $\{X_t\}$  into the geometric Brownian motion on the torus.

The rest of this paper is organized as follows. In Section 2 (resp. Section 3), we give a proof of Theorem 1.4 (resp. Theorem 1.5). In Appendix, we prepare some tools for these proofs.

**2. A proof of Theorem 1.4**

In Case I, the statement follows from Theorem 1.2, for  $\{X_t^{\tau^-}\}$  is always a  $P_x$ -supermartingale being bounded from below. Henceforth we consider Case II.

Suppose  $\{X_t\}$  is a  $P_x$ -submartingale and let  $z < x$ . Then  $\{X_t^{\tau_z}\}$  is bounded from below so that it is a  $P_x$ -martingale. For  $\lambda > 0$ , let  $f_-^z$  (resp.  $f_+^z$ ) be the positive increasing (resp. positive decreasing) solution to the equation  $\mathcal{L}f = \lambda f$  such that  $f_-^z(z) = 0$ . Then we have

$$f_-^z(x) = f_-(x) - \frac{f_-(z)}{f_+(z)}f_+(x), \quad f_+^z(x) = f_+(x).$$

Since  $f_+^z$  is increasing, we have

$$f_-^z(x) = f_-^z(x) + \frac{f_-(z)}{f_+(z)}f_+^z(x) \geq f_-^z(x) + \frac{f_-(z)}{f_+(z)}f_+^z(z), \quad x \in (z, \infty).$$

Applying Theorem 1.2 to  $\{X_t^{\tau_z}\}$  yields  $\lim_{t \rightarrow \infty} f_-^z(t) = \infty$  and thus  $\lim_{t \rightarrow \infty} f_-^z(t) = \infty$ .

Conversely, suppose  $\lim_{t \rightarrow \infty} f_-^z(t) = \infty$  and let  $z < x$ . Then

$$\begin{aligned} \lim_{z \rightarrow \infty} z \int_0^\infty e^{-\lambda t} P_x(\tau_z < t) dt &= \lim_{z \rightarrow \infty} \frac{z}{\lambda} E_x[e^{-\lambda \tau_z}] = \lim_{z \rightarrow \infty} \frac{z}{\lambda} \frac{f_-(x)}{f_-(z)} \\ &= \lim_{z \rightarrow \infty} \frac{f_-(x)}{\lambda} \frac{1}{f_-^z(z)} = 0 \end{aligned}$$

where we used Lemma 4.3 and l'Hospital's rule. By Fatou's lemma,

$$\int_0^\infty e^{-\lambda t} \liminf_{z \rightarrow \infty} z P_x(\tau_z < t) dt = 0.$$

Hence  $\liminf_{z \rightarrow \infty} z P_x(\tau_z < t) = 0$  so that we can find a sequence  $\{z_n\} \subset (x, \infty)$  with  $\lim_{n \rightarrow \infty} z_n = \infty$  such that

$$\lim_{n \rightarrow \infty} z_n P_x(\tau_{z_n} < t) = 0.$$

On the other hand  $\{X_t^{\tau_{z_n}}\}$  is a  $P_x$ -submartingale being bounded from above and

$$x \leq E_x[X_{t \wedge \tau_{z_n}}] = z_n P_x(\tau_{z_n} < t) + E_x[X_t; \tau_{z_n} \geq t].$$

Since  $\lim_{n \rightarrow \infty} P_x(\tau_{z_n} \geq t) = 1$ ,  $x \leq E_x[X_t]$ . Markov property implies  $\{X_t\}$  is a  $P_x$ -submartingale. □

**3. A proof of Theorem 1.5**

Without losing generality, we may suppose  $L_- < 0$ . For  $\lambda > 0$ , let  $f_-$  (resp.  $f_+$ ) be the positive increasing (resp. positive decreasing) solution to the equation  $\mathcal{L}f = \lambda f$  such that  $f_-(L_-) = 0$ . Let  $G$  be Green's function of  $\mathcal{L}$ :

$$\begin{aligned} G(x, y, \lambda) &:= \begin{cases} \frac{1}{h} f_-(y) f_+(x) & (y < x) \\ \frac{1}{h} f_-(x) f_+(y) & (x \leq y) \end{cases} \\ h &:= f_+(x) f_-'(x) - f_-(x) f_+'(x). \end{aligned}$$

Then we have

$$(3.1) \quad \int_{L_-}^{\infty} G(x, y, \lambda)(y - L_-)m(dy) = E_x \left[ \int_0^{\infty} e^{-\lambda t} (X_{t \wedge \tau_-} - L_-) dt \right].$$

Let  $\alpha_+ := \lim_{t \rightarrow \infty} f_+(t)$ . Then  $f'_+ \in L^1(a, \infty)$  for  $a \in (L_-, \infty)$  and

$$f_+(x) = \alpha_+ - \int_x^{\infty} f'_+(y) dy.$$

Therefore  $\lim_{x \rightarrow \infty} f'_+(x) = 0$ . The equation  $\mathcal{L}f_+ = \lambda f_+$  yields

$$\begin{aligned} f'_+(x) &= -\lambda \int_x^{\infty} f_+(y) m(dy) \\ f_+(x) &= \alpha_+ + \lambda \int_x^{\infty} (y - x) f_+(y) m(dy) \end{aligned}$$

so that we have

$$\lambda \int_x^{\infty} y f_+(y) m(dy) = f_+(x) - \alpha_+ - x f'_+(x).$$

Similarly,

$$\begin{aligned} f'_-(y) &= f'_-(L_-) + \lambda \int_{L_-}^y f_-(z) m(dz) \\ f_-(x) &= f'_-(L_-)(x - L_-) + \lambda \int_{L_-}^x (x - y) f_-(y) m(dy) \\ \lambda \int_{L_-}^x y f_-(y) m(dy) &= f'_-(L_-)(x - L_-) - f_-(x) + \lambda x \int_{L_-}^x f_-(y) m(dy). \end{aligned}$$

Substituting them into (3.1) yields

$$(3.2) \quad \int_0^{\infty} e^{-\lambda t} E_x [X_{t \wedge \tau_-} - L_-] dt = \frac{x - L_-}{\lambda} - \frac{\alpha_+ f_-(x)}{\lambda h}.$$

We note that (3.2) and Lemma 4.1 also proves Theorem 1.1 in Case I.

Suppose  $\{X_t^{\tau_-}\}$  is a strict  $P_x$ -supermartingale. The discussion above implies  $\alpha_+ > 0$ . We shall show below that

$$(3.3) \quad \lim_{\lambda \rightarrow 0} \left( x - L_- - \frac{\alpha_+ f_-(x)}{h} \right) = 0.$$

Let  $\phi, \psi$  be the solution to  $\mathcal{L}f = \lambda f$  with the initial condition

$$\begin{aligned} \phi(0) &= 1, & \phi'(0) &= 0 \\ \psi(0) &= 0, & \psi'(0) &= 1. \end{aligned}$$

Then  $f_{\pm}$  satisfy

$$f_+(x) = \phi(x) - \left( \lim_{x \rightarrow \infty} \frac{\phi(x)}{\psi(x)} \right) \psi(x), \quad f_-(x) = \phi(x) - \left( \lim_{x \rightarrow L_-} \frac{\phi(x)}{\psi(x)} \right) \psi(x).$$

$\psi, \psi$  can be decomposed by the method of successive approximation :

$$\begin{aligned} \phi(x) &= 1 + \sum_{n=1}^{\infty} \lambda^n \phi_n(x), & \phi_0(x) &= 1, & \phi_n(x) &= \int_0^x (x-y)\phi_{n-1}(y)m(dy) \\ \psi(x) &= x + \sum_{n=1}^{\infty} \lambda^n \psi_n(x), & \psi_0(x) &= x, & \psi_n(x) &= \int_0^x (x-y)\psi_{n-1}(x)m(dy) \end{aligned}$$

which is convergent locally uniformly w.r.t.  $\lambda$  [4], which yields

$$\lim_{\lambda \rightarrow 0} \phi(x) = 1, \quad \lim_{\lambda \rightarrow 0} \phi'(x) = 0, \quad \lim_{\lambda \rightarrow 0} \psi(x) = x, \quad \lim_{\lambda \rightarrow 0} \psi'(x) = 1.$$

Moreover

$$\lim_{\lambda \rightarrow 0} \left( - \lim_{x \rightarrow l_-} \frac{\psi(x)}{\phi(x)} \right) = \lim_{\lambda \rightarrow 0} \left( \int_{l_-}^0 \frac{1}{(\phi(x))^2} dx \right) = \int_{l_-}^0 dx = -l_-$$

implies

$$\lim_{\lambda \rightarrow 0} f_-(x) = 1 - \frac{x}{l_-}, \quad \lim_{\lambda \rightarrow 0} f'_-(x) = -\frac{1}{l_-}.$$

On the other hand, by  $\alpha_+ > 0$  and by Lemma 4.2, we have  $\int_r^\infty xm(dx) < \infty$ ,  $r \in (l_-, \infty)$  so that we can find  $g$  satisfying

$$g(x) = 1 + \lambda \int_x^\infty (y-x)g(y)m(dy)$$

by successive approximation. Using  $\alpha_+ > 0$ ,  $\lim_{t \rightarrow \infty} f'_+(t) = 0$ ,  $\lim_{t \rightarrow \infty} g(t) = 1$  and  $\lim_{t \rightarrow \infty} g'(t) = 0$ , we have

$$f_+(x)g'(x) - f'_+(x)g(x) = 0$$

which implies  $f_+(x) = Cg(x)$  for some positive constant  $C$ . Because  $\lim_{\lambda \rightarrow 0} g(x) = 1$ ,  $\lim_{\lambda \rightarrow 0} g'(x) = 0$ , we have

$$\lim_{\lambda \rightarrow 0} f_+(x) = C, \quad \lim_{\lambda \rightarrow 0} f'_+(x) = 0.$$

Therefore

$$\lim_{\lambda \rightarrow 0} \left( x - l_- - \frac{\alpha_+ f_-(x)}{h} \right) = x - l_- - \frac{C \left( 1 - \frac{x}{l_-} \right)}{C \cdot \left( \frac{-1}{l_-} \right) - 0 \cdot \left( 1 - \frac{x}{l_-} \right)} = 0$$

proving (3.3). Since  $X_{t \wedge \tau_-}$  is a supermartingale,  $f(t) := E_x[X_{t \wedge \tau_-} - l_-] \in C^1[0, \infty)$  is monotone decreasing which shows that  $\lim_{t \rightarrow \infty} f(t)$  exists and  $f' \in L^1(0, \infty)$ . Thus by (3.2) and Lemma 4.4

$$\lim_{t \rightarrow \infty} E_x[X_{t \wedge \tau_-} - l_-] = 0.$$

Conversely, suppose that  $\lim_{t \rightarrow \infty} E_x[X_{t \wedge \tau_-} - l_-] = 0$ . Then

$$\lim_{\lambda \rightarrow 0} \lambda \int_0^\infty e^{-\lambda t} E_x[X_{t \wedge \tau_-} - l_-] dt = 0$$

which implies  $\alpha_+ > 0$  since otherwise it would contradict to (3.2), (3.3). Therefore  $\{X_t^{\tau_-}\}$  is not a martingale. □

#### 4. Appendix

**Lemma 4.1** (Lemma 2 in [1]). *Suppose  $\{X_{t \wedge \tau_-}\}$  is a  $P_\mu$ -martingale for some  $\mu \in P(l_-, \infty)$ . Then for any  $t \geq 0$ ,  $x \in (l_-, \infty)$ ,*

$$(4.1) \quad E_x[X_{t \wedge \tau_-}] = x.$$

*Conversely, if (4.1) is valid, then  $\{X_{t \wedge \tau_-}\}$  is a  $P_\mu$ -martingale for any  $\mu \in P(l_-, \infty)$  with  $\int_{l_-}^{\infty} |x| \mu(dx) < \infty$ .*

**Lemma 4.2.** *Let  $\lambda > 0$  and let  $f_+$  be the positive decreasing solution to  $\mathcal{L}f = \lambda f$  with  $\alpha_+ := \lim_{x \rightarrow \infty} f_+(x)$ . Then the following three conditions are equivalent.*

- (1)  $\alpha_+ = 0$
- (2)  $\int_a^{\infty} y m(dy) = \infty$
- (3)  $\lambda \int_x^{\infty} (y - x) f_+(y) m(dy) = f_+(x)$ .

**Lemma 4.3.** *Let  $f_{\pm}$  be the ones defined in the proof of Theorem 1.5. Then*

$$E_x[e^{-\lambda \tau_a}] = \frac{f_+(x)}{f_+(a)}, \quad a < x$$

$$E_x[e^{-\lambda \tau_b} : \tau_b < \tau_-] = \frac{f_-(x)}{f_-(b)}, \quad -\infty \leq l_- < x < b.$$

**Lemma 4.4.** *Suppose  $f \in C^1[0, \infty)$  and  $f' \in L^1(0, \infty)$ . Then*

- (1)  $\lim_{t \rightarrow \infty} f(t)$  exists, and
- (2)  $\lim_{t \rightarrow \infty} f(t) = \lambda \lim_{\lambda \downarrow 0} \int_0^{\infty} e^{-\lambda t} f(t) dt$ .

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