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SPECTRUM OF DARBOUX TRANSFORMATION OF DIFFERENTIAL OPERATOR

Dedicated to Professor Shunichi Tanaka on his 60th birthday

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1. Introduction

The purpose of the present paper is to clarify how the spectrum of the second order ordinary differential operator

$$H(u) = -\partial^2 + u(x), \quad x \in D$$

undergoes changes by the Darboux transformation, where $u(x)$ is a meromorphic function defined in the complex domain D, and $\partial = {}' = d/dx$ denotes the differentiation with respect to the complex variable x. The Darboux transformation of the operator $H(u)$ with the spectral parameter λ is defined as follows: Suppose $f(x,\lambda) \in \ker(H(u) - \lambda) \setminus \{0\}$ and put $q(x,\lambda) = \partial \log f(x,\lambda) = f_x(x,\lambda)/f(x,\lambda)$ and $A_\pm = \pm\partial + q(x,\lambda)$, then the factorization

$$H(u) = A_+ \cdot A_- + \lambda$$

follows, where $A \cdot B$ denotes the product of the operators A and B. Interchanging the factors $A_\pm$, we obtain the another operator

$$\hat{H}(u) = A_- \cdot A_+ + \lambda.$$

One immediately verifies

$$\hat{H}(u) = H(\hat{u}) = -\partial^2 + \hat{u}(x),$$

where

$$\hat{u}(x) = u(x) - 2q_x(x,\lambda) = u(x) - 2\partial^2 \log f(x,\lambda).$$

We call the operator $\hat{H}(u)$ and the coefficient $\hat{u}(x)$ the Darboux transformations of $H(u)$ and $u(x)$ with the spectral parameter λ by the solution $f(x,\lambda)$.

The method of Darboux transformation, which was originated more than 110 years ago by J. G. Darboux [2], was used by M. M. Crum [1] as an algorithm for adding or removing eigenvalues of the selfadjoint Sturm-Liouville operator considered in $L^2(a,b)$. For instance, the procedure of removing an eigenvalue is as follows: Suppose $u(x)$, $-\infty < x < \infty$, be the real valued measurable potential such that

$$\int_{-\infty}^{\infty} (1+|x|)|u(x)|dx < \infty,$$

then the selfadjoint operator $H(u)$ considered in $L^2(-\infty,\infty)$ has purely absolutely continuous spectrum $[0,\infty)$ and a finite number of the discrete spectrum $\sigma_p(H(u)) = \{-\beta_n^2 < -\beta_{n-1}^2 < \cdots < -\beta_1^2\} \subset (-\infty, 0)$. Let $H(\hat{u})$ be the Darboux transformation by the Jost solution $f(x, i\beta_n)$, i.e., $\hat{u}(x) = u(x) - 2\partial^2 \log f(x, i\beta_n)$, then $\sigma_p(H(\hat{u})) = \sigma_p(H(u)) \setminus \{-\beta_n^2\}$. Thus, the minimum discrete eigenvalue $-\beta_n^2$ of $H(u)$ is removed by the Darboux transformation by $f(x, i\beta_n)$. See also [4] and [19]. Therefore, if one apply successively this procedure n times to the operator with the Bargmann potential such that the reflection coefficient identically vanishes and $\sharp\sigma_p(H(u)) = n$ (cf. [9] and [16]), then $H(u)$ is reduced to the simple operator $H(0) = -\partial^2$. These procedures are clearly described in [13]. On the other hand, one can see easily that if $u(x) = 2\wp(x+\omega)$, $-\infty < x < \infty$ then none of simple discrete eigenvalues of $H(u)$ considered in the class of functions of period 2π are removed by the Darboux transformation, where $\wp(z)$, $z \in \boldsymbol{C}$ is the Weierstrass elliptic function with the real primitive period π and the imaginary primitive period 2ω. Thus, there are two kinds of eigenvalues, namely, the one can be removed by the Darboux transformation, and the other can not be. The purpose of the present paper is to establish the algebraic criteria to distinguish them. Of course, it is quite difficult to obtain such criteria for the operator with the potential of general type. Therefore we treat of only the operator with the potential which satisfies the higher order stationary KdV equation. The Bargmann potentials [11], the finite-zone potentials [5], and certain kind of rational potentials [17] belong to this class. In this case, it is appropriate to consider the problem for the differential operator in the complex domain with the meromorphic potential, since we can treat the problem in the unified way by this approach different from the method of the real analysis.

In our approach, the integro-differential operator

$$\Lambda(u) = \partial^{-1} \cdot (\frac{1}{2}u'(x) + u(x)\partial - \frac{1}{4}\partial^3), \tag{1}$$

which is usually called the Λ-operator, plays crucial role. First of all, we regard the Λ-operator as the generator of the infinite sequence of the differential polynomials $Z_n(u(x))$, $n \in \boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$, which are defined by the recurrence relation $Z_\nu(u) = \Lambda(u)Z_{\nu-1}(u)$, $\nu \in \boldsymbol{N}$ with $Z_0(u) = 1$, and called the KdV polynomials. Let $V(u)$ be the vector space over the complex number field $\boldsymbol{C}$ spanned by

$Z_\nu(u(x))$, $\nu \in \boldsymbol{Z}_+$. Then $\Lambda(u)$ can be regarded as the linear operator in $V(u)$. Since the potential $u(x)$ satisfies the higher order stationary KdV equation if and only if $V(u)$ is the finite dimensional vector space, the methods of elementary linear algebra are quite effective for our investigation. Some preliminary studies in this direction have been done in [18] and [20]. By the way, the notion of the Λ-operator has a quite long history. Particularly, the Λ-operator played the essential role from the very start of the inverse scattering method for the KdV equation [7]. See [6] for more information about the another use of the Λ-operator together with the historical remark, in which the Λ-operator for the NLS model was extensively used for the study of the hierarchy of Poisson structure.

In [18], two kinds of spectra $\Gamma(u)$ and $\Gamma_0(u)$ are introduced. The outlines of their definitions are as follows : Suppose $n = \dim V(u) - 1 < \infty$ then there uniquely exist the polynomials $a_\nu(\lambda; u)$, $\nu = 0, 1, \cdots, n$ in the spectral parameter λ such that

$$Z_{n+1}(u(x) - \lambda) = \sum_{\nu=0}^{n} a_\nu(\lambda; u) Z_\nu(u(x) - \lambda).$$

Put

$$F(x, \lambda; u) = Z_n(u(x) - \lambda) - \sum_{\nu=1}^{n} a_\nu(\lambda; u) Z_{\nu-1}(u(x) - \lambda)$$

and define the Λ-spectral discriminant $\Delta(\lambda; u)$ by

$$\Delta(\lambda; u) = F_x(x, \lambda; u)^2 - 2F(x, \lambda; u) F_{xx}(x, \lambda; u) + 4(u(x) - \lambda) F(x, \lambda; u)^2,$$

which is the polynomial of degree $2n+1$ in λ with constant coefficients. Moreover put

$$\Omega(\mu; u) = \mu^{n+1} - \sum_{\nu=0}^{n} a_\nu(0; u) \mu^\nu.$$

Then the spectra $\Gamma(u)$ and $\Gamma_0(u)$ are defined by

$$\begin{aligned} \Gamma(u) &= \{\lambda \,|\, \Delta(\lambda; u) = 0\}, \\ \Gamma_0(u) &= \{\mu \,|\, \Omega(\mu; u) = 0\}. \end{aligned}$$

More precise definitions and meanings of $\Delta(\lambda; u)$ and $\Gamma(u)$ are stated in the body of the paper. On the other hand, $\Gamma_0(u)$ is nothing but the spectrum of $\Lambda(u) \in \operatorname{End} V(u)$ [18, p.427, Corollary 16], and plays only auxiliary role in our method. The main purpose of the present paper is to clarify how the spectrum $\Gamma(u)$ undergoes changes by the Darboux transformation. More precisely, we prove that the multiplicity of $\lambda_* \in \Gamma(u)$

can be reduced by the Darboux transformation by the specific eigenfunction if $\lambda = \lambda_*$ is the multiple root of the Λ-spectral discriminant $\Delta(\lambda; u)$. Moreover, in that case, the multiplicity of λ_* as the root of $\Delta(\lambda; u)$ decreases by 2.

The contents of the present paper are as follows. §2 is devoted to the preliminaries such as the KdV polynomials, the notion of Λ-rank, the spectra $\Gamma(u), \Gamma_0(u)$, the fundamental equality of the Darboux transformation, and the Kuperschmidt-Wilson factorization of $\Lambda(u)$. At the end of §2, we state Theorem, which is the main result of this paper. In §3, we prove some lemmas which are necessary for the proof of the main theorem. In §4, Theorem is proved. In §5, the illustrative examples are investigated.

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2. Preliminaries and Main Theorem

In this section the necessary materials are summarized mainly from [18] and [20].

Since the Λ-operator $\Lambda(u)$ includes the indefinite integral ∂^{-1} as its factor, the expression (1) is somewhat formal one. Hence, first of all, it is necessary for our purpose to define $\Lambda(u)$ strictly in order to avoid the ambiguity of the integration constant.

Let $\mathcal{A}$ be a differential algebra over the complex number field $\boldsymbol{C}$ of polynomials in infinite formal symbols $u_\nu, \nu \in \boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$ with the derivation $\delta = \sum_{\nu=0}^{\infty} u_{\nu+1}\partial/\partial u_\nu$. We denote its subalgebra of polynomials without free term by $\mathcal{A}_0$. Put $\mathcal{A}_0^* = \delta\mathcal{A}_0$ and let δ_* be the restriction of δ to $\mathcal{A}_0$, then $\delta_* : \mathcal{A}_0 \to \mathcal{A}_0^*$ is the bijection. Hence there exists the inverse mapping $\delta_*^{-1} : \mathcal{A}_0^* \to \mathcal{A}_0$. On the other hand, put

$$\hat{K} = \frac{1}{2}u_1 + u_0\delta - \frac{1}{4}\delta^3,$$

then, by the complete parallel discussion to [17, p.621 Lemma 3.1] (see also [22]), one can show that $\hat{K} \cdot (\delta_*^{-1} \cdot \hat{K})^{n-1} P_0$ is uniquely defined and belong to $\mathcal{A}_0^*$ for all $n \in \boldsymbol{N}$, where $P_0 = 1 \in \mathcal{A}$. Hence $\hat{\Lambda}^n P_0$ are the polynomials in $u_0, u_1, \cdots, u_{2n-2}$, $n \in \boldsymbol{N}$, where $\hat{\Lambda} = \delta_*^{-1} \cdot \hat{K}$. We denote them by $P_n(u_0, u_1, \cdots, u_{2n-2})$, $n \in \boldsymbol{N}$;

$$\hat{\Lambda}^n P_0 = P_n(u_0, u_1, \cdots, u_{2n-2}).$$

On the other hand, let $u(x)$ be a meromorphic function of the one complex variable x defined in the domain $D \subset \boldsymbol{C}$. Put $Z_0(u(x)) = P_0 = 1$ and

$$Z_n(u(x)) = P_n(u(x), u'(x), \cdots, u^{(2n-2)}(x)), \quad n \in \boldsymbol{N},$$

where $u^{(k)}(x)$ is the k-th derivative of $u(x)$. We call $Z_n(u(x))$ the KdV polynomials in $u(x)$. Let $V(u)$ be the vector space over $\boldsymbol{C}$ spanned by $Z_\nu(u(x))$, $\nu \in \boldsymbol{Z}_+$;

$$V(u) = \bigcup_{\nu=0}^{\infty} \boldsymbol{C} Z_\nu(u(x)),$$

where $\boldsymbol{C}\boldsymbol{v}$ denotes the 1-dimensional vector space generated by the single vector $\boldsymbol{v}$ over $\boldsymbol{C}$. Moreover let $\Lambda(u)$ be the endomorphism of $V(u)$ defined by

$$Z_n(u(x)) = \Lambda(u) Z_{n-1}(u(x)), \quad n \in \boldsymbol{N}. \tag{2}$$

Let $K(u)$ be the 3rd order ordinary differential operator defined by

$$K(u) = \frac{1}{2}u'(x) + u(x)\partial - \frac{1}{4}\partial^3.$$

Then, by the definition, one verifies

$$\partial Z_n(u(x)) = K(u) Z_{n-1}(u(x)), \quad n \in \boldsymbol{N}. \tag{3}$$

Thus, we can formally express $\Lambda(u)$ as (1), i.e., $\Lambda(u) = \partial^{-1} \cdot K(u)$. Note that the operator $\Lambda(u-\lambda)$ does not coincide with the operator $\Lambda(u) - \lambda$ in general, while $K(u-\lambda) = K(u) - \lambda\partial$. In fact one has

$$\Lambda(u-\lambda)Z_0(u(x)) = Z_1(u(x)-\lambda) = \frac{1}{2}(u(x)-\lambda),$$
$$(\Lambda(u)-\lambda)Z_0(u(x)) = Z_1(u(x)) - \lambda Z_0(u(x)) = \frac{1}{2}u(x) - \lambda,$$

because $Z_0(u(x)) = Z_0(u(x)-\lambda) = 1$.

By the way, the ordinary differential operator $K(u)$ has the following significant property [21, p.23, Theorem 7]: if $f(x), g(x) \in \ker H(u)$, then $f(x)g(x) \in \ker K(u)$. According to the historical remark of [23, p.298], we call this fact Appell's lemma.

For $u(x)$ such that $\dim V(u) = n+1$, $n \in \boldsymbol{Z}_+$, the Λ-rank of $u(x)$ is defined by

$$\mathrm{rank}_\Lambda u(x) = n.$$

If $n = \mathrm{rank}_\Lambda u(x) < \infty$, then $Z_0(u(x)), Z_1(u(x)), \cdots, Z_n(u(x))$ are the basis of the vector space $V(u)$ [18, p.416, lemma 5];

$$V(u) = \bigoplus_{\nu=0}^{n} \boldsymbol{C} Z_\nu(u(x)).$$

Hence there uniquely exist the constants ρ_ν, $\nu = 0, 1, \cdots, n$ such that

$$Z_{n+1}(u(x)) = \sum_{\nu=0}^{n} \rho_\nu Z_\nu(u(x)).$$

We call ρ_ν, $\nu = 0, 1, \cdots, n$ the Λ-characteristic coefficients of $u(x)$. On the other hand, define the coefficients $\alpha_\nu^{(n)}$, $\nu = 0, 1, \cdots, n$ by the recurrence relation

$$\alpha_\nu^{(n)} = \begin{cases} 1 & \text{for } \nu = n \\ \alpha_{\nu-1}^{(n-1)} + \alpha_\nu^{(n-1)} & \text{for } \nu = 1, 2, \cdots, n-1 \\ \dfrac{(2n)!}{2^{2n}(n!)^2} & \text{for } \nu = 0, \end{cases} \tag{4}$$

then we have the expansion formula [18, p.414, Theorem 3]

$$Z_m(u(x) - \lambda) = \sum_{\nu=0}^{m} (-1)^{m-\nu} \alpha_\nu^{(m)} Z_\nu(u(x)) \lambda^{m-\nu}. \tag{5}$$

This implies that

$$V(u) = V(u - \lambda)$$

and

$$\mathrm{rank}_\Lambda u(x) = \mathrm{rank}_\Lambda (u(x) - \lambda)$$

hold for any $\lambda \in \boldsymbol{C}$. Hence if $n = \mathrm{rank}_\Lambda u(x) < \infty$ then there exist the Λ-characteristic coefficients $a_\nu(\lambda; u)$, $\nu = 0, 1, \cdots, n$ of $u(x) - \lambda$ such that

$$Z_{n+1}(u(x) - \lambda) = \sum_{\nu=0}^{n} a_\nu(\lambda; u) Z_\nu(u(x) - \lambda) \tag{6}$$

for any $\lambda \in \boldsymbol{C}$. Of course $\rho_\nu = a_\nu(0; u)$, $\nu = 0, 1, \cdots, n$ follow. More generally, $a_\nu(\lambda; u)$, $\nu = 0, 1, \cdots, n$ are the polynomials of degree $n - \nu + 1$ in λ [18, p.417, lemma 7];

$$a_\nu(\lambda; u) = -\alpha_\nu^{(n+1)} \lambda^{n-\nu+1} + \sum_{\mu=\nu}^{n} \alpha_\nu^{(\mu)} \rho_\mu \lambda^{\mu-\nu}. \tag{7}$$

On the other hand, put

$$F(x, \lambda; u) = Z_n(u(x) - \lambda) - \sum_{\nu=1}^{n} a_\nu(\lambda; u) Z_{\nu-1}(u(x) - \lambda)$$

then $F(x,\lambda;u)$ does not identically vanish as a function of x for any $\lambda \in \boldsymbol{C}$, and, by (3),

$$K(u)F(x,\lambda;u) = \lambda F_x(x,\lambda;u)$$

follows. Hence, by Appell's lemma mentioned above, it turns out that $F(x,\lambda;u)$ can be expressed as the quadratic form of $f_1(x,\lambda)$ and $f_2(x,\lambda)$, which are the fundamental system of solutions of the eigenvalue problem

$$(8) \qquad H(u)f(x,\lambda) = -f''(x,\lambda) + u(x)f(x,\lambda) = \lambda f(x,\lambda).$$

Suppose that $F(x,\lambda;u)$ is expressed as

$$F(x,\lambda;u) = A(\lambda)f_1(x,\lambda)^2 + B(\lambda)f_1(x,\lambda)f_2(x,\lambda) + C(\lambda)f_2(x,\lambda)^2.$$

Then the discriminant $\Delta(\lambda;u) = B(\lambda)^2 - 4A(\lambda)C(\lambda)$ of the quadratic form $F(x,\lambda;u)$ is expressed as

$$(9) \quad \Delta(\lambda;u) = F_x(x,\lambda;u)^2 - 2F(x,\lambda;u)F_{xx}(x,\lambda;u) + 4(u(x)-\lambda)F(x,\lambda;u)^2,$$

which is the polynomial of degree $2n+1$ in λ with the constant coefficients [18, p.419-420, lemma 8 and Corollary 10]. We call the polynomial $\Delta(\lambda;u)$ the Λ-spectral discriminant. If λ_* is the root of the algebraic equation

$$(10) \qquad \Delta(\lambda;u) = 0,$$

then the function $\sqrt{F(x,\lambda_*;u)}$ solves the eigenvalue problem (8) for $\lambda = \lambda_*$ [18, p.420, Theorem 11]. Put

$$\Gamma(u) = \{\lambda \in \boldsymbol{C} \mid \Delta(\lambda;u) = 0\},$$

then $\Gamma(u)$ corresponds to the set of simple periodic eigenvalues in the case of periodic potentials [14, p 236], or the set of discrete eigenvalues in the case of Bargmann potentials [16, p 67]. Moreover, let $\Gamma^\natural(u)$ be the collection of all complex roots of the Λ-spectral discriminant $\Delta(\lambda;u)$ arranged same times as its multiplicity, i.e.,

$$\Gamma^\natural(u) = \{\lambda_0, \lambda_1, \cdots, \lambda_{2n}\}$$

for

$$\Delta(\lambda;u) = -4\prod_{j=0}^{2n}(\lambda - \lambda_j).$$

On the other hand, put

$$\Omega(\lambda; u) = \lambda^{n+1} - \sum_{\nu=0}^{n} \rho_\nu \lambda^\nu \tag{11}$$

then

$$\Lambda(u)F(x, \lambda; u) = \lambda F(x, \lambda; u) - \Omega(\lambda; u) \tag{12}$$

follows [18, p.425, Theorem 15]. Since

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \rho_0 \\ 1 & 0 & \cdots & 0 & \rho_1 \\ 0 & 1 & \cdots & 0 & \rho_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \rho_n \end{pmatrix}$$

is the matrix of $\Lambda(u) \in \mathrm{End}V(u)$ relative to the basis $Z_0(u), Z_1(u), \cdots, Z_n(u)$ of the vector space $V(u)$, one has

$$\det(\lambda - \Lambda(u)) = \Omega(\lambda; u).$$

In other words, $\Omega(\lambda; u)$ is the characteristic polynomial of $\Lambda(u) \in \mathrm{End}V(u)$.

Here we define the Darboux transformation with spectral parameter for a general meromorphic potential $u(x)$ not assuming $\mathrm{rank}_\Lambda u(x) < \infty$: Let $f(x, \lambda_*)$ be a nontrivial solution of the eigenvalue problem (8) for $\lambda = \lambda_*$. Then

$$g(x, \lambda_*) = \frac{1}{f(x, \lambda_*)}(s + t \int f(x, \lambda_*)^2 dx), \quad s, t \in \boldsymbol{C}$$

is the general solution of

$$H(\hat{u})g(x, \lambda_*) = -g''(x, \lambda_*) + \hat{u}(x)g(x, \lambda_*) = \lambda_* g(x, \lambda_*),$$

where $\hat{u}(x) = u(x) - 2\partial^2 \log f(x, \lambda_*)$. This implies that $u(x) - \lambda_*$ itself is also the Darboux transformation of $\hat{u}(x) - \lambda_*$ by the solution $\dfrac{1}{f(x, \lambda_*)}$. These facts are follow immediately from Darboux's lemma [21, p.88, lemma 1].

Particularly, suppose that $\mathrm{rank}_\Lambda u(x) < \infty$ and $\lambda_* \in \Gamma(u)$, then, by [18, p.420, Theorem 11], we can define $f_j^0(x, \lambda_*)$, $j = 1, 2$, the fundamental system of solutions of the eigenvalue problem (8) for $\lambda = \lambda_*$ by

$$f_1^0(x, \lambda_*) = \sqrt{F(x, \lambda_*; u)}, \tag{13}$$

$$f_2^0(x,\lambda_*) = \sqrt{F(x,\lambda_*;u)} \int \frac{1}{F(x,\lambda_*;u)}dx, \tag{14}$$

where the integration constant of the above is arbitralily fixed. In what follows, we fix these fundamental system of solutions $f_j^0(x,\lambda_*)$, $j=1,2$. Note

$$W[f_1^0(x,\lambda_*), f_2^0(x,\lambda_*)] = 1,$$

where $W[f,g] = fg' - f'g$ is the Wronskian. For $\alpha \in \boldsymbol{P}_1 = \boldsymbol{C} \cup \{\infty\}$, put

$$f(x,\lambda_*;\alpha) = \begin{cases} f_1^0(x,\lambda_*) + \alpha f_2^0(x,\lambda_*), & \alpha \in \boldsymbol{C} \\ f_2^0(x,\lambda_*), & \alpha = \infty \end{cases} \tag{15}$$

and

$$u^*_{\lambda_*,\alpha} = u^*(x,\lambda_*;\alpha) = u(x) - 2\partial^2 \log f(x,\lambda_*;\alpha). \tag{16}$$

Moreover, put

$$q(x,\lambda_*;\alpha) = \partial \log f(x,\lambda_*;\alpha)$$

then we have immediately

$$q(x,\lambda_*;\alpha)^2 + q_x(x,\lambda_*;\alpha) = u(x) - \lambda_*, \tag{17}$$

and

$$\begin{aligned} q(x,\lambda_*;\alpha)^2 - q_x(x,\lambda_*;\alpha) &= u(x) - 2q_x(x,\lambda_*;\alpha) - \lambda_* \\ &= u^*(x,\lambda_*;\alpha) - \lambda_* \end{aligned} \tag{18}$$

by (16). The relations (17) and (18) imply

$$H(u) - \lambda_* = A_+(\lambda_*,\alpha) \cdot A_-(\lambda_*,\alpha)$$

and

$$H(u^*_{\lambda_*,\alpha}) - \lambda_* = A_-(\lambda_*,\alpha) \cdot A_+(\lambda_*,\alpha)$$

respectively, where $A_\pm(\lambda_*,\alpha)$ are the first order differential operators defined by

$$A_\pm(\lambda_*,\alpha) = \pm\partial + q(x,\lambda_*;\alpha).$$

As briefly mentioned in the introduction, the operator

$$H(u^*_{\lambda_*,\alpha}) = -\partial^2 + u^*(x,\lambda_*;\alpha)$$

is the Darboux transformation of $H(u)$ by the solution $f(x,\lambda_*;\alpha)$.

Moreover let us introduce another first order ordinary differential operators

$$B_\pm(\lambda_*,\alpha) = \pm\partial + 2q(x,\lambda_*;\alpha), \quad \alpha \in \boldsymbol{P}_1.$$

The Kupershmidt-Wilson (K-W) factorization [10]

$$\Lambda(u-\lambda_*) = \frac{1}{4}\partial^{-1}\cdot B_+(\lambda_*,\alpha)\cdot\partial\cdot B_-(\lambda_*,\alpha), \tag{19}$$

$$\Lambda(u^*_{\lambda_*,\alpha}-\lambda_*) = \frac{1}{4}\partial^{-1}\cdot B_-(\lambda_*,\alpha)\cdot\partial\cdot B_+(\lambda_*,\alpha), \tag{20}$$

and the fundamental equality of the Darboux transformation [20, p 6, Theorem 2]

$$B_+(\lambda_*,\alpha)Z_m(u^*(x,\lambda_*;\alpha)-\lambda_*) = B_-(\lambda_*,\alpha)Z_m(u(x)-\lambda_*) \tag{21}$$

play the essential roles in this paper. The fundamental equality (21) was firstly obtained in [17, p.623, Theorem 3.2]. An alternative simple proof was given in [20, p.20, Theorem 2]. On the other hand, the operators $B_\pm(\lambda_*,\alpha)$ can be regarded as the Fréchet derivatives of the Miura transformations (17) and (18). Recently, the study from this viewpoint is developed in [15]. As the application of the fundamental equality (21), one has

$$n-1 \le \mathrm{rank}_\Lambda u^*(x,\lambda_*;\alpha) \le n+1 \tag{22}$$

(see [20, p 13, Theorem 10]).

Now, fix the fundamental system of the solutions $f_1^0(x,\lambda_*)$ and $f_2^0(x,\lambda_*)$ defined by (13) and (14). As mentioned in the introduction, the purpose of the present work is to understand how the structure of the spectrum of the operator $H(u)$ is simplified by the Darboux transformation. On the other hand, the Λ-rank of $u(x)$ is closely related to the complexity of the spectrum of the operator $H(u)$. In fact, we have immediately

$$\sharp\Gamma^\natural(u) = 2\mathrm{rank}_\Lambda u(x) + 1.$$

Thus, for our purpose, it is the most essential problem to clarify the condition for $\lambda_* \in \Gamma(u)$ such that

$$\mathrm{rank}_\Lambda u^*(x,\lambda_*;\alpha) = n-1$$

for some $\alpha \in \boldsymbol{P}_1$. In other words, we obtain the necessary and sufficient condition for $\lambda_* \in \Gamma(u)$ such that $\Sigma_-(u;\lambda_*) \neq \emptyset$, where

$$\Sigma_-(u;\lambda_*) = \{\alpha \in \boldsymbol{P}_1 \mid \mathrm{rank}_\Lambda u^*(x,\lambda_*;\alpha) = n-1\}$$

for $\lambda_* \in \Gamma(u)$.

Now we can state the main theorem of this paper.

Theorem. *Assume $n = \mathrm{rank}_\Lambda u(x) < \infty$. Then, $\Sigma_-(u;\lambda_*) \neq \emptyset$ for $\lambda_* \in \Gamma(u)$, if and only if $\lambda = \lambda_*$ is the multiple root of the Λ-spectral discriminant $\Delta(\lambda;u)$. Moreover, if $\Sigma_-(u;\lambda_*) \neq \emptyset$ then $\Sigma_-(u;\lambda_*) = \{0\}$, and we have*

$$\Delta(\lambda; u^*_{\lambda_*,0}) = \frac{\Delta(\lambda;u)}{(\lambda-\lambda_*)^2}. \tag{23}$$

In other words, if λ_* ia the multiple root of $\Delta(\lambda;u)$, we have

$$\mathrm{rank}_\Lambda(u(x) - \partial^2 \log F(x,\lambda_*;u)) = n-1.$$

Before proving this theorem, we consider the meaning of this results. First of all, the spectral theoretical meaning of the above theorem is that if $\lambda = \lambda_*$ is the multiple root of the discriminant $\Delta(\lambda;u)$, i.e., $\{\lambda_*,\lambda_*\} \subset \Gamma^\natural(u)$, then

$$\Gamma^\natural(u^*_{\lambda_*,0}) = \Gamma^\natural(u) \setminus \{\lambda_*,\lambda_*\}.$$

In particular, if the eigenvalue λ_* is the double root of the discriminant $\Delta(\lambda;u)$, then λ_* is removed from the spectrum $\Gamma(u)$ by the Darboux transformation by the solution $\sqrt{F(x,\lambda_*;u)}$, i.e., $\Gamma(u^*_{\lambda_*,0}) = \Gamma(u) \setminus \{\lambda_*\}$. This fact corresponds to Crum's algorithm for the reflectionless (Bargmann) potentials. Therefore, we call the procedure constructing the Darboux transformation

$$u^*(x,\lambda_*;0) = u(x) - \partial^2 \log F(x,\lambda_*;u)$$

from $u(x)$ for the multiple root $\lambda_* \in \Gamma(u)$ of $\Delta(\lambda;u)$ *Darboux's removing algorithm* at λ_*. Moreover, we define the *repeated* Darboux's removing algorithm at λ_* as follows: Suppose that the multiplicity of $\lambda_* \in \Gamma(u)$ is $m \geq 4$ and let $k \in \boldsymbol{N}$ be $k \leq [\frac{m}{2}]$, where $[\alpha]$ denotes the greatest integer not exceeding α. Put

$$u^*_k = u^*_k(x;\lambda_*) = u^*_{k-1}(x;\lambda_*) - \partial^2 \log F(x,\lambda_*;u^*_{k-1}),$$

where $u^*_0(x;\lambda_*) = u(x)$, and

$$F(x,\lambda_*;u^*_k) = Z_{n-k}(u^*_k(x;\lambda_*) - \lambda_*) - \sum_{\nu=1}^{n-k} a_\nu(\lambda_*;u^*_k) Z_{\nu-1}(u^*_k(x;\lambda_*) - \lambda_*)$$

for the Λ-characteristic coefficients $a_\nu(\lambda_*; u_k^*)$ of $u_k^*(x;\lambda_*)$. Here, note that we have

$$\mathrm{rank}_\Lambda u_k^*(x;\lambda_*) = n - k$$

by Theorem. We call $u_k^*(x;\lambda_*)$ the k-th repeated Darboux transformation of $u(x)$ at $\lambda = \lambda_*$, and the procedure constructing $u_k^*(x;\lambda_*)$ from $u(x)$ the k-times repeated Darboux's removing algorithm at λ_*.

In what follows, by applying the repeated Darboux's removing algorithm at each multiple roots of $\Delta(\lambda; u)$, we show that the differential operator $H(u)$ such that $\mathrm{rank}_\Lambda u(x) < \infty$ can be transformed to the most reduced one such that all of its removable roots of $\Delta(\lambda; u)$ are completely removed by Darboux transformation.

We say that the root $\lambda_* \in \Gamma(u)$ is *odd* or *even* according as the multiplicity of λ_* as the root of the discriminant $\Delta(\lambda; u)$ is an odd number or an even number. Let $\Gamma_{\mathrm{odd}}(u)(\subset \Gamma(u))$ be the set of all odd roots of $\Delta(\lambda; u)$ and $\Gamma_{\mathrm{even}}(u)(\subset \Gamma(u))$ be the set of all even roots of $\Delta(\lambda; u)$. Since the cardinality $\sharp\Gamma^\natural(u)$ is an odd number, one can see immediately that $\sharp\Gamma_{\mathrm{odd}}(u)$ is also an odd number. Suppose that

$$\sharp\Gamma_{\mathrm{odd}}(u) = 2g - 1, \quad g \in \boldsymbol{N}$$

and

$$\Gamma_{\mathrm{odd}}(u) = \{\lambda_0, \lambda_1, \ldots, \lambda_{2g-2}\},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$.

Firstly, by Theorem, it turns out that if the multiplicity of $\lambda_* \in \Gamma_{\mathrm{even}}(u)$ is $2l, l \in \boldsymbol{N}$, then λ_* can be removed by the l times repeated Darboux's removing algorithm at λ_*. Therefore, by applying this procedure at each $\lambda_* \in \Gamma_{\mathrm{even}}(u)$, all of $\lambda_* \in \Gamma_{\mathrm{even}}(u)$ can be completely removed.

Secondly, by Theorem, one can see easily that if the multiplicity of $\lambda_* \in \Gamma_{\mathrm{odd}}(u)$, is $2l + 1, l \in \boldsymbol{N}$ then, by the l times repeated Darboux's removing algorithm at λ_*, the operotor $H(u)$ is reduced to the operator $H(u_l^*)$ such that λ_* is the simple root of $\Delta(\lambda; u_l^*)$. Therefore, applying succesively this procedure at each $\lambda_j \in \Gamma_{\mathrm{odd}}(u)$ $(j = 0, 1, \cdots, 2g-2)$, all of $\lambda_j \in \Gamma_{\mathrm{odd}}(u)$ can be reduced to the simple roots. Thus we have the following.

Corollary. *Suppose that* $\mathrm{rank}_\Lambda u(x) < \infty$ *and*

$$\Gamma_{\mathrm{odd}}(u) = \{\lambda_0, \lambda_1, \cdots, \lambda_{2g-2}\}$$

then, by applying the repeated Darboux's removing algorithm at each multiple roots of $\Delta(\lambda; u)$, *the operator* $H(u)$ *can be reduced to the operator* $H(u^*)$ *such that*

$$\mathrm{rank}_\Lambda u^*(x) = \frac{1}{2}(\sharp\Gamma_{\mathrm{odd}}(u) - 1) = g - 1,$$

$$\Delta(\lambda; u^*) = -4 \prod_{j=0}^{2g-2} (\lambda - \lambda_j),$$

i.e., $\Gamma(u^*) = \Gamma_{\mathrm{odd}}(u)$ *and all of the roots of the discriminant* $\Delta(\lambda; u^*)$ *are simple.*

REMARK. It is well known that the discrete spectrum does not determine the potential $u(x)$ itself even in the case of the Bargmann potential and the finite-zone potential. See e.g. [9], [14] [22]. Therefore we cannot conclude the uniqueness of such a reduced potential $u^*(x)$ as in Corollary. The potential $u^*(x)$ resulting from the repeated Darboux's removing algorithm seems to depend on the order of the roots as the removing object. We will treat this problem in the forthcoming paper.

3. Lemmas

For the proof of the main theorem stated in the preceding section, it is necessary to prove some lemmas.

First we prove the following formulae concerned with the binomial coefficients $\alpha_\nu^{(n)}$ defined by the recurrence relation (4).

Lemma 1. *If* $0 \le j \le n$ *then we have*

$$\begin{aligned} \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} \alpha_j^{(\nu-1)} &= 1, \\ \sum_{\nu=j}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} \alpha_j^{(\nu)} &= 0. \end{aligned}$$

Proof. These formulae have been already proved for $j = 0$ in [18, p.415 Proposition 4]. Now suppose $j < n$ and put

$$\begin{aligned} A_j^{(n)} &= \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} \alpha_j^{(\nu-1)}, \\ B_j^{(n)} &= \sum_{\nu=j}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} \alpha_j^{(\nu)}. \end{aligned}$$

By the recurrence relation (4), one immediately verifies

$$\begin{aligned} B_j^{(n)} &= \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} (\alpha_{j-1}^{(\nu-1)} + \alpha_j^{(\nu-1)}) - \alpha_j^{(n)} \alpha_j^{(j)} \\ &= \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} \alpha_{j-1}^{(\nu-1)} + \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1} \alpha_\nu^{(n)} \alpha_j^{(\nu-1)} - \alpha_j^{(n)} \alpha_{j-1}^{(j-1)} \end{aligned}$$

$$= \quad -A_{j-1}^{(n)} + A_j^{(n)}.$$

Here we used $\alpha_j^{(j)} = \alpha_{j-1}^{(j-1)} = 1$. Hence the recurrence relation

$$A_j^{(n)} - B_j^{(n)} = A_{j-1}^{(n)} \tag{24}$$

follows. On the other hand, similarly to the above, one has

$$\begin{aligned} A_j^{(n)} - B_j^{(n)} &= \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1}(\alpha_\nu^{(n)} + \alpha_{\nu-1}^{(n)})\alpha_j^{(\nu-1)} + (-1)^{n-j}\alpha_n^{(n)}\alpha_j^{(n)} \\ &= \sum_{\nu=j+1}^{n} (-1)^{\nu-j-1}\alpha_\nu^{(n+1)}\alpha_j^{(\nu-1)} + (-1)^{n-j}\alpha_{n+1}^{(n+1)}\alpha_j^{(n)} \\ &= A_j^{(n+1)}. \end{aligned}$$

Thus we have

$$A_j^{(n)} - B_j^{(n)} = A_j^{(n+1)}. \tag{25}$$

By (24) and (25), we have

$$A_j^{(n+1)} = A_{j-1}^{(n)}, \quad j = 0, 1, \cdots, n-1.$$

On the other hand, since $A_0^{(n)} = 1$ [18, Proposition 4], $A_j^{(n)} = 1$ follows. Hence, $B_j^{(n)} = 0$ follows immediately from (24). □

By lemma 1, we can extremely simplify the expression of $F(x, \lambda; u)$ as follows. This simplification is not only essential for our consideration but also useful for the explicit construction of certain kind of solutions to the evolution equations of KdV type (cf. [20, p.14 Theorem 11]).

Lemma 2. *Suppose that $n = \mathrm{rank}_\Lambda u(x)$, $n \in \boldsymbol{N}$ and ρ_ν, $\nu = 0, 1, \cdots, n$ are the Λ-characteristic coefficients of $u(x)$;*

$$Z_{n+1}(u(x)) = \sum_{\nu=0}^{n} \rho_\nu Z_\nu(u(x)).$$

Put

$$p_j(\lambda; u) = \lambda^{n-j} - \sum_{\nu=j+1}^{n} \rho_\nu \lambda^{\nu-j-1},$$

then we have

$$F(x,\lambda;u) = Z_n(u(x)) + \sum_{j=0}^{n-1} p_j(\lambda;u) Z_j(u(x)).$$

Proof. By the expansion formula (5), one easily verifies

$$\begin{aligned}
&F(x,\lambda;u)\\
&= Z_n(u(x)-\lambda) - \sum_{\nu=1}^{n} a_\nu(\lambda;u) Z_\nu(u(x)-\lambda)\\
&= \sum_{j=0}^{n}(-1)^{n-j}\alpha_j^{(n)} Z_j(u(x))\lambda^{n-j} - \sum_{\nu=1}^{n}(-\alpha_\nu^{(n+1)}\lambda^{n-\nu+1}\\
&\quad + \sum_{\mu=\nu}^{n}\alpha_\nu^{(\mu)}\rho_\mu\lambda^{\mu-\nu})\sum_{j=0}^{\nu-1}(-1)^{\nu-j-1}\alpha_j^{(\nu-1)} Z_j(u(x))\lambda^{\nu-j-1}\\
&= \sum_{j=0}^{n}(-1)^{n-j}\alpha_j^{(n)} Z_j(u(x))\lambda^{n-j}\\
&\quad + \sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}\{(-1)^{\nu-j-1}\alpha_\nu^{(n+1)}\alpha_j^{(\nu-1)}\lambda^{n-j}\\
&\quad - \sum_{\mu=\nu}^{n}(-1)^{\nu-j-1}\alpha_\nu^{(\mu)}\alpha_j^{(\nu-1)}\rho_\mu\lambda^{\mu-j-1}\} Z_j(u(x))\\
&= Z_n(u(x)) + \sum_{j=0}^{n-1}\{((-1)^{n-j}\alpha_j^{(n)} + \sum_{\nu=j+1}^{n}(-1)^{\nu-j-1}\alpha_\nu^{(n+1)}\alpha_j^{(\nu-1)})\lambda^{n-j}\\
&\quad - \sum_{\nu=j+1}^{n}\sum_{\mu=\nu}^{n}(-1)^{\nu-j-1}\alpha_\nu^{(\mu)}\alpha_j^{(\nu-1)}\rho_\mu\lambda^{\mu-j-1}\} Z_j(u(x)).
\end{aligned}$$

Let $j < n$ and put

$$\begin{aligned}
p_j(\lambda;u) &= ((-1)^{n-j}\alpha_j^{(n)} + \sum_{\nu=j+1}^{n}(-1)^{\nu-j-1}\alpha_\nu^{(n+1)}\alpha_j^{(\nu-1)})\lambda^{n-j}\\
&\quad - \sum_{\nu=j+1}^{n}\sum_{\mu=\nu}^{n}(-1)^{\nu-j-1}\alpha_\nu^{(\mu)}\alpha_j^{(\nu-1)}\rho_\mu\lambda^{\mu-j-1}.
\end{aligned}$$

Note $\alpha_{j+1}^{(n+1)} = 1$, then, by lemma 1, we have

$$p_j(\lambda;u) = \sum_{\nu=j+1}^{n+1}(-1)^{\nu-j-1}\alpha_\nu^{(n+1)}\alpha_j^{(\nu-1)}\lambda^{n-j}$$

$$- \sum_{\mu=j+1}^{n} \left(\sum_{\nu=j+1}^{\mu} (-1)^{\nu-j-1} \alpha_{\nu}^{(\mu)} \alpha_{j}^{(\nu-1)} \right) \rho_{\mu} \alpha^{\mu-j-1}$$

$$= \lambda^{n-j} - \sum_{\mu=j+1}^{n} \rho_{\mu} \lambda^{\mu-j-1}.$$

This completes the proof. □

4. Proof of Theorem

In this section we prove Theorem stated in §2. Throughout this section, we assume that $n = \mathrm{rank}_{\Lambda} u(x)$, $n \in \boldsymbol{N}$ and $\lambda_* \in \Gamma(u)$.

By the definition, we have

$$Z_{n+1}(u(x) - (k+\lambda)) - \sum_{\nu=0}^{n} a_{\nu}(k+\lambda; u) Z_{\nu}(u(x) - (k+\lambda)) = 0$$

and

$$Z_{n+1}((u(x) - k) - \lambda) - \sum_{\nu=0}^{n} a_{\nu}(\lambda; u-k) Z_{\nu}((u(x) - k) - \lambda) = 0.$$

Hence, by the uniqueness of the Λ-characteristic coefficients, we have

$$a_{\nu}(\lambda; u-k) = a_{\nu}(\lambda + k; u), \quad \nu = 0, 1, \cdots, n$$

for any λ, $k \in \boldsymbol{C}$. This also implies

$$F(x, \lambda; u-k) = F(x, \lambda + k; u)$$

and

$$\Delta(\lambda; u-k) = \Delta(\lambda + k; u). \tag{26}$$

Hence if $\lambda_* \in \Gamma(u)$ then $0 \in \Gamma(u - \lambda_*)$ follows.

By lemma 2, one can express $F(x, \lambda; u - \lambda_*)$ as

$$F(x, \lambda; u - \lambda_*) = \sum_{\nu=0}^{n} G_{\nu}(x) \lambda^{\nu}.$$

Now let us compute the first two coefficients $G_\nu(x)$, $\nu = 0, 1$. Firstly, by lemma 2, one verifies

$$\begin{aligned} F(x,\lambda;u-\lambda_*) &= Z_n(u(x)-\lambda_*) + \sum_{j=0}^{n-1} p_j(\lambda;u-\lambda_*)Z_j(u(x)-\lambda_*) \\ &= \sum_{j=0}^{n} Z_j(u(x)-\lambda_*)\lambda^{n-j} - \sum_{j=0}^{n-1}\sum_{\mu=j+1}^{n} a_\mu(\lambda_*;u)Z_j(u(x)-\lambda_*)\lambda^{\mu-j-1}. \end{aligned}$$

Hence one has immediately

$$G_0(x) = Z_n(u(x)-\lambda_*) - \sum_{j=0}^{n-1} a_{j+1}(\lambda_*;u)Z_j(u(x)-\lambda_*) = F(x,\lambda_*;u) \tag{27}$$

and

$$G_1(x) = Z_{n-1}(u(x)-\lambda_*) - \sum_{j=0}^{n-2} a_{j+2}(\lambda_*;u)Z_j(u(x)-\lambda_*). \tag{28}$$

Since $\Delta(\lambda;u-\lambda_*)$ is the polynomial of degree $2n+1$ in λ with the constant coefficients [18, p.420 Corollary 10], put

$$\Delta(\lambda;u-\lambda_*) = \sum_{j=0}^{2n+1} \Delta_j\lambda^j.$$

Then, by direct calculation, one verifies

$$\begin{aligned} \Delta_0 &= G_0'(x)^2 - 2G_0(x)G_0''(x) + 4(u(x)-\lambda_*)G_0(x)^2, \\ \Delta_1 &= 2G_0'(x)G_1'(x) - 2G_0(x)G_1''(x) \\ &\quad - 2G_0''(x)G_1(x) + 8(u(x)-\lambda_*)G_0(x)G_1(x) - 4G_0(x)^2. \end{aligned} \tag{29}$$

By (19),(26) and (27), we have

$$\Delta_0 = \Delta(0;u-\lambda_*) = \Delta(\lambda_*;u) = 0.$$

Now let us assume that $\lambda = \lambda_*$ is the multiple root of the Λ-spectral discriminant $\Delta(\lambda;u)$. Then, by (26), $\lambda = 0$ is the multiple root of

$$\Delta(\lambda;u-\lambda_*) = 0.$$

Hence $\Delta_1 = 0$ follows.

Put

$$J(x,\alpha) = Z_n(u^*(x,\lambda_*;\alpha) - \lambda_*) - \sum_{\nu=0}^{n-1} a_{\nu+1}(\lambda_*;u) Z_\nu(u^*(x,\lambda_*;\alpha) - \lambda_*)$$

for $(x,\alpha) \in D \times \boldsymbol{P}_1$. Then we have the following.

Lemma 3. *If $\lambda = \lambda_*$ is the multiple root of the Λ-spectral discriminant $\Delta(\lambda;u)$, then $J(x,0)$ does not depend on x.*

Proof. By the fundamental equality (21) and the expression (28) of $G_1(x)$, we have

$$\begin{aligned}&B_-(\lambda_*,0)G_1(x)\\ &\quad = B_+(\lambda_*,0)(Z_{n-1}(u^*(x,\lambda_*;0) - \lambda_*) - \sum_{\nu=0}^{n-2} a_{\nu+2}(\lambda_*;u) Z_\nu(u^*(x,\lambda_*;0) - \lambda_*)).\end{aligned}$$

Operate with $\frac{1}{4}B_-(\lambda_*,0) \cdot \partial$ on the both sides of the above, then, by the K-W factorization (20) and the recurrence relation (2), the right hand side becomes

$$\begin{aligned}&(\partial \cdot \Lambda(u^*_{\lambda_*,0} - \lambda_*))(Z_{n-1}(u^*(x,\lambda_*;0) - \lambda_*)\\ &\qquad\qquad - \sum_{\nu=0}^{n-2} a_{\nu+2}(\lambda_*;u) Z_\nu(u^*(x,\lambda_*;0) - \lambda_*))\\ &= \partial(Z_n(u^*(x,\lambda_+;0) - \lambda_*) - \sum_{\nu=1}^{n-1} a_{\nu+1}(\lambda_*;u) Z_\nu(u^*(x,\lambda_*;0) - \lambda_*))\\ &= \partial(J(x,0) + a_1(\lambda_*;u))\\ &= J_x(x,0).\end{aligned}$$

On the other hand, by the definition, one immediately verifies

$$B_\pm(\lambda_*,0) = \pm\partial + \frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)}.$$

Moreover, by the definition of the operators $B_\pm(\lambda_*,\alpha)$ and the K-W factorization (19), the left hand side becomes

$$\frac{1}{4}(B_-(\lambda_*,0) \cdot \partial \cdot B_-(\lambda_*,0))G_1(x)$$

$$= -\frac{1}{4}((B_+(\lambda_*,0) - 2\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)}) \cdot \partial \cdot B_-(\lambda_*,0))G_1(x)$$
$$= -(\partial \cdot \Lambda(u-\lambda_*))G_1(x) + \frac{1}{2}\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)}(\partial \cdot B_-(\lambda_*,0))G_1(x).$$

By the recurrence relation (2), one verifies

$$\begin{aligned} \Lambda(u-\lambda_*)G_1(x) &= Z_n(u(x)-\lambda_*) - \sum_{j=0}^{n-2} a_{j+2}(\lambda_*;u)Z_{j+1}(u(x)-\lambda_*) \\ &= F(x,\lambda_*;u) + a_1(\lambda_*;u). \end{aligned}$$

Hence, by straightforward calculation, one obtains

$$J_x(x,0) = -F_x(x,\lambda_*;u) + \frac{1}{2}\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)^3}(-G_0(x)^2G_1''(x)$$
$$-G_0(x)G_0''(x)G_1(x) + G_0(x)G_0'(x)G_1'(x) + 4(u(x)-\lambda_*)G_0(x)^2G_1(x)).$$

On the other hand, by (29), we have

$$\frac{1}{2}G_0(x)\Delta_1 = G_0(x)G_0'(x)G_1'(x) - G_0(x)G_0''(x)G_1(x)$$
$$-G_0(x)^2G_1''(x) + 4(u(x)-\lambda_*)G_0(x)^2G_1(x) - 2G_0(x)^3 = 0,$$

where we used $\Delta_1 = 0$. Hence, by (27), one has immediately

$$J_x(x,0) = -F_x(x,\lambda_*;u) + \frac{1}{2}\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)^3}(\frac{1}{2}G_0(x)\Delta_1 + 2G_0(x)^3) = 0.$$

This completes the proof. □

Next we have the following.

Lemma 4. *We have*

$$\Sigma_-(u;\lambda_*) = \{\alpha \in \boldsymbol{P}_1 \mid J(x,\alpha) \equiv 0 \textit{ for all } x \in D\}.$$

Proof. Put

$$Y = \{\alpha \in \boldsymbol{P}_1 \mid J(x,\alpha) \equiv 0 \text{ for all } x \in D\}.$$

Firstly, suppose $\alpha \in \Sigma_-(u;\lambda_*)$. Then, there uniquely exist the Λ-characteristic coefficients ρ_ν^*, $\nu = 0, 1, \cdots, n-1$ of $u^*_{\lambda_*,\alpha} - \lambda$ such that

$$Z_n(u^*(x,\lambda_*;\alpha) - \lambda_*) - \sum_{\nu=0}^{n-1} \rho_\nu^* Z_\nu(u^*(x,\lambda_*;\alpha) - \lambda_*) = 0.$$

By the fundamental equality (21), we have

$$\begin{aligned} &B_-(\lambda_*,\alpha)(Z_n(u(x)-\lambda_*) - \sum_{\nu=0}^{n-1} \rho_\nu^* Z_\nu(u(x)-\lambda_*)) \\ &\quad = B_+(\lambda_*,\alpha)(Z_n(u^*(x,\lambda_*;\alpha)-\lambda_*) - \sum_{\nu=0}^{n-1} \rho_\nu^* Z_\nu(u^*(x,\lambda_*;\alpha)-\lambda_*)) = 0. \end{aligned}$$

Operate with $\dfrac{1}{4}\partial^{-1} \cdot B_+(\lambda_*,\alpha) \cdot \partial$ on the both sides of the above, then, by the K-W factorization (19) and the recurrence relation (2), one verifies that there is a constant κ such that

$$Z_{n+1}(u(x)-\lambda_*) - \sum_{\nu=0}^{n-1} \rho_\nu^* Z_{\nu+1}(u(x)-\lambda_*) = \kappa.$$

By (6) and the uniqueness of the Λ-characteristic coefficients, we have

$$a_\nu(\lambda_*;u) = \begin{cases} \rho_{\nu-1}^*, & 1 \le \nu \le n \\ \kappa, & \nu = 0. \end{cases} \tag{30}$$

This implies that $J(x,\alpha) \equiv 0$ for $\alpha \in \Sigma_-(u;\lambda_*)$. Thus we have

$$\Sigma_-(u;\lambda_*) \subset Y.$$

Conversely, if $\alpha \in Y$ then

$$\mathrm{rank}_\Lambda u^*(x,\lambda_*;\alpha) = n-1,$$

i.e. $Y \subset \Sigma_-(u;\lambda_*)$. Therefore

$$\Sigma_-(u;\lambda_*) = Y$$

follows. □

By the fundamental equality (21), one can see immediately

$$\begin{aligned}B_{+}(\lambda_*,\alpha)(Z_n(u^*(x,\lambda_*;\alpha)-\lambda_*)-\sum_{\nu=0}^{n-1}a_{\nu+1}(\lambda_*;u)Z_\nu(u^*(x,\lambda_*;\alpha)-\lambda_*))\\ =B_{-}(\lambda_*,\alpha)(Z_n(u(x)-\lambda_*)-\sum_{\nu=0}^{n-1}a_{\nu+1}(\lambda_*;u)Z_\nu(u(x)-\lambda_*)).\end{aligned}$$

Hence, by the definitions of $J(x,\alpha)$ and $F(x,\lambda_*;u)$, we have

$$B_{+}(\lambda_*,\alpha)J(x,\alpha)=B_{-}(\lambda_*,\alpha)F(x,\lambda_*;u). \tag{31}$$

The relation (31) can be regarded as the 1-st order nonhomogeneous ordinary differential equation for $J(x,\alpha)$. Therefore, by integrating (31), we have

$$J(x,\alpha)=F(x,\lambda_*;u)-\frac{1}{f(x,\lambda_*;\alpha)^2}\int f(x,\lambda_*;\alpha)^2F_x(x,\lambda_*;u)dx, \tag{32}$$

where $f(x,\lambda_*;\alpha)$ is defined by (15) and the integration constant is appropriately chosen. Now suppose $\alpha\in\Sigma_{-}(u;\lambda_*)$, i.e., $J(x,\alpha)\equiv 0$. Moreover, multiply the both sides of (32) by $f(x,\lambda_*;\alpha)^2$, and differentiate the both sides of it with respect to x, then we have

$$\begin{aligned}&2F(x,\lambda_*;u)f(x,\lambda_*;\alpha)f_x(x,\lambda_*;\alpha)\\&\quad+F_x(x,\lambda_*;u)f(x,\lambda_*;\alpha)^2-2F_x(x,\lambda_*;u)f(x,\lambda_*;\alpha)^2=0.\end{aligned}$$

Thus

$$\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)}=2\frac{f_x(x,\lambda_*;\alpha)}{f(x,\lambda_*;\alpha)}$$

follows. If $\alpha_1,\ \alpha_2\in\Sigma_{-}(u;\lambda_*)$ and $\alpha_1,\ \alpha_2\neq\infty$ then, since the left hand side of the above does not depend on α, one verifies

$$\begin{aligned}0&=\frac{f_x(x,\lambda_*;\alpha_1)}{f(x,\lambda_*;\alpha_1)}-\frac{f_x(x,\lambda_*;\alpha_2)}{f(x,\lambda_*;\alpha_2)}\\&=\frac{(\alpha_1-\alpha_2)W[f_1^0(x,\lambda_*),f_2^0(x,\lambda_*)]}{f(x,\lambda_*;\alpha_1)f(x,\lambda_*;\alpha_2)}\\&=\frac{\alpha_1-\alpha_2}{f(x,\lambda_*;\alpha_1)f(x,\lambda_*;\alpha_2)}.\end{aligned}$$

This implies $\alpha_1=\alpha_2$. Similarly to the above, one can show that $\alpha_1=\alpha_2$ holds even in the case $\alpha_1=\infty$. Thus we have the following.

Lemma 5. *If $\Sigma_-(u;\lambda_*)\neq\emptyset$ and $\alpha\in\Sigma_-(u;\lambda_*)$ then $\Sigma_-(u;\lambda_*)=\{\alpha\}$ follows.*

On the other hand, by (31), we have

$$B_+(\lambda_*,0)J(x,0)=B_-(\lambda_*,0)F(x,\lambda_*;u).$$

Since

$$B_-(\lambda_*,0)F(x,\lambda_*;u)=-F_x(x,\lambda_*;u)+\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)}F(x,\lambda_*;u)=0,$$

we have

$$J_x(x,0)+\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)}J(x,0)=0.$$

On the other hand, it is easy to see that $F(x,\lambda_*;u)$ is a constant if and only if $\mathrm{rank}_\Lambda u(x)=0$, i.e., $u(x)$ is the constant function. Hence, by lemma 3, $J(x,0)\equiv 0$ follows. This implies that if we choose $f_1^0(x,\lambda_*)$ and $f_2^0(x,\lambda_*)$ defined by (13) and (14) as the fundamental system of solutions of the eigenvalue problem (8) for $\lambda=\lambda_*$, then $0\in\Sigma_-(u;\lambda_*)$ follows. Hence, by lemma 5, $\Sigma_-(u;\lambda_*)=\{0\}$ also follows. Thus, we have proved that if $\lambda=\lambda_*$ is the multiple root of the discriminant $\Delta(\lambda;u)$ then $\Sigma_-(u;\lambda_*)=\{0\}\neq\emptyset$.

Conversely, now assume that $\Sigma_-(u;\lambda_*)\neq\emptyset$, and let us compute the polynomial $\Delta(\lambda;u^*_{\lambda_*,\alpha})$ for $\alpha\in\Sigma_-(u;\lambda_*)$. However, the computation is very complicated. Hence we must devise a method to arrange the expressions of $\Delta(\lambda;u)$ and $\Delta(\lambda;u^*_{\lambda_*,\alpha})$. For this purpose, firstly we prove the following.

Lemma 6. *If $\Sigma_-(u;\lambda_*)=\{0\}$ then we have*

$$B_-(\lambda_*,0)F(x,\lambda;u-\lambda_*)=\lambda B_+(\lambda_*,0)F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*)$$

for any $\lambda\in\boldsymbol{C}$.

Proof. By lemma 4, we have

$$J(x,0)=Z_n(u^*(x,\lambda_*;0)-\lambda_*)-\sum_{\nu=0}^{n-1}a_{\nu+1}(\lambda_*;u)Z_\nu(u^*(x,\lambda_*;0)-\lambda_*)\equiv 0.$$

This implies

$$\Omega(\mu;u^*_{\lambda_*,0}-\lambda_*)=\mu^n-\sum_{\nu=0}^{n-1}a_{\nu+1}(\lambda_*;u)\mu^\nu,$$

where $\Omega(\lambda; u)$ is defined by (11). On the other hand, by(11), one has

$$\begin{aligned}\Omega(\mu; u-\lambda_*) &= \mu^{n+1} - \sum_{\nu=0}^{n} a_\nu(\lambda_*; u)\mu^\nu \\ &= \mu\Omega(\mu; u^*_{\lambda_*,0} - \lambda_*) - a_0(\lambda_*; u).\end{aligned} \tag{33}$$

Moreover, since $\mathrm{rank}_\Lambda u^*(x, \lambda_*; 0) = n-1$ by the assumption, lemma 2 implies

$$\begin{aligned}&F(x, \lambda; u^*_{\lambda_*,0} - \lambda_*) \\ &\quad = Z_{n-1}(u^*(x, \lambda_*; 0) - \lambda_*) + \sum_{j=0}^{n-2} p_j(\lambda; u^*_{\lambda_*,0} - \lambda_*) Z_j(u^*(x, \lambda_*; 0) - \lambda_*).\end{aligned}$$

By (30) and lemma 2, one verifies

$$\begin{aligned}p_j(\lambda; u^*_{\lambda_*,0} - \lambda_*) &= \lambda^{n-j-1} - \sum_{\nu=j+1}^{n-1} \rho^*_\nu \lambda^{\nu-j-1} \\ &= \lambda^{n-j-1} - \sum_{\nu=j+1}^{n-1} a_{\nu+1}(\lambda_*; u)\lambda^{\nu-j-1} \\ &= \lambda^{n-(j+1)} - \sum_{\nu=j+2}^{n} a_\nu(\lambda_*; u)\lambda^{\nu-(j+2)} \\ &= p_{j+1}(\lambda; u-\lambda_*)\end{aligned} \tag{34}$$

for $j = 0, 1, \cdots, n-2$. Moreover, by (33), we have

$$\begin{aligned}p_0(\lambda; u-\lambda_*) &= \lambda^n - \sum_{\nu=1}^{n} a_\nu(\lambda_*; u)\lambda^{\nu-1} \\ &= \frac{1}{\lambda}(\Omega(\lambda; u-\lambda_*) + a_0(\lambda_*; u)) \\ &= \Omega(\lambda; u^*_{\lambda_*,0} - \lambda_*).\end{aligned} \tag{35}$$

By (12) and (35), we have

$$\begin{aligned}\Lambda(u^*_{\lambda_*,0} - \lambda_*)F(x, \lambda; u^*_{\lambda_*,0} - \lambda_*) &= \lambda F(x, \lambda; u^*_{\lambda_*,0} - \lambda_*) - \Omega(\lambda; u^*_{\lambda_*,0} - \lambda_*) \\ &= \lambda F(x, \lambda; u^*_{\lambda_*,0} - \lambda_*) - p_0(\lambda; u-\lambda_*).\end{aligned} \tag{36}$$

On the other hand, by (34), we have

$$\Lambda(u^*_{\lambda_*,0} - \lambda_*)F(x, \lambda; u^*_{\lambda_*,0} - \lambda_*)$$

$$(37)\qquad = Z_n(u^*(x,\lambda_*;0)-\lambda_*)+\sum_{j=0}^{n-2} p_j(\lambda;u^*_{\lambda_*,0}-\lambda_*)Z_{j+1}(u^*(x,\lambda_*;0)-\lambda_*)$$
$$= Z_n(u^*(x,\lambda_*;0)-\lambda_*)+\sum_{j=0}^{n-2} p_{j+1}(\lambda;u-\lambda_*)Z_{j+1}(u^*(x,\lambda_*;0)-\lambda_*).$$

Comparing the right hand sides of (36) and (37), we have

$$(38)\qquad \lambda F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*)$$
$$= Z_n(u^*(x,\lambda_*;0)-\lambda_*)+\sum_{j=0}^{n-1} p_j(\lambda;u-\lambda_*)Z_j(u^*(x,\lambda_*;0)-\lambda_*).$$

Hence, by the fundamental equality (21) and (38), we have finally

$$B_-(\lambda_*,0)F(x,\lambda;u-\lambda_*)$$
$$= B_-(\lambda_*,0)(Z_n(u(x)-\lambda_*)+\sum_{j=0}^{n-1} p_j(\lambda;u-\lambda_*)Z_j(u(x)-\lambda_*))$$
$$= B_+(\lambda_*,0)(Z_n(u^*(x,\lambda_*;0)-\lambda_*)+\sum_{j=0}^{n-1} p_j(\lambda;u-\lambda_*)Z_j(u^*(x,\lambda_*;0)-\lambda_*))$$
$$= \lambda B_+(\lambda_*,0)F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*)$$

This completes the proof. □

Now put

$$\phi(x,\lambda) = F(x,\lambda;u-\lambda_*)-\lambda F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*)$$

and

$$(39)\qquad \psi(x,\lambda) = F(x,\lambda;u-\lambda_*)-\frac{1}{2}\phi(x,\lambda).$$

Then we have the following lemma, which extremely simplifies the computation of $\Delta(\lambda;u^*_{\lambda_*,0})$.

Lemma 7. *If* $\Sigma_-(u;\lambda_*)=\{0\}$, *then we have*

$$(40)\qquad B_+(\lambda_*,0)\phi(x,\lambda) = 2F_x(x,\lambda;u-\lambda_*),$$

$$(41)\qquad B_-(\lambda_*,0)\phi(x,\lambda) = 2\lambda F_x(x,\lambda;u^*_{\lambda_*,0}-\lambda_*),$$

$$(42)\qquad \psi(x,\lambda) = \lambda F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*)+\frac{1}{2}\phi(x,\lambda)$$

for any $\lambda \in \boldsymbol{C}$.

Proof. By lemma 6, one verifies

$$
\begin{aligned}
&B_+(\lambda_*,0)\phi(x,\lambda)\\
&\quad = B_+(\lambda_*,0)F(x,\lambda;u-\lambda_*) - \lambda B_+(\lambda_*,0)F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*)\\
&\quad = B_+(\lambda_*,0)F(x,\lambda;u-\lambda_*) - B_-(\lambda_*,0)F(x,\lambda;u-\lambda_*)\\
&\quad = 2F_x(x,\lambda;u-\lambda_*).
\end{aligned}
$$

Thus, (40) is proved. Similarly to the above, one can prove (41). On the other hand, (42) follows immediately by (40) and (41). □

Now let us rewrite the discriminants $\Delta(\lambda;u-\lambda_*)$ and $\Delta(\lambda;u^*_{\lambda_*,0}-\lambda_*)$ in terms of $\phi(x,\lambda)$ and $\psi(x,\lambda)$. By the definitions of $\Delta(\lambda;u-\lambda_*)$ and $\Delta(\lambda;u^*_{\lambda_*,0}-\lambda_*)$ and lemma 7, we have

$$
\begin{aligned}
&\Delta(\lambda;u-\lambda_*)\\
&\quad = \frac{1}{4}(B_+(\lambda_*,0)\phi(x,\lambda))^2 - (\frac{1}{2}\phi(x,\lambda)+\psi(x,\lambda))\partial\cdot B_+(\lambda_*,0)\phi(x,\lambda)\\
&\qquad + 4(u(x)-\lambda_*-\lambda)(\frac{1}{2}\phi(x,\lambda)+\psi(x,\lambda))^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda^2\Delta(\lambda;u^*_{\lambda_*,0}-\lambda_*)\\
&\quad = \frac{1}{4}(B_-(\lambda_*,0)\phi(x,\lambda))^2 - (-\frac{1}{2}\phi(x,\lambda)+\psi(x,\lambda))\partial\cdot B_-(\lambda_*,0)\phi(x,\lambda)\\
&\qquad + 4(u(x)-2q_x(x,\lambda_*;0)-\lambda_*-\lambda)(-\frac{1}{2}\phi(x,\lambda)+\psi(x,\lambda))^2,
\end{aligned}
$$

where we used (18). Hence, by straightforward calculation, one verifies

$$
\begin{aligned}
&\Delta(\lambda;u-\lambda_*) - \lambda^2\Delta(\lambda;u^*_{\lambda_*,0}-\lambda_*)\\
&\quad = 8(q_x(x,\lambda_*;0)\psi(x,\lambda) - q_x(x,\lambda_*;0)\phi(x,\lambda)\\
&\qquad + (u(x)-\lambda_*-\lambda)\phi(x,\lambda) - \frac{1}{4}\phi_{xx}(x,\lambda))\psi(x,\lambda)\\
&\quad = (4(\partial\cdot B_-(\lambda_*,0))F(x,\lambda;u-\lambda_*) - 8\lambda\phi(x,\lambda))\psi(x,\lambda).
\end{aligned}
$$

Put

$$
g(x) = 4(\partial\cdot B_-(\lambda_*,0))F(x,\lambda;u-\lambda_*) - 8\lambda\phi(x,\lambda),
$$

then, by (19), (12) and (40), we have

$$\frac{1}{4}B_+(\lambda_*,0)g(x)$$
$$= 4(\partial\cdot\Lambda(u-\lambda_*))F(x,\lambda;u-\lambda_*) - 2\lambda B_+(\lambda_*,0)\phi(x,\lambda) = 0.$$

This implies that $g(x)$ solves the homogeneous 1-st order ordinary differential equation

$$g'(x) + 2q(x,\lambda_*;0)g(x) = 0.$$

Since

$$q(x,\lambda_*;0) = \frac{1}{2}\frac{F_x(x,\lambda_*;u)}{F(x,\lambda_*;u)},$$

one can easily see

$$\frac{1}{F(x,\lambda_*;u)} \in \ker B_+(\lambda_*,0)\setminus\{0\}.$$

Hence there exists the rational function $c_1(\lambda)$ of λ such that

$$g(x) = \frac{c_1(\lambda)}{F(x,\lambda_*;u)}.$$

Hence we have

$$\Delta(\lambda;u-\lambda_*) - \lambda^2\Delta(\lambda;u^*_{\lambda_*,0}-\lambda_*) = \frac{c_1(\lambda)\psi(x,\lambda)}{F(x,\lambda_*;u)}.$$

Note that the left hand side of the above does not depend on x. Hence, if $c_1(\lambda)$ does not identically vanish, then there exists the rational function $c_2(\lambda)$ of λ, which is independent of x, such that

$$\psi(x,\lambda) = c_2(\lambda)F(x,\lambda_*;u).$$

Therefore, by lemma 6,

$$B_-(\lambda_*,0)\psi(x,\lambda) = 0$$

follows. On the other hand, by (39) and (42), we have

$$\psi(x,\lambda) = \frac{1}{2}(F(x,\lambda;u-\lambda_*) + \lambda F(x,\lambda;u^*_{\lambda_*,0}-\lambda_*))$$

Hence, by lemma 6, one verifies

$$\begin{aligned}
0 &= B_-(\lambda_*, 0)\psi(x,\lambda) \\
&= \frac{1}{2}B_-(\lambda_*, 0)(F(x,\lambda; u-\lambda_*) + \lambda F(x,\lambda; u^*_{\lambda_*,0} - \lambda_*)) \\
&= \frac{1}{2}(\lambda B_+(\lambda_*, 0)F(x,\lambda; u^*_{\lambda_*,0} - \lambda_*) + \lambda B_-(\lambda_*, 0)F(x,\lambda; u^*_{\lambda_*,0} - \lambda_*)) \\
&= 2\lambda q(x,\lambda_*; 0)F(x,\lambda; u^*_{\lambda_*,0} - \lambda_*).
\end{aligned}$$

This is contradiction. Hence $c_1(\lambda) \equiv 0$ follows. Thus we have proved

$$\Delta(\lambda; u-\lambda_*) = \lambda^2\Delta(\lambda; u^*_{\lambda^*,0} - \lambda_*).$$

By (26), we have

$$\Delta(\lambda; u) = (\lambda-\lambda_*)^2\Delta(\lambda; u^*_{\lambda_*,0}).$$

Hence, (23) in Theorem is proved. Moreover, this formula implies simultaneously that if $\Sigma_-(u;\lambda_*) \neq \emptyset$ then $\lambda = \lambda_*$ is the multiple root of the Λ-spectral discriminant $\Delta(\lambda; u)$. This completes the proof of Theorem.

5. Illustrative Examples

In this section let us investigate some illustrative examples. Throughout this section, we make extensive use of the computer algebra system *Mathematica* 2.2 [24]. Several computations appeared in this section are impossible to do without such computer algebra system.

First we investigate the elliptic function. Let $\wp(x)$ be the Weierstrass elliptic function with the invariants g_2 and g_3. Then $\wp(x)$ satisfies the following two differential equations [12];

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3, \tag{43}$$

$$\wp''(x) = 6\wp(x)^2 - \frac{1}{2}g_2. \tag{44}$$

Put $u(x) = 2\wp(x)$ then, by [18, p.424 Corollary 14], $\mathrm{rank}_\Lambda u(x) = 1$ follows. In fact, by (44), one can see

$$\frac{1}{8}(3u(x)^2 - u''(x)) - \frac{1}{8}g_2 = 0. \tag{45}$$

Since

$$Z_0(u(x)) = 1, \quad Z_1(u(x)) = \frac{1}{2}u(x), \quad Z_2(u(x)) = \frac{1}{8}(3u(x)^2 - u''(x)),$$

(45) implies

$$Z_2(u(x)) - \frac{1}{8}g_2 Z_0(u(x)) = 0.$$

Thus the Λ-characteristic coefficients of $u(x)$ are

$$\rho_0 = \frac{1}{8}g_2, \quad \rho_1 = 0.$$

Hence, by (7), one verifies

$$\begin{aligned} a_1(\lambda; u) &= -\alpha_1^{(2)}\lambda + \alpha_1^{(1)}\rho_1 = -\frac{3}{2}\lambda, \\ a_0(\lambda; u) &= -\alpha_0^{(2)}\lambda^2 + \alpha_0^{(1)}\rho_1\lambda + \alpha_0^{(0)}\rho_0 = -\frac{3}{8}\lambda^2 + \frac{1}{8}g_2. \end{aligned}$$

Therefore one has immediately

$$\begin{aligned} F(x, \lambda; u) &= Z_1(u(x) - \lambda) - a_1(\lambda; u)Z_0(u(x) - \lambda) \\ &= \frac{1}{2}(u(x) - \lambda) + \frac{3}{2}\lambda \\ &= \frac{1}{2}u(x) + \lambda \end{aligned}$$

Hence, using (43), one verifies

$$\Delta(\lambda; u) = -4\lambda^3 + g_2\lambda - g_3.$$

Since $16(g_2^3 - 27g_3^2) \neq 0$ is the discriminant of the equation $\Delta(\lambda; u) = 0$, this cubic polynomial has no multiple roots. Therefore, by Theorem, we have

$$\mathrm{rank}_\Lambda u^*(x, \lambda_*; \alpha) \geq 1$$

for any $\lambda_* \in \Gamma(u)$ and $\alpha \in \boldsymbol{P}_1$.

Actually, we can show this fact directly as follows: Assume

$$\mathrm{rank}_\Lambda u^*(x, \lambda_*; \alpha) = 0$$

for some $\lambda_* \in \Gamma(u)$ and $\alpha \in \boldsymbol{P}_1$. Then, one can see immediately that $u^*(x, \lambda_*; \alpha) \equiv c$ holds for some constant $c \in \boldsymbol{C}$. Hence we have

$$\begin{aligned} u(x) &= u^*(x, \lambda_*; \alpha) + 2\partial^2 \log g(x, \lambda_*; \alpha) \\ &= c + 2\partial^2 \log g(x, \lambda_*; \alpha). \end{aligned}$$

On the other hand, by Darboux's lemma [21, pp 88-91],

$$g(x) = \frac{1}{g(x, \lambda_*; \alpha)}$$

solves the differential equation

$$H(c)g(x) = -g''(x) + cg(x) = \lambda_* g(x).$$

Hence, if $c = \lambda_*$ then $g(x) = sx + t$, and if $c \neq \lambda_*$ then $g(x) = se^{\gamma x} + te^{-\gamma x}$, where $s, t \in \boldsymbol{C}$ and $\gamma = \sqrt{c - \lambda_*}$. This is obviously contradiction.

Next we investigate the Bargmann potential(cf. [9], see also [11] and [16]). Put

$$u(x) = -\frac{8e^{2x}}{(1 + e^{2x})^2}$$

then we have

$$Z_2(u(x)) + Z_1(u(x)) = 0,$$

that is to say, $\mathrm{rank}_\Lambda u(x) = 1$ and the Λ-characteristic coefficients of $u(x)$ are

$$\rho_0 = 0, \quad \rho_1 = -1.$$

Hence, by (7), we have

$$a_1(\lambda; u) = -\alpha_1^{(2)}\lambda + \alpha_1^{(1)}\rho_1 = -\frac{3}{2}\lambda - 1,$$

$$a_0(\lambda; u) = -\alpha_0^{(2)}\lambda^2 + \alpha_0^{(1)}\rho_1\lambda + \alpha_0^{(0)}\rho_0 = -\frac{3}{8}\lambda^2 - \frac{1}{2}\lambda$$

and

$$\begin{aligned} F(x, \lambda; u) &= Z_1(u(x) - \lambda) - a_1(\lambda; u) Z_0(u(x) - \lambda) \\ &= \frac{1}{2}u(x) + \lambda + 1. \end{aligned}$$

Therefore we have

$$\Delta(\lambda; u) = -4\lambda^3 - 8\lambda^2 - 4\lambda = -4\lambda(\lambda+1)^2.$$

Hence $\Gamma(u) = \{0, -1\}$ and $\Gamma^{\natural}(u) = \{0, -1, -1\}$ follow. Now let us calculate the Darboux transformations $u^*(x, \lambda_*; \alpha)$ for $\lambda_* = 0, -1$. Since

$$F(x, -1; u) = \frac{1}{2}u(x) = -\frac{4e^{2x}}{(1+e^{2x})^2},$$

$$F(x, 0; u) = \frac{1}{2}u(x) + 1 = (\frac{1-e^{2x}}{1+e^{2x}})^2,$$

one obtains

$$\begin{aligned} u^*(x, -1; \alpha) &= \frac{32\alpha e^{2x}(1+e^{2x})((\alpha x + \alpha + 2) - (\alpha x - \alpha + 2)e^{2x})}{(\alpha - (\alpha x + 8)e^{2x} - \alpha e^{4x})^2}, \\ u^*(x, 0; \alpha) &= 2\frac{\alpha^2 + (4\alpha^2 x^2 + 8(\alpha^2+\alpha)x + 6\alpha^2 + 8\alpha + 4)e^{2x} + \alpha^2 e^{4x}}{((\alpha x + 2\alpha + 1) - (\alpha x + 1)e^{2x})^2}. \end{aligned}$$

Particularly, $u^*(x, -1; 0) \equiv 0$ follows. Thus we have $0 \in \Sigma_-(u; -1)$. On the other hand, we have immediately $\Sigma_-(u; 0) = \emptyset$ from the above expression.

Finally we investigate the rational potential

$$u(x) = \frac{2}{x^2}.$$

One easily verifies that $u(x)$ solves the differential equations

$$u'(x)^2 - 2u(x)^3 = 0$$

and

$$u''(x) - 3u(x)^2 = 0.$$

This implies

$$Z_2(u(x)) = 0,$$

namely, $\text{rank}_{\Lambda} u(x) = 1$ and the Λ-characteristic coefficients of $u(x)$ are

$$\rho_0 = \rho_1 = 0.$$

Hence, by (7), we have

$$a_1(\lambda; u) = -\frac{3}{2}\lambda, \quad a_0(\lambda; u) = -\frac{3}{8}\lambda^2$$

and

$$F(x, \lambda; u) = \frac{1}{2}u(x) + \lambda.$$

Therefore

$$\Delta(\lambda; u) = -4\lambda^3$$

follows. This implies $\mathrm{rank}_\Lambda u^*(x, 0; 0) = 0$. Actually, since

$$F(x, 0; u) = \frac{1}{x^2},$$

we have immediately

$$u^*(x, 0; \alpha) = \begin{cases} 6\dfrac{\alpha^2 x^4 - 6\alpha x}{(\alpha x^3 + 3)^2}, & \alpha \neq \infty \\ \dfrac{6}{x^2}, & \alpha = \infty. \end{cases}$$

Hence $\Sigma_-(u; 0) = \{0\}$ follows. This example was investigated by the present author [17] from the viwpoint of the monodromy presearving deformation of Fuchsian differential equation.

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