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ON CYCLIC AND ITERATED CYCLIC PRODUCTS OF SPHERES

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1. G-products

Let G be a subgroup of the full symmetric group $S(m)$ on m objects. Then a *G-product* on a based space (X, e) will mean a based map $f: X^m = X\times\cdots\times X\to X$ satisfying

$$f(x_1, \cdots, x_m) = f(x_{\tau(1)}, \cdots, x_{\tau(m)}) \qquad \text{for all } \tau\in G.$$

Let $i: X\to X^m$ denote the embedding $i(x)=(x, e, \cdots, e)$. Two G-products f_1, f_2: $X^m\to X$ are *equivalent* if the composites $f_1 i$ and $f_2 i$ are (based) homotopic.

We consider the particular case $X=S^n$, and here equivalence of G-products f is determined by the degree of fi. The latter is called the *type* of f and is an integer. On S^n we study (equivalence classes of) G-products for G any p-Sylow subgroup of $S(p^r)$ with p an odd prime. The following result gives a complete determination of such G-products on S^n:

Theorem 1.1. *Let G be a p-Sylow subgroup of $S(p^r)$ with p an odd prime. For $n=2t+1$ S^n admits a G-product of type q if and only if q is a multiple of p^{rt}.*

Note it is well known that even dimensional spheres do not admit G-products (for any subgroup G of $S(p^r)$). G-products on any space X are in 1–1 correspondence with maps $X^m/G\to X$ where X^m/G is the space of G-orbits with the quotient topology. Furthermore, if G_1 and G_2 are conjugate subgroups of $S(m)$, X^m/G_1 and X^m/G_2 are homeomorphic. Thus in the proof of Theorem 1.1 we may select G to be the r-fold Wreath product of the cyclic group of order p Z_p with itself. The corresponding orbit space X^m/G is just the usual r^{th} iterated p-fold cyclic product of X.

The proof of the 'only if' half of Theorem 1.1 given in paragraph 2 is a minor modification of Landweber's proof for $G=S(m)$. Sections 3–5 are devoted to the proof of remaining half of 1.1 and the procedure follows in outline that of [17]. Suspension-order considerations enter in a crucial way; in particular we require a result of Mimura, Nishida and Toda on the mod p decomposability of $EL_0^n(p)$, the suspension of the lens space $L_0^n(p)$ [8].

2. Application of equivariant K-theory

Let G^1 be the subgroup of the symmetric group $S(p)$ generated by the p-cycle $(1, 2, \cdots, p)$. G^1 is cyclic of order p and a p-Sylow subgroup of $S(p)$. Suppose inductively that the subgroup G^r of $S(p^r)$ has been defined. Partition the ordered set $\{1, 2, \cdots, p^{r+1}\}$ into p ordered subsets $S_1=\{1, \cdots, p^r\}, \cdots, S_p=\{(p-1)p^r+1, \cdots, p^{r+1}\}$ and let G^r_k be the subgroup of $S(p^{r+1})$ isomorphic to G^r via the order-preserving identification of the sets $\{1, 2, \cdots, p^r\}$ and $S_k=\{(k-1)p^r+1, \cdots, kp^r\}$. Finally let $\tau\in S(p^{r+1})$ be the permutation of order p which permutes the subsets $S_1, \cdots, S_p$ cyclically, sending the i^{th} element of S_k to the i^{th} element of S_{k+1} (or S_1 if $k=p$). G^{r+1} is then defined to be the subgroup of $S(p^{r+1})$ generated by all the subgroups G^r_k, $k=1, 2, \cdots, p$, and the element τ.

By definition G^r is the r-fold Wreath product of $G^1(\cong Z_p)$ with itself. G^r satisfies a short exact sequence of groups

$$1 \to \prod_{k=1}^{p} G_k^{r-1} \to G^r \to Z_p \to 1 \tag{2.1}$$

From an easy counting argument (see [10]) we have that the order $|G^r|$ of G^r is $p^{N(r)}$ where $N(r)=1+p+p^2+\cdots+p^{r-1}$ and so G^r is a p-Sylow subgroup of $S(p^r)$.

Lemma 2.1. *The number of p^r-cycles in G^r is $(p-1)^{M(r)}p^{N(r)-r}$ where $M(r)=p^{r-1}$ and $N(r)=1+p+p^2+\cdots+p^{r-1}$.*

Proof. For any p^r-cycle $\tau\in S(p^r)$ $\tau^p=\tau\tau\cdots\tau$ is a product of p disjoint p^{r-1} cycles $\tau^p=\tau_1\tau_2\cdots\tau_p$. From the exactness of (2.1) it follows that such a p^r-cycle τ belongs to G^r exactly when τ_i in $\tau^p=\tau_1\tau_2\cdots\tau_p$ is a p^{r-1}-cycle in G_i^{r-1} (after possibly reordering the factors). So we can proceed by induction on r.

For $r=1$ there are obviously $(p-1)$ p-cycles. Assume the number of p^{r-1}-cycles in G^{r-1} is $(p-1)^{M(r-1)}\cdot p^{N(r-1)-(r-1)}$. By the above remark the number of p^r-cycles in G^r is the number of products of p disjoint p^{r-1}-cycles, one from each G_i^r, times the number of p^r-cycles whose p^{th} power gives such a product. This number is $((p-1)^{M(r-1)}\cdot p^{N(r-1)})^p\cdot(p^{r-1})^{p-1}$ where the last factor enters as follows: given such a product $\tau_1\tau_2\cdots\tau_p$ and any τ with $\tau^p=\tau_1\tau_2\cdots\tau_p$ we may write τ with initial entry 1. We can also assume that τ_i consists of the $(kp^{r-1}+i)^{th}$ entries of τ for $k=0, 1, \cdots, p-1$ and so τ_1 contains the entry 1. With 1 as initial entry of τ, it follows that the $(kp^{r-1}+1)^{st}$ entries of τ are uniquely determined. Further for $i\neq 1$ τ_i only determines the $(kp^{r-1}+i)^{th}$ entries of τ module the p^{r-1} different possible initial entries. Thus the p^{r-1} different ways of describing each of $\tau_2, \tau_3, \cdots, \tau_p$ (via different initial entries) determine $(p^{r-1})^{p-1}$ different $\tau\in G^r$ with $\tau^p=\tau_1\tau_2\cdots\tau_p$. As $((p-1)^{M(r-1)}\cdot p^{N(r-1)})^p\cdot(p^{r-1})^{p-1}=(p-1)^{M(r)}\cdot p^{N(r)}$, the induction is complete.

Lemma 2.1 is all that is required to adapt Landweber's application of equivariant K-theory to the study of G^r-maps. A brief resumé of his procedure follows. For any $G\subset S(m)$ let $P\colon G\subset S(m)\to R^m$ be the usual permutation representation of G. P splits as $Q\oplus \boldsymbol{R}$ where Q acts on the hyperplane $\{x\in R^m \mid \Sigma x_i=0\}$ and $\boldsymbol{R}$ is the trivial 1-dimensional representation acting on the diagonal $\{x\in R^m \mid x_i=x_j \text{ all } i, j\}$. Via the generalized Hopf construction Landweber obtains from a G-map $f\colon S^n\times\cdots\times S^n\to S^n$ ($n=2t+1$) an equivariant map

$$g\colon \Sigma((n+1)Q_C\oplus \boldsymbol{C}^{t+1})\to \Sigma(Q_C\oplus \boldsymbol{C}^{t+1})$$

where Q_C is the complexification of Q and ΣW is the one-point compactification of the G-representation W. Since g maps the fixed point set to the fixed point set, there is a commutative diagram of maps

(2.2)
$$\begin{array}{ccc} \Sigma(\boldsymbol{C}^{t+1}) & \longrightarrow & \Sigma((t+1)Q_C\oplus \boldsymbol{C}^{t+1}) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \Sigma(\boldsymbol{C}^{t+1}) & \longrightarrow & \Sigma(Q_C\oplus \boldsymbol{C}^{t+1}) \end{array}$$

An easy calculation shows that $\deg(g')=m\cdot\text{type}(f)$. Hence to obtain the 'only if' part of 1.1. we must show that $p^{r(t+1)}\mid\deg(g')$ for $G=G^r\subset S(p^r)$.

For any finite group G let K_G denote the complex equivariant K-theory functor. Then $K_G(\text{point})=R(G)$, the complex representation ring. Further for complex G-modules $W=W_1\oplus W_2$ there is an isomorphism

$$K_GW_1\xrightarrow{\lambda_{W_2}}K_G(W_1\oplus W_2)$$

defined by multiplication by a certain class $\lambda_{W_2}\in K_GW_2$ which restricts to $\lambda_{-1}W_2=\Sigma(-1)^i\lambda^i W_2\in R(G)=K_G\{0\}$. In particular for $W_1=\{0\}$ we have that K_GW is a free $R(G)$-module on the one generator λ_W. Applying this information to (2.2) we obtain the commutative diagram

$$\begin{array}{ccc} R(G) & \xleftarrow{\cdot\lambda_{Q_C}^{t+1}} & R(G) \\ {\scriptstyle (g')^*}\uparrow & & \downarrow{\scriptstyle g^*} \\ R(G) & \xleftarrow[\cdot\lambda]{} & R(G) \end{array}$$

giving the equation

(2.3)
$$g^*(1)\cdot\lambda_{Q_C}^{t+1}=\deg(g')\cdot\lambda_{Q_C}$$

Statement (ii) below is needed in the application of this equation. Its proof depends on (i)—of which it is a converse.

(i) (Atiyah [1]) The $S(m)$ representation $\lambda_{-1}Q_C$ as a class function vanishes on composites (i.e., products of cycles) and assumes the value m on any m-cycle.

(ii) (Landweber [7]) If $\alpha \in R(S(m))$, as a class function, vanishes on composites, then α is a multiple of $\lambda_{-1}Q_C$.
Both statements extend easily with G^r in place of $S(p^r)$. In particular the G^r representation '$\lambda_{-1}Q_C$' is just the composite $G^r \subset S(p^r) \to C$, i.e., inclusion followed by the $S(p^r)$ representation $\lambda_{-1}Q_C$. Hence '$\lambda_{-1}Q_C$' also vanishes on composites (in G^r) and assumes the value p^r on any p^r-cycle, giving the desired extension of (i). Similarly using Lemma 2.1 we can prove: If $\alpha \in R(G^r)$ vanishes on composites and assumes the same value on p^r-cycles, then α is a multiple of $\lambda_{-1}Q_C$ and so $p^r | \lambda_{-1}Q_C(\sigma)$ for any p^r-cycle σ in G^r. In fact for 1 the principal character of G^r, we know that the inner product

$$(\alpha, 1) = \frac{1}{|G^r|} \sum_{x \in G^r} \alpha(x) 1(x)$$

is integral. The only nonzero terms occur for x a p^r-cycle and on each of these $\alpha(x)$ assumes the same value. $|G^r| = p^{N(r)}$ and there are $(p-1)^{M(r)} \cdot p^{N(r)-r}$ p^r-cycles in G^r, so for x any p^r-cycle $\alpha(x) \cdot (p-1)^{M(r)}/p^r$ is an integer and so $p^r | \alpha(x)$.

Now equation (2.3) evaluated on any p^r-cycle σ becomes $g^*(1)(\sigma) \cdot (p^r)^{t+1} = \deg(g')p^r$ and so $g^*(1)(\sigma) \cdot p^{rt} = \deg(g')$, hence all that remains to show is that $g^*(1)$ vanishes on composites. Landweber proves this for $g^*(1)$ viewed as an $S(m)$ representation, making use of naturality and a calculation of the Adams operation ψ^k. His proof (namely Lemmas 4.4, 4.5 and the final paragraph of [7]) carries over without change for $g^*(1)$ considered as a G^r-representation.

Remarks. 1. For $r=1$ (i.e., the cyclic product CP^pS^n) a separate proof that p^t divides the *type* of any map $CP^pS^n \to S^n$ $(n=2t+1)$ can be given along the lines of [16] using Liao's computation of $H^*(CP^pS^n; Z)$ [8]. Such a procedure should also work for all $r \geq 1$ (and for $p=2$ as well), given a strong will to evaluate the higher differentials in the coefficient K-theory spectral sequence for $CP_r^pS^n$.

2. If in fact one succeeds in the higher differentials approach of remark 1, it then becomes reasonable to ask if these differentials are related to the representation theory of the groups G^r.

3. Of course the argument in this paragraph works equally well for the prime 2 and so proves that the type of any map $CP_r^2 S^n \to S^n$ $(n=2t+1)$ must be a multiple of 2^{rt}. Just as in the case $r=1$, however, we expect some improvement of this result using real K-theory instead of complex K-theory.

3. Geometry

The p-fold cyclic product CP^pX of a based space (X, e) is the based quotient space $(CP^pX, [e, \cdots, e])$ where CP^pX is the orbit space X^p/G^1 under the action of the cyclic subgroup $G^1 \subset S(p)$ of order p. The r^{th} iterated p-fold cyclic product $CP^p_r X$ of X is defined inductively as $CP^p_r X = CP^p(CP^p_{r-1} X)$ for $r \geq 2$. One verifies easily that $CP^p_r X$ is homeomorphic to the orbit space X^{p^r}/G^r, where G^r is the r-fold Wreath product of G^1 with itself (recall G^r is a p-Sylow subgroup of $S(p^r)$).

Here we develop the geometry of $CP^p S^n$ analogous to that of the m-fold symmetric product $SP^m S^n$ of S^n [17] in order to construct maps $CP^p S^n \to S^n$. Via the cyclic product functor we obtain maps for each i $CP^p_i S^n \to CP^p_{i-1} S^n$ which under composition provide the desired map $CP^p_r S^n \to S^n$ meeting the requirements of Theorem 1.1.

Let D^n denote the n-disc in R^n and S^{n-1} its boundary $(n-1)$-sphere ∂D^n. For each $\tau \in G^1$ there is a permutation homeomorphism $h_\tau \colon (D^n)^p \to (D^n)^p$. Set $A^n_{p,l} = (D^n)^{p-l} \times (S^{n-1})^l$, $0 \leq l \leq p$. Then $\tilde{X}^n_{p,l} = \bigcup_\tau h_\tau A^n_{p,l}$, τ running over G^1, is a G^1-invariant subspace of $A^n_{p,0} = (D^n)^p$ and so the quotient $X^n_{p,l} = \tilde{X}^n_{p,l}/G^1$ is defined. Similarly for $B^n_{p,l} = (S^n)^{p-l} \times \{e\}^l$, $0 \leq l \leq p$, $\tilde{Y}^n_{p,l} = \bigcup_\tau h_\tau B^n_{p,l}$ is a G^1-invariant subspace of $B^n_{p,0} = (S^n)^p$ and so the $Y^n_{p,l} = \tilde{Y}^n_{p,l}/G^1$ is also defined.

Lemma 3.1. *$X^n_{p,0}$ is homeomorphic to the cone $CX^n_{p,1} = X^n_{p,1} \times I / X^n_{p,1} \times \{0\}$.*

Proof. The map $X^n_{p,1} \times I \to X^n_{p,0}$ given by $([x_1, \cdots, x_p], t) \to [tx_1, \cdots, tx_p]$ induces a topological map

$$\alpha \colon CX^n_{p,1} \to X^n_{p,0} \tag{3.1}$$

sending $X^n_{p,1} \times \{1\}$ onto the subspace $X^n_{p,1} \subset X^n_{p,0}$.

A relative homeomorphism $\tilde{h} \colon (D^n, S^{n-1}) \to (S^n, e)$ induces a relative hoemomorphism

$$h \colon (X^n_{p,0}, X^n_{p,1}) \to (Y^n_{p,0}, Y^n_{p,1}) . \tag{3.2}$$

From Lemma 3.1 and (3.2) we obtain

Lemma 3.2. *$Y^n_{p,0} = CP^p S^n$ is homeomorphic to the adjunction space $Y^n_{p,1} \cup CX^n_{p,1}$ with attaching map given by $h \mid X^n_{p,1}$.*

Let $W^{n-1}_l \subset (S^{n-1})^l$ be the subspace $\{x = (x_1, \cdots, x_l) \in (S^{n-1})^l \mid x_i = e \text{ for some } i\}$ and set $\tilde{Z}^n_{p,l} = (D^n)^{p-l} \times W^{n-1}_l \subset (D^n)^{p-l} \times (S^{n-1})^l$ and $\tilde{Z}^n_{p,l}$ the image of $\tilde{Z}^n_{p,l}$ under the canonical the projection

$$(D^n)^{p-l} \times (S^{n-1})^l \to X^n_{p,l} .$$

The surjective map $(D^n)^{p-l} \times (D^{n-1})^l \to X^n_{p,l}$, $1 \leq l \leq p$, given by the composite

$$(D^n)^{p-l} \times (D^{n-1})^l \xrightarrow{id \times (\tilde{h})^l} (D^n)^{p-l} \times (S^{n-1})^l \to X^n_{p,l}$$

sends $\partial((D^n)^{p-l}) \times (D^{n-1})^l$ to $X^n_{p,l+1}$ and $(D^n)^{p-l} \times \partial((D^{n-1})^l)$ to $Z^n_{p,l}$ and defines a relative homeomorphism between the pairs $((D^n)^{p-l}\times(D^{n-1})^l, \partial((D^n)^{p-l})\times(D^{n-1})^l))$ and $(X^n_{p,l}, X^n_{p,l+1}\cup Z^n_{p,l})$. Similarly the map $(D^n)^{p-l}\times(S^{n-1})^l\to X^n_{p,l}$ is surjective and defines a relative homeomorphism between the pairs $((D^n)^{p-l}\times(S^{n-1})^l, \partial((D^n)^{p-l}\times(S^{n-1})^l))$ and $(X^n_{p,l}, X^n_{p,l+1})$. Thus we obtain

Lemma 3.3. *Let* $1\leq l\leq p$. *Then*
(i) $X^n_{p,l}$ *is homeomorphic to the adjunction space*

$$(X^n_{p,l+1}\cup Z^n_{p,l})\cup(D^n)^{p-l}\times(D^{n-1})^l; \textit{ and}$$

(ii) $X^n_{p,l}$ *is also homeomorphic to the adjunction space*

$$X^n_{p,l+1}\cup(D^n)^{p-l}\times(S^{n-1})^l.$$

(The attaching maps are described above.)

Recall that the join $X*Y$ of X and Y is the subspace $X\times CY\cup CX\times Y$ of the product $CX\times CY$. The map of pairs

$$c\colon (C(X*Y), X*Y)\to(CX\times CY, X*Y)$$

defined by

$$\begin{aligned} c[(x, [y, t]), u] &= [[x, y], [y, tu]] && \text{for } (x, [y, t])\in X\times CY\\ c[([x, t], y), u] &= [[x, tu], [y, u]] && \text{for } ([x, t], y)\in CX\times Y \end{aligned}$$

is a homeomorphism whose restriction to $(X*Y)\times\{1\}$ is the identity map. As a consequence of Lemma 3.3 we have

Lemma 3.4. *Let* $1\leq l\leq p$. *Then*
(i) $X^n_{p,l}/(X^n_{p,l+1}\cup Z^n_{p,l})\cong E(S^{n(p-l)-1}*S^{(n-1)l-1})\cong S^{np-l}$
(ii) $X^n_{p,l}/X^n_{p,l+1}\sim E(S^{n(p-l)-1})\vee E(S^{n(p-l)-1}\wedge S^{(n-1)l})\cong S^{n(p-l)}\vee S^{np-l}$

Proof. (i) is immediate from 3.3(i) and the homeomorphism c. The second homeomorphism of (i) is well known. Similarly (ii) follows from 3.3 (ii) and the following two facts: (1) $(CX\times Y, X\times Y)$ and $(EX\times Y, e\times Y)$ are relatively homeomorphic and (2) $EX\times Y/e\times Y$ and $EX\vee E(X\wedge Y)$ have the same homotopy type.

For $l=p-1$ 3.3 (i) becomes $X^n_{p,p-1}\cong(X^n_{p,p}\cup Z^n_{p,p-1})\cup e^{np-l}$. As $X^n_{p,p}\cap Z^n_{p,p-1}=Y^{n-1}_{p,1}$, collapsing $Z^n_{p,p-1}$ in $X^n_{p,p-1}$ gives a relative homeomorphism $X^n_{p,p-1}/Z^n_{p,p-1}\cong(X^n_{p,p}/Y^{n-1}_{p,1})\cup e^{np-l}$. But $X^n_{p,p}=Y^{n-1}_{p,0}$ and by Lemma 3.2 $Y^{n-1}_{p,0}/Y^{n-1}_{p,1}\cong EX^{n-1}_{p,1}$, whence we obtain

$$X^n_{p,p-1}/Z^n_{p,p-1}\cong EX^{n-1}_{p,1}\cup e^{np-l} \tag{3.3}$$

From this formula one can deduce the following result.

Proposition 3.5. $X^n_{p,p-1}/Z^n_{p,p-1}$ *has the homotopy type of a CW complex of the form* $S^{n-1}*K$ *for some finite CW complex* K. *In particular, for* $n\geq 2$ $X^n_{p,p-1}/Z^n_{p,p-1}$ *twice desuspends up to homotopy type.*

We give only the idea of the proof. First show that $X^{n-1}_{p,1}$ is homeomorphic to a space of the form $S^{n-2}*K'$ (essentially already done in Toda [15]). Then show that the attaching map of (3.3) factors as fg, g a homotopy equivalence and f a map of the form $E(id*f')\colon E(S^{n-2}*L')\to E(S^{n-2}*K')$.

Recall that to construct a map g from an adjunction space $X\cup_f CA$ extending a given map $\varphi\colon X\to Y$ what is needed is a nullhomotopy $N_t\colon \in\sim\varphi f$ of the composite φf.

(3.4)
$$
\begin{tikzcd}
 & X\cup_f CA \arrow[r, dashed, "g"] & Y \\
A \arrow[r, "f"] & X \arrow[u, "i"] \arrow[ur, "\varphi"'] &
\end{tikzcd}
$$

For then the extension g is defined by $g|X=\varphi$ and $g|CA=N_t$. This idea is already sufficient to construct maps $SP^2S^n\to S^n$ on the symmetric square of S^n, since by Theorem 2.3 of [4] we have $SP^2S^n\cong S^n\cup_f CX$ and so a nullhomotopy $\varepsilon\sim\varphi f$ provides a map g as in (3.4). In particular type $(g)=$degree(φ) and so any map $\varphi\colon S^n\to S^n$ of degree q such that $\varepsilon\sim\varphi f$ produces a map $g\colon SP^2S^n\to S^n$ of type q.

A generalization of this procedure for the construction of maps $SP^mS^n\to S^n$ on the m-fold symmetric product of S^n is given in [17]. Almost without modification this generalization carries over to the case of cyclic products. We summarize the gist of this generalization for cyclic products, omitting the details which the reader can recover from [17]. By restriction the attaching map $h\colon X^n_{p,1}\to Y^n_{p,1}$ of (3.2) defines maps $h_i\colon X^n_{p,p-i}\to Y^n_{p,p-i}$, $\tilde h_i\colon X^n_{p,p-i}\cup Z^n_{p,p-i-1}\to Y^n_{p,p-i-1}$ and a commutative diagram of maps.

(3.5)
$$
\begin{tikzcd}
X^n_{p,1} \arrow[r, "h_{p-1}"] & Y^n_{p,1} \\
X^n_{p,2}\cup Z^n_{p,1} \arrow[u] \arrow[ur, "\tilde h_{p-2}"'] & \\
X^n_{p,2} \arrow[u] \arrow[r, "h_{p-2}"'] & Y^n_{p,2} \arrow[uu] \\
\vdots \arrow[u] & \vdots \arrow[u] \\
X^n_{p,p-2} \arrow[u] \arrow[r, "h_2"] & Y^n_{p,p-2} \arrow[u] \\
X^n_{p,p-1}\cup Z^n_{p,p-2} \arrow[u] \arrow[ur, "\tilde h_1"'] & \\
X^n_{p,p-1} \arrow[u] \arrow[r, "h_1"'] & Y^n_{p,p-1}=S^n \arrow[uu]
\end{tikzcd}
$$

where the vertical maps are the obvious inclusions. The procedure is now to find a map $\varphi_1\colon Y^n_{p,p-1}=S^n\to S^n$ of degree q such that φh_1 is nullhomotopic, for then the geometry of this situation enables one to construct further maps $\varphi_i\colon Y^n_{p,p-i}\to S^n$ such that $\varphi_i h_i$ is nullhomotopic (and also the intermediate composites $\varphi_i \hat{h}_{i-1}\sim\varepsilon$) and so arrive at a map $CP^p S^n\to S^n$ of type q. This geometry is given below in statements (i)′ and (ii)′, which are analogues of statements (i) and (ii) on page 541 of [17]. Set $C_i=X^n_{p,p-i}$ and $D^n_i=X^n_{p,p-i}\cup Z^n_{p,p-i-1}$. Lemma 3.1 (i) for $l=p-i-1$ gives immediately

$$\text{(i)}' \qquad C_{i+1}\cong D_i\cup(D^n)^{p-l}\times(D^{n-1})^l\cong D_i\cup C(S^{n(p-l)-1}*S^{(n-1)l-1})$$

Further $X^n_{p,p-i}\cap Z^n_{p,p-i+1}=Z^n_{p,p-i+2}$ so we need to express $Z^n_{p,p-i+1}$ as an adjunction space obtained from $Z^n_{p,p-i+2}$. The discussion preceding Lemma 3.3 is relevant here and shows that the obvious map $(D^n)^{i-1}\times(S^{n-1})^{p-i}\to Z^n_{p,p-i+1}$ sends $\partial((D^n)^{i-1}\times(S^{n-1})^{p-i})=(\partial(D^n)^{i-1})\times(S^{n-1})^{p-i}$ to $Z^n_{p,p-i+2}$ and induces a relative homeomorphism between pairs. Hence

$$\begin{aligned} Z^n_{p,p-i+1}&\cong Z^n_{p,p-i+2}\cup(D^n)^{i-2}\times(S^{n-1})^{p-i}\\ &\cong Z^n_{p,p-i+2}\cup C(S^{n(i-1)-1})\times(S^{n-1})^{p-i}\end{aligned}$$

It then follows easily that

$$\text{(ii)}' \qquad D_i\cong C_i\cup C(S^{n(i-1)-1})\times(S^{n-1})^{p-i}$$

(i)′ and (ii)′ provide the geometry needed to generalize the proof of Lemma 3.1 of [17] to the diagram (3.5) for $CP^p S^n$. Thus we obtain

Lemma 3.6. *If $\varphi\colon S^n\to S^n$ is a map of degree q such that $fh_1\colon X^n_{p,p-1}\to S^n$ is nullhomotopic, then the above construction provides a map $g\colon CP^p S^n\to S^n$ of type q.*

4. Lens spaces and suspension-order

Let X be a based CW complex X. The order of the class $\iota_{EX}\in[EX, EX]$ of the identity map of EX is called the *suspension-order* of X (Toda [14]) or the *characteristic* of X (Barratt [2]). It is a homotopy type invariant. The suspension-order of $E^{-1}X^{2t+1}_{2,1}\sim E^{2t}P^{2t}$, the $(2t-1)^{st}$ suspension of real projective $2t$-space, plays an essential role in the construction of maps $SP^2S^n\to S^n$ of least positive type ($n=2t+1$). Its computation is given in [14]. In [17] a similar computation is made for a related suspension complex for maps $SP^3S^n\to S^n$. In our case of maps $CP^pS^n\to S^n$ we require the suspension-order of $E^{2t-1}L_0^{t(p-1)}(p)$, where $L^n_0(p)$ denotes the $2n$-skeleton of the lens space $L^n(p)=S^{2n+1}/Z_p$.

Recall $L^n(p)$ has a cell structure given by $L^n(p)=S^1\cup e^2\cup\cdots\cup e^{2n+1}$ and integral cohomology

$$H^i(L^n(p);Z)\cong\begin{cases}Z_p & i=2,4,\cdots,2n\\ Z & i=0,2n+1\\ 0 & \text{other } i\end{cases}$$

and so the integral cohomology of $L_0^n(p)$ is given by

$$H^i(L_0^n(p);Z)\cong\begin{cases}Z_p & i=2,4,\cdots,2n\\ Z & i=0\\ 0 & \text{other } i\end{cases}$$

Lemma 4.1. *For each $l=1, 2, \cdots, (p-1)/2$ the pair of spaces $X^n_{p,2l}$ and $E^nL_0^r(p)$, $r=(n(p-1)/2)-l$, have the same homotopy type.*

Proof. $E^nL_0^r(p)$ is the $n+2r$ skeleton of $E^nL^r(p)$ and by Lemma 3.3 (ii) $X^n_{p,2l}$ is the $pn-2l$ skeleton of $X^n_{p,1}$. Thus to prove Lemma 4.1 it suffices to produce a cellular map $f\colon X^n_{p,1}\to E^nL^r(p)$ such that the restriction of f to the $pn-2l$ skeleton of $X^n_{p,1}$ (as a map into $E^nL_0^r(p)$) is a homotopy equivalence for each $l=1, 2, \cdots, (p-1)/2$. As both $(X^n_{p,1})^{(pn-2l)}=X^n_{p,2l}$ and $E^nL_0^r(p)$ are simply connected it is enough to require that f induce isomorphisms of all integral cohomology groups in dimensions $\leq pn-2l$. From [15] or [18] we have that $X^n_{p,1}$ is homeomorphic to a join $S^{n-1}*Y$, Y a finite CW complex for which there exists a cellular map $f'\colon Y\to L^s(p)$ inducing isomorphisms of all mod p cohomology groups. For $(X^n_{p,1})^{(pn-2l)}$ and $L^s(p)^{(pn-2l)}$ reduction from integral to mod p cohomology is an isomorphism in all positive even dimensions. This implies that f' also induces isomorphisms on integral cohomology groups. The result follows.

To calculate the suspension-order of $E^{2t-1}L_0^{t(p-1)}(p)$ we follow a suggestion of Toda, making use of [9] as follows.

Proposition 4.2. *The suspension-order of $L_0^n(p)$, $n=s(p-1)+r$, $0\leqslant r<p-1$, is p^s if $r=0$ and is p^{s+1} if $r>0$.*

Proof. Proposition 9.6 of [9] asserts that $EL^n(p)$ is mod p decomposable into a wedge of $p-1$ spaces. The same proof establishes a like result for $EL_0^n(p)$. Thus $EL_0^{p-1}(p)$ is mod p decomposable into a $(p-1)$-fold wedge of Moore spaces $\bigvee_{i=1}^{p-1} M_p^{2i}$. In fact, as reduction mod p in the cohomology of $EL_0^{p-1}(p)$ is an isomorphism in positive even dimensions, any p-equivalence $EL_0^{p-1}(p)\to\bigvee_{i=1}^{p-1} M_p^{2i}$ is a homotopy equivalence. Theorems 4.4 and 1.2 of [14] imply that the suspension-order of $L_0^n(p)$, $1\leqslant n\leqslant p-1$, is a divisor of p. From Kambe [5] $K\tilde{U}^0(L_0^n(p))$ contains an element of order p, so by Theorem 1.1 of [14] the proposition is true for $s=0$, $1\leqslant r<p-1$ and $s=1$, $r=0$.

Assume it is true for $s=k$, $0\leqslant r<p-1$ and $s=k+1$, $r=0$. Set $n=(k+1)\cdot(p-1)+r$ and $m=k(p-1)+r$, $1\leqslant r\leqslant p-1$. Then the proof of Proposition 9.6 [9] provides a cellular map of pairs ψ^k: $(L_0^n(p), L_0^m(p))\to(L_0^n(p), L_0^m(p))$ satisfying condition D_p of paragraph 9 [9]. This map induces a map of quotient spaces $L_0^n(p)/L_0^m(p)\to L_0^n(p)/L_0^m(p)$ which also satisfies condition D_p, and so by Theorem 9.3 [9] $E(L_0^n(p)/L_0^m(p))$ is p-equivalent (and hence homotopy equivalent) to a $(p-1)$-fold wedge of Moore spaces. Now repeating the argument, i.e. [14] and [5] plus the induction assumption for $EL_0^m(p)$, we obtain the result that the suspension order of $L_0^n(p)$ is p^{k+2} if $r>0$, or if $r=p-1$, i.e., $m=(k+2)(p-1)$. This completes the induction.

5. Proof of Theorem 1.1

Let H_p: $[E^2K, S^{2m+1}; p]\to[E^2K, S^{2pm+1}; p]$ be the mod p Hopf invariant, where K is a finite CW complex and $[E^2K, L; p]$ denotes the p-primary component of the group of homotopy classes of maps $E^2K\to L$. For f_q: $S^k\to S^k$ a map of degree q there is a homorphism ψ_q: $[EK, S^k]\to[EK, S^k]$ defined by $\psi_q[g]=[f_q\circ g]$. In [17] the following generalizations of results in paragraph 4 of [4] were obtained:

Lemma 5.1. *Suppose $H^i(E^2K; Z)=0$ for all $i>pn-(p-1)$, $n=2t+1$ and $qH^{pn-(p-1)}(E^2K; Z)=0$ for some integer q. Then Ker H_p contains the subgroups $q[E^2K, S^{2t+1}; p]$ and $\psi_q[E^2K, S^{2t+1}; p]$.*

Lemma 5.2. *Let $r=kq^2$ and assume $[EK, S^{2pt-1}; p]=0$ in addition to the hypotheses of* 5.1. *The ψ_r acts on $[E^2K, S^{2t+1}; p]$ as multiplication by r.*

The proof of 1.1 is given in two steps.

Step 1: $r=1$. S^1 is an abelian topological group whose group multiplication $S^1\times\cdots\times S^1\to S^1$ defines a map $CP^pS^1\to S^1$ of type 1. For $n=3$ $X_{p,p-1}^3$ and $E^3L_0^{p-1}(p)$ have the same homotopy type and $E^3L_0^{p-1}(p)$ has the homotopy type of the wedge $X=\bigvee_{i=2}^{p}M_p^{2i}$. Thus the attaching map $X_{p,p-1}^3\to S^3$ can be viewed, up to homotopy, as a map $X\to S^3$. To apply the construction of paragraph 3 we need only show that the composite $f_p\circ\varphi$: $X\to S^3$ ($\deg(f_p)=p$) is nullhomotopic. However, this is clearly the case for $f_p\circ\varphi$ restricted to each M_p^{2i} if $i<p$. For $i=p$ the effect of ψ_p on $[S^{2p}, S^3; p]\cong Z_p$ is multiplication by p (see (13.13) of [13]) and so $f_p\circ\varphi\,|\,M_p^{2p}$ is also nullhomotopic. This implies the existence of a map $CP^pS^3\to S^3$ of type p.

If $n=2t+1\geq 5$, then $t\geq 2$ and so we may apply 5.2. Since $\dim X_{p,p-1}^{2t+1}=p(2t+1)-(p-1)$, then for all $i>p(2t+1)-(p-1)$ $H^i(X_{p,p-1}^n; Z)=0$. Also $pH^j(X_{p,p-1}^n; Z)=p\cdot Z_p=0$ for $j=p(2t+1)-(p-1)$. Finally $H^{2pt}(E^{-1}X_{p,p-1}^{2t+1}; Z_2)=0$ and $H^{2pt-1}(E^{-1}X_{p,p-1}^{2t+1}; Z)=0$ so by the Steenrod Classification Theorem

[4, p. 460] we have that $[E^{-1}X^{2t+1}_{p,p-1}, S^{2pt-1}; p]=0$. So by 5.2 $\psi_t[\varphi]=p^t[\varphi]$ which is zero in $[X^{2t+1}_{p,p-1}, S^{2t+1}; p]$— as p^t is also the suspension-order of $E^{-1}X^{2t+1}_{p,p-1}$. So by paragraph 3 we obtain a map $CP^pS^n\to S^n$ $(n=2t+1)$ of type p^t.

Step 2: Let f: $CP^pS^n\to S^n$ be a map of type p^t where $n=2t+1$. We claim that the composite $CP^p_{r-1}f\circ\cdots\circ CP^pf\circ f$: $CP^p_rS^n\to S^n$ has type $(p^t)^r=p^{rt}$. Here CP^pg is the map $CP^pX\to CP^pY$ induced from g: $X\to Y$ via the covariant functor CP^p. The proof of this claim requires a closer look at a generator of $H^n(CP^p_iS^n; Z)$ and its relation to $H^n((CP^p_{i-1}S^n)^p; Z)$.

Let $u\in H^n(K; Z)$ be a generator of infinite order. If $u_i=\pi_i^*u$ where π_i: $K^p\to K$ is the i^{th} projection, then $\Sigma u_i=u_1+\cdots+u_p\in H^n(K^p; Z)$ is $p^*[u]$ where p: $K^p\to CP^pK$ is the usual quotient map and $[u]$ is the class defined by Σu_i. For $K=S^n$ we take $u\in H^n(S^n; Z)$ to be a generator and $[u]$ is then a generator of infinite order of $H^n(CP^pS^n; Z)$. By induction the class $[u]_s=[[\cdots[u]\cdots]]$ (s brackets) is a generator of infinite order of $H^n(CP^p_sS^n; Z)$ satisfying $p_s^*[u]_s=\sum_{i=1}^{p}[u_{s-1}]_i$, where p_s: $(CP^p_{s-1}S^n)^p\to CP^p_sS^n$. For any map g: $CP^pS^n\to S^n$ the induced maps CP^p_ig: $CP^p_{i+1}S^n\to CP^p_iS^n$ satisfy the commutative diagram

$$\begin{array}{ccc} H^nCP^p_{s-1}S^n & \xrightarrow{p^*_{s-1}} & H^n(CP^p_{s-2}S^n)^p \\ {\scriptstyle (CP^p_{s-1}g)^*}\downarrow & & \downarrow{\scriptstyle (CP^p_{s-2}g)^*} \\ H^nCP^p_sS^n & \xrightarrow[p^*_s]{} & H^n(CP^p_{s-1}S^n)^p \end{array}$$

Commutativity of this diagram and induction on s gives $(CP^p_sg)^*[u]_{s-1}=q[u]_s$ where q=type of g. Thus the type of the composite $CP^p_{r-1}f\circ\cdots\circ CP^pf\circ f$ is $(p^t)^r=p^{rt}$ and the proof of Theorem 1.1 is complete.

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