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KOLMOGOROV EQUATIONS IN HILBERT SPACES WITH APPLICATION TO ESSENTIAL SELF-ADJOINTNESS OF SYMMETRIC DIFFUSION OPERATORS

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1. Introduction

The essential self-adjointness of differential operators over infinite dimensional spaces has been extensively studied. For historical comments and literature on this topic see the monograph by Berezanskii [3] and a recent paper by Albeverio, Kondratiev and Röckner [2]. For some generalizations to certain Banach spaces we refer to Long [13]. Sometimes essential self-adjointness is also called strong uniqueness. There is another kind of uniqueness for symmetric diffusion operators, i.e. Markovian uniqueness, which means that one has uniqueness only within the class of self-adjoint operators which generate sub-Markovian semigroups. Obviously essential self-adjointness implies Markovian uniqueness. For details we refer to [2] and some references therein. In this paper, we aim to prove the essential self-adjointness of a certain class of perturbed Ornstein-Uhlenbeck operators associated to stochastic evolution equations (SEE) in a separable Hilbert space, by using a general parabolic criterion of Berezanskii [3]. In [4], Berezanskii and Samoilenko established the essential self-adjointness of Ornstein-Uhlenbeck operators with a certain potential perturbation by using the finite-dimensional approximation approach. In [17], Shigekawa proved the essential self-adjointness of perturbed Ornstein-Uhlenbeck operators by using Malliavin calculus. Our method is completely different from Shigekawa's. For our purpose, we need first to establish the existence and uniqueness of classical solutions to the Kolmogorov equations associated to the perturbed Ornstein-Uhlenbeck operators. The definition of classical solution will be given in Section 2 , following Cannarsa and Da Prato [5]. In [7], Da Prato proved the existence and regularity of classical solutions to Kolmogorov equations associated to Ornstein-Uhlenbeck operators.

We consider the semilinear SEE :

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + Q^{\frac{1}{2}}dW(t) \\ \qquad X(0) = x \in H \end{cases} \tag{1.1}$$

on a separable Hilbert space H with norm $\|\cdot\|$ and inner product $<\cdot,\cdot>$. We denote

by $\mathcal{L}(H)$ the Banach space of all bounded linear operators from H into H, with norm $\|\cdot\|_{\mathcal{L}(H)}$. We always assume that:
(A.1) A is a self-adjoint and negative unbounded linear operator on H , which generates a contraction C_0-semigroup $S(t), t \geq 0$ satisfying $\|S(t)\|_{\mathcal{L}(H)} \leq e^{\omega t}$ for some constant $\omega < 0$, and Q is a symmetric bounded operator on H. Moreover, A and Q have discrete spectrum with a common set of normalized eigenvectors $\{e_n; n \in \mathbf{N}\}$, which form an orthonormal basis of H with corresponding finite eigenvalues $\alpha_n < 0$, $\lambda_n > 0$, $n \in \mathbf{N}$ such that

$$\sum_{n=1}^{\infty}(-\alpha_n^{-1})\lambda_n < +\infty.$$

(A.2) W is a cylindrical Wiener process on H with identity covariance.
Under the assumption (A.1), in [14], Mück proved the fact that

$$\sup_{t\geq 0}\int_0^t \mathrm{Tr} S(r)QS^*(r)dr < \infty.$$

Therefore, under the assumptions (A.1) and (A.2), the linear version of (1.1)

$$\left\{\begin{aligned} dZ(t) &= AZ(t)dt + Q^{\frac{1}{2}}dW(t) \\ Z(0) &= x \in H \end{aligned}\right. \tag{1.2}$$

has a well-defined H-valued mild solution given by

$$Z(t,x) = S(t)x + \int_0^t S(t-s)Q^{\frac{1}{2}}dW(s). \tag{1.3}$$

Moreover, there exists a unique invariant measure for $Z(t,x)$, which is the Gaussian measure $\gamma = \mathcal{N}(0, -\frac{1}{2}A^{-1}Q)$. If F is a Lipschitz mapping from H to H, then (1.1) has a unique mild solution $X(t,x), \quad t \geq 0$

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)Q^{\frac{1}{2}}dW(s), \quad t \geq 0 \tag{1.4}$$

If we further assume that
(A.3) for some $T > 0$, there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\int_0^T s^{-2\alpha}\|S(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(H)}ds < +\infty, \tag{1.5}$$

where $\mathcal{L}_2(H)$ stands for the space of all Hilbert-Schmidt operators on H with the corresponding norm $\|\cdot\|_{\mathcal{L}_2(H)}$, then the mild solution $X(t,x)$ of (1.1) has a continuous version.

Let L be the following operator

$$(1.6) \qquad L\varphi(x) = \frac{1}{2}\mathrm{Tr}[QD^2\varphi(x)] + < x, A^*D\varphi(x) > + < F(x), D\varphi(x) >$$

with domain
$D(L) = \{\varphi \in C_b^2(H) : QD^2\varphi(x) \in \mathcal{L}_1(H)$ with $\sup_{x\in H} \|QD^2\varphi(x)\|_{\mathcal{L}_1(H)} < +\infty$ and there exists $\psi \in C_b^2(H)$ such that $\varphi(x) = \psi(A^{-1}x)\ \}$.
Here we have denoted by $\mathcal{L}_1(H)$ the Banach space of all nuclear operators on H. As usual we denote by $C_b^k(H)$ the space of all bounded k-times differentiable functions on a Hilbert space H with the derivatives continuous and bounded. From the arguments in Section 2 of Da Prato and Zabczyk [11], we know that $D(L)$ is dense in $L^2(H, \gamma)$.

The contents of this paper are organized as follows. In section 2, we consider the Kolmogorov equation

$$(1.7) \begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\mathrm{Tr}[QD^2u(t,x)] + < x, A^*Du(t,x) > + < F(x), Du(t,x) >, t \geq 0, \\ \qquad u(0,x) = \varphi(x) \in D(L), x \in H \end{cases}$$

and show the existence and uniqueness of classical solution of (1.7) under the assumption that the non-linear term F satisfies the following condition
(A.4) there exists a mapping $G : H \to H$, Lipschitz continuous on H and twice Fréchet differentiable with bounded and continuous derivatives, such that $A^{-1}F(x) = G(A^{-1}x)$.
In section 3, we shall first establish an approximation criterion for the essential self-adjointness of the symmetric Kolmogorov-type operator $(L, D(L))$ by using the general parabolic criterion in Berezanskii [3]. We use this criterion and the result in section 2 to prove that if $(L, D(L))$ is symmetric on $L^2(H, \mu)$, $\|x\|$ is in $L^2(H, \mu)$ and F is Lipschitz continuous on H, then $(L, D(L))$ is essentially self-adjoint on $L^2(H, \mu)$. Then we prove that if $F = QDU$, where U is Gâteaux differentiable on H and such that $V = e^U \in W^{1,2}(H, \gamma)$, then $(L, D(L))$ is symmetric on $L^2(H, \mu)$ with $\mu(dx) = e^{2U(x)}\gamma(dx)$ and as a consequence we obtain the essential self-adjointness of the symmetric operator $(L, D(L))$ on $L^2(H, \mu)$ provided that $F = QDU \in \mathrm{Lip}(H, H)$.

2. Existence and uniqueness of classical solutions to Kolmogorov equations

Consider the transition semigroup $P_t, \quad t \geq 0$ determined by the SEE (1.1)

$$(2.1) \qquad P_t\varphi(x) = \mathbf{E}[\varphi(X(t,x))], \quad t \geq 0, \ x \in H,$$

where φ belongs to the space $B_b(H)$ of all real bounded Borel functions on H. Our first task is to prove that P_t preserves the domain $D(L)$. Set $u(t,x) = P_t\varphi(x)$.

Proposition 2.1. *Under the assumptions (A.1)-(A.4), P_t maps $D(L)$ into itself for any $t \geq 0$.*

Proof. From the assumption (A.4) on F, we know that

$$A^{-1}F(x) = G(A^{-1}x).$$

Define a new stochastic process $Y(t) = A^{-1}X(t)$. Then it is easy to see that $Y(t)$ is a mild solution to the following SEE

$$\begin{cases} dY(t) = [AY(t) + G(Y(t))]dt + A^{-1}Q^{\frac{1}{2}}dW(t) \\ Y(0) = A^{-1}x. \end{cases} \tag{2.2}$$

Since $Y(t) \in D(A)$ $a.s.$ and $AY(t) = X(t) \in L^1([0,T], H)$, $\forall T > 0, a.s.$, we can conclude that $Y(t)$ is a strong solution to the SEE (2.2) (Grecksch and Tudor [12, Theorem 1.1, Chapter 2]).
For $\varphi \in D(L)$, there exists a $\psi \in C_b^2(H)$ such that $\varphi(x) = \psi(A^{-1}x)$. Then

$$\begin{aligned} u(t,x) &= P_t\varphi(x) = \mathbf{E}[\varphi(X(t,x))] = \mathbf{E}[\psi(A^{-1}X(t,x))] = \mathbf{E}[\psi(Y(t,A^{-1}x))] \\ &:= (T_t\psi)(A^{-1}x), \end{aligned}$$

where T_t is the semigroup associated to the process $Y(t)$. From Theorem 9.17 in Da Prato and Zabczyk [10], under our assumptions on the coefficients of (1.1), we know that T_t is a semigroup which preserves $C_b^2(H)$. Now set $v(t,y) = T_t\psi(y)$. Then

$$Du(t,x) = (A^{-1})^* Dv(t, A^{-1}x),$$

which implies that $A^*Du(t,x)$ is uniformly bounded on H for any $t > 0$. On the other hand, it is easy to see that

$$QD^2u(t,x) = (A^{-1})^* QD^2v(t, A^{-1}x)(A^{-1}).$$

From this we can get the following estimate

$$\sup_{x\in H} \|QD^2u(t,x)\|_{\mathcal{L}_1(H)} \leq \sup_{y\in H} \|D^2v(t,y)\|_{\mathcal{L}(H)} \cdot \|A^{-1}\|_{\mathcal{L}(H)} \|A^{-1}Q\|_{\mathcal{L}_1(H)} < +\infty.$$

This completes the proof.

We now consider the Kolmogorov equation (1.7). Following Cannarsa and Da Prato [5], we say that $u(t,x)$ is a classical solution to the Kolmogorov equation (1.7),

if $u(t,x)$ satisfies the following conditions
(1) For any $t>0$, $u(t,\cdot)\in C_b^2(H)$ and for any $h,k\in H$, the function

$$[0,\infty)\times H\to\mathbf{R},\quad (t,x)\mapsto< QD^2u(t,x)h,k>$$

is continuous.
(2) For any $t>0$ and for any $x\in H$, $QD^2u(t,x)$ is of trace class and the function

$$[0,\infty)\times H\to\mathbf{R},\quad (t,x)\mapsto \mathrm{Tr}QD^2u(t,x)$$

is continuous.
(3) For any $t>0$ and for any $x\in H$, $Du(t,x)\in D(A^*)$ and the function

$$[0,+\infty)\times H\to\mathbf{R},\quad (t,x)\mapsto< x,A^*Du(t,x)>$$

is continuous.
(4) $u(t,x)$ satisfies (1.7).
Note that the so-called classical solutions are different from the strict solutions in Da Prato and Zabczyk [10] and the $\mathcal{K}$-strong solutions in Cerrai and Gozzi [6].

Theorem 2.1. *Suppose that the assumptions (A.1)-(A.4) hold. If $\varphi\in D(L)$, then $u(t,x)=P_t\varphi(x)$ is the unique classical solution to the Kolmogorov equation (1.7).*

Proof. The proof is analogous to that of Theorem 9.16 in [10]. From the arguments in the proof of Proposition 2.1, we know that $Y(t)=A^{-1}X(t,x)$ is a strong solution to the SEE (2.2). For $\varphi\in D(L)$, there exists a function $\psi\in C_b^2(H)$ such that $\varphi(x)=\psi(A^{-1}x)$. By applying Ito's formula (cf. Theorem 4.17 in [10]) to the function ψ and the process $Y(t)$ we get

$$\begin{aligned}
&\psi(Y(t))-\psi(Y(0))\\
&=\int_0^t< D\psi(Y(s)),A^{-1}Q^{\frac12}dW(s)>+\int_0^t< D\psi(Y(s)),AY(s)>ds\\
&+\int_0^t< D\psi(Y(s)),G(Y(s))>ds+\frac12\int_0^t \mathrm{Tr}[(A^{-1}Q^{\frac12})^*D^2\psi(Y(s))(A^{-1}Q^{\frac12})]ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\varphi(X(t))-\varphi(x)\\
&=\int_0^t< A^*D\varphi(X(t)),A^{-1}Q^{\frac12}dW(s)>+\int_0^t< A^*D\varphi(X(s)),X(s)>ds\\
&+\int_0^t< A^*D\varphi(X(s)),A^{-1}F(X(s))>ds+\frac12\int_0^t \mathrm{Tr}[QD^2\varphi(X(s))]ds.
\end{aligned}\tag{2.3}$$

From this it follows that

$$\begin{aligned}
& \mathbf{E}[\varphi(X(t))] - \varphi(x) \\
= \; & \mathbf{E}[\int_0^t < A^* D\varphi(X(s)), X(s) > ds] \\
& + \mathbf{E}[\int_0^t < D\varphi(X(s)), F(X(s)) > ds] + \mathbf{E}[\int_0^t \frac{1}{2}\mathrm{Tr}(QD^2\varphi(X(s)))ds].
\end{aligned}$$

Therefore,

$$\text{(2.4)} \quad \frac{\partial^+ u(t,x)}{\partial t}\Big|_{t=0} = \frac{1}{2}\mathrm{Tr}[QD^2\varphi(x)] + < A^* D\varphi(x), x > + < D\varphi(x), F(x) > .$$

Now by using the fact that $u(s,x) \in D(L)$ and the Markov property, we get

$$\text{(2.5)} \quad \frac{\partial^+ u(s,x)}{\partial s} = \frac{1}{2}\mathrm{Tr}[QD^2 u(s,x)] + < A^* Du(s,x), x > + < Du(s,x), F(x) > .$$

Since the right-hand side is a continuous function of s, we can conclude that $u(s,x)$ is differentiable with respect to s and that

$$\text{(2.6)} \qquad \frac{\partial u(s,x)}{\partial s} = L(u(s,x)), \quad \forall s \geq 0, \quad \forall x \in H.$$

This completes the proof of the existence part. The uniqueness follows from Theorem 9.17 in Da Prato and Zabczyk [10], since our classical solution is also a strict solution in the sense of the definition in [10].

We shall use the existence and uniqueness of classical solutions to equation (1.7) to deal with the essential self-adjointness of a certain class of symmetric diffusion operators in the next section.

3. Essential self-adjointness of symmetric diffusion operators

In this section we always assume that $(L, D(L))$ is symmetric on $L^2(H, \mu)$, where μ is the invariant measure of the diffusion process $X(t)$ associated to the diffusion operator $(L, D(L))$. In order to deal with the essential self-adjointness of $(L, D(L))$ on $L^2(H, \mu)$, we need the following parabolic criterion for self-adjointness.

Theorem 3.1 (Berezanskii [3, Theorem 6.13, Chapter 2]). *Let B be a symmetric operator with domain $D(B)$ in a separable Hilbert space $\mathcal{H}$ and semibounded below. Assume that there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of operators in $\mathcal{H}$ with domain $D(B_n)$,*

a dense subset Φ of $\mathcal{H}$, and $b \in (0, \infty]$ with the following properties:
(1) *For any $T \in (0, b)$ and any $\varphi_0 \in \Phi$, there exists a sequence $\{\varphi_{0,n}\}_{n=1}^{\infty}$ of vectors in $\mathcal{H}$ with $\varphi_{0,n} \to \varphi_0$ in $\mathcal{H}$ as $n \to \infty$ along with the corresponding strong solutions of the Cauchy problems*

$$\left\{ \begin{array}{l} \frac{d\varphi_n(t)}{dt} - B_n\varphi_n(t) = 0, \;\; t \in [0, T] \\ \varphi_n(T) = \varphi_{0,n}, \varphi_n(t) \in D(B), \end{array} \right. \tag{3.1}$$

(2) *For any $T \in (0, b)$ and any $\varphi_0 \in \Phi$*

$$\int_0^T (u(t), (B_n - B)\varphi_n(t))_{\mathcal{H}} dt \to 0 \quad \text{as } n \to \infty, \tag{3.2}$$

where $u(t)$ is an arbitrary strong solution of the following Cauchy problem

$$\frac{du(t)}{dt} + B^* u(t) = 0 \quad (t \in [0, \infty)), u(0) = 0. \tag{3.3}$$

Then B is essentially self-adjoint with domain $D(B)$ on $\mathcal{H}$.

Now we shall prove the following approximation criterion for the essential self-adjointness of the symmetric operator $(L, D(L))$ on $L^2(H, \mu)$. For convenience, we shall denote by $\|u\|_2$ the L^2-norm of u in $L^2(H, \mu)$.

Theorem 3.2. *Assume that (A.1)-(A.3) hold. Let $(L, D(L))$ be symmetric on $L^2(H, \mu)$ and suppose that*

$$\int_H \|x\|^2 \mu(dx) < +\infty.$$

Assume that there exists a sequence of mappings $\{F_n(\cdot)\}_{n \in \mathbf{N}}$, $F_n \in Lip(H, H)$ and F_n twice Fréchet differentiable with bounded and continuous derivatives, $\forall n \in \mathbf{N}$ such that
(1) *$\|F_n - F\| \to 0$ in $L^2(H, \mu)$ as $n \to \infty$.*
(2) *For each $n \in \mathbf{N}$, there exists a mapping $G_n : H \to H$ Lipschitz continuous on H and twice Fréchet differentiable with bounded and continuous derivatives, such that $A^{-1}F_n(x) = G_n(A^{-1}x)$.*
(3) *There exists a constant $C > 0$ such that*

$$\sup_{x \in H} \|DF_n(x)\|_{\mathcal{L}(H)} \leq C \quad \text{for all} \quad n \in \mathbf{N}.$$

Then $(L, D(L))$ is essentially self-adjoint on $L^2(H, \mu)$.

Proof. Let L_n be defined by

$$L_n\varphi(x) = \frac{1}{2}\mathrm{Tr}[QD^2\varphi(x)] + < x, A^*D\varphi(x) > + < F_n(x), D\varphi(x) > . \tag{3.4}$$

with domain $D(L)$. For $\varphi \in D(L)$, let $u_n(t,x) = \mathbf{E}(\varphi(X_n(t,x))$ where $X_n(t,x)$ is the unique mild solution of the following semilinear SEE

$$dX_n(t,x) = [AX_n(t,x) + F_n(X_n(t,x))]dt + Q^{\frac{1}{2}}dW_t, X_n(0,x) = x. \tag{3.5}$$

From equality (2.3) it follows that for all $x \in H$,

$$\frac{\partial u_n(t,x)}{\partial t} = \mathbf{E}\left((L_n\varphi)(X_n(t,x))\right)$$

and from this we get that there exists a constant C_1 such that for all $x \in H$ and all $t \in [0,1]$

$$\left|\frac{\partial u_n(t,x)}{\partial t}\right|^2 \leq C_1\left(1 + \|x\|^2\right).$$

This, together with Theorem 2.1 shows that the function $u_n : [0,1] \to L^2(H,\mu)$, defined by $u_n(t) = \mathbf{E}(\varphi(X_n(t,\cdot))$, is a strong solution of the Cauchy problem

$$\begin{cases} \frac{\partial u_n(t,x)}{\partial t} = L_n u_n(t,x), t \geq 0 \\ u_n(0,x) = \varphi(x) \end{cases} \tag{3.6}$$

Now, we follow the same procedure as in Albeverio, Kondratiev and Röckner [2]. By using Theorem 3.1 we only need to prove the following convergence

$$\int_0^1 \|(L - L_n)u_n(t,\cdot)\|_2 dt \to 0 \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned} & \int_0^1 \|(L - L_n)u_n(t,\cdot)\|_2 dt \\ \leq & \int_0^1 \| < F_n - F, Du_n(t) > \|_2 dt \\ \leq & \int_0^1 \|(\|F_n - F\|) \cdot (\|Du_n(t)\|)\|_2 dt. \end{aligned}$$

To estimate $\|Du_n(t)\|$, we can borrow some results from Da Prato, Elworthy and Zabczyk [9]. From Proposition 2.1 of [9], we know that $X_n(t,x)$ is differentiable along any direction $h \in H$ in the mean-square sense and $V_{n,x}^h(t) = D_h X_n(t,x)$ satisfies the following estimate

$$\begin{aligned} \|V_{n,x}^h(t)\| &\leq \|h\| e^{t[\omega + \sup_{x\in H} \|DF_n(x)\|_{\mathcal{L}(H)}]} \\ &\leq \|h\| e^{t[\omega + C]}. \end{aligned}$$

It easily follows that

$$\begin{aligned} \|Du_n(t,x)\| &= \sup_{\|h\|=1} |\mathbf{E}[< D\varphi(X_n(t,x)), V_{n,x}^h(t) >]| \\ &\leq \sup_{\|h\|=1} \mathbf{E}[\|D\varphi(X_n(t,x))\| \cdot \|V_{n,x}^h(t)\|] \qquad (3.7) \\ &\leq \sup_{y\in H} \|D\varphi(y)\| \cdot e^{t[\omega+C]}. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_0^1 \|(L - L_n)u_n(t,\cdot)\|_2 dt \\ \leq\ & \sup_{y\in H} \|D\varphi(y)\| \cdot e^{[\omega+C]} \, \|(\|F_n - F\|)\|_2 , \end{aligned}$$

which tends to zero as $n \to \infty$. This completes the proof.

By using the finite dimensional approximation procedure due to Peszat and Zabczyk [15], we can prove the following result.

Theorem 3.3. *Assume that (A.1)-(A.3) hold. Let $(L, D(L))$ be symmetric on $L^2(H,\mu)$. Suppose that F is Lipschitz continuous on H, i.e. there exists a constant $K > 0$ such that $\|F(x) - F(y)\| \leq K\|x - y\|$ for any $x, y \in H$ and*

$$\int_H \|x\|^2 \mu(dx) < +\infty.$$

Then $(L, D(L))$ is essentially self-adjoint on $L^2(H,\mu)$.

Proof. According to Theorem 3.2, we only need to construct a sequence of mappings $\{F_n(\cdot)\}_{n\in\mathbf{N}}$ which satisfies all the conditions (1)-(3) of Theorem 3.2. We adopt the approximation technique used in [15] by Peszat and Zabczyk. We choose a sequence of non-negative smooth functions $\{\rho_n\}$ such that

$$\text{supp}(\rho_n) \subseteq \{\xi \in \mathbf{R}^n : \|\xi\|_{\mathbf{R}^n} \leq \frac{1}{n}\}$$

and

$$\int_{\mathbf{R}^n} \rho_n(\xi)d\xi = 1.$$

Let P_n be the orthonormal projection of H onto $\operatorname{span}\{e_1, \cdots, e_n\}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of H as given in (A.1). We will identify $\mathbf{R}^n$ with $\operatorname{span}\{e_1, \cdots, e_n\}$ and set $T_n\xi = \sum_{i=1}^n \xi_i e_i$ for $\xi \in \mathbf{R}^n$. Now we define the mapping $F_n : H \to H$ by

$$F_n(x) = \int_{\mathbf{R}^n} \rho_n(\xi - P_n x)F(T_n\xi)d\xi. \tag{3.8}$$

Then F_n is twice Fréchet differentiable with bounded and continuous derivatives for each $n \in \mathbf{N}$. Moreover, for all $x,\ y \in H$ and $n \in \mathbf{N}$,

$$\begin{aligned}
&\|F_n(x) - F_n(y)\| \\
= \ & \|\int_{\mathbf{R}^n} \rho_n(\xi)[F(T_n\xi + P_n x) - F(T_n\xi + P_n y)]d\xi\| \\
\leq \ & K\|P_n(x-y)\| \int_{\mathbf{R}^n} \rho_n(\xi)d\xi \\
\leq \ & K\|x-y\|.
\end{aligned}$$

This implies that $\sup_{x\in H} \|DF_n(x)\|_{\mathcal{L}(H)} \leq K$, i.e. $\{F_n\}_{n\in\mathbf{N}}$ satisfies condition (3). For condition (1), we have

$$\begin{aligned}
&\|F_n - F\|_2^2 \\
= \ & \int_H \left\| \int_{\mathbf{R}^n} \rho_n(\xi - P_n x)F(T_n\xi)d\xi - F(x) \right\|^2 d\mu(x) \\
= \ & \int_H \left\| \int_{\mathbf{R}^n} (\rho_n(\xi - P_n x)F(T_n\xi) - \rho_n(\xi)F(x))d\xi \right\|^2 d\mu(x) \\
\leq \ & \int_H \left[\int_{\mathbf{R}^n} \rho_n(\xi)\ \|F(T_n\xi + P_n x) - F(x)\|\ d\xi \right]^2 d\mu(x) \\
\leq \ & \int_H \left[\int_{R^n} K\ \|T_n\xi + P_n x - x\|\ \rho_n(\xi)d\xi \right]^2 d\mu(x) \\
\leq \ & 2K^2 \left[\int_{\mathbf{R}^n} \rho_n(\xi) \cdot \|\xi\| d\xi \right]^2 \cdot \mu(H) \\
& +2K^2 \int_H \|P_n x - x\|^2 d\mu(x) \\
\leq \ & \frac{2K^2}{n^2} \cdot \mu(H) + 2K^2 \int_H \|P_n x - x\|^2 d\mu(x),
\end{aligned}$$

which tends to zero as $n \to \infty$. For condition (2), we only need to show that there exists a sequence of mappings $\{G_n(\cdot)\}_{n\in\mathbf{N}}$ such that $G_n \in Lip(H,H)$, G_n twice Fréchet differentiable with bounded and continuous derivatives for each $n \in \mathbf{N}$, and $A^{-1}F_n(x) = G_n(A^{-1}x)$ for all $x \in H$. In fact, by setting

$$G_n(y) = \int_{\mathbf{R}^n} \rho_n(\xi - AP_n y))A^{-1}F(T_n\xi)d\xi, \tag{3.9}$$

we can easily check that $\{G_n(\cdot)\}_{n\in\mathbf{N}}$ is the required sequence. This completes the proof.

In the following we consider a certain class of symmetric diffusion operators which satisfy the conditions of Theorem 3.2. Suppose that F is of gradient form, i.e. $F = QDU$, where U is Gâteaux differentiable on H. We shall prove that $(L, D(L))$ is symmetric on $L^2(H,\mu)$ with $\mu(dx) = e^{2U(x)}\gamma(dx)$, under some conditions on U. For this we need an integration by parts formula with respect to the Gaussian measure γ on the Hilbert space H (cf. Lemma 2.1 in Berezanskii and Samoilenko [4]).

Lemma 3.1. *Let f and g be bounded differentiable functions on H with bounded first order derivative. Then for $h \in D(Q^{-1}A)$*

$$\begin{aligned}
&\int_H < Df(x), h > g(x)\gamma(dx) \\
&= \int_H f(x)[- < Dg(x), h > -2 < x, Q^{-1}Ah > g(x)]\gamma(dx).
\end{aligned} \tag{3.10}$$

We denote by D_k the derivative in the direction e_k, $k \in \mathbf{N}$. From Lemma 3.1, it easily follows that D_k is closable for any $k \in \mathbf{N}$. For simplicity, we shall still denote by D_k ($k \in \mathbf{N}$) the closure. Now we introduce the Sobolev space $W^{1,2}(H,\gamma)$. We denote by $W^{1,2}(H,\gamma)$ the linear space of all functions $f \in L^2(H,\gamma)$ such that $D_k f \in L^2(H,\gamma)$ for all $k \in \mathbf{N}$ and

$$\int_H \|Df\|^2\gamma(dx) = \sum_{k=1}^{\infty} \int_H |D_k f(x)|^2\gamma(dx) < +\infty.$$

Then it is easy to see that Lemma 3.1 holds for $f,\ g \in W^{1,2}(H,\gamma)$.

Now we prove the following result.

Theorem 3.4. *Let (A.1) be fulfilled. If $F = QDU$, where U is Gâteaux differentiable on H and such that $V = e^U \in W^{1,2}(H,\gamma)$, then $(L, D(L))$ is symmetric on $L^2(H,\mu)$ with $\mu(dx) = e^{2U(x)}\gamma(dx)$.*

Proof. From the assumption $V = e^U \in W^{1,2}(H,\gamma)$, it follows that

$$\int_H \|DU\|^2 \mu(dx) = \int_H \|DU\|^2 V^2 \gamma(dx) = \int_H \|DV\|^2 \gamma(dx) < +\infty.$$

Moreover, from Theorem 2.4 in Aida, Masuda and Shigekawa [1] or Proposition 2.4 in Da Prato [8], we have

$$\int_H \|x\|^2 \mu(dx) < +\infty.$$

Therefore, from Lemma 3.1 and by using the same arguments as in the proof of Proposition 2.1 in Röckner and Zhang [16], we obtain for $h \in D(Q^{-1}A)$ and all $f, g \in C_b^1(H)$

$$\begin{aligned}(3.11)\qquad &\int_H < Df(x), h > g(x)\mu(dx) \\ &= \int_H f(x)[- < Dg(x), h > -2(< x, Q^{-1}Ah > + < DU(x), h >)g(x)]\mu(dx).\end{aligned}$$

For fixed $e \in D(A)$ and any $f, g \in D(L)$, we have

$$< QD^2 f(x)e, e >=< D(< Df(x), Q^{\frac{1}{2}}e >), Q^{\frac{1}{2}}e >, \ x \in H.$$

By using (3.11), we obtain

$$\begin{aligned} &\int_H < QD^2 f(x)e, e > g(x)\mu(dx) \\ = &\int_H < D(< Df(x), Q^{\frac{1}{2}}e >), Q^{\frac{1}{2}}e > g(x)\mu(dx) \\ (3.12)\qquad = &\int_H < Df(x), Q^{\frac{1}{2}}e > [- < Dg(x), Q^{\frac{1}{2}}e >]\mu(dx) \\ &-2\int_H < Df(x), Q^{\frac{1}{2}}e > < x, Q^{-1}AQ^{\frac{1}{2}}e > g(x)\mu(dx) \\ &-2\int_H < Df(x), Q^{\frac{1}{2}}e >< DU(x), Q^{\frac{1}{2}}e > g(x)\mu(dx). \end{aligned}$$

Note that if e is an eigenvector of A and Q with the eigenvalue α and λ respectively, then

$$\begin{aligned} < Df(x), Q^{\frac{1}{2}}e >< x, Q^{-1}AQ^{\frac{1}{2}}e > &= \alpha < Df(x), e >< x, e > \\ &= < Df(x), \alpha e >< x, e > \\ &= < A^* Df(x), e >< x, e > . \end{aligned}$$

Then we have

$$
\begin{aligned}
&\int_H \mathrm{Tr}[QD^2 f(x)]g(x)\mu(dx)\\
&= \int_H \sum_{j=1}^{\infty} < QD^2 f(x)e_j, e_j > g(x)\mu(dx)\\
&= \sum_{j=1}^{\infty} \int_H < QD^2 f(x)e_j, e_j > g(x)\mu(dx)\\
&= -\sum_{j=1}^{\infty} \int_H < Df(x), Q^{\frac{1}{2}}e_j >< Dg(x), Q^{\frac{1}{2}}e_j > \mu(dx)\\
&\quad -2\sum_{j=1}^{\infty} \int_H < A^* Df(x), e_j >< x, e_j > g(x)\mu(dx)\\
&\quad -2\sum_{j=1}^{\infty} \int_H < Df(x), Q^{\frac{1}{2}}e_j >< DU(x), Q^{\frac{1}{2}}e_j > g(x)\mu(dx)\\
&= -\int_H < Q^{\frac{1}{2}}Df(x), Q^{\frac{1}{2}}Dg(x) > \mu(dx) - 2\int_H < x, A^* Df(x) > g(x)\mu(dx)\\
&\quad -2\int_H < Q^{\frac{1}{2}}DU(x), Q^{\frac{1}{2}}Df(x) > g(x)\mu(dx).
\end{aligned}
\tag{3.13}
$$

We have interchanged summation and integration four times. This is valid because of the following estimates:

$$
\sum_{j=1}^{n} | < QD^2 f(x)e_j, e_j > | \leq \sup_{x\in H} \|QD^2 f(x)\|_{\mathcal{L}_1(H)} \tag{3.14}
$$

$$
\sum_{j=1}^{n} < Q^{\frac{1}{2}}Df(x), e_j >< Q^{\frac{1}{2}}Dg(x), e_j > \leq \|Q^{\frac{1}{2}}Df(x)\| \cdot \|Q^{\frac{1}{2}}Dg(x)\| \tag{3.15}
$$

$$
|\sum_{j=1}^{n} < A^* Df(x), e_j >< x, e_j > | \leq \|A^* Df(x)\| \cdot \|x\| \tag{3.16}
$$

and

$$
|\sum_{j=1}^{n} < Df(x), Q^{\frac{1}{2}}e_j >< DU(x), Q^{\frac{1}{2}}e_j > | \leq \|Df(x)\| \cdot \|QDU(x)\|, \tag{3.17}
$$

where $x \in H$, and $n = 1, 2, \cdots$. From the expression (1.6) for the operator L and (3.13), we obtain

$$\int_H Lf(x)\, g(x)\mu(dx) = -\frac{1}{2}\int_H < Q^{\frac{1}{2}}Df(x), Q^{\frac{1}{2}}Dg(x) > \mu(dx). \tag{3.18}$$

This shows that $(L, D(L))$ is symmetric on $L^2(H, \mu)$.

Combining Theorem 3.3 and Theorem 3.4, we immediately conclude the following result.

Corollary 3.1. *Suppose that (A.1)-(A.3) hold and the condition of Theorem 3.4 is fulfilled. If we further assume that* $F = QDU \in \mathrm{Lip}(H, H)$, *then* $(L, D(L))$ *is essentially self-adjoint on* $L^2(H, \mu)$.

REMARK 3.1. As the authors of [2] pointed out, it is often important to choose a domain of essential self-adjointness for L which only contains cylinder functions. We denote by $\mathcal{F}C_b^\infty(H)$ the set of all functions f on H such that there exist $n \in \mathbf{N}$, $l_1, \cdots, l_n \in D(A^*)$ and $\varphi \in C_b^\infty(\mathbf{R}^n)$ such that

$$f(x) = \varphi(< x, l_1 >, \cdots, < x, l_n >), \quad x \in H.$$

We can use similar arguments to the ones in the proof of Lemma 6 in Albeverio, Kondratiev and Röckner [2] to show that the closure of $(L, \mathcal{F}C_b^\infty(H))$ coincides with the closure of $(L, D(L))$ in $L^2(H, \mu)$. From this, we can immediately conclude from Theorem 3.2, Theorem 3.3 and Corollary 3.1 that the set $\mathcal{F}C_b^\infty(H)$ is a domain of essential self-adjointness for the symmetric diffusion operator L on $L^2(H, \mu)$.

Finally we provide two examples.

EXAMPLE 3.1. Consider an open cube $\Lambda := (0, \pi)^{\otimes d}$ in $\mathbf{R}^d$ and set $H = L^2(\Lambda, d\xi)$, where $d\xi$ denotes d-dimensional Lebesgue measure. Let $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial \xi_i^2}$ denote the d-dimensional Laplacian and let $(\Delta_\Lambda, D(\Delta_\Lambda))$ denote the Friedrich's extension of Δ on $C_0^\infty(\Lambda)$. Let $m \geq 1$ and set $B = (m - \Delta_\Lambda), D(B) = D(\Delta_\Lambda)$. Then B has a complete set of eigenvectors

$$Be_k = (m + |k|^2)e_k := \sigma_k e_k, k \in \mathbf{N}^d,$$

where

$$|k|^2 = k_1^2 + \cdots + k_d^2, (k_1, \cdots, k_d) \in \mathbf{N}^d,$$

and

$$e_k(\xi) = (\frac{2}{\pi})^{\frac{d}{2}} \sin(k_1\xi_1)\cdots\sin(k_d\xi_d).$$

Let $\beta_A \geq 0$ and let $\beta+\beta_A > \frac{d}{2}$. Set $A = -(m-\Delta_\Lambda)^{\beta_A}$ and $Q = (m-\Delta_\Lambda)^{-\beta}$. Then, according to Lemma 28 of [14], the assumption (A.1) holds under the above conditions on β and β_A. Now we verify the assumption (A.3), i.e. there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\int_0^T s^{-2\alpha}\|S(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(H)} ds < +\infty.$$

For this, we need to decompose the above conditions on β and β_A into the following three cases
(i) $\beta > \frac{d}{2}$ and $\beta_A \geq 0$; in this case, for any $q > 1$, we always have

$$\beta + \frac{\beta_A}{q} > \frac{d}{2}.$$

(ii) $\beta = \frac{d}{2}$ and $\beta_A > 0$; in this case, for any $q > 1$, we also have

$$\beta + \frac{\beta_A}{q} > \frac{d}{2}.$$

(iii) $\beta < \frac{d}{2}$ and $\beta_A > \frac{d}{2} - \beta$; in this case, for any $q \in (1, \frac{\beta_A}{\frac{d}{2}-\beta})$, we have

$$\beta + \frac{\beta_A}{q} > \frac{d}{2}.$$

Let p be the conjugate index of q, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In any case of (i)-(iii), by using Hölder inequality, we have

$$\begin{aligned}
&\int_0^T s^{-2\alpha}\|S(s)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(H)} ds \\
= &\int_0^T s^{-2\alpha} \sum_{k\in\mathbf{N}^d} < e^{sA}Q^{\frac{1}{2}}e_k, e^{sA}Q^{\frac{1}{2}}e_k > ds \\
= &\sum_{k\in\mathbf{N}^d} \sigma_k^{-\beta} \int_0^T s^{-2\alpha} e^{-2s\sigma_k^{\beta_A}} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k\in\mathbf{N}^d} \sigma_k^{-\beta}\{\int_0^T s^{-2p\alpha} e^{-ps\sigma_k^{\beta_A}} ds\}^{\frac{1}{p}} \cdot \{\int_0^T e^{-qs\sigma_k^{\beta_A}} ds\}^{\frac{1}{q}} \\
&\leq \{\int_0^T s^{-2p\alpha} e^{-ps(m+d)^{\beta_A}} ds\}^{\frac{1}{p}} \sum_{k\in\mathbf{N}^d} \sigma_k^{-\beta}\{\frac{1-e^{-qT\sigma_k^{\beta_A}}}{q\sigma_k^{\beta_A}}\}^{\frac{1}{q}} \\
&\leq \{\int_0^T s^{-2p\alpha} e^{-ps(m+d)^{\beta_A}} ds\}^{\frac{1}{p}} \sum_{k\in\mathbf{N}^d} \sigma_k^{-(\beta+\frac{\beta_A}{q})}.
\end{aligned}
$$

Since

$$\beta + \frac{\beta_A}{q} > \frac{d}{2},$$

thus according to Lemma 28 of [14], we know that

$$\sum_{k\in\mathbf{N}^d} \sigma_k^{-(\beta+\frac{\beta_A}{q})} < +\infty.$$

On the other hand, if we choose $\alpha > 0$ such that $0 < 2p\alpha < 1$, i.e. $\alpha \in (0, \frac{q-1}{2q})$, then

$$\int_0^T s^{-2p\alpha} e^{-ps(m+d)^{\beta_A}} ds < +\infty.$$

Therefore the assumption (A.3) holds for $\alpha \in (0, \frac{q-1}{2q})$.

Let $x = x(\xi)$ for $\xi \in \Lambda$. Define a differential operator L with domain $D(L)$ as in (1.6) with F defined by

$$F(x)(\xi) = Qf'(x(\xi)), \forall \xi \in \Lambda,$$

where f satisfies the following conditions
(1) $f(\cdot) \in C^2(\mathbf{R})$.
(2) For any $x \in H$, $f \circ x \in L^1(\Lambda)$ and $f(t) \leq -K_1 t^2 + K_2$ for some positive constants K_1 and K_2.
(3) $|f'(t)| \leq K_3(1 + |t|)$ for some constant $K_3 > 0$.
(4) $\sup_{t\in\mathbf{R}} |f''(t)| \leq K_4 < \infty$ for some constant $K_4 > 0$.
Then the coefficients of the differential operator L satisfy the conditions of Corollary 3.1. In fact, by setting $U(x) = \int_\Lambda f(x(\xi))d\xi$ and $V(x) = e^{U(x)}$, then obviously $U \in C^1(H)$ and from the condition (4) it follows that $F = QDU$ is Lipschitz continuous on H. On the other hand, for $h \in H$ we have

$$|D_h V(x)| = |\int_\Lambda f'(x(\xi))h(\xi)d\xi \cdot \exp(\int_\Lambda f(x(\xi))d\xi|$$

$$\begin{aligned}
&\leq e^{-K_1\|x\|^2+K_2}\{2K_3(1+\|x\|^2)\}^{\frac{1}{2}} \cdot \|h\| \\
&\leq \frac{e^{K_2}}{1+K_1\|x\|^2}\sqrt{2K_3}(1+\|x\|^2)^{\frac{1}{2}}\|h\| \\
&\leq K_5\|h\|,
\end{aligned}$$

where K_5 is a positive constant depending only on K_1, K_2 and K_3. This shows that $V \in C_b^1(H)$. Thus according to Corollary 3.1, we know that the differential operator $(L, D(L))$ defined as above is essentially self-adjoint on $L^2(H, \mu)$ with $\mu(dx) = e^{2U(x)}\gamma(dx)$.

EXAMPLE 3.2. Let $H = L^2(\Lambda, d\xi)$ and let the operator A, Q and the function F be defined as in Example 3.1, with $f \in C_b^2(\mathbf{R})$. Then the hypotheses of Corollary 3.1 are satisfied.

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