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VERY AMPLENESS OF $K_X \otimes \mathcal{L}^{\dim X}$ FOR AMPLE AND SPANNED LINE BUNDLES $\mathcal{L}$

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Introduction

Let X be a normal projective variety and let $\mathcal{L}$ be an ample and spanned line bundle on X, i.e. assume that there is a finite to one holomorphic map ψ: $X \to \boldsymbol{P}_C$ with $\mathcal{L} = \psi^* \mathcal{O}_{P_C}(1)$.

The study of $K_X \otimes \mathcal{L}^t$ has been a powerful technique ([So1], [VdV], [So2], [S-VdV]) in understanding the original pair $(X, \mathcal{L})$ or equivalently (X, ψ), when X is smooth and $\mathcal{L}$ is very ample (i.e. ψ is an embedding).

In the article [A-S] a detailed study was made of the spannedness properties of $K_X \otimes \mathcal{L}^{n-1}$ where $n = \dim X$ and K_X denotes the dualizing sheaf. This article, building on these results, works out the very ampleness properties of $K_X \otimes \mathcal{L}^t$, $t \geqslant n = \dim X$ on *reg*(X), the smooth points of X.

For simplicity of description we assume in the introduction $\mathrm{cod}_X Sing(X) \geqslant 3$, $Sing(X)$ being the set of singular points.

Theorem. *If $t \geqslant n+1$, then $K_X \otimes \mathcal{L}^t$ is very ample on reg(X) unless $t = n+1$ and $(X, \mathcal{L}) = (\boldsymbol{P}^n, \mathcal{O}_{P^n}(1))$.*

This result partially relies on the following

Theorem. *If $t \geqslant n$, then $\Gamma(K_X \otimes \mathcal{L}^t)$ spans $K_X \otimes \mathcal{L}^t$ on reg(X) unless $t = n$ and $(X, \mathcal{L}) = (\boldsymbol{P}^n, \mathcal{O}_{P^n}(1))$.*

The above results are actually shown to be true in the stronger form with $\mathcal{K}_X$, the Grauert-Riemenschneider canonical sheaf, replacing K_X, the dualizing sheaf. In [So4], [So3], the spannedness of $K_X \otimes \mathcal{L}^t$ for $t \geqslant n$ was shown on all of X when X is Gorenstein with a finite set of non-rational singulartities. The techniques used there and developed further in [A-S] are used to prove the above results. These techniques plus Reider's theorem on $K_X \otimes \mathcal{L}$ ([Re], see (0.6.2)) and the methods of [So5] lead to the next result. To state it, we need to recall that in [So5] $\mathfrak{X}_2$, a very special class of well understood varieties described in (0.3), is characterized by the condition $h^0(K_X\otimes\mathcal{L}^{n-1})=0$.

Theorem. *Let X be a Cohen-Macaulay normal projective variety of dimension $n\geqslant 2$ with $\mathrm{cod}_X\,Sing(X)\geqslant 3$ and only a finite set of non-rational singularities. Let $\mathcal{L}$ be an ample and spanned line bundle and assume that $(X,\mathcal{L})\notin\mathcal{X}_2$. If either $c_1(\mathcal{L})^n\geqslant 5$ or $c_1(\mathcal{L})^n\geqslant 3$ and $\Gamma(\mathcal{L})$ gives a generically one ot to one map, then $K_X\otimes\mathcal{L}^n$ is very ample on $reg(X)$.*

The assertion is no longer true without the finiteness assumption. A counterexample can be found in [So5], (0.2.4). We have to recall that in the smooth case some aspects of our results can be deduced by Mori's theory [M].

The above results have various corollaries. Some of them are concerned with the ramification divisor of branched covers of $\boldsymbol{P}^n$ and hyperquadrics by a normal projective variety X with $\mathrm{cod}_X\,Sing\,(X)\geqslant 3$.

Theorem. *Let $\psi: X\to\boldsymbol{P}^n$ be a branched cover of $\boldsymbol{P}^n$. If $\deg\psi\geqslant 2$, then the ramification divisor of ψ is very ample on $reg(X)$.*

Theorem. *Let $\psi: X\to Q^n$ be a branched cover of a hyperquadric. Assume that X is Cohen-Macaulay with only a finite set of non-rational singularities. If $\deg\psi\geqslant 3$, then the ramification divisor of ψ is very ample on $reg(X)$.*

To finish with, in the following applications $\mathcal{L}$ is an ample and spanned line bundle on a smooth projective n-fold X.

Theorem. *Assume that $K_X^N=\mathcal{O}_X$ for some $N\geqslant 1$. If $c_1(\mathcal{L})^n\geqslant 5$, then $\mathcal{L}^n$ is very ample.*

Theorem. *If $K_X\otimes\mathcal{L}^{n-1}$ is numrically effective and $c_1(\mathcal{L})^n\geqslant 5$, then $(X,\mathcal{L})$ admits a reduction $(X',\mathcal{L}')$ such that $(K_{X'}\otimes\mathcal{L}'^{n-1})^2$ is very ample.*

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0. Notation and background

(0.0) We work over the complex numbers. All spaces are complex analytic and all maps are holomorphic. By **variety** we mean an irreducible and reduced complex analytic space. If X is a complex analytic space, $\mathcal{O}_X$ stands for its holomorphic structure sheaf. We denote by $reg(X)$, $Sing(X)$, $Irr(X)$ respectively the set of smooth points, of singular points, of non-rational singularities. In other words, the **irrational locus** $Irr(X)$ of X is the union of the supports of the sheaves $\pi_{(i)}(\mathcal{O}_{\bar{X}})$ for $i>0$, where $\pi: \bar{X}\to X$ is a desingularization of X.

(0.1) Let X be a normal variety. If $\mathcal{D}$ is a Cartier divisor, $[\mathcal{D}]$ denotes the associated invertible sheaf. Let now $\mathcal{D}$ be a Weil divisor on X. By $[\mathcal{D}]$ we mean the reflexive divisorial sheaf associated to $\mathcal{D}$, i.e. $[\mathcal{D}]=i_*[\mathcal{D}]_{reg(X)}$, where $i\colon reg(X)\to X$ is the obvious inclusion. Note that $[\mathcal{D}]$ is a reflexive rank-1 sheaf.

Given any reflexive rank-1 sheaf $\mathcal{S}$ on X and an integer $t>0$, $\mathcal{S}^t$ will stand for

$$(\underbrace{\mathcal{S}\otimes\cdots\otimes\mathcal{S}}_{\text{t-times}})^{**}$$

and $\mathcal{S}^{-t}$ for $\mathcal{S}^{*t}$, where $\mathcal{S}^*$ is the dual of $\mathcal{S}$. If $\mathcal{S}^t$ is invertible for some $t>0$, then $c_1(\mathcal{S})\in H^2(X, \boldsymbol{Q})$ is well defined as $c_1(\mathcal{S}^t)/t$. Such an $\mathcal{S}$ is said to be **big** if $c_1(\mathcal{S})^{\dim X}>0$ and is said to be **numerically effective** (**nef** for short) if $c_1(\mathcal{S})[C]\geqslant 0$ for all effective curves C on X.

Let $\mathcal{S}$ be a reflexive rank-1 sheaf on X. Let $U\in X$ be a Zariski open set on which $\mathcal{S}$ is invertible. By saying that $\mathcal{S}$ is **spanned on** U we mean that $\Gamma(\mathcal{S})$ spans $\mathcal{S}$ at every point $x\in U$; by saying that $\mathcal{S}$ is **very ample** on U we mean that $\mathcal{S}$ is spanned on U and that the map associated to $\Gamma(\mathcal{S})$ gives an embedding on U. In particular, if $\mathcal{D}$ is a Weil divisor, since it is a Cartier divisor on $reg(X)$, we say that $\mathcal{D}$ or $[\mathcal{D}]$ is very ample on a Zariski open set $U\subseteq X$ if $\Gamma([\mathcal{D}])$ spans $[\mathcal{D}]$ at every point $x\in U$ and the associated map is an embedding on U.

The **dualizing sheaf** of the normal variety X is, by definition, $K_X=i_*K_{reg(X)}$ where $i\colon reg(X)\to X$ is the obvious inclusion and $K_{reg(X)}$ is the canonical sheaf of $reg(X)$. K_X is a reflexive rank-1 sheaf on X. If X is a normal Cohen-Macaulay variety of dimension $n\geqslant 2$ and A is an effective normal Cartier divisor on X, we have by [A-K], p. 7,

$$(0.1.1) \qquad (K_X\otimes[A])_A = K_A\,.$$

(0.2) Let X be a normal variety. Let $\pi\colon \bar{X}\to X$ be a desingularization. The **Grauert-Riemenschneider sheaf,** which we denote by $\mathcal{K}_X$, is defined as $\pi_*K_{\bar{X}}$, [G-R]. As is known, the sheaf $\mathcal{K}_X$ is independent of the desingularization chosen. One of the fundamental facts is the Grauert-Riemenschneider vanishing theorem, which says that $\pi_{(i)}(K_{\bar{X}})=0$, $i>0$. In particular, given an ideal sheaf $\mathcal{J}_F$, where $F\subseteq reg(X)$, and any line bundle $\mathcal{L}$, from the Leray spectral sequence for π, we get

$$(0.2.1) \qquad H^i(K_{\bar{X}}\otimes\pi^*\mathcal{L}\otimes\mathcal{J}_{\pi^{-1}(F)}) = H^i(\mathcal{K}_X\otimes\mathcal{L}\otimes\mathcal{J}_F)\,.$$

Note also that there is a canonical sequence

$$0\to\mathcal{K}_X\to K_X\to\mathcal{S}\to 0\,,$$

where $\mathcal{S}$ is supported on $Irr(X)$. As a consequence, for any line bundle $\mathcal{L}$ and any integer t,

(0.2.2) if $\Gamma(\mathcal{K}_X \otimes \mathcal{L}^t)$ spans $\mathcal{K}_X \otimes \mathcal{L}^t$ at $x \in reg(X)$, then $\Gamma(K_X \otimes \mathcal{L}^t)$ spans $K_X \otimes \mathcal{L}^t$ at x;
(0.2.3) if $\mathcal{K}_X \otimes \mathcal{L}^t$ is very ample on $reg(X)$, then $K_X \otimes \mathcal{L}^t$ is very ample on $reg(X)$.

Another useful consequence is the following corollary of the Grauert-Riemenschneider [G-R] vanishing theorem and of the Kawamata [Ka] and Viehweg [Vi] vanishing theorem (for a proof see [So5], Theorem (0.2.1))

Theorem. (0.2.4) *Let $\mathcal{L}$ be a nef and big line bundle on a normal projective variety X. Then*

$$H^i(X, \mathcal{L}^{-1}) = 0 \quad \textit{for} \quad i < \min(\dim X, 2)$$
$$H^i(X, K_X \otimes \mathcal{L}) = 0 \quad \textit{for} \quad i > \max(0, \dim Irr(X)).$$

(0.3) An important role will be played in the sequel by a special class $\mathcal{X}_2$ of pairs $(X, \mathcal{L})$, where X is a normal Cohen-Macaulay projective variety of dimension $n \geqslant 2$ and $\mathcal{L}$ is an ample and spanned line bundle on X. Class $\mathcal{X}_2$ consists of the following pairs:
a) $(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$;
b) $(Q^n, \mathcal{O}_{Q^n}(1))$, where $Q^n \subset \boldsymbol{P}^{n+1}$ is a quadric hypersurface;
c) scrolls over a smooth curve C, i.e. pairs $(X, \mathcal{L})$ where X is a holomorphic $\boldsymbol{P}^{n-1}$-bundle over C and the restriction $\mathcal{L}_f$ of $\mathcal{L}$ to a fibre f is $\mathcal{O}_{\boldsymbol{P}^{n-1}}(1)$;
d) possibly degenerated generalized cones on one of the following pairs:
a scroll over $\boldsymbol{P}^1$, $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2))$, $(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(e))$, $e \geqslant 3$; for the definition of generalized cone see [So5], (3.4).
This class was characterized by the third author as follows.

Theorem. (0.3.1) *Let $\mathcal{L}$ be an ample and spanned line bundle on a normal Cohen-Macaulay projective variety X of dimension $n \geqslant 2$. Assume that $Irr(X)$ is finite. Then $(X, \mathcal{L}) \in \mathcal{X}_2$ if and only if $h^0(K_X \otimes \mathcal{L}^{n-1}) = 0$.*

(0.4) Let X be a normal variety and $\mathcal{L}$ an ample and spanned line bundle on X. By **line** we mean a curve $l \subset X$ isomorphic to a smooth $\boldsymbol{P}^1$ such that $c_1(\mathcal{L})[l] = 1$. The **vertex set** $\mathcal{V}(X)$ is defined as the set of those $x \in X$ such that there is a non-empty variety $Z \subseteq Sing(X)$ with $\mathrm{cod}_X Z = 2$ and for each point $\zeta \in Z$ a line l containing x and ζ, ie. x is the vertex of a cone on Z. Note that if $\mathrm{cod}_X Sing(X) \geqslant 3$, then $\mathcal{V}(X) = \emptyset$.

The main technical tool we need in section 2 is the following result

Theorem. (0.4.1) ([A-S], Theorem (2.1)). *Let $\mathcal{L}$ be an ample and spanned line bundle on a normal Cohen-Macaulay projective variety X of dimension $n \geqslant 2$ with $Irr(X)$ finite. Assume that $c_1(\mathcal{L})^n \geqslant 5$ or that $\mathcal{L}$ gives a generically one to one map. Then the following facts are equivalent*

α) $(X, \mathcal{L}) \notin \mathcal{X}_2$;
β) $\Gamma(K_X \otimes \mathcal{L}^{n-1})$ *spans* $K_X \otimes \mathcal{L}^{n-1}$ *at all Gorenstein points and at every point* $x \in reg(X) \setminus \mathcal{V}(X)$.

(0.5) Let X be a normal projective variety. If $\mathcal{L}$ is a line bundle on X, $|\mathcal{L}|$ will denote the corresponding complete linear system. We will often denote by L a divisor associated to $\mathcal{L}$. If $x_1, \cdots, x_r$ are some smooth points on X and $k_1, \cdots, k_r$ are positive integers, we will denote by

$$|\mathcal{L} - k_1 x_1 - \cdots - k_r x_r|$$

the linear subsystem of $|\mathcal{L}|$ consisting of the effective divisors vanishing of order $\geqslant k_i$ at x_i. Note that if $\mathcal{L}$ is ample and spanned, $|\mathcal{L} - x|$ has a finite base locus. This follows because the map associated to $\Gamma(\mathcal{L})$ is finite to one.

Let $\pi: \bar{X} \to X$ be a morphism between two normal projective varieties. We will frequently use the following notation. If $\mathcal{L}$ is any line bundle on X, $\bar{\mathcal{L}}$ will stand for $\pi^* \mathcal{L}$ and $\bar{L}$ will be any divisor associated to $\bar{\mathcal{L}}$.

(0.5.0) If π is a desingularization of X, note that $\bar{\mathcal{L}}$ is nef if $\mathcal{L}$ is nef.

We will also need the following technical fact.

Lemma. (0.5.1) *Let $\mathcal{L}$ be a nef line bundle on a normal projective variety X of dimension $n \geqslant 2$ and let Δ be a linear system contained in $|\mathcal{L}|$. If the base locus of Δ is a finite set F, then*

$$\bar{L} - \sum_{i=1}^{r} \mathcal{P}_i$$

is nef, where $\pi: \bar{X} \to X$ is the blow-up of X at $F' = \{x_1, \cdots, x_r\} \subseteq F$ with $F' \subset reg(X)$ and $\mathcal{P}_i = \pi^{-1}(x_i)$. If furthermore each element of Δ is singular at x_i to the t_i-th order at least, $i = 1, \cdots, r$, then

$$\bar{L} - \sum_{i=1}^{r} t_i \mathcal{P}_i$$

is nef.

Proof. Let $C \subset \bar{X}$ be an irreducible reduced curve. If $C \subset \mathcal{P}_i$ for an index i, then $\mathcal{P}_i \cdot C < 0$ and so $(\bar{L} - \sum_{i=1}^{r} \mathcal{P}_i) \cdot C > 0$. Now assume that $C \not\subset \pi^{-1}(F')$. In this case there exists a reduced $D \in \Delta$ such that $\pi(C) \not\subset Supp(D)$ as Δ has a finite base locus. Then

$$\pi^* D = \tilde{D} + \sum_{i=1}^{r} k_i \mathcal{P}_i ,$$

where $\tilde{D} = \overline{\pi^{-1}(reg(D))}$ and $k_i > 0$ is the multiplicity of D at x_i. We thus have

$$(\bar{L} - \sum_{i=1}^{r} \mathcal{P}_i) \cdot C = (\tilde{D} + \sum_{i=1}^{r} (k_i - 1) \mathcal{P}_i) \cdot C \geqslant \tilde{D} \cdot C \geqslant 0 .$$

The second part follows in the same way, when we consider that $k_i \geqslant t_i$.

(0.6) In section 2 we shall study the behaviour of the map associated to $K_X \otimes \mathcal{L}^n$. When $c_1(\mathcal{L})^n \leqslant 2$ the situation is very easy as the following lemma shows

Lemma. (0.6.1) *Let $\mathcal{L}$ be an ample and spanned line bundle on a normal projective variety X of dimension n and let Φ be the map associated to $K_X \otimes \mathcal{L}^n$.*
a) If $c_1(\mathcal{L})^n=1$, then $(X, \mathcal{L})=(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(1))$ and Φ is not defined.
b) If $c_1(\mathcal{L})^n=2$, then either $(X, \mathcal{L})=(\underset{\sim}{Q}^n, \mathcal{O}_{Q^n}(1))$ and Φ is trivial or $p: X \to \boldsymbol{P}^n$ is a double cover, $\mathcal{L}=p^*\mathcal{O}_{\boldsymbol{P}^n}(1)$ and Φ factors through p. In particular X is Gorenstein.

Proof. The assertion about the structure of $(X, \mathcal{L})$ is obvious. In fact, if $\varphi: X \to \boldsymbol{P}_{\boldsymbol{C}}$ is the morphism associated to $|\mathcal{L}|$, then

$$c_1(\mathcal{L})^n = \deg \varphi \cdot \deg \varphi(X).$$

It only remains to prove the assertion about Φ in the last case. If $n=1$ this is a standard fact about hyperelliptic curves. Therefore, without loss of generality, we can assume that $n \geqslant 2$. Let $B \in |\mathcal{O}_{\boldsymbol{P}^n}(2k)|$ be the branch locus of p and note that every irreducible component of B is reduced, X being normal. Since $K_X \otimes \mathcal{L}^n = p^*\mathcal{O}_{\boldsymbol{P}^n}(k-1)$, it is enough to show that

$$h^0(p^*\mathcal{O}_{\boldsymbol{P}^n}(k-1)) = h^0(\mathcal{O}_{\boldsymbol{P}^n}(k-1)).$$

Let $\bar{B}$ denote the ramification divisor of p. Since $\bar{B} \in |p^*\mathcal{O}_{\boldsymbol{P}^n}(k)|$, we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & p^*\mathcal{O}_{\boldsymbol{P}^n}(-1) & \to & p^*\mathcal{O}_{\boldsymbol{P}^n}(k-1) & \xrightarrow{r_1} & (p^*\mathcal{O}_{\boldsymbol{P}^n}(k-1))_{\bar{B}} & \to & 0 \\
 & & \uparrow & & \uparrow \alpha & & \uparrow \beta & & \\
0 & \to & \mathcal{O}_{\boldsymbol{P}^n}(-k-1) & \to & \mathcal{O}_{\boldsymbol{P}^n}(k-1) & \xrightarrow{r_2} & (\mathcal{O}_{\boldsymbol{P}^n}(k-1))_B & \to & 0.
\end{array}$$

Since $p|_{\bar{B}}$ is an isomorphism, β induces an isomorphism at the H^0-cohomology level. The same fact happens for r_1 and r_2 in view of the vanishing theorem (0.2.4). Hence, this is true for α too and the assertion is proved.

A key role in the study of the map associated to $K_X \otimes \mathcal{L}^n$ will be played in section 2 by Reider's theorem, which we recall in the following form.

Theorem (0.6.2) [Re], see also [Be]). *Let X be a smooth projective surface and let $\mathcal{L}=\mathcal{O}_X(L)$ be a nef line bundle on X. If $c_1(\mathcal{L})^2 \geqslant 9$ and $K_X \otimes \mathcal{L}$ is not very ample, then there exists an effective divisor D satisfying one of the following conditions*

$$L\cdot D=0\,,\quad D^2=-1 \text{ or } -2;$$
$$L\cdot D=1\,,\quad D^2=0 \quad\text{ or } -1;$$
$$L\cdot D=2\,,\quad D^2=0;$$

$D^2=1$ and L is numerically equivalent to $3D$.

1. Some general results on $\mathcal{K}_X\otimes\mathcal{L}^t$

Throughout this section $\mathcal{L}$ will be an ample and spanned line bundle on a normal projective variety X of dimension n. Recall that the very ampleness or the spannedness of $\mathcal{K}_X\otimes\mathcal{L}^t$ on $reg(X)$ is stronger than the analogous statement for $K_X\otimes\mathcal{L}^t$ (see (0.2.2), (0.2.3)).

Theorem. (1.1) *If $t\geqslant n$, $\Gamma(\mathcal{K}_X\otimes\mathcal{L}^t)$ spans $\mathcal{K}_X\otimes\mathcal{L}^t$ on $reg(X)$ with the exception $(X,\mathcal{L})=(\boldsymbol{P}^n,\mathcal{O}_{\boldsymbol{P}^n}(1))$ and $t=n$.*

Proof. Clearly, by (0.6.1) we have $(X,\mathcal{L})=(\boldsymbol{P}^n,\mathcal{O}_{\boldsymbol{P}^n}(1))$ if and only if $c_1(\mathcal{L})^n=1$. We can thus assume that $c_1(\mathcal{L})^n\geqslant 2$. We can also assume $t=n$. We shall show that $\mathcal{K}_X\otimes\mathcal{L}^n$ is spanned at $x\in reg(X)$ by proving that

$$H^1(\mathcal{K}_X\otimes\mathcal{L}^n\otimes\mathcal{J}_x)=0\,, \tag{1.1.1}$$

$\mathcal{J}_x$ being the ideal sheaf of x. Let $\tilde{X}$ be a desingularization of X and $\sigma\colon\tilde{X}\to X$ the corresponding morphism. Let $b\colon X^\sharp\to X$ be the blowing-up of X at x. If $\bar{X}$ stands for the fibre product of σ and b, we get the following commutative diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{B} & \tilde{X} \\ \pi\downarrow & & \downarrow\sigma \\ X^\sharp & \xrightarrow{b} & X\,. \end{array} \tag{1.1.2}$$

Note that $\bar{X}$ is a desingularization of X since $x\in reg(X)$. Let $\mathcal{P}=\pi^{-1}(b^{-1}(x))$, $\bar{\mathcal{L}}=\pi^*(b^*\mathcal{L})$ and $\tilde{\mathcal{L}}=\sigma^*\mathcal{L}$. Since $K_{\bar{X}}=B^*K_{\tilde{X}}\otimes[\mathcal{P}]^{n-1}$, (0.2.1) yields

$$H^1(\mathcal{K}_X\otimes\mathcal{L}^n\otimes\mathcal{J}_x)=H^1(K_{\tilde{X}}\otimes\tilde{\mathcal{L}}^n\otimes\mathcal{J}_{\sigma^{-1}(x)})=H^1(K_{\bar{X}}\otimes\bar{\mathcal{L}}^n\otimes[\mathcal{P}]^{-n})\,.$$

Therefore, in view of (0.2.4), (1.1.1) will be proved once we have shown that $\bar{L}-\mathcal{P}$ is nef and big. Since $|\mathcal{L}-x|$ has a finite base locus, $b^*\mathcal{L}\otimes[b^{-1}(x)]^{-1}$ is nef in view of (0.5.1). On the other hand, $\pi\colon\bar{X}\to X^\sharp$ is a desingularization and so (0.5.0) says that $\bar{\mathcal{L}}\otimes[\mathcal{P}]^{-1}$ is nef too. The bigness follows from the inequality

$$(\bar{L}-\mathcal{P})^n=c_1(\bar{\mathcal{L}})^n-1=c_1(\mathcal{L})^n-1\geqslant 1\,.$$

The spannedness of $K_X\otimes\mathcal{L}^n$ at every point of X (with the obvious exception of $(\boldsymbol{P}^n,\mathcal{O}_{\boldsymbol{P}^n}(1))$) has already been proved by the third author under the additional assumption that X is Gorenstein and $Irr(X)$ is finite ([So4], Theorem (0.4)).

Theorem. (1.2) *If $t\geqslant n+1$, then $\mathcal{K}_X\otimes\mathcal{L}^t$ is very ample on $reg(X)$ with the exception $(X, \mathcal{L})=(\boldsymbol{P}^n, \mathcal{O}_{P^n}(1))$ and $t=n+1$.*

Proof. As before we can assume that $c_1(\mathcal{L})^n\geqslant 2$ by (0.6.2). We can also assume $t=n+1$.

First we prove that $\Gamma(\mathcal{K}_X\otimes\mathcal{L}^{n+1})$ gives an immersion on $reg(X)$. To see this let $x\in reg(X)$. We shall show that

$$H^1(\mathcal{K}_X\otimes\mathcal{L}^{n+1}\otimes\mathcal{J}_x^2)=0,$$

$\mathcal{J}_x$ being the ideal sheaf of x. To do this consider the same diagram (1.1.2) as in the proof of (1.1). With the same notation as before, we have

$$H^1(\mathcal{K}_X\otimes\mathcal{L}^{n+1}\otimes\mathcal{J}_x^2)=H^1(K_{\tilde{X}}\otimes\tilde{\mathcal{L}}^{n+1}\otimes\mathcal{J}^2_{\sigma^{-1}(x)})=H^1(K_{\bar{X}}\otimes\bar{\mathcal{L}}^{n+1}\otimes[\mathcal{P}]^{-n-1}).$$

Therefore, in view of (0.2.4) it is enough to show that $\bar{L}-\mathcal{P}$ is nef and big. Both facts have already been proved in the proof of (1.1).

To see that $\Gamma(\mathcal{K}_X\otimes\mathcal{L}^{n+1})$ separates $x, y\in reg(X)$, $x\neq y$, note that if $|\mathcal{L}-x|\neq|\mathcal{L}-y|$, then $(\mathcal{K}_X\otimes\mathcal{L}^n)\otimes\mathcal{L}$ separates x and y by (1.1). So we can assume that $|\mathcal{L}-x|=|\mathcal{L}-y|$. We shall show that

$$H^1(\mathcal{K}_X\otimes\mathcal{L}^{n+1}\otimes\mathcal{J}_x\otimes\mathcal{J}_y)=0,$$

$\mathcal{J}_x, \mathcal{J}_y$ being the ideal sheaves of x, y. To this end, blow-up X at x and y and get a diagram similar to (1.1.2)

$$\begin{array}{ccc} \bar{X} & \xrightarrow{B} & \tilde{X} \\ \pi\downarrow & & \downarrow\sigma \\ X^\# & \xrightarrow{b} & X \end{array}$$

Let $\mathcal{P}=\pi^{-1}(b^{-1}(x))$ and $Q=\pi^{-1}(b^{-1}(y))$. As before we have

$$\begin{aligned} H^1(\mathcal{K}_X\otimes\mathcal{L}^{n+1}\otimes\mathcal{J}_x\otimes\mathcal{J}_y) &= H^1(K_{\tilde{X}}\otimes\tilde{\mathcal{L}}^{n+1}\otimes\mathcal{J}_{\sigma^{-1}(x)}\otimes\mathcal{J}_{\sigma^{-1}(y)}) \\ &= H^1(K_{\bar{X}}\otimes\bar{\mathcal{L}}^{n+1}\otimes[\mathcal{P}]^{-n}\otimes[Q]^{-n}). \end{aligned}$$

Once again, it is enough to show that $(n+1)\bar{L}-n\mathcal{P}-nQ$ is nef and big. Since $|\mathcal{L}-x-y|=|\mathcal{L}-x|=|\mathcal{L}-y|$ has a finite base locus, the same argument using (0.5.1) as in the proof of (1.1) shows that $\bar{L}-\mathcal{P}-Q$ is nef and so $(n+1)\bar{L}-n\mathcal{P}-nQ=\bar{L}+n(\bar{L}-\mathcal{P}-Q)$ is nef too. As to the bigness, we have

$$\begin{aligned} ((n+1)\bar{L}-n\mathcal{P}-nQ)^n &\geqslant \bar{L}^n+n^n(\bar{L}-\mathcal{P}-Q)^n= \\ &= c_1(\mathcal{L})^n+n^n(c_1(\mathcal{L})^n-2)>0. \end{aligned}$$

REMARK (1.3). The above proofs can be modified so as to get a more general statement.

Let $\mathcal{L}_1, \cdots, \mathcal{L}_t$ *be ample and spanned line bundles on* X *and let* $\mathcal{L}$ *be a nef line bundle on* X.

(1.3.1) *If* $t \geqslant n$, *then* $\Gamma(\mathcal{K}_X \otimes \prod_{i=1}^{t} \mathcal{L}_i \otimes \mathcal{L})$ *spans* $\mathcal{K}_X \otimes \prod_{i=1}^{t} \mathcal{L}_i \otimes \mathcal{L}$ *on* $reg(X)$ *with the exception* $t=n$, $X=\boldsymbol{P}^n$, $\mathcal{L}_i=\mathcal{O}_{P^n}(1)$, $i=1, \cdots, n$ *and* $\mathcal{L}=\mathcal{O}_X$.

(1.3.2) *If* $t \geqslant n+1$, *then* $\mathcal{K}_X \otimes \prod_{i=1}^{t} \mathcal{L}_i \otimes \mathcal{L}$ *is very ample on* $reg(X)$ *with the exception* $t=n+1$, $X=\boldsymbol{P}^n$, $\mathcal{L}_i=\mathcal{O}_{P^n}(1)$, $i=1, \cdots, n+1$ *and* $\mathcal{L}=\mathcal{O}_X$.

The proof goes along the same lines as in (1.1) and (1.2). As to (1.3.1) we need to show the vanishing of $H^1(\mathcal{K}_{\bar{X}} \otimes \prod_{i=1}^{n} \bar{\mathcal{L}}_i \otimes \bar{\mathcal{L}} \otimes [\mathcal{P}]^{-n})$. The nefness of $\sum_{i=1}^{n} \boldsymbol{L}_i + \boldsymbol{L} - n\mathcal{P}$ is obvious. Moreover,

$$(\sum_{i=1}^{n} \boldsymbol{L}_i + \boldsymbol{L} - n\mathcal{P})^n \geqslant \sum_{i=1}^{n} \boldsymbol{L}_i^n + \boldsymbol{L}^n - n^n \geqslant n^n + \boldsymbol{L}^n - n^n \geqslant 0 .$$

If the last inequality is an equality, then $c_1(\mathcal{L})^n=0$. If, in addition, the last but one inequality is an equality, then we also have $c_1(\mathcal{L}_i)^n=1$ for any $i=1, \cdots, n$, which proves (1.3.1). As to (1.3.2) the immersion part of the proof goes as usual and the exception comes out when $\sum_{i=1}^{n+1} \boldsymbol{L}_i + \boldsymbol{L} - (n+1)\mathcal{P}$ is not big. As for the second part, $\mathcal{K}_X \otimes \prod_{i=1}^{n+1} \mathcal{L}_i \otimes \mathcal{L} = (\mathcal{K}_X \otimes \mathcal{L}_1 \otimes \cdots \otimes \hat{\mathcal{L}}_j \otimes \cdots \mathcal{L}_{n+1} \otimes \mathcal{L}) \otimes \mathcal{L}_j$ (where $\wedge$ stands for suppression) separates x, y if $|\mathcal{L}_j - x| \neq |\mathcal{L}_j - y|$ for one j at least in view of (1.3.1) unless $X=\boldsymbol{P}^n$, $\mathcal{L}_i=\mathcal{O}_{P^n}(1)$, $i \neq j$, $\mathcal{L}=\mathcal{O}_X$; but this yields $\mathcal{L}_j=\mathcal{O}_{P^n}(1)$ too. Now assume that $|\mathcal{L}_i - x| = |\mathcal{L}_i - y|$ for all i. We have to prove the vanishing of $H^1(\mathcal{K}_{\bar{X}} \otimes \prod_{i=1}^{n+1} \bar{\mathcal{L}}_i \otimes \bar{\mathcal{L}} \otimes [\mathcal{P}]^{-n} \otimes [\mathcal{Q}]^{-n})$. This happens when $\sum_{i=1}^{n+1} \boldsymbol{L}_i + \boldsymbol{L} - n\mathcal{P} - n\mathcal{Q}$ is nef and big. Nefness is proved as usual. As to the bigness,

$$(\sum_{i=1}^{n+1} \boldsymbol{L}_i + \boldsymbol{L} - n\mathcal{P} - n\mathcal{Q})^n = (\sum_{i=1}^{n+1} \boldsymbol{L}_i + \boldsymbol{L})^n - 2n^n \geqslant (n+1)^n - 2n^n > 0 .$$

2. The very ampleness of $K_X \otimes \mathcal{L}^n$

(2.0) In this section $\mathcal{L}$ is an ample and spanned line bundle on a Cohen-Macaulay normal projective variety X of dimension $n \geqslant 2$ with $Irr(X)$ finite. We also assume that $(X, \mathcal{L}) \neq (\boldsymbol{P}^n, \mathcal{O}_{P^n}(1))$.

Let $\Phi: X \to \boldsymbol{P}_C$ be the map associated to $\Gamma(K_X \otimes \mathcal{L}^n)$. Then Φ is well defined on $reg(X)$ in view of (1.1).

(2.1) Notice that $\dim \Phi(X) < \dim X$ implies $\Gamma(K_X \otimes \mathcal{L}^{n-1})=0$, and by (0.3.1) this happens only if $(X, \mathcal{L})$ is in the class $\mathcal{X}_2$ described in (0.3).

REMARK (2.1.1). If $(X, \mathcal{L})\in\mathcal{X}_2$, then $\dim\Phi(X)\leqslant 1$. More precisely,

$\dim\Phi(X) = 0$ if and only if $(X, \mathcal{L}) = (Q^n, \mathcal{O}_{Q^n}(1))$,

$\dim\Phi(X) = 1$ if and only if $(X, \mathcal{L})$ is a scroll or a generalized cone.

REMARK (2.1.2). Let $\dim\Phi(X)=n$. Then $c_1(\mathcal{L})^n\geqslant 2$, by (0.6.1), equality holding if and only if $p\colon X\to \boldsymbol{P}^n$ is a double cover and $\mathcal{L}=p^*\mathcal{O}_{\boldsymbol{P}^n}(1)$. In this case Φ factors through p.

In view of the above results we shall assume that

(2.2) $(X, \mathcal{L})$ is a pair as in (2.0), not in the class $\mathcal{X}_2$ and with $c_1(\mathcal{L})^n\geqslant 3$.

Main Theorem. (2.3) *Let $(X, \mathcal{L})$ be as in* (2.2). *Assume that either $c_1(\mathcal{L})^n\geqslant 5$ or $\Gamma(\mathcal{L})$ gives a generically one to one map. Then Φ is an embedding on $reg(X)\setminus\mathcal{CV}(X)$.*

Since $\mathcal{CV}(X)=\emptyset$ by definition if $\operatorname{cod}_X Sing(X)\geqslant 3$, we get the following

Corollary. (2.3.1) *Assume that $(X, \mathcal{L})$ is as in* (2.2) *and that either $c_1(\mathcal{L})^n\geqslant 5$ or $\Gamma(\mathcal{L})$ gives a generically one to one map. If $\operatorname{cod}_X Sing(X)\geqslant 3$, then $\Phi\colon reg(X)\to \boldsymbol{P}_C$ is an embedding.*

We shall need the following

Lemma. (2.4) *Let $(X, \mathcal{L})$ be as in* (2.0). *Let $x, y\in reg(X)$, $x\neq y$ and assume that $c_1(\mathcal{L})^n\geqslant 3$.*

(2.4.1) *If $|\mathcal{L}-x|=|\mathcal{L}-y|$, then $\Gamma(K_X\otimes\mathcal{L}^n)$ separates x and y.*

(2.4.2) *If $|\mathcal{L}-x|=|\mathcal{L}-2x|$, then $\Gamma(K_X\otimes\mathcal{L}^n)$ gives an immersion at x.*

Proof. We prove that $\Gamma(K_X\otimes\mathcal{L}^n)$ separates x and y by showing that

$$H^1(K_X\otimes\mathcal{L}^n\otimes\mathcal{J}_x\otimes\mathcal{J}_y) = 0\,,$$

$\mathcal{J}_x$, $\mathcal{J}_y$ being the ideal sheaves of x and y. Let $b\colon \bar{X}\to X$ be the blowing-up of X at x and y and put $\mathcal{P}=b^{-1}(x)$, $Q=b^{-1}(y)$. We have

$$H^1(K_X\otimes\mathcal{L}^n\otimes\mathcal{J}_x\otimes\mathcal{J}_y) = H^1(K_{\bar{X}}\otimes\bar{\mathcal{L}}^n\otimes[\mathcal{P}]^{-n}\otimes[Q]^{-n})\,.$$

Therefore, in view of (0.2.4) it is enough to show that $\bar{L}-\mathcal{P}-Q$ is nef and big. Since $|\mathcal{L}-x-y|=|\mathcal{L}-x|=|\mathcal{L}-y|$ has a finite base locus, the nefness follows from (0.5.1). The bigness is immediate, as

$$(\bar{L}-\mathcal{P}-Q)^n = c_1(\mathcal{L})^n-2>0\,.$$

Similarly, in order to show that $\Gamma(K_X\otimes\mathcal{L}^n)$ gives an immersion at x we prove that

$$H^1(K_X\otimes\mathcal{L}^n\otimes\mathcal{J}_x^2) = 0\,.$$

Let $b: \bar{X} \to X$ be the blowing-up of X at x. As

$$H^1(K_X \otimes \mathcal{L}^n \otimes \mathcal{J}_x^2) = H^1(K_{\bar{X}} \otimes \bar{\mathcal{L}}^n \otimes [\mathcal{P}]^{-(n+1)}),$$

in view of (0.2.4) it is enough to show that $n\bar{L}-(n+1)\mathcal{P}$ is nef and big. Due to the assumption and by (0.5.1), $\bar{L}-\mathcal{P}$ and $\bar{L}-2\mathcal{P}$ are nef and then $n\bar{L}-(n+1)\mathcal{P}$ $=(n-1)(\bar{L}-\mathcal{P})+\bar{L}-2\mathcal{P}$ is nef too. Moreover an easy check shows that

$$(n\bar{L}-(n+1)\mathcal{P})^n = n^n c_1(\mathcal{L})^n-(n+1)^n>0$$

since $c_1(\mathcal{L})^n \geqslant 3$.

Proof of (2.3). Let $x, y \in reg(X) \setminus \mathcal{V}(X)$, $x \neq y$. If $\Gamma(\mathcal{L})$ separates x and y, then, since $K_X \otimes \mathcal{L}^{n-1}$ is spanned on $reg(X) \setminus \mathcal{V}(X)$ by (0.4.1), we see that $\Phi(x) \neq \Phi(y)$. If $\Gamma(\mathcal{L})$ does not separate x and y, then $|\mathcal{L}-x|=|\mathcal{L}-y|$ and (2.4.1) implies $\Phi(x) \neq \Phi(y)$ again.

We now proceed by induction to prove that Φ is an immersion at $x \in reg(X) \setminus \mathcal{V}(X)$. First assume that X has dimension 2 and let $\pi: \bar{X} \to X$ be a desingularization. Assume, by contradiction, that Φ is not an immersion at x. Then $K_{\bar{X}} \otimes \bar{\mathcal{L}}^2$ does not give an immersion at $\pi^{-1}(x)$. Since $\bar{\mathcal{L}}$ is nef and $c_1(\bar{\mathcal{L}}^2)^2$ $=4c_1(\mathcal{L})^2>9$, this implies, by (0.6.2), that $\bar{X}$ contains an effective divisor D with $\pi^{-1}(x) \in D$ satisfying either

$$\begin{aligned} 2\bar{L}\cdot D &= 0, \quad D\cdot D = -2, -1, \\ 2\bar{L}\cdot D &= 1, \quad D\cdot D = -1, 0, \quad \text{or} \\ 2\bar{L}\cdot D &= 2, \quad D\cdot D = 0. \end{aligned}$$

As $2\bar{L}\cdot D$ is even, it can only happen that $\bar{L}\cdot D=0$ or $\bar{L}\cdot D=1$. In the former case $\pi(D)$ is a finite set contained in $Sing(X)$, but this gives a contradiction as $\pi^{-1}(x) \in D$. Let $\bar{L}\cdot D=1$. Then, $\mathcal{D}$, the divisor part of $\pi(D)$, is a line by the projection formula and $\mathcal{D} \subset reg(X)$, otherwise x would be in $\mathcal{V}(X)$. We have $D=\hat{D}+J$, where $\hat{D}=\pi^{-1}(\mathcal{D})=$the proper transform of $\mathcal{D}$ and J is an effective (possibly trivial) divisor contracted by π to a finite set in $Sing(X)$. Then $J\cdot\hat{D}=$ 0, $J\cdot J \leqslant 0$ and the equality $D\cdot D=0$ shows that $t=\mathcal{D}\cdot\mathcal{D}=\hat{D}\cdot\hat{D} \geqslant 0$. If $t=0$, then $(X, \mathcal{L})$ is a scroll, but this contradicts the assumption $(X, \mathcal{L}) \notin \mathcal{X}_2$. Let $t>0$. Then $(\mathcal{D}-tL)\cdot\mathcal{D}=0$ and the Hodge index theorem implies that $(\mathcal{D}-tL)^2$ $=t(tL\cdot L-1) \leqslant 0$. Hence $t=1$ and $L\cdot L=1$. Thus (0.6.1) shows that $(X, \mathcal{L})=$ $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(1))$, which is again a contradiction.

Now assume that $\dim X \geqslant 3$. Since $x \in reg(X)$, the general element of $|\mathcal{L}-x|$ is normal ([So5], (0.4.1) a)). If all the elements of $|\mathcal{L}-x|$ were singular at x, then $|\mathcal{L}-x|=|\mathcal{L}-2x|$ and by (2.4.2) Φ would give an immersion at x. So we can assume that there is a normal element $A \in |\mathcal{L}-x|$ smooth at x. By (0.1.1) and (0.2.4) the cohomology sequence of

$$0 \to K_X \otimes \mathcal{L}^{n-1} \to K_X \otimes \mathcal{L}^n \to K_A \otimes \mathcal{L}_A^{n-1} \to 0$$

shows hat the restricted map $\Phi|_A$ is nothing else than the map defined by $\Gamma(K_A\otimes\mathcal{L}_A^{n-1})$, which, by induction, is an immersion at x along A. We can thus choose some sections $s_0, \cdots, s_{n-1}\in\Gamma(K_X\otimes\mathcal{L}^n)$ with $s_0(x)\neq 0, s_i(x)=0$ for $i\neq 0$ and such that $s_1/s_0, \cdots, s_{n-1}/s_0$ give local coordinates on A at x. Now since $x\in reg(X)\setminus\mathcal{V}(X)$, (0.4.1) says that there exists a section $t\in\Gamma(K_X\otimes\mathcal{L}^{n-1})$ with $t(x)\neq 0$. Let $s_A\in\Gamma(\mathcal{L})$ be the tautological section defining A. Then $s_A\otimes t\in\Gamma(K_X\otimes\mathcal{L}^n)$ and

$$s_1/s_0, \cdots, s_{n-1}/s_0, s_A\otimes t/s_0$$

give local coordinates on X at x, as can immediately be seen. This proves that Φ is an immersion at x.

REMARK (2.5). The above proof can be slightly modified in order to get the following more general statement.

Let $\mathcal{L}_1, \cdots, \mathcal{L}_n$ be ample and spanned line bundles on X and assume that each pair $(X, \mathcal{L}_i)$ is as in (2.2) *and* $\mathrm{cod}_X\, Sing(X)\geqslant 3$. *Furthermore assume that* $c_1(\mathcal{L}_i)^n\geqslant 5$ *for some i. Then*

$$K_X\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n\otimes\mathcal{L}$$

is very ample on $reg(X)$ for every nef line bundle $\mathcal{L}$.

In view of (1.3.1) and [A-S], Theorem (2.5), the proof can run along the same lines as the above one provided that Lemma (2.4) is replaced by the following assertions

(2.5.1) If $|\mathcal{L}_i-x|=|\mathcal{L}_i-y|$ for all i, then $\Gamma(K_X\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n\otimes\mathcal{L})$ separates x and y.

(2.5.2) If $|\mathcal{L}_i-x|=|\mathcal{L}_i-2x|$ for all i, then $\Gamma(K_X\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n\otimes\mathcal{L})$ gives an immersion at x.

Then, looking over the proof of (2.3) one realizes that the only possible troubles are:

a) the two-dimensional step to prove that $\Gamma(K_X\otimes\mathcal{L}_1\otimes\mathcal{L}_2\otimes\mathcal{L})$ gives an immersion at x;

b) the bigness assertion for $\sum_{i=1}^n L_i-n\mathcal{P}-nQ+L$ and $\sum_{i=1}^n L_i-(n+1)\mathcal{P}+L$.

As to a), with the same notation as before, we have

$$c_1(\bar{\mathcal{L}}_1\otimes\bar{\mathcal{L}}_2\otimes\bar{\mathcal{L}})^2\geqslant c_1(\bar{\mathcal{L}}_1)^2+c_1(\bar{\mathcal{L}}_2)^2+2c_1(\bar{\mathcal{L}}_1)\cdot c_1(\bar{\mathcal{L}}_2)\geqslant 8+2\sqrt{15}\,,$$

in view of the Hodge index theorem. Then, by Reider's theorem (0.6.2), there is no additional case to consider and the proof runs exactly as before. As to b) we recall the following fact

$$c_1(\sum_{i=1}^n \bar{\mathcal{L}}_i)^n\geqslant(\sum_{i=1}^n (c_1(\bar{\mathcal{L}}_i)^n)^{1/n})^n\,.$$

Indeed

$$c_1(\sum_{i=1}^n \bar{\mathcal{L}}_i)^n = \sum_{|j|=n} \binom{n}{j} c_1(\bar{\mathcal{L}}^{\cdot})^j ,$$

where $j=(j_1, \cdots, j_n)$ is a multiindex, $|j|$ its length and $c_1(\bar{\mathcal{L}}^{\cdot})^j := \prod_{i=1}^n c_1(\bar{\mathcal{L}}_i)^{j_i}$. As

$$c_1(\bar{\mathcal{L}}^{\cdot})^j \geqslant \prod_{i=1}^n (c_1(\bar{\mathcal{L}}_i)^n)^{j_i/n}$$

by the generalized Hodge inequality [B-B-S], Theorem (1.4), we have the desired inequality. We thus get

$$\begin{aligned}(\sum_{i=1}^n L_i - n\mathcal{P} - nQ + L)^n &\geqslant (\sum_{i=1}^n L_i)^n - 2n^n + L^n \geqslant \\ &\geqslant (5^{1/n} + (n-1)\,3^{1/n})^n - 2n^n .\end{aligned}$$

Similarly

$$\begin{aligned}(\sum_{i=1}^n L_i - (n+1)\,\mathcal{P} + L)^n &\geqslant (\sum_{i=1}^n L_i)^n - (n+1)^n + L^n \geqslant \\ &\geqslant (5^{1/n} + (n-1)\,3^{1/n})^n - (n+1)^n\end{aligned}$$

and then the bigness follows from an easy computation.

We conclude this section with a conjecture.

CONJECTURE (2.6). *Let E be an ample and spanned rank-n holomorphic vector bundle on a smooth projective n-fold X. If $c_1(E)^n \geqslant (n+1)^n + 1$, then $K_X \otimes \det E$ is very ample apart from pairs (X, E) where $(X, \det E)$ is a conic bundle.*

3. Applications

For simplicity we assume $\mathrm{cod}_X\, Sing(X) \geqslant 3$ in this section.

Theorem. (3.1) *Let $\psi: X \to \boldsymbol{P}^n$ be a finite to one cover of $\boldsymbol{P}^n$ by a normal projective variety X. Assume that* $\deg\psi \geqslant 2$ *and that* $\mathrm{cod}_X\, Sing(X) \geqslant 3$. *Then the ramification divisor R of ψ is very ample on $reg(X)$.*

Proof. This is an easy consequence of (1.2) as $R \in |K_X \otimes \psi^* \mathcal{O}_{P^n}(n+1)|$.

Theorem (3.2) *Let $\psi: X \to Q^n$ be a branched cover of a quadric by a normal projective variety X and let R be the ramification divisor. Then* $\Gamma([R])$ *spans* [R] *at every point $x \in reg(X)$. Furthermore, if $n \geqslant 3$, X is Cohen-Macaulay,* $\mathrm{cod}_X\, Sing(X) \geqslant 3$, *$Irr(X)$ is finite and* $\deg\psi \geqslant 3$, *then* [R] *is very ample on $reg(X)$.*

Proof. As $R \in |K_X \otimes \psi^* \mathcal{O}_{Q^n}(n)|$, the former assertion follows from (1.1), whereas the latter one is a consequence of (2.3) when we consider that

$c_1(\psi^*\mathcal{O}_{Q^n}(1))^n=2\deg\psi\geqslant 6$.

The above corollaries provide a wide generalization of results contained in [E], Theorem 1, [L-P], Theorem 3.2, [Io], Corollary 11.

Theorem. (3.3) *Let X be a smooth projective n-fold, let $\mathcal{L}_1, \cdots, \mathcal{L}_n$ be ample and spanned line bundles and $\mathcal{L}$ a nef line bundle on X. Assume that $c_1(\mathcal{L}_i)^n\geqslant 5$ for at least one i and that X is not isomorphic to a $\boldsymbol{P}^{n-1}$-bundle. Then $K_X\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n\otimes\mathcal{L}$ is very ample.*

Proof. The assertion follows from (2.5) recalling (0.3).

REMARK. (3.3.1) Let X, $\mathcal{L}_i$, $\mathcal{L}$ be as in (3.3). For eaxmple, if $K_X^N=\mathcal{O}_X$ for some $N\geqslant 1$, then (3.3) says that

$$\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n = K_X\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n\otimes K_X^{N-1}$$

is very ample. Here is another example. Let X be as in (3.3) and let $\mathcal{L}$ be an ample and spanned line bundle on X with $c_1(\mathcal{L})^n\geqslant 5$. If $(K_X\otimes\mathcal{L}^{n-2})^N=\mathcal{O}_X$ for some $N\geqslant 1$ (e.g. if $(X, \mathcal{L})$ is a Mukai n-fold), then (3.3) says that

$$\mathcal{L}^2 = K_X\otimes\mathcal{L}^n\otimes(K_X\otimes\mathcal{L}^{n-2})^{N-1}$$

is very ample.

(3.4) To state the last application we need to recall the concept of reduction. Let $\mathcal{L}$ be an ample line bundle on a smooth projective n-fold X. A reduction of $(X, \mathcal{L})$ is a pair $(X', \mathcal{L}')$ consisting of an ample line bundle $\mathcal{L}'$ on a smooth projective n-fold X' such that
a) there is a map π: $X\to X'$ expressing X as X' blown-up at a finite set F;
b) $K_X\otimes\mathcal{L}^{n-1}=\pi^*(K_{X'}\otimes\mathcal{L}'^{n-1})$.
Note that the elements $A\in|\mathcal{L}|$ are in a one to one correspondence with the elements $A'\in|\mathcal{L}'|$ passing through F. Apart from some well understood pairs (see [So5], table with $\sigma\leqslant 2$), $K_X\otimes\mathcal{L}^{n-1}$ can be assumed nef and big. In this case $(X, \mathcal{L})$ admits a reduction $(X', \mathcal{L}')$ on which $K_{X'}\otimes\mathcal{L}'^{n-1}$ is ample (e.g. see [Io], (1.6)).

A general and useful fact is that

(3.4.1) *The intersection of generic elements of $|\mathcal{L}'|$ is smooth.*

Proof. Use Bertini's theorem on X and the fact that the fibres of the reduction are linear.

Theorem. (3.5) *Let $\mathcal{L}$ be an ample and spanned line bundle on a smooth projective n-fold X and assume that $K_X\otimes\mathcal{L}^{n-1}$ is nef and big and that $c_1(\mathcal{L})^n\geqslant 5$. Let $(X', \mathcal{L}')$ be the reduction of $(X, \mathcal{L})$. Then $(K_{X'}\otimes\mathcal{L}'^{n-1})^2$ is very ample.*

Proof. First note that this result is an immediate consequence of Reider's theorem (0.6.2) if $n=2$. Indeed $K_{X'}\otimes\mathcal{L}'$ is ample; furthermore

$$((K_{X'}+L')+L')^2=(K_{X'}+L')^2+2(K_{X'}+L')\cdot L'+L'^2\geqslant 1+2\cdot 3+5=12\,,$$

where the cross term is estimated by the Hodge index theorem. Thus, if D is the curve in (0.6.2) that exists if $K_{X'}\otimes(K_{X'}\otimes\mathcal{L}')\otimes\mathcal{L}'$ is not very ample, $(K_{X'}+L'+L')\cdot D\leqslant 2$. Therefore $L'\cdot D=1=(K_{X'}+L')\cdot D$. Thus, since $K_{X'}\otimes\mathcal{L}'$ is ample and spanned by (0.4.1), D is a smooth rational curve and then its selfintersection is -2. This settles the case $n=2$, by (0.6.2).

Now assume $n\geqslant 3$. We would like to apply (2.5) with $\mathcal{L}_1=K_{X'}\otimes\mathcal{L}'^{n-1}$ and $\mathcal{L}_i=\mathcal{L}'$ for all $i\geqslant 2$. At first sight we have the problem that $\mathcal{L}'$ is spanned off a finite set. However looking over the proof of (2.3), replacing $K_X\otimes\mathcal{L}^{n-1}$ with $K_{X'}\otimes(K_{X'}\otimes\mathcal{L}'^{n-1})\otimes\mathcal{L}'^{n-2}$ and noting (3.4.1), we see that this is no problem if we know that

(a) $K_{X'}\otimes(K_{X'}\otimes\mathcal{L}'^{n-1})\otimes\mathcal{L}'^{n-2}$ *is spanned.*

Now we shall prove (a). Of course we can assume that

(3.5.1) $\mathcal{L}'$ is not spanned.

We will need two lemmas. For simplicity we put $\mathcal{H}=K_{X'}\otimes\mathcal{L}'^{n-1}$. Note that, under the assumption of (3.5), it follows from (0.4.1) that

(3.5.2) $\mathcal{H}$ is spanned.

Lemma. (3.5.3) *Either $H^2\cdot L'^{n-2}\geqslant 5$ or the theorem is true.*

Proof. Note that $c_1(\mathcal{L}')^n\geqslant 6$, otherwise $(X,\mathcal{L})=(X',\mathcal{L}')$ and so $\mathcal{L}'$ would be spanned, contradicting (3.5.1). Furthermore $c_1(\mathcal{H})^n=(K_{X'}+(n-1)\,L')^n\geqslant 3$. Actually if $(K_{X'}+(n-1)\,L')^n\leqslant 2$, then by (3.5.2) and (0.6.1) X' is either $\boldsymbol{P}^n$, a quadric, or a double cover of $\boldsymbol{P}^n$ and in all these cases $\mathcal{L}'$ would be spanned, contradicting (3.5.1).

In view of the above inequalties we have

$$H\cdot H\cdot L'^{n-2}\geqslant(c_1(\mathcal{H})^n)^{2/n}\,(c_1(\mathcal{L}')^n)^{(n-2)/n}\geqslant 3^{2/n}\,6^{(n-2)/n}=3\cdot 2^{(n-2)/n}\,.$$

If $n\geqslant 4$ then $H\cdot H\cdot L'^{n-2}>4$ and we are done. Let $n=3$ and assume that the lemma is not true. Then we must have

$$(*)\qquad H\cdot H\cdot L'=4\,,\quad c_1(\mathcal{L}')^3=6\,,\quad c_1(\mathcal{H})^3=3\,.$$

Let S be a general element of $|\mathcal{L}'|$; then S is a smooth surface by (3.4.1) and $\mathcal{H}_S=K_S\otimes\mathcal{L}'_S$. Note that $K_S\otimes\mathcal{H}_S$ is nef, otherwise $(S,\mathcal{H}_S)$ would be either $(\boldsymbol{P}^2,\mathcal{O}_{P^2}(e))$, $e=1,2$, or $(Q^2,\mathcal{O}_{Q^2}(1))$ or a scroll. In the first two cases X' is either $\boldsymbol{P}^3$ or a quadric and $\mathcal{L}'$ is spanned, contradicting (3.5.1). In the last case S is a $\boldsymbol{P}^1$-bundle and $\mathcal{L}'_f=\mathcal{O}_{P^1}(3)$ for a general fiber f of the ruling. But this is impossible due to a result by Badescu [B], Thms 1 and 3. The nefness of $K_S\otimes\mathcal{H}_S$ implies that

$$0\leqslant c_1(K_S\otimes\mathcal{H}_S)^2=4(K_S+L'_S)\cdot K_S+L'_S\cdot L'_S\ ,$$

hence

$$(**)\qquad (K_S+L'_S)\cdot K_S\geqslant -1\ .$$

On the other hand the Hodge index theorem and (*) give

$$((K_S+L'_S)\cdot L'_S)^2\geqslant (K_S+L'_S)^2\, L'^2_S=c_1(\mathcal{H}_S)^2\, c_1(\mathcal{L}'_S)^2=24\ ,$$

which, by the even parity of $(K_S+L'_S)\cdot L'_S$ implies $(K_S+L'_S)\cdot L'_S\geqslant 6$. Thus we get from $(K_S+L'_S)^2=4$ that $(K_S+L'_S)\cdot K_S\leqslant -2$, contradicting (**). This proves the Lemma.

Lemma. (3.5.4) *$(X', \mathcal{H})\notin\mathcal{X}_2$ except possibly if $n=3$ and $(X', \mathcal{H})$ is a scroll over a smooth curve. In this case $(S, \mathcal{H}_S)$ is not in class $\mathcal{X}_2$ for any irreducible element $S\in|\mathcal{L}'|$.*

Proof. Let $(X', \mathcal{H})\in\mathcal{X}_2$. Looking over the list in (0.3) and recalling (3.5.1) we see that the only possibility is that $(X', \mathcal{H})$ is a scroll over a smooth curve. Let f be a general fibre of the ruling. We have $K_{X'f}=\mathcal{O}_{P^{n-1}}(-n)$ and $\mathcal{L}'_f=\mathcal{O}_{IP^{n-1}}(k)$, with $k\geqslant 2$ since $(X', \mathcal{L}')\notin\mathcal{X}_2$. Then $\mathcal{H}_f=\mathcal{O}_{P^{n-1}}(k(n-1)-n)$ can be $\mathcal{O}_{P^{n-1}}(1)$ only if $n=3$, $k=2$. In this case let $S\in|\mathcal{L}'|$ be an irreducible element. Then the restriction to S of the scroll projection of $(X', \mathcal{H})$ defines a ruling of S. Note that S cannot be a $\boldsymbol{P}^1$-bundle by Badescu's result [B]. Therefore $(S, \mathcal{H}_S)$ cannot belong to $\mathcal{X}_2$.

Now we are ready to prove (a).

Proof of (a). Let $x\in X'$ and choose $n-2$ general elements of $|\mathcal{L}'-x|$. By the usual arguments their intersection S is an irreducible, normal, Gorenstein surface and the restriction to S induces a surjection

$$\Gamma(K_{X'}\otimes\mathcal{H}\otimes\mathcal{L}'^{n-2})\rightarrow\Gamma(K_S\otimes\mathcal{H}_S)\rightarrow 0\ .$$

So, if we show that $K_S\otimes\mathcal{H}_S$ is spanned at $x\in S$, then we will be done. Note that $(S, \mathcal{H}_S)$ is not in class $\mathcal{X}_2$, by (3.5.4); moreover $\mathcal{H}_S$ is spanned by (3.5.2), ample, and $c_1(\mathcal{H}_S)^2\geqslant 5$, by (3.5.3), so (0.4.1) applies. As a first thing, if x is a singular point of S, then $K_S\otimes\mathcal{H}_S$ is spanned at x, by (0.4.1), since S is Gorenstein. Next assume that x is a smooth point of S. If $K_S\otimes\mathcal{H}_S$ is not spanned at x, then by (0.4.1) x is a vertex point of $(S, \mathcal{H}_S)$: this means that there is an $\mathcal{H}_S$-line ℓ on S passing through x and meeting $Sing(S)$. The singular points of S are in the finite base locus B of $|\mathcal{L}'-x|$, by Bertini's theorem. Note that an $\mathcal{H}_S$-line in S is also an $\mathcal{H}$-line in X'. So, since $\mathcal{H}$ is ample and spanned, any such line must be the inverse image under the map $\Phi\colon X'\rightarrow\boldsymbol{P}$ associated to $\Gamma(\mathcal{H})$ of one of the finite number of lines in $\boldsymbol{P}$ connecting $\Phi(x)$ with $\Phi(B)$.

Thus the $\mathcal{H}_S$-line l must be one of a finite number of curves on X' that are independent of S. Since $|\mathcal{L}'-x|$ has finite base locus, a surface obtained by intersecting $n-2$ general members of $|\mathcal{L}'-x|$ cannot contain one of a preassigned finite set of curves. This shows that x cannot be a vertex point of $(S, \mathcal{H}_S)$.

REMARK (3.6) The result in (3.5) holds true also for $(K_{X'}\otimes\mathcal{L}'^{n-1})^2\otimes\mathcal{M}$ where $\mathcal{M}$ is nef.

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