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ISOMORPHISMS OF β-AUTOMORPHISMS TO MARKOV AUTOMORPHISMS

YŌICHIRŌ TAKAHASHI

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0. Introduction

The purpose of the present paper is to construct an isomorphism which shows the following:

Theorem. *A β-automorphism is isomorphic to a mixing simple Markov automorphism in such a way that their futures are mutually isomorphic.*

Though the state of this Markov automorphism is countable and not finite, we obtain immediately from the proof of the theorem:

Corollary 1. *The invariant probability measure of β-transformation is unique under the condition that its metrical entropy coincides with topological entropy* $\log \beta$.

An extention of Ornstein's isomorphism theorem for countable generating partitions ([2]) shows the following known result (Smorodinsky [5], Ito-Takahashi [3]):

Corollary 2. *A β-automorphism is Bernoulli.*

We now give the definition of β-automorphism and auxiliary notions. Let β be a real number >1.

DEFINITION. A *β-transformation* is a transformation T_β of the unit interval [0, 1] into itself defined by the relation

$$(1) \qquad T_\beta t \equiv \beta t \qquad (\mathrm{mod}\ 1) \ (0 \leqq t < 1)$$

and by $T_\beta^n 1 = \lim_{t \to 1} T_\beta^n t$.

This transformation has been studied by A. Renyi, W. Parry, Ito-Takahashi et al. Parry [3] showed that there is an invariant probability measure for a

β-transformation which is absolutely continuous with respect to the ordinary Lebesgue measure dt and whose density is given by

$$f_\beta(t) = \sum_{n\geqq 0} 1_{[0, T_\beta^n 1)}(t)\beta^{-n-1} / \sum_{n\geqq 0} T_\beta^n 1 \cdot \beta^{-n-1} .$$

The measure preserving transformation ([0, 1]), T_β, $f_\beta dt$) will be called *β-endomorphism*, and its natural extension β-automorphism. But we will give a concrete definition for the latter in terms of symbolic dynamics. Let s be an integer such that $s<\beta\leqq s+1$, and $A=\{0, 1, \cdots, s\}$.

DEFINITION. A subsystem (X_β, σ) of the (topological) shift transformation $(\mathrm{A}^{\boldsymbol{Z}}, \sigma)$ over symbol set A(where σ denotes the one-step shift transformation to the left) is called *β-shift* if there exists an element ω_β of X_β such that

$$(2) \qquad X_\beta = \mathrm{cl.}\, \{\omega\in \mathrm{A}^{\boldsymbol{Z}} \,|\, \sigma^n\omega<\omega_\beta (n\in \boldsymbol{Z})\}$$

and that the number β is the unique positive solution of equation

$$(3) \qquad \sum_{n\geqslant 0} \omega_\beta(n) t^{-n-1} = 1.$$

Here cl. denotes the closure operation in the product space $\mathrm{A}^{\boldsymbol{Z}}$ and the symbol $<$ denotes the partial order defined as follows: $\omega<\eta$ if there is an n such that

$$\omega(k) = \eta(k) \qquad (0\leqq k<n) \text{ and } \omega(n)<\eta(n) ,$$

where $\omega(n)$ denotes the n-th coordinate of $\omega\in \mathrm{A}^{\boldsymbol{Z}}$. We note that $\omega_\beta(0)=s$.

This partial order plays an essential role when the β-shifts are proved in [3] to be realizations of β-transformations: Let us define a map ρ_β of X_β into the unit interval [0, 1] by

$$(4) \qquad \rho_\beta(\omega) = \sum_{n\geqslant 0} \omega(n)\beta^{-n-1} ,$$

then the map ρ_β is continuous and defines a homomorphism (as endomorphism) of subshift (X_β, σ) onto the Borel dynamical system ([0, 1], T_β) which is invertible except for a countable subset of X_β. It is now obvious that the map ρ_β induces from $f_\beta dt$ an invariant probability measure μ_β for (X_β, σ), which can be expressed in the symbolical form:

$$(5) \qquad \mu_\beta(\{\omega' \colon \omega'<\omega\}) = C_\beta \sum_{n\geqslant 0} \beta^{-n-1} \min\{\rho_\beta(\omega), \rho_\beta(\sigma^n\omega_\beta)\}$$

where C_β is normalizing constant.

DEFINITION. The invertible measure preserving transformation $(X_\beta, \sigma, \mu_\beta)$ will be called *β-automorphism*.

According to the result of W. Parry [4] for the β-endomorphisms, the metrical entropy $h(\mu_\beta)$ of $(X_\beta, \sigma, \mu_\beta)$ is equal to $\log \beta$, and it is proved in [3] that the topological entropy $e(X_\beta, \sigma)$ is also $\log \beta$.

DEFINITION. An invariant probability measure of a topological dynamics will be called of maximal entropy, or simply, maximal if the metrical entropy coincides with the topological entropy of the dynamics.

REMARK 1). The theorem does not assert the Markovian-ness of the β-endomorphism, which is identified with the image of $(X_\beta, \sigma, \mu_\beta)$ under the projection π_+: $\pi_+(\omega)(n)=\omega(n)$, $n\geqq 0$. It seems that β-endomorphisms are not Markov except for those β's such that

$$1-\beta^{-p-1} = \sum_{n=0}^{p} a_i \beta^{-n-1} \qquad \text{for some} \quad a_j \in \mathrm{A} \text{ and } p \geqq 0,$$

which are proved in [3] to exhaust the Markovian cases with canonical generator.

2). What we will study essentially in the following is the *dual β-endomorphism.* The notion of the "dual" depends in general upon the choice of the "present" and in our case it is defined as follows: Let π_- be the projcetion A^Z onto A^Z defined by the relation:

$$\pi_-(\omega)(n) = \omega(-n) \qquad (n \geqq 0).$$

Then the map π_- induces a homomorphism of $(X_\beta, \sigma^{-1}, \mu_\beta)$ (considered as endomorphism) into $(\mathrm{A}^N, \sigma, \pi_-(\mu_\beta))$. The dual β-endomorphism is its image $(X_\beta^*, \sigma, \mu_\beta^*)$ by the map π_-. Then what we will show is the following:

Theorem. *A dual β-endomorphism is isomorphic to a mixing simple Markov endomorphism.*

1. A class of Markov subshifts

Before studying β-automorphisms we are concerned with a class of Markov subshifts over a countable symbol set $\mathrm{I}=\{-r, -(r-1), \cdots, 0, 1, \cdots, \infty\}$ $(r\geqq 0)$. Let $M=(M_{ij})_{i,j\in \mathrm{I}}$ be a matrix with the following properties:

(i) $M_{ij}\in\{0, 1\}$ for all $i, j\in \mathrm{I}$

(ii) $M_{ij}=1$ if $i=j+1<\infty$, if $i\leqq 1$ and $j\leqq 0$, or if $i=j=\infty$

(iii) $M_{ij}=0$ if $1<i\leqq\infty$ and $i\neq j+1$, if $i=1$ and $1\leqq j\leqq\infty$ or if $i=\infty$ and $j<\infty$

(iv) $M_{i\infty}=\limsup_{j\to\infty} M_{ij}$ if $i\leqq 0$

The undetermined entries are M_{ij}, $i\leqq 0$, $1\leqq j<\infty$. We set $M_0=r+1=\sum_{i=-r}^{0} M_{ij}$ $(j\leqq 0)$ and $M_j=\sum_{i=-r}^{0} M_{ij}$ for $j\geqq 1$. We will consider the class of Markov subshifts $(\mathfrak{M}(M), \sigma)$ with structure matrices M whose entries are given by (i)–(iv) where

$$\mathfrak{M}(M) \equiv \{\eta \in \mathrm{I}^{\boldsymbol{Z}} : M_{\eta(n)\eta(n+1)} = 1 \quad \text{for any} \quad n \in \boldsymbol{Z}\}. \tag{6}$$

(The details of Markov subshift will be discussed in [3])

Let $\lambda \neq 0$ and

$$\sum_j M_{ij} x_j = \lambda x_i, \quad \sum_i y_i M_{ij} = \lambda y_j$$

for some non-zero vectors $x = (x_i)_{i \in \mathrm{I}}$ and $y = (y_i)_{i \in \mathrm{I}}$. Then it is easy to see that

$$\begin{aligned} x_i &= \lambda^{-i} \ (i \geqq 1), \quad = \sum_{k \geqq 0} M_{ik} \lambda^{-k-1} \ (i \leqq 0) \\ y_j &= \lambda^j \sum_{k \geqq j} M_k \lambda^{-k-1} \ (j \geqq 1), \quad = 1 \quad (j \leqq 0) \end{aligned} \tag{7}$$

up to scalar multiplication and that

$$1 = \sum_{j \geqq 0} M_j \lambda^{-j-1}. \tag{8}$$

Conversely if λ satisfies (8), then (7) gives right and left eigenvectors x and y corresponding to the eigenvalue λ. It is obvious that (8) has a unique positive solution $\rho = \rho(M)$, which is of maximal modulus among the solutions of (8). Let us define a transition matrix $P = (P_{ij})_{i,j \in \mathrm{I}}$ and $\pi = (\pi_i)_{i \in \mathrm{I}}$ by the relation:

$$\mathrm{P}_{ij} = M_{ij} x_j / \rho x_i \quad \text{and} \quad \pi_i = x_i y_i / \sum_j x_j y_j. \tag{9}$$

Then it is obvious that the metrical entropy of the Markov automorphism defined by this pair P and π is equal to $\log \rho(M)$. We show that $\log \rho(M)$ is also the topological entropy of the subshift.

Lemma 1. 1) *The topological entropy* $e(\mathfrak{M}(M)\sigma)$, *is equal to the value* $\log \rho(M)$, *where* $\rho(M)$ *is the unique positive solution of the equation* (8).

2) *There uniquely exist a transition matrix* $P^* = P(M) = (P^*_{ij})_{i,j \in \mathrm{I}}$ *and a row probability vector* $\pi^* = (\pi^*_i)_{i \in \mathrm{I}}$ *which maximize the function*

$$H(\pi, P) = -\sum_{i,j} \pi_i P_{ij} \log P_{ij} \tag{10}$$

subject to the conditions $0 \leqq P_{ij} \leqq M_{ij}$, $\pi_i \geqq 0$, $\sum \pi_i = 1$ *and* $\pi P = P$. *Furthermore* $H(\pi^*, P^*) = \log \rho(M)$.

Proof. In the case when $\sum_{i \leqq 0} M_{i\infty} = 0$, the set I may be identified with the finite set $\{-r, -r+1, \cdots, 0\}$ and the proof is trivial. Thus we assume that $\sum_{i \leqq 0} M_{i\infty} > 0$. Let $M^{(n)}$ be the "cut-off" matirx defined as follows ($n \geqq 0$):

$$M_{ij}^{(n)} = \begin{cases} M_{ij} & \text{if } \ -r \leqq j < n, \\ 1 & \text{if } \ j=n \ \text{ and } \ \sum_{k \geqq n} M_{ik} > 0, \\ & \text{or if } \ i=j=n \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where i runs over the set $\{-r, \cdots, 0, \cdots, n\}$. We note that $\mathfrak{M}(M^{(n)})$ is the factor space of $\mathfrak{M}(M)$ with respect to the partition $\{\{-r\}, \cdots, \{0\}, \cdots, \{n-1\}, \{m: m \geqq n\}\}$ which is also an open cover of I. It follows from an elementary computation that each matrix $M^{(n)}$ is irreducible and that its eigenvalue ρ_n of maximal modulus is the unique positive solution of the algebraic equation

$$1 = \sum_{j=0}^{n-1} M_j \lambda^{-j-1} + \sum_{i=-r}^{0} M_{in}^{(n)} \lambda^{-n} (\lambda - 1)^{-1}. \tag{12}$$

Consequently from the definition of topological entropy we obtain

$$e(\mathfrak{M}(M), \sigma) = \sup_{n \geqq 0} e(\mathfrak{M}(M^{(n)}), \sigma) = \sup_{n \geqq 0} \log \rho_n \tag{13}$$

(See [1], [3], for example). Moreover from (12) it follows that the sequence $(\rho_n)_{n \geqq 0}$ converges as $n \to \infty$ to the unique positive solution of (8). Thus we proved 1).

The maximizing problem in the statement 2) is equivalent to maximize the value

$$H(X) = -\sum_{i,j} M_{ij} X_{ij} \log X_{ij} + \sum_i \Big(\sum_j M_{ij} X_{ij}\Big) \log \Big(\sum_j M_{ij} X_{ij}\Big) \tag{14}$$

among the matrices $X = (X_{ij})_{i,j \in \mathrm{I}}$ satisfying (a) $X_{ij} \geqq 0$, (b) $\sum_j X_{ij} = \sum_j X_{ji}$ and (c) $\sum_{i,j} X_{ij} = 1$. In fact if $M_{ij} X_{ij} = \pi_i P_{ij}$, then $H(X) = H(\pi, P)$. In order to solve this problem we appeal to the Lagrange's multiplier method. Let λ_i and κ be the Lagrange constants corresponding to the conditions (b) and (c). Then we obtain

$$\sum_k M_{ik} X_{ik}^* = e^{\lambda_j - \lambda_i - \kappa} X_{ij}^* \qquad \text{for} \quad M_{ij} = 1 \tag{15}$$

if $H(X^*) = \max H(X)$. This equality implies that the vector $x = (x_i)_{i \in \mathrm{I}}$ with $x_i = e^{-\lambda_i}$ is a right eigenvector of matrix M corresponding to the eigenevalue $\lambda = e^{-\kappa}$ and that the vector $y = (y_i)_{i \in \mathrm{I}}$ with $y_i = e^{\lambda_i} \sum_j X_{ij}^*$ is a left eigenvector corresponding to λ. Thus the local maximum of the original problem is given by

$$P_{ij}^* = X_{ij}^* / \pi_i^* = M_{ij} x_j / \lambda x_i$$

and

$$\pi_i^* = \sum_j X_{ij}^* = x_i y_i / \sum_{j \in \mathrm{I}} x_j y_j$$

and the value is

$$H(X^*) = H(\pi^*, P^*) = \log \lambda .$$

But we already know that the eignevalue of maximum modulus $\rho(M)$ is simple. Hence the statement 2) is proved.

REMARK. 1) All non-zero eigenvalues of the matrix M are solutions of equation (8) and vice versa.

2) Any eigenvalue of the matrix $P(M)$ is of the form $\lambda/\rho(M)$ where λ is an eigenvalue of the matrix M; In particular, the eigenvalue 1 is simple. In fact if $P(M)z=\kappa z$, then $Mu=\rho(M)\kappa u$ where $u_i=x_i z_i$.

Corollary. *There exists one and only one maximal invariant probability measure μ for the Markov subshift $(\mathfrak{M}(M), \sigma)$, which is necessarily Markovian and mixing.*

Proof. We recall the Parry's result: an invariant probability measure μ for a transformation σ is Markovian with respect to a countable partition α if and only if $H_\mu(\alpha/\sigma^{-1}\alpha)=h(\mu)$. Let μ be a maximal invariant probability measure for our system and α the partition whose atoms are $\{\omega\colon \omega(0)=i\}$, $i\in\mathrm{I}$. Then from 1) of Lemma 1 it follows that

$$\log \rho(M) = h(\mu)$$

But

$$h(\mu) = H_\mu(\alpha \mid \bigvee_{n\geqq 1} \sigma^{-n}\alpha) \leqq H_\mu(\alpha \mid \sigma^{-1}\alpha)$$

and the last term minorizes $\log \rho(M)$ as we stated in 2) of Lemma 1. Consequently we have the equality

$$h(\mu) = H\mu(\alpha \mid \sigma^{-1}\alpha) = \log \rho(M)$$

which asserts the Markov property of μ and the uniqueness according again to 2) of Lemma 1.

Finally the mixing property follows from the ergodicity and the absence of cyclic states.

2. Construction of isomorphism

We are now to construct an isomorphism ϕ_β of β-shift into a Markov subshift in the class which is investigated in the previous paragraph. We begin with the definition of a number $\tau(\omega)$ $(\leqq\infty)$ for $\omega\in X_\beta$:

$$(16) \qquad \tau(\omega)=\begin{cases}\sup\{i:\ i\geqq 1,\ \omega\in B_i\} \\ 0 \qquad \text{if} \quad \omega\in X_\beta\backslash \bigcup_{i\geqq 1} B_i\end{cases}$$

where

$$(17) \quad B_i=\{\omega\in X_\beta\colon (\omega(-),\cdots,\omega(-1))=(\omega_\beta(0),\cdots,\omega_\beta(i-1)),\} \qquad (i\geqq 1).$$

We note that $\tau(\omega)$ is the first hitting time to the set $X_\beta\backslash \bigcup_{i\geqq 1} B_i$, which will be justified below by Lemma 2. Let

$$(18) \qquad C_i=\begin{cases}\{\omega\in X_\beta\colon \tau(\omega)=i\} & (1\leqq j\leqq\infty) \\ \{\omega\in X_\beta\colon \tau(\omega)=0,\ \omega(-1)=-i\} & (-s<i\leqq 0)\end{cases}$$

where $s<\beta\leqq s+1$ and let $\mathrm{I}=\{-(s-1),\cdots,0,1,2,\cdots,\infty\}$. Then the sets C_i, $i\in \mathrm{I}\backslash\{\infty\}$ are closed in X_β and form a partition of the set $X_\beta\backslash C_\infty$.

Let us define a map ϕ_β of X_β into the infinite product space $\mathrm{I}^{\boldsymbol{Z}}$ by the relation:

$$(19) \qquad \phi_\beta(\omega)(n)=i \qquad \text{if} \quad \omega\in\sigma^n C_i \ (n\in\boldsymbol{Z},\ i\in\mathrm{I}),$$

then it is easy to see that the map ϕ_β is Borel, is injective on $X_\beta\backslash C_\infty$ and anti-commutes with the shift transformation, i.e., $\phi_\beta\circ\sigma=\sigma^{-1}\circ\phi_\beta$. Furthermore the inverse ϕ_β^{-1} of the map ϕ_β coincides on the set $\phi_\beta(X_\beta\backslash C)_\infty$ with the map ψ_β of $\mathrm{I}^{\boldsymbol{Z}}$ into $\mathrm{A}^{\boldsymbol{Z}}$:

$$(20) \quad \psi_\beta(\eta)(-n)=\begin{cases}|\eta(n-1)| & \text{if} \quad \eta(n-1)\leqq 0 \\ \omega_\beta(\eta(n-1)-1) & \text{if} \quad \eta(\mathrm{n}-1)\geqq 1 \ (n\in\boldsymbol{Z},\ \eta\in\mathrm{I}^{\boldsymbol{Z}}).\end{cases}$$

Lemma 2. *The image $\phi_\beta(X_\beta\backslash C_\infty)$ by the map ϕ_β is contained in the set $\mathfrak{M}(M^\beta)$ where the matrix M^β is determined by the conditions* (i)–(iv) *with $r=s-1$ and, for $i\leqq 0$ and $i\leqq j<\infty$,*

$$(21) \qquad M^\beta_{i,j}=\begin{cases}1 & \text{if} \quad \omega_\beta(j)>|i|=-i, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Assume first that there is an element $\omega\in X_\beta\backslash C_\infty$ such that $\phi_\beta(\omega)(0)=i\geqq 2$ and $\phi_\beta(\omega)(1)=j$, i.e. $\omega\in C_i\cap\sigma C_j$. Then the condition $\omega\in C_i$ implies that

$$(22) \qquad \sup\{k\colon (\omega(-k),\cdots,\omega(-1))=(\omega_\beta(0),\cdots,\omega_\beta(k-1))\}=i$$

while $\sigma^{-1}\omega\in C_j$ implies that

$$(23) \qquad \sup\{l\colon (\omega(-l-1),\cdots,\omega(-2))=(\omega_\beta(0),\cdots,\omega_\beta(l-1))\}=j$$

Consequently $j\geqq i-1$. We must prove $j=i-1$. Suppose the contrary: $j\geqq i$. Then by (23) $\omega(-i+k)=\omega_\beta(j-i+1+k)$ for $0\leqq k\leqq i-2$. Combining this with

(22) it follows that $\omega_\beta(k)=\omega_\beta(j-i+1+k)$ for $0\leqq k\leqq i-2$. Since $\sigma^{j-i+1}\omega_\beta\leqq\omega_\beta$ by the definition of ω_β itself, we obtain $\omega_\beta(i-1)\geqq\omega_\beta(j)$. On the other hand from $\sigma^{j+1}\omega\leqq\omega_\beta$ and (23) we can deduce that $\omega(-1)\leqq\omega_\beta(j)$. But $\omega(-1)=\omega_\beta(i-1)$ by (22), so that $\varphi(-1)=\varphi_\beta(j)$. Thus we had $\omega(j+k)=\omega_\beta(k)$ for $0\leqq k\leqq j$, which contradicts to (22).

In particular, if $\phi_\beta(\omega)(0)=i$ and $\phi_\beta(\omega)(1)=j\leqq 0$, then $i\leqq 1$.
Finally if $\phi_\beta(\omega)(0)=i\leqq 0$ and $\phi_\beta(\omega)(1)=j\geqq 1$, then

$$(\omega(-j-1), \cdots, \omega(-2)) = \omega_\beta((0), \cdots, \omega_\beta(j-1)),$$
$$(\omega(-j-1), \cdots, \omega(-1) \neq (\omega_\beta(0), \cdots, \omega_\beta(j))$$

and

$$\omega(-1) = |i| .$$

But, $(\omega(-j-1), \cdots, \omega(-1)\leqq(\omega_\beta(0), \cdots, \omega_\beta(j))$ (lexicographical order) since $\omega\in X_\beta$. Hence $\omega_\beta(j)>|i|$.

Thus we have proved that $M^\beta_{\phi_\beta(\omega)(0),\phi_\beta(\omega)(1)}=1$ if $\omega\in X_\beta$. Now the Lemma 2 follows from the shift-invariance of the set $\phi_\beta(X_\beta\backslash C_\infty)$.

Proof of Theorem. Let ν be an arbitrary maximal ergodic measure for the β-shift. We note that $\nu(C_\infty)=0$. In fact let $\omega\in C_\infty$ and

$$n_0(\omega) = 0 ,$$
$$n_k(\omega) = \min\{n>n_{k-1}(\omega)\colon (\omega(-n), \cdots, \omega(-1)) = (\omega_\beta(0), \cdots, \omega_\beta(n-1))\} ,$$

for $k=1, 2, \cdots$. Then $n_k(\omega)$, $k\geqq 0$ are well-defined and tend to infinity as $k\to\infty$ since $\omega\in C_\infty$. Furthermore

$$n_{k+1}(\omega) = f(n_k((\omega))$$

where

$$f(m) \equiv \min\{n>m\colon \omega_\beta(n-m+k)\omega_\beta(k) , \qquad 0\leqq k<m\}.$$

In particular the number $n_1(\omega)$ determines the sequence $(n_k(\omega))_{k\geqq 1}$, and so the sequence $(\omega(n))_{n\leqq 0}$. But

$$C_\infty = \bigcup_{n\geqq 0}\{\omega\in C_\infty\,|\,n_1(\omega) = n\} .$$

Thus $\mu\{\omega\in C_\infty\,|\,n_1(\omega)=n\}=0$ since ν is an ergodic measure with positive entropy and therefore non-atomic.

Now we recall that the map ϕ_β defined by (19) is a Borel injection of $X_\beta\backslash C_\infty$ into $\mathfrak{M}(M^\beta)$ and anti-commuting with the shift transformation and that $e(X_\beta, \sigma)=\log\beta$. The topological entropy of the Markov subshift $(\mathfrak{M}(M^\beta), \sigma)$ is also $\log\beta$; Indeed $M_j=\sum_{i\geqq 0} M_{ij}=\omega_\beta(j)$, and consequently the equations (3) and (8) coincide.

Let ν' be the invariant probability measure of $(\mathfrak{M}(M^{\beta}), \sigma)$ induced by the map ϕ_{β} from ν on X_{β}, which is concentrated on the set $X\backslash C_{\infty}$ as we have seen above. Since ϕ_{β} is invertible ν-almost everywhere, the metrical entropy $h(\nu')$ of the measure ν' is $h(\nu)=\log\beta$. But we already know the uniqueness of maximal invariant measure for $(\mathfrak{M}(M^{\beta}), \sigma)$ in Corollary to Lemma 1. Consequently the map ϕ_{β} is an isomorphism (mod 0) of (X_{β}, σ, ν) onto $(\mathfrak{M}(M^{\beta}), \sigma, \lambda_{\beta})$, λ_{β} being the unique maximal invariant measure, and $\nu=\psi_{\beta}(\lambda_{\beta})$ is unique. Thus we completed the proof of Theorem and automatically the proof of Corollary.

REMARK. Let us denote by $\boldsymbol{P}$ the maximal invariant measure of $(\mathfrak{M}(M^{\beta}), \sigma)$ and the coordinate function by ξ_{n}.
Then $(\boldsymbol{P}, \xi_{n})$ is a Markov chain. Let

$$\begin{aligned} \tau_{0} &= \inf\{k>0:\ \xi_{k}\in\{-(s--1), \cdots, 0\}\} \\ \tau_{n} &= \inf\{k>\tau_{n-1}:\ \xi_{k}\in\{-(s-1), \cdots, 0\}\} \qquad (n\geqq 1) \\ \tau_{n} &= \sup\{k<\tau_{n+1}:\ \xi_{k}\in\{-(s-1), \cdots, 0\}\} \qquad (n\leqq -1) \end{aligned}$$

It will be interesting that $(\boldsymbol{P}, (\xi_{\tau_{u}}, \tau_{n+1}))$ and $(\boldsymbol{P}, \xi_{\tau_{n}})$ are both Bernoulli; the former is in one-to-one correspondence with $(\boldsymbol{P}, \xi_{n})$ and the latter is isomorphic to the Bernoulli scheme $B(x_{-s+1}, \cdots, x_{0})$ (See (7)). In particular the Markov chain $(\boldsymbol{P}, \xi_{n})$ can be obtained as an automorphism based upon the Bernoulli scheme $B(x_{-(s-1)}, \cdots, x_{0})$ under a random function f, which is independent of $B(x_{-(s-1)}, \cdots, x_{0})$ under the conditioning by $\xi_{\tau_{0}}$, where

$$x_{i} = \sum_{k\leqq 0} M_{i,k}\beta^{-k-1} \quad \text{and} \quad \boldsymbol{P}(f=n\,|\,\xi_{\tau_{0}}=i) = \frac{M_{i,n-1}\beta^{-n}}{x_{i}}$$

On the other hand a Bernoulli automorphism $B(p_{0}, p_{1}, \cdots, p_{n})$ $(p_{i}\geqq 0, \sum_{i=1}^{n} p_{i}=1)$ is obtained as an automorphism based upon Bernoulli scheme $B\left(\frac{1-p_{0}}{p_{1}}, \cdots, \frac{p_{n}}{1-p_{0}}\right)$ under a random function g which is independent of the basic automorphism and such that

$$\boldsymbol{P}(g=n) = \frac{p_{0}^{n-1}}{1-p_{0}} \qquad (n\geqq 1)$$

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HIROSHIMA UNIVERSITY

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