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POSITIVE GENERALIZED WIENER FUNCTIONS AND POTENTIAL THEORY OVER ABSTRACT WIENER SPACES

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Introduction

The notion of generalized Wiener functions (or "functionals" if the underlying space is a function space) has been introduced in a development of the Malliavin calculus ([17] [20] [21]). This is a natural infinite dimensional analogy of the Schwartz distribution theory in which the role of the Lebesgue measure on a Euclidean space $\boldsymbol{R}^n$ is now replaced by a Gaussian measure on a Banach space. (Such a measure space is called an abstract Wiener space.) In this paper, we will show that generalized Wiener functions which are positive, i.e., those which yield non-negative values when they act upon positive test functions, are measures on the underlying Banach space. It is an analogue of the fact that positive Schwartz distributions are measures.

The class of measures corresponding to positive generalized Wiener functions contains many measures which are singular with respect to the original Gaussian measure and yet, in contrast with finite dimensional cases, it constitutes a rather small class in the totality of finite Borel measures on the Banach space. Many properties of this class can be stated in terms of the potential theory over the abstract Wiener space, particularly, in terms of *capacities*. Such a potential theory has been discussed, among others, by Malliavin [10], Fukushima [3], Fukushima-Kaneko [4] and Takeda [19]. We will show in this paper that these measures can not have their mass in any set of capacities zero and that, on the contrary, for any set of non-zero capacity, there exists a non-trivial measure in this class which is supported on the closure of the set.

In many probelms of extending results in finite dimensional spaces to those in infinite dimensional spaces, a difficulty usually occurs from the fact that an infinite dimensional vector space is *not* locally compact. Indeed, this is the case in our problem, too. However, this difficulty can be fortunately overcome by the fact that a probability measure on a complete separable metric space is

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tight, i.e., for any $\varepsilon>0$, we can take a compact set whose measure is not less than $1-\varepsilon$. Especially, in the case of a Gaussian measure on a separable Banach space, a theorem of Gross (Lemma 1.1 below) is essential. In addition, we have a powerful tool of the so-called *Ornstein-Uhlenbeck semigroup* by which we can connect the Gross theorem to the Malliavin calculus. It will play an important role throughout the present paper as something like a *mollifier* on the abstract Wiener space (Lemma 2.1$\sim$2.5).

In section 1, we will introduce basic notions in the theory of abstract Wiener spaces and the Malliavin calculus including the Gross theorem. In section 2, we will establish several lemmas involving the Ornstein-Uhlenbeck semigroup. In section 3, the properties of capacities will be summarized and, among others, we will prove the capacitability of Borel sets, i.e., any Borel set can be approximated in capacities by compact sets from below. In section 4, we will prove that positive generalized Wiener functions are measures and that such measures never have their mass in any set of capacities zero. In section 5, we will investigate the equilibrium measures of sets, which are special examples of our measures. In the last section, we will discuss positive generalized Wiener functions from viewpoints other than the potential theory. In conclusion, we can say that the program of Fukushima [2] or Maz'ya-Khavin [11] in finite dimensional cases could be realized in our infinite case as well.

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1. Preliminaries

General notation

"$A:=B$" or "$B=:A$" means "A is defined by B".

$\boldsymbol{N}:=$the set of all positive integers.

$\boldsymbol{R}:=$the set of all real numbers.

$x\vee y:=\max(x, y)$, $x\wedge y:=\min(x, y)$, $x, y\in\boldsymbol{R}$.

If A and B are sets, $A\backslash B:=\{x\in A; x\notin B\}$.

$\mathcal{D}(\boldsymbol{R}^n)$ and $\mathcal{D}'(\boldsymbol{R}^n)$ are the n-dimensional Schwartz spaces of test functions and distributions respectively.

$\mathcal{B}(X)$ is the topological σ-algebra of a topological space X.

If X is a Banach space, $\|\cdot\|_X$ denotes its norm. But if X is a Hilbert space, the norm is denoted by $|\cdot|_X$ and the inner product by $\langle\cdot, \cdot\rangle_X$. In particular, if X is a Euclidean space $\boldsymbol{R}^n$, they are denoted simply by $|\cdot|$ and $\langle\cdot, \cdot\rangle$ respectively.

If X is a topological linear space, X^* denotes its topological dual space, and the pairing of X^* and X is denoted by $(\cdot, \cdot)$.

If V is a linear operator densely defined on a topological linear space, V^* is its dual operator.

A symbol "$\subsetneq$" stands for a continuous imbedding.

Sometimes we omit the domain of integration, if it is the whole space.

Abstract Wiener space

First we have to make our framework clear. Let W be a real separable Banach space. If ν is a measure on a measurable space $(W, \mathcal{B}(W))$ and if E is a real separable Hilbert space, we use the term "E-valued ν-measurable mapping" to mean "E-valued mapping defined on W which is $\overline{\mathcal{B}(W)}^{\nu}|\mathcal{B}(E)$-measurable", where $\overline{\mathcal{B}(W)}^{\nu}$ stands for the completion of $\mathcal{B}(W)$ by the measure ν. We define $L_p(\nu; E)=L_p(\nu; W\to E)$ for $1\leqq p<\infty$ as the totality of E-valued ν-measurable mappings F such that $|F|_E^p$ are ν-integrable. As usual, two elements of $L_p(\nu; E)$ which coincide ν-a.e. are identified. Hence $L_p(\nu; E)$ is a Banach space with a norm $||F||_{L_p(\nu; E)}:=\{\int_W |F(w)|_E^p\, \nu(dw)\}^{1/p}$, $F\in L_p(\nu; E)$. It is convenient to define $L_{1+}(\nu; E):=\bigcup_{1<p<\infty} L_p(\nu; E)$ and $L_{\infty-}(\nu; E):=\bigcap_{1<p\leq\infty} L_p(\nu; E)$. In case $E=\boldsymbol{R}$, we simply denote $L_p(\nu; E)$ by $L_p(\nu)$ for $1\leqq p<\infty$ or $p=1+, \infty-$. Note that $L_{\infty-}(\nu)$ is an algebra.

Let μ be a *Gaussian measure* on $(W, \mathcal{B}(W))$ with mean vector 0 whose topological support spreads over the whole space W. Then there is a unique real separable Hilbert space H continuously and densely imbedded in W such that $\int \exp(\sqrt{-1}(l, w))\, \mu(dw)=\exp(-|l|_H^2/2)$ holds for each $l\in W^*$. (Since $H\subset W$ implies $W^*\subset H^*$ and H^* is identified with H by the Riesz theorem, an element l of W^* is regarded as an element of H. It is readily seen that $W^*\subsetneq H$ and that W^* is dense in H.) The triplet (W, H, μ) is called an *abstract Wiener space*. Accordingly, E-valued μ-measurable mappings are called *E-valued Wiener functions* (or *"functionals"* when W is a function space). The following lemma is crucial in our theory.

Lemma 1.1. *There exists a separable Banach space W_1 such that*

(i) $H\subsetneq W_1\subsetneq W$, *where both imbeddings are compact,*

(ii) $\mu(W_1)=1$, (iii) $||\cdot||_{W_1}\in L_{\infty-}(\mu)$.

For (i) and (ii), see [6]. As for (iii), we should notice that although $||w||_{W_1}$ is not defined for $w\in W\setminus W_1$, $||\cdot||_{W_1}$ is defined μ-a.e. on account of (ii). Its measurability is clear from the construction of W_1 (indeed, it is $\mathcal{B}(W)$-measurable) and finally its integrability is due to Fernique [1]. See also an excellent review of Kuo [8].

This lemma tells us that (W_1, H, μ) is again an abstract Wiener space. Consequently, the Banach space W is not essential for the Gaussian measure μ. It

is merely a support of μ, and we can always replace it by a smaller support W_1. But anyway, *we fix such a W_1 from now on.*

EXAMPLE 1.1. (The classical Wiener space) Let W_0^d be a Banach space consisting of all continuous functions w of $[0, T]$, $T>0$, into $\boldsymbol{R}^d$ such that $w(0)=0$ with the usual maximum norm, and H^1 be a Hilbert space consisting of all absolutely continuous functions h of $[0, T]$ into $\boldsymbol{R}^d$ such that $h(0)=0$ and that $|h|_{H^1}^2 := \int_0^T |\frac{d}{dt}h(t)|^2 dt < \infty$. Then together with the *standard Wiener measure* P_0 on W_0^d, the triplet (W_0^d, H^1, P_0) forms an abstract Wiener space, which is called the *classical d-dimensional Wiener space.* Then a smaller Banach space W_1, is taken, for example, as follows; Let $0<\alpha<1/2$ and $m\in\boldsymbol{N}$. We put

$$(1.1)\quad W^{\alpha,2m} := \{w\in L_{2m}(dt; [0, T]) \to \boldsymbol{R}^d) ;$$
$$||w||_{\alpha,2m} := \left[\int_0^T\int_0^T \frac{|w(t)-w(s)|^{2m}}{|t-s|^{1+2m\alpha}}\, dsdt\right]^{1/2m} < \infty\} .$$

Suppose $2m\alpha>1$. Then each element of $W^{\alpha,2m}$ has a continuous version and a Banach space $W_1 := \{w\in W^{\alpha,2m}; w(0)=0\}$ with the norm $||\cdot||_{W_1} := ||\cdot||_{\alpha,2m}$ is well-defined as a subspace of W_0^d. It can be shown that this W_1 plays the required role in Lemma 1.1 (for the compact imbedding $W_1 \subseteq\!\!\!\rightarrow W_0^d$, see Muramatsu [14]).

Sobolev spaces over an abstract Wiener space

Now let us proceed to the definition of the Sobolev spaces over (W, H, μ). First we introduce polynomials. We put

$$\boldsymbol{P} := \{F: W\to\boldsymbol{R}; \text{ there exist } n\in\boldsymbol{N}, l_1, \cdots, l_n\in W^*, \text{ and a polynomial}$$
$$f: \boldsymbol{R}^n\to\boldsymbol{R} \text{ such that } F(w) = f((l_1, w), \cdots, (l_n, w)), w\in W\} ,$$

and

$$\boldsymbol{P}_n := \{F\in\boldsymbol{P}; \text{ the polynomial } f \text{ in the above expression is of degree at most } n\},\quad n = 0, 1, 2, \cdots.$$

Note that $\boldsymbol{P}$ is a dense subspace of $L_p(\mu)$, $1\leqq p<\infty$. Let $\boldsymbol{Z}_0 := \boldsymbol{P}_0$=the space of all constant functions, and $\boldsymbol{Z}_n := \bar{\boldsymbol{P}}_n^{L_2(\mu)} \cap \boldsymbol{P}_{n-1}^{\perp}$, $n=1, 2, \cdots$ ("$\perp$" stands for the orthogonal complement). Then we get an orthogonal direct sum decomposition of $L_2(\mu)$, known as the *Wiener homogeneous chaos decomposition.* The *Ornstein-Uhlenbeck operator* L is defined by a spectral decomposition $L := \sum_{n=0}^{\infty} (-n) J_n$, where J_n is the orthogonal projection of $L_2(\mu)$ onto $\boldsymbol{Z}_n$. Obviously, L maps $\boldsymbol{P}$ into $\boldsymbol{P}$. Furthermore, we define operators $(I-L)^{r/2}$, $r\in\boldsymbol{R}$, mapping $\boldsymbol{P}$ into $\boldsymbol{P}$ by $(I-L)^{r/2} := \sum_{n=0}^{\infty}(1+n)^{r/2} J_n$. In order to introduce E-valued Sobolev spaces, we define E-valued polynomials as finite sums of functions $F(w)\, e$, $w\in W$, $e\in E$ and $F\in\boldsymbol{P}$. The totality of E-valued polynomials is denoted by $\boldsymbol{P}(E)$. If

S is a linear mapping from $\boldsymbol{P}$ into itself, it naturally induces a linear mapping $\bar{S}$ from $\boldsymbol{P}(E)$ into itself defined by $\bar{S}(F(\cdot)\,e)\,(w) := (SF)\,(w)\,e$, $w \in W$, $e \in E$ and $F \in \boldsymbol{P}$. $\bar{S}$ will be denoted by the same letter S.

DEFINITION 1.1. For $1<p<\infty$ and $r \in \boldsymbol{R}$, we define a Sobolev space $\boldsymbol{D}_p^r(E)$ of E-valued Wiener functions as the completion of $\boldsymbol{P}(E)$ with respect to a norm $\|\cdot\|_{p,s\,;\,E} := \|(I-L)^{r/2}\cdot\|_{L_p(\mu\,;\,E)}$. For simplicity, we denote $\boldsymbol{D}_p^r(\boldsymbol{R})$ and $\|\cdot\|_{p,r\,;\,\boldsymbol{R}}$ by $\boldsymbol{D}_p^r$ and $\|\cdot\|_{p,r}$, respectively.

p is of course the integrability index, while r can be interpreted as the differentiability index. In fact, in case r is a positive integer, the norm $\|\cdot\|_{p,r\,;\,E}$ is equivalent to another Sobolev norm defined in terms of the *H-differential* ([16]). To be precise, let $\mathcal{H}(E)$ be the set of all linear mappings $H \to E$ of Hilbert-Schmidt type. It is a separable Hilbert space with the Hilbert-Schmidt norm. We inductively define $\mathcal{H}^n(E)$ by $\mathcal{H}(\mathcal{H}^{n-1}(E))$, $n=1, 2, \cdots$, where $\mathcal{H}^1(E) := \mathcal{H}(E)$. Since $\mathcal{H}(R)$ is nothing but H^*, it is usually identified with H. Now *H-differential operator* D mapping $\boldsymbol{P}(E)$ into $\boldsymbol{P}(\mathcal{H}(E))$ is defined by

$$DF(w)\,[h] := \lim_{t \downarrow 0} \frac{1}{t}\,(F(w+th)-F(w)), \quad w \in W,\ h \in H \qquad \text{for} \quad F \in \boldsymbol{P}(E)\,.$$

Obviously, n-times iteration of D, $n \in \boldsymbol{N}$, yields the mapping $D^n\colon \boldsymbol{P}(E) \to \boldsymbol{P}(\mathcal{H}^n(E))$. Then it is known that the norm $\|\cdot\|_{p,n\,;\,E}$ is equivalent to a norm $\|\cdot\|_{L_p(\mu\,;\,E)} + \|D^n\cdot\|_{L_p(\mu\,;\,\mathcal{H}^n(E))}$. In other words, $\boldsymbol{D}_p^n(E)$ is again obtained by completing $\boldsymbol{P}(E)$ with respect to the latter norm ([17], Th.2.4]).

Our Sobolev spaces have the following properties ([17] [21]).

(i) *The system of norms* $\{\|\cdot\|_{p,r\,;\,E};\ 1<p<\infty, r \in \boldsymbol{R}\}$ *is compatible. Furthermore, if* $p \leqq p'$ *and* $r \leqq r'$, *we have* $\|\cdot\|_{p,r\,;\,E} \leqq \|\cdot\|_{p',r'\,;\,E}$ *and hence* $\boldsymbol{D}_{p'}^{r'}(E) \subseteq \boldsymbol{D}_p^r(E)$.

(ii) *For* $r \in \boldsymbol{R}$ *and* $1<p, q<\infty$ *satisfying* $1/p+1/q=1$, *we have* $(\boldsymbol{D}_p^r(E))^* = \boldsymbol{D}_q^{-r}(E)$ *under the standard identification of* $(L_p(\mu; E))^* = L_q(\mu; E)$. *In particular, it holds that* $(\Phi, F) \leqq \|\Phi\|_{q,-r\,;\,E}\,\|F\|_{p,r\,;\,E}$ *for* $\Phi \in \boldsymbol{D}_q^{-r}(E)$ *and* $F \in \boldsymbol{D}_p^r(E)$.

By the property (i), the following definitions make sense.

DEFINITION 1.2.

$$\boldsymbol{D}^{+\infty}(E) := \bigcup_{1<p<\infty}\ \bigcup_{r>0} \boldsymbol{D}_p^r(E)\,, \quad \boldsymbol{D}^{-\infty}(E) := \bigcup_{1<q<\infty}\ \bigcup_{r>0} \boldsymbol{D}_q^{-r}(E)\,.$$

$\boldsymbol{D}^{+\infty}(\boldsymbol{R})$ and $\boldsymbol{D}^{-\infty}(\boldsymbol{R})$ are denoted simply by $\boldsymbol{D}^{+\infty}$ and $\boldsymbol{D}^{-\infty}$ respectively.

Giving $\boldsymbol{D}^{+\infty}(E)$ the topology induced by the norms $\|\cdot\|_{n,n\,;\,E}$, $n=1, 2, \cdots$, it is a complete countable normed space and then the property (ii) implies that $\boldsymbol{D}^{-\infty}(E)$ is identified with $(\boldsymbol{D}^{+\infty}(E))^*$. Hence it is natural to call an element of $\boldsymbol{D}^{-\infty}(E)$ an *E-valued generalized Wiener function* (or *"functional"* when W is a function space). The term "generalized Wiener funciton(-al)" will be abbre-

viated to "*GWF*". It is important to notice that $\boldsymbol{D}^{+\infty}$ is an algebra.

Now, the operators defined on $\boldsymbol{P}(E)$ above are closable and hence, can be extended uniquely to operators on $\boldsymbol{D}^{-\infty}(E)$; Namely, $L: \boldsymbol{D}^{-\infty}(E)\to\boldsymbol{D}^{-\infty}(E)$, $(I-L)^{r/2}: \boldsymbol{D}^{-\infty}(E)\to\boldsymbol{D}^{-\infty}(E)$ and $D: \boldsymbol{D}^{-\infty}(E)\to\boldsymbol{D}^{-\infty}(\mathcal{H}(E))$ are all well-defined and $L: \boldsymbol{D}_p^s(E)\to\boldsymbol{D}_p^{s-2}(E)$, $(I-L)^{r/2}: \boldsymbol{D}_p^s(E)\to\boldsymbol{D}_p^{s-r}(E)$ and $D: \boldsymbol{D}_p^s(E)\to\boldsymbol{D}_p^{s-1}(\mathcal{H}(E))$, $1<p<\infty$, $s\in\boldsymbol{R}$, $r\in\boldsymbol{R}$, are all continuous. In particular, the operator $(I-L)^{-r/2}$ gives an isometry from $L_p(\mu; E)$ onto $\boldsymbol{D}_p^r(E)$. By the duality, $D^*: \boldsymbol{D}^{-\infty}(\mathcal{H}(E))\to\boldsymbol{D}^{-\infty}(E)$ is well-defined and $D^*: \boldsymbol{D}_p^s(\mathcal{H}(E))\to\boldsymbol{D}_p^{s-1}(E)$ is continuous for all $1<p<\infty$, $s\in\boldsymbol{R}$. Furthermore, it holds that $L=-D^*D$ (cf. [17] [21]).

2. The Ornstein-Uhlenbeck semigroup

The *Ornstein-Uhlenbeck semigroup* $\{T_t\}_{t>0}$ is a semigroup generated by the operator L, i.e., $T_t:=e^{tL}=\sum_{n=0}^{\infty}e^{-nt}J_n$. Obviously, T_t, $t>0$, maps $\boldsymbol{P}(E)$ and $L_2(\mu; E)$ into $\boldsymbol{P}(E)$ and $L_2(\mu; E)$, respectively. For a bounded continuous function $F: W\to\boldsymbol{R}$, the following expression is known;

$$T_tF(w)=\int F(e^{-t}w+\sqrt{1-e^{-2t}}v)\,\mu(dv),\quad w\in W,\ t>0\,. \tag{2.1}$$

The next lemma is nearly directly derived from this expression.

Lemma 2.1.

(i) *T_t, $t>0$, is uniquely extended to a continuous linear operator on $L_p(\mu)$, $1\leqq p<\infty$, with operator norm* 1.

(ii) *For each $F\in L_p(\mu)$, $1\leqq p<\infty$, T_tF converges to F in $L_p(\mu)$ as $t\downarrow 0$.*

(iii) *If F is bounded and continuous, then $T_tF(w)$ is also bounded and continuous and it converges to $F(w)$ for all $w\in W$ as $t\downarrow 0$.*

(iv) *T_t, $t>0$, is a positive operator, i.e., if $F\in L_p(\mu)$ is non-negative μ-a.e., then so is T_tF. Hence T_t is Markovian, i.e., if $0\leqq F\leqq M$, μ-a.e. for some $M>0$, then so is T_tF.*

(v) *T_t, $t>0$, is μ-invariant, i.e., $\int T_tF\,d\mu=\int F\,d\mu$, $F\in L_1(\mu)$.*

Note that the equality (2.1) holds for all $F\in L_1(\mu)$ that are continuous on account of (i) in the lemma.

The role $\{T_t\}_{t>0}$ will play in the sequel may be compared to the one that the mollifier has played in the finite dimensional analysis. Namely we have the following, in addition to Lemma 2.1 (it can be seen in [5] with an incomplete proof).

Lemma 2.2.

(i) *For any $t>0$, any $1<p<\infty$ and any pair $r<s$, T_t maps $\boldsymbol{D}_p^r(E)$ into $\boldsymbol{D}_p^s(E)$ continuously.*

(ii) *If $t>0$ and F belongs to $L_{\infty-}(\mu; E)$, then T_tF belongs to $\boldsymbol{D}^{+\infty}(E)$. In*

particular, if $F: W \to \boldsymbol{R}$ *is* μ*-measurable and bounded, then* $T_t F \in \boldsymbol{D}^{+\infty}$.

Proof. (ii) is obvious from (i) and we show (i) only. For this, it is enough to show when $E=\boldsymbol{R}$ ([17], Lem. 2.2). We first put $\mathcal{F}C_b^1 := \{F: W \to \boldsymbol{R};\ F(w) = f((l_1, w), \cdots, (l_n, w))$ for some $n \in \boldsymbol{N}$, $l_1, \cdots, l_n$ and $f: \boldsymbol{R}^n \to \boldsymbol{R}$ of C^1-class bounded with its derivatives$\}$ and prove that

$$\|DT_t F\|_{L_p(\mu; H)} \leqq c \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \|F\|_{L_p(\mu)} \tag{2.2}$$

for all $F \in \mathcal{F}C_b^1$, where c is a positive constant independent of F and t.

Let $\{l_i\}_{i=1}^{\infty}$ be a complete orthonormal system (abbreviated to CONS) of H each element l_i of which is taken from W^*. We calculate the following;

$$\begin{aligned}
\langle DT_t F(w), l_i \rangle_H &= \langle D\left[\int F(e^{-t}\cdot + \sqrt{1-e^{-2t}}\, v)\, \mu(dv)\right](w), l_i \rangle_H \\
&= \int \langle DF(e^{-t}\cdot + \sqrt{1-e^{-2t}}\, v)(w), l_i \rangle_H \, \mu(dv) \\
&= e^{-t} \int \langle (DF)(e^{-t}w + \sqrt{1-e^{-2t}}\, v), l_i \rangle_H \, \mu(dv) \\
&= \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int \langle DF(e^{-t}w + \sqrt{1-e^{-2t}}\,\cdot)(v), l_i \rangle_H \, \mu(dv) \\
&= \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int F(e^{-t}w + \sqrt{1-e^{-2t}}\, v)(D^* l_i)(v)\, \mu(dv)
\end{aligned}$$

Noticing that $(D^* l_i)(v) = (l_i, v)$, we know $\{D^* l_i\}_{i=1}^{\infty}$ forms a CONS of the Wiener homogeneous chaos $\boldsymbol{Z}_1$ of order one ([17], Th.3.3). Consequently, we have

$$\begin{aligned}
|DT_t F(w)|_H^2 &= \sum_{i=1}^{\infty} \langle DT_t F(w), l_i \rangle_H^2 \\
&= \frac{e^{-2t}}{1-e^{-2t}} \sum_{i=1}^{\infty} \left[\int F(e^{-t}w + \sqrt{1-e^{-2t}}\, v)(l_i, v)\, \mu(dv)\right]^2 \\
&= \frac{e^{-2t}}{1-e^{-2t}} \|J_1 F(e^{-t}w + \sqrt{1-e^{-2t}}\,\cdot)\|_{L_2(\mu)}^2 .
\end{aligned}$$

Since $L_2(\mu)$-norm and $L_p(\mu)$-norm are equivalent on the subspace $\boldsymbol{Z}_1$, and since J_1 is a bounded operator also on $L_p(\mu)$, we can find $c>0$ so that $\|J_1 G\|_{L_2(\mu)} \leqq c \|G\|_{L_p(\mu)}$ for all $G \in L_p(\mu)$ ([17], Lem. 1.1, Th. 2.3). Hence we have

$$\begin{aligned}
\int |DT_t F(w)|_H^p \, \mu(dw) &\leqq c^p \left[\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\right]^p \int \|F(e^{-t}w + \sqrt{1-e^{-2t}}\,\cdot)\|_{L_p(\mu)}^p \, \mu(dw) \\
&= c^p \left[\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\right]^p \int \mu(dw) \int |F(e^{-t}w + \sqrt{1-e^{-2t}}\, v)|^p \, \mu(dv) \\
&= c^p \left[\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\right]^p \int \mu(dw)\, T_t(|F|^p)(w)
\end{aligned}$$

$$= c^p \left[\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\right]^p \|F\|^p_{L_p(\mu)}.$$

The last "=" is due to Lemma 2.1 (v). Thus (2.2) is proved.

Since $\mathcal{F}C_b^1$ is dense in $L_p(\mu)$, (2.2) implies that T_t maps $L_p(\mu)$ into $\boldsymbol{D}_p^1$ continuously. For any $n\in\boldsymbol{N}$, replacing t by t/n and using the semigroup property, we see T_t maps $L_p(\mu)$ into $\boldsymbol{D}_p^n$ continuously. Now assertion (i) can be concluded by recalling that T_t commutes with $(I-L)^{\pm r/2}$. Q.E.D.

REMARK. By Meyer's result ([13] Th.2), the inequality (2.2) implies that $\|\sqrt{-L}T_tF\|_{L_p(\mu)} \leqq c' \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \|F\|_{L_p(\mu)}$, $F\in L_p(\mu)$, where $\sqrt{-L}=\sum_{n=1}^{\infty}\sqrt{n}\,J_n$ and c' is a certain positive constant. Therefore we have

$$\|(\sqrt{-L})^k T_tF\|_{L_p(\mu)} = \|(\sqrt{-L}\,T_{t/k})^k F\|_{Lp(\mu)} \leqq c'^k \left[\frac{e^{-t/k}}{\sqrt{1-e^{-2t/k}}}\right]^k \|F\|_{L_p(\mu)},$$

for $F\in L_p(\mu)$. Then it follows from another Meyer's result ([13] Th.3) that

$$\begin{aligned}(2.3)\qquad \|D^k T_tF\|_{L_p(\mu\,;\,\mathcal{H}^k(\boldsymbol{R}))} &\leqq c'' \left[\frac{e^{-t/k}}{\sqrt{1-e^{-2t/k}}}\right]^k \|F\|_{L_p(\mu)} \\ &\leqq c''' t^{-k/2} \|F\|_{L_p(\mu)}, \quad F\in L_p(\mu),\end{aligned}$$

where c'' and c''' are positive constants independent of t and F and we used here an inequality $e^{-t}/(1-e^{-2t})^{1/2}<(2t)^{-1/2}$ for $t>0$.

We present here two approximation lemmas. The former one is easily derived from Lemma 2.1, Lemma 2.2 and the fact that T_t commutes with $(I-L)^{\pm r/2}$.

Lemma 2.3. *Let $1<p<\infty$ and $r\in\boldsymbol{R}$. For each $F\in\boldsymbol{D}_p^r(E)$, $T_tF(\in\boldsymbol{D}_p^{+\infty}(E):=\bigcap_{r>0}\boldsymbol{D}_p^r(E))$ converges to F in $\boldsymbol{D}_p^r(E)$ as $t\downarrow 0$.*

Before proceeding to the next lemma, we give some remarks on the *resolvents* $(\alpha I-L)^{-1}$, $\alpha>0$, or more generally, their fractional powers $(\alpha I-L)^{-r/2}$, $\alpha>0$, $r>0$. Every theorem and lemma of this article involving approximation arguments can be rewritten by using these operators instead of T_t, $t>0$. The operator $(\alpha I-L)^{-r/2}$ has the following integral expression;

$$(2.4)\qquad (\alpha I-L)^{-r/2} = \frac{1}{\Gamma(r/2)} \int_0^{\infty} s^{r/2-1} e^{-\alpha s} T_s\, ds, \quad \alpha>0, r>0.$$

Most properties of this operator are derived from the above expression and the properties of T_t, $t>0$. For example, $\alpha^r(\alpha I-L)^{-r/2}$ is a positive Markovian operator, bounded on $L_p(\mu)$ with operator norm 1 and it converges strongly to

the identity operator on $L_p(\mu)$ as $\alpha\to\infty$. The only difference from T_t, $t>0$, is that it maps $\boldsymbol{D}_p^s(E)$ into $\boldsymbol{D}_p^{s+r}(E)$ continuously, but not into $\boldsymbol{D}_p^{+\infty}(E)$.

Lemma 2.4. *Let $F\in\boldsymbol{D}_p^r$, $1<p<\infty$, $r>0$, be non-negative μ-a.e. Then for any $\varepsilon>0$, there exists $G\in\boldsymbol{D}^{+\infty}$ such that it is non-negative μ-a.e. and that $||F-G||_{p,r}<\varepsilon$.*

Proof. Put $F_M:=(I-L)^{-r/2}\{((-M)\vee(I-L)^{r/2}F)\wedge M\}$, $M>0$. Clearly $F_M\in\boldsymbol{D}_p^r$ and F_M converges to F in $\boldsymbol{D}_p^r$, hence in $L_p(\mu)$, as $M\to\infty$. Because $F\geqq 0$ μ-a.e., $F_M\vee 0$ also converges to F in $L_p(\mu)$. Therefore we can take $M>0$ and $t>0$ by Lemma 2.2 and Lemma 2.3 so that the two conditions below are fulfilled;

$$||T_t(F_M\vee 0)-T_tF||_{p,r}<\varepsilon/2$$
$$||T_tF-F||_{p,r}<\varepsilon/2$$

Thus we have $||T_t(F_M\vee 0)-F||_{p,r}<\varepsilon$, but $F_M\vee 0$ is bounded (due to the remarks just mentioned before this lemma) and non-negative, hence $T_t(F_M\vee 0)\in\boldsymbol{D}^{+\infty}$ and it is non-negative μ-a.e. Q.E.D.

We will state one more rather interesting lemma. It asserts that any two closed sets which are separated from each other with a positive distance can be separated by a function in $\boldsymbol{D}^{+\infty}$.

Lemma 2.5. *Let A be an arbitrary closed set of W and ε be an arbitrary positive number. We put $A_\varepsilon:=\{w\in W;\ \mathrm{dist}(w,A)<\varepsilon\}$, where $\mathrm{dist}(w,A):=\inf\{||w-v||_W;\ v\in A\}$. Then we can find a function $F_\varepsilon\colon W\to\boldsymbol{R}$ which satisfies the following four conditions.*

(i) *F_ε is continuous,*
(ii) *$0\leqq F_\varepsilon(w)\leqq 1$ for all $w\in W$,*
(iii) *$F_\varepsilon(w)=1$ if $w\in A$ and $F_\varepsilon(w)=0$ if $w\in W\setminus A_\varepsilon$,*
(iv) *$F_\varepsilon\in\boldsymbol{D}^{+\infty}$.*

Proof. *Step* 1: *The case of bounded A*; We define a function $F\colon W\to\boldsymbol{R}$ by $F(w):=0\vee(1-\frac{2}{\varepsilon}\,\mathrm{dist}(w,A))$. Obviously, F has the following properties;

(2.5) F is Lipschitz continuous, i.e., $|F(w)-F(v)|\leqq\frac{2}{\varepsilon}||w-v||_W$, $w,v\in W$,

(2.6) $$0\leqq F(w)\leqq 1,\quad w\in W,$$

(2.7) $$F(w)=1 \text{ if } w\in A \text{ and } F(w)=0 \text{ if } w\in W\setminus A_{\varepsilon/2}.$$

Now let $R>0$ be such that $A_\varepsilon\subset\{w\in W;\ ||w||_W<R\}$. Then if $w\in A$, we have

$$(2.8)\qquad |T_tF(w)-1|\leqq\int|F(e^{-t}w+\sqrt{1-e^{-2t}}\,v)-F(w)|\,\mu(dv)$$
$$\leqq\frac{2}{\varepsilon}\int||e^{-t}w+\sqrt{1-e^{-2t}}\,v-w||_W\,\mu(dv)$$

$$\leqq \frac{2}{\varepsilon}\left[(1-e^{-t})\,\|w\|_W + \sqrt{1-e^{-2t}}\int \|v\|_W\,\mu(dv)\right]$$
$$\leqq \frac{2}{\varepsilon}\left[(1-e^{-t})R + \sqrt{1-e^{-2t}}\int \|v\|_W\,\mu(dv)\right].$$

Consequently, we can choose $\tau_1>0$ so small that $0<t\leqq\tau_1$ implies that $T_tF(w)>2/3$ for each $w\in A$.

On the other hand, if $w\in W\setminus A_\varepsilon$, we have

$$\begin{aligned}0\leqq T_tF(w) &= \int F(e^{-t}w+\sqrt{1-e^{-2t}}\,v)\,\mu(dv)\\ &\leqq \int I_{A_{\varepsilon/2}}(e^{-t}w+\sqrt{1-e^{-2t}}\,v)\,\mu(dv)\\ &= \mu(\{v\in W;\, e^{-t}w+\sqrt{1-e^{-2t}}\,v\in A_{\varepsilon/2}\})\\ &= \mu\big((1-e^{-2t})^{-1/2}(A_{\varepsilon/2}-e^{-t}w)\big),\end{aligned}$$

where $I_{A_{\varepsilon/2}}$ is the indicator function of the set $A_{\varepsilon/2}$. Take $\tau_2>0$ so that

$$R\{\exp(\tau_2)-1\}<\frac{\varepsilon}{6} \quad\text{and}\quad \exp(-\tau_2)>\frac{3}{4}. \tag{2.9}$$

Then we see easily that $\operatorname{dist}(e^{-t}w, A_{\varepsilon/2})>\dfrac{\varepsilon}{4}$ for $0<t\leqq\tau_2$ and $w\in W\setminus A_\varepsilon$. Hence we have

$$0\leqq T_tF(w)\leqq\mu(\{v\in W;\,\|v\|_W\geqq(1-e^{-2t})^{-1/2}\cdot\frac{\varepsilon}{4}\}),\quad 0<t\leqq\tau_2,\ w\in W\setminus A_\varepsilon.$$

Suppose that R' is a positive number such that $\mu(\{v\in W;\,\|v\|_W\geqq R'\})<1/3$. Then taking $\tau_3>0$ so that $0<t\leqq\tau_3$ implies $(1-e^{-2t})^{-1/2}\cdot\dfrac{\varepsilon}{4}\geqq R'$, we have $T_tF(w)<1/3$ for $0<t\leqq\tau_2\wedge\tau_3$ and $w\in W\setminus A_\varepsilon$.

Therefore if $0<t\leqq\tau_1\wedge\tau_2\wedge\tau_3$, it holds that $T_tF(w)>2/3$ for $w\in A$ and that $T_tF(w)<1/3$ for $w\in W\setminus A_\varepsilon$. Now we take a function $\phi\colon \boldsymbol{R}\to\boldsymbol{R}$ satisfying

$$\begin{cases}\phi \text{ is of } C^\infty\text{-class},\ 0\leqq\phi(x)\leqq 1,\ x\in\boldsymbol{R},\\ \phi(x)=1 \text{ if } x>2/3 \quad\text{and}\quad \phi(x)=0 \text{ if } x<1/3.\end{cases} \tag{2.10}$$

We finally define F_ε by using this function as follows;

$$F_\varepsilon(w) := \phi(T_tF(w)),\quad t := \tau_1\wedge\tau_2\wedge\tau_3,\ w\in W.$$

It is easy to see that F_ε satisfies all the conditions (i)$\sim$(iv).

Step 2: *The general case*; Let $A_n := \{w\in A;\, n-1\leqq\|w\|_W\leqq n\}$ and $F_n(w) := 0\vee(1-\dfrac{2}{\varepsilon}\operatorname{dist}(w, A_n))$. Since each A_n is bounded, we can take $t_n>0$ so that $F_{n,\varepsilon}(w) := \phi(T_{t_n}F_n(w))$ satisfies all the required conditions (i)$\sim$(iv) for A_n, $n=$

1, 2, ⋯. It easily follows from (2.8) (2.9) in the proof of Step 1 that $t_n=O(1/n)$ will do for large n. Now we consider the following sum.

$$\sum_{n=1}^{\infty} F_{n,\varepsilon}(w), \quad w\in W$$

Note that it is actually a sum of finite terms for each $w\in W$. Clearly it is bounded and continuous. Moreover we can show that it is an element of $\boldsymbol{D}^{+\infty}$. What we will prove for this is that $\sum_{n=1}^{\infty}\|F_{n,\varepsilon}\|_{p,k}<\infty$ for each $1<p<\infty$ and $k\in\boldsymbol{N}$.

To this end, it is enough to see that

$$\sum_{n=1}^{\infty}\|F_{n,\varepsilon}\|_{L_p(\mu)}<\infty, \quad 1<p<\infty, \tag{2.11}$$

$$\sum_{n=1}^{\infty}\|D^k F_{n,\varepsilon}\|_{L_p(\mu;\,\mathcal{H}^k(\boldsymbol{R}))}<\infty, \quad 1<p<\infty,\ k\in\boldsymbol{N}. \tag{2.12}$$

The latter condition may be replaced by the following.

$$\sum_{n=1}^{\infty}\|D^k T_{t_n} F_n\|_{L_p(\mu;\,\mathcal{H}^k(\boldsymbol{R}))}<\infty, \quad 1<p<\infty,\ k\in\boldsymbol{N}. \tag{2.13}$$

Indeed, suppose (2.13) holds and let $k=2$ for example. (The same method works for any other k.) By the chain rule,

$$D^2F_{n,\varepsilon}=\phi'(T_{t_n}F_n)\,D^2T_{t_n}F_n+\phi''(T_{t_n}F_n)\,DT_{t_n}F_n\otimes DT_{t_n}F_n$$

and it holds by the Hölder inequality that

$$\begin{aligned}\|D^2F_{n,\varepsilon}\|_{L_p(\mu;\,\mathcal{H}^2(\boldsymbol{R}))}\leqq{}&\|\phi'(T_{t_n}F_n)\|_{L_{2p}(\mu)}\|D^2T_{t_n}F_n\|_{L_{2p}(\mu;\,\mathcal{H}^2(\boldsymbol{R}))}+\\&+\|\phi''(T_{t_n}F_n)\|_{L_{3p}(\mu)}\|DT_{t_n}F_n\|^2_{L_{3p}(\mu;\,H)}.\end{aligned} \tag{2.14}$$

Observing that

$$\|\phi'(T_{t_n}F_n)\|_{L_{2p}(\mu)}\leqq\sup_{x\in\boldsymbol{R}}|\phi'(x)|, \quad \|\phi''(T_{t_n}F_n)\|_{L_{3p}(\mu)}\leqq\sup_{x\in\boldsymbol{R}}|\phi''(x)|$$

and that $\sum_{n=1}^{\infty}\|DT_{t_n}F_n\|_{L_{3p}(\mu)}<\infty$ implies $\sum_{n=1}^{\infty}\|DT_{t_n}F_n\|^2_{L_{3p}(\mu)}<\infty$, we can conclude that $\sum_{n=1}^{\infty}\|D^2F_{n,\varepsilon}\|_{L_p(\mu)}<\infty$.

(2.11) is obvious. So we have to prove (2.13) only, but it is also easy. In fact, the estimate (2.3) says that

$$\begin{aligned}\sum_{n=1}^{\infty}\|D^kT_{t_n}F_n\|_{L_p(\mu;\,\mathcal{H}^k(\boldsymbol{R}))}&\leqq c'''\sum_{n=1}^{\infty}t_n^{-k/2}\|F_n\|_{L_p(\mu)}\\&\leqq c'''\sum_{n=1}^{\infty}t_n^{-k/2}\,\mu(\{w\in W;\,F_n(w)>0\})\\&\leqq c'''\sum_{n=1}^{\infty}t_n^{-k/2}\,\mu(\{w\in W;\,\|w\|_W>n-1-\varepsilon\}).\end{aligned}$$

Since $t_n=O(1/n)$ and $\mu(\{w\in W;\,\|w\|_W>n-1-\varepsilon\})=O(e^{-\alpha n^2})$ for some $\alpha>0$ by a theorem of Fernique [1] (cf. also [8]), the above sum is finite. Thus $\sum_{n=1}^{\infty}F_{n,\varepsilon}\in\boldsymbol{D}^{+\infty}$ is valid.

We finally define F_ε by $F_\varepsilon(w):=\phi\big(\sum_{n=1}^{\infty}F_{n,\varepsilon}(w)\big)$, $w\in W$. It obviously

satisfies all the required conditions. Q.E.D.

3. Capacities over an abstract Wiener space

In this section, we will summarize the properties of capacities which are defined in accordance with our Sobolev spaces $\boldsymbol{D}_p^r$. Let us start with their definition following Malliavin [10], Fukushima-Kaneko [4] and others.

DEFINITION 3.1. Let $1<p<\infty$ and $r>0$. For an open set O of W, we define its (p, r)-*capacity* $C_p^r(O)$ by

$$C_p^r(O) := \inf \{\|U\|_{p,r}^p;\ U\in\boldsymbol{D}_p^r,\ U\geqq 1\ \mu\text{-a.e. on } O\} .$$

For each subset A of W, we define its (p, r)-*capacity* $C_p^r(A)$ by

$$C_p^r(A) := \inf \{C_p^r(O);\ O \text{ is open and } O\supset A\} .$$

These capacities were originally introduced to discuss the regularity of functions of $\boldsymbol{D}_p^r$ ([10]). They are more subtle scales to estimate the size of sets than μ, i.e., a set of (p, r)-capacity zero is always μ-measure zero, but the converse is false in general. Now let us introduce some terms and notations to describe some known results exactly. We will use a term "(p, r)-*quasi-everywhere*" or simply "(p, r)-*q.e.*" to mean "except on a set of (p, r)-capacity zero". If (p, r)-capacity of a set A vanishes for every $1<p<\infty$ and every $r>0$, the set A is said to be *slim*. By "∞-*quasi-everywhere*" or simply "∞-*q.e.*", we mean "except on a slim set". A function $F: W\to\boldsymbol{R}$ is said to be (p, r)-*quasi-continuous*, if for any $\varepsilon>0$ there exists an open set O with $C_p^r(O)<\varepsilon$ such that F is continuous on the complement $W\setminus O$ of O. If F is (p, r)-quasi-continuous for every $1<p<\infty$ and every $r>0$, F is said to be ∞-*quasi-continuous*.

Lemma 3.1.

(A) *Let* $1<p<\infty$ *and* $r>0$.

(i) *For each* $F\in\boldsymbol{D}_p^r$, *there exists a function* $\widetilde{F}$ *such that* $\widetilde{F}=F$ μ-*a.e. and* $\widetilde{F}$ *is* (p, r)-*quasi-continuous.* $\widetilde{F}$ *is uniquely defined* (p, r)-*q.e.*

(ii) *Let* $F\in\boldsymbol{D}_p^r$ *and* $F_n\in\boldsymbol{D}_p^r$, $n=1, 2, \cdots$. *If* F_n *converges to* F *in* $\boldsymbol{D}_p^r$, *we can take a suitable subsequence* $\{F_{n_j}\}$ *of* $\{F_n\}$ *so that* $\widetilde{F}_{n_j}$ *converges to* $\widetilde{F}$ (p, r)-*q.e.* (*Here the symbol "~" stands for the* (p, r)-*quasi-continuous version stated in* (i).)

(iii) *If* $F\in\boldsymbol{D}_p^r$ *is non-negative* μ-*a.e., then* $\widetilde{F}$ *is non-negative* (p, r)-*q.e.*

(B) (i) *For each* $F\in\boldsymbol{D}^{+\infty}$, *there exists a function* $\widetilde{F}$ *such that* $\widetilde{F}=F$ μ-*a.e. and* $\widetilde{F}$ *is* ∞-*quasi-continuous.* $\widetilde{F}$ *is uniquely defined* ∞-*q.e.*

(ii) *Let* $F\in\boldsymbol{D}^{-\infty}$ *and* $F_n\in\boldsymbol{D}^{+\infty}$, $n=1, 2, \cdots$. *If* F_n *converges to* F *in* $\boldsymbol{D}^{+\infty}$, *we can take a suitable subsequence* $\{F_{n_j}\}$ *of* $\{F_n\}$ *so that* $\widetilde{F}_{n_j}$ *converges to* $\widetilde{F}$ ∞-*q.e.* (*Here the symbol "~" stands for the* ∞-*quasi-continuous version stated in* (i).)

(iii) *If* $F\in\boldsymbol{D}^{+\infty}$ *is non-negative* μ-*a.e., then* $\widetilde{F}$ *is non-negative* ∞-*q.e.*

For the proof, see [2], [4] and [10].

From now on, the (p,r)-(*or* ∞-) *quasi-continuous version of* $F\in\boldsymbol{D}_p^r$ (*or* $\boldsymbol{D}^{+\infty}$) *will be denoted by* $\tilde{F}$ *as Lemma* 3.1, *and the pair of indices* p *and* r *will be arbitrarily fixed so that* $1<p<\infty$ *and* $r>0$.

The following is another result of Fukushima-Kaneko [4].

Lemma 3.2.

(i) *For an arbitrary subset* A *of* W, *it holds that*
$C_p^r(A)=\inf\{\|U\|_{p,r}^p;\ U\in\boldsymbol{D}_p^r \text{ and } \tilde{U}\geqq 1\ (p,r)\text{-q.e. on } A\}$.

(ii) *There exists a unique element* $U_A=U_{A;p,r}\in\boldsymbol{D}_p^r$ *such that*
$\tilde{U}_A\geqq 1\ (p,r)$-*q.e. on* A *and* $C_p^r(A)=\|U_A\|_{p,r}^p$.

(iii) $\tilde{U}_A\geqq 0\ (p,r)$-*q.e.*

DEFINITION 3.2. U_A of Lemma 3.2 is called the (p,r)-*equilibrium potential or* the (p,r)-*capacity potential* of the set A.

U_A is clearly non-zero if $C_p^r(A)>0$. The equilibrium potentials will play a fundamental role in the subsequent sections.

Now our next aim is to establish the *capacitability* of Borel sets by compact sets. Namely, we will show the following.

Theorem 3.1. *For* $B\in\mathcal{B}(W)$, *it holds that*

$$C_p^r(B)=\sup\{C_p^r(K);\ K\subset B,\ K \text{ is compact}\}\ .$$

This theorem can not be proved by directly applying the general theory of Choquet (cf. [11]), because W is not locally compact if $\dim W=\infty$. However, we can instead apply the following lemma which asserts the *tightness* of capacities, i.e., capacities are almost supported by compact sets. It can be seen in [5] with an uncompleted proof. (In the case of the *Dirichlet space,* i.e., $p=2$ and $r=1$, it has been rigorously proved by Kusuoka [9] and Takeda [19].)

Lemma 3.3. *Given an arbitrary* $\varepsilon>0$, *there exists a compact set* K *of* W *such that* $C_p^r(W\setminus K)<\varepsilon$.

Proof. *Step* 1; First we will construct a Wiener function F: $W\to\boldsymbol{R}\cup\{\infty\}$ (we allow the value ∞ for convenience's sake) satisfying the following conditions.

$$\text{(3.1)}\quad\begin{cases}\text{(i)}\ \ F\in\boldsymbol{D}^{+\infty}\\ \text{(ii)}\ \ \text{For each } M>0,\text{ the set } \{w\in W;\ F(w)\leqq M\} \text{ is}\\ \qquad\text{relatively compact.}\\ \text{(iii)}\ \ \text{For each } \varepsilon>0,\text{ there exists } M>0 \text{ such that}\\ \qquad\mu(\{w\in W;\ F(w)>M\})<\varepsilon\ .\end{cases}$$

Actually, the function $T_t||\cdot||_{W_1}$, $t>0$, (recall Lemma 1.1) is a candidate. It belongs to $\boldsymbol{D}^{+\infty}$ by virtue of Lemma 1.1 (iii) and Lemma 2.2 (ii). $t>0$ being fixed, let us specify one of its versions by

$$F(w) := \begin{cases} T_t||w||_{W_1} = \int_{W_1} ||e^{-t}w+\sqrt{1-e^{-2t}}\,v||_{W_1}\,\mu(dv), & w\in W_1 \\ \infty\,, & w\in W\setminus W_1 \end{cases} \tag{3.2}$$

Suppose $w\in W_1$. Since we have

$$\begin{aligned} |T_t||w||_{W_1}-||w||_{W_1}| &\leqq \int_{W_1} |\,||e^{-t}w+\sqrt{1-e^{-2t}}\,v||_{W_1}-||w||_{W_1}|\,\mu(dv) \\ &\leqq (1-\delta_1)\,||w||_{W_1}+\delta_2\,, \end{aligned}$$

where we put $\delta_1:=1-e^{-t}$ and $\delta_2:=\sqrt{1-e^{-2t}}\int_{W_1}||v||_{W_1}\,\mu(dv)$, it holds that

$$(1-\delta_1)\,||w||_{W_1}-\delta_2 \leqq T_t||w||_{W_1} \leqq (1+\delta_1)\,||w||_{W_1}+\delta_2\,, \quad w\in W_1\,. \tag{3.3}$$

Therefore we see that $\{w\in W;\ F(w)\leqq M\}$ is contained in $\{w\in W;\ ||w||_{W_1}\leqq (M+\delta_2)/(1-\delta_1)\}$, and hence it is relatively compact by Lemma 1.1 (i). On the other hand, (3.3) also implies that $\{w\in W;\ F(w)>M\}$ is contained in $\{w\in W;\ ||w||_{W_1}>(M-\delta_2)/(1+\delta_1)\}$, and hence it follows from Lemma 1.1 (ii) that $\mu(\{w\in W;\ F(w)>M\})$ can be arbitrarily small if we take M sufficiently large. Thus the function F satisfies all the required conditions (3.1).

Step 2; Take a family of functions $\{\phi_R\}_{R>0}$ so that

$$\begin{cases} \phi_R\in C^\infty(\boldsymbol{R}\cup\{\infty\}\to\boldsymbol{R}),\ 0\leqq\phi_R(x)\leqq 1,\ x\in\boldsymbol{R},\ \phi_R(\infty)=1\,, \\ \phi_R(x)=1 \quad \text{if} \quad |x|>R+\delta_3\ (\delta_3>0 \text{ being fixed}), \\ \phi_R(x)=0 \quad \text{if} \quad |x|<R\,, \\ \sup_{x\in\boldsymbol{R}}\sup_{R>0}\left|\left(\frac{d}{dx}\right)^n\phi_R(x)\right| =: M_n<\infty, \quad n=1,2,\cdots\,, \end{cases} \tag{3.4}$$

and define $F_R\in\boldsymbol{D}^{+\infty}$, $R>0$, by

$$F_R(w) := \phi_R(F(w))\,, \quad w\in W\,, \tag{3.5}$$

where F is the function defined by (3.2). Then we have the following estimate

$$||F_R||_{p,k}\leqq c\ \mu(\{w\in W;\ F(w)>R\})\,. \tag{3.6}$$

Here $k\in\boldsymbol{N}$ and $c>0$ is a constant independent of R. Let us prove (3.6) when $k=2$, for example. (The same method works in other cases.) As was remarked in section 1, it is sufficient to show the following;

$$||F_R||_{L_p(\mu)}\leqq c_1\ \mu(\{w\in W;\ F(w)>R\})\,, \tag{3.7}$$

$$\|D^2F_R\|_{L_p(\mu\,;\,\mathcal{H}^2(\boldsymbol{R}))} \leqq c_2\,\mu(\{w\in W;\, F(w)>R\})\,. \tag{3.8}$$

The former is obvious for $c_1=1$. To prove the latter, we use the chain rule and the Hölder inequality. That is, we get the following estimate in the same way as (2.14).

$$\|D^2F_R\|_{L_p(\mu\,;\,\mathcal{H}^2(\boldsymbol{R}))} \leqq \|\phi_R'(F)\|_{L_{2p}(\mu)}\|D^2F\|_{L_{2p}(\mu\,;\,\mathcal{H}^2(\boldsymbol{R}))} + \\ + \|\phi_R''(F)\|_{L_{3p}(\mu)}\|DF\|^2_{L_{3p}(\mu\,;\,H)}$$

Consequently, we have (3.8) for

$$c_2 = M_1\|D^2F\|_{L_{2p}(\mu\,;\,\mathcal{H}^2(\boldsymbol{R}))} + M_2\|DF\|^2_{L_{3p}(\mu\,;\,H)}\,.$$

Step 3; Now let K_R be the topological closure of the set $\{w\in W;\, F(w)\leqq R+\delta_3\}$. K_R is a compact set on account of (3.1) (ii). In addition, by the definition of (p,r)-capacity and its monotone property, we have $C_p^r(W\backslash K_R)\leqq C_p^r(\{w\in W;\, F(w)>R+\delta_3\})\leqq\|F_R\|_{p,r}^p$. Observe that this inequality is still valid when we replace $\|F_R\|_{p,r}$ by $\|F_R\|_{p,k}$ provided k is a positive integer not less than r. On the other hand, (3.6) and (3.1) (iii) implies that $\|F_R\|_{p,k}$ can be made arbitrarily small by taking R sufficiently large. Q.E.D.

REMARK. In the case of the classical Wiener space (Example 1.1), it can be verified that a functional $F(w)$ defined by

$$F(w) := \|w\|^{2m}_{\alpha,2m} = \int_0^T\int_0^T \frac{|w(t)-w(s)|^{2m}}{|t-s|^{1+2m\alpha}}\,dsdt,\quad m\in\boldsymbol{N},\ 0<\alpha<1/2,\ 2m\alpha>1\,,$$

satisfies all the conditions of (3.1).

Corollary. *$W\backslash W_1$ is a slim set.*

Proof. Obvious from $C_p^r(W\backslash W_1)\leqq C_p^r(\{w\in W;\, F(w)>R+\delta_3\})$. Q.E.D.

Proof of Theorem 3.1. Let $\mathcal{C}$ be the set of all compact sets of W. The followings hold.

(i) If K_1, $K_2\in\mathcal{C}$ and $K_1\subset K_2$, then $C_p^r(K_1)\leqq C_p^r(K_2)$.

(ii) If $K_n\in\mathcal{C}$, $K_n\subset K_{n+1}$, $n=1,2,\cdots$, then $C_p^r(\cup K_n)=\sup_n C_p^r(K_n)$.

(iii) If $K_n\in\mathcal{C}$, $K_n\supset K_{n+1}$, $n=1,2,\cdots$, then $C_p^r(\cap K_n)=\inf_n C_p^r(K_n)$.

(i) is obvious and (ii) was proved by Fukushima-Kaneko [4] without compactness. To show (iii), it is enough to see that $C_p^r(\cap K_n)\geqq\inf_n C_p^r(K_n)$. For an arbitrary $\varepsilon>0$, we can find an open set O such that $O\supset\cap K_n$ and that $C_p^r(O)\leqq C_p^r(\cap K_n)+\varepsilon$. But since K_n's are compact, $K_n\subset O$ holds for sufficiently large n. Consequently, we have $C_p^r(\cap K_n)\geqq\inf_n C_p^r(K_n)$.

Let $B\in\mathcal{B}(W)$ and ε be an arbitrary positive number again. We first take

a compact set K so that $C_p^r(W\backslash K)<\varepsilon/2$. Since $B\cap K$ is a Borel set of K in the induced topology, it is $\mathcal{C}$-analytic. Therefore the properties (i) (ii) and (iii) stated above assure that there exists a compact set $K'\subset B\cap K$ such that $C_p^r(B\cap K)\leqq C_p^r(K')+\varepsilon/2$ ([12], III, T19). Then we have, by the subadditivity of the capacity, $C_p^r(B)\leqq C_p^r(B\cap K)+C_p^r(W\backslash K)<C_p^r(K')+\varepsilon$, which completes the proof. Q.E.D.

4. Positive generalized Wiener functions

The following is well-known; Positive Schwartz distributions are measures. Namely, let $T\in\mathcal{D}'(\boldsymbol{R}^n)$ satisfy that $(T,f)\geqq 0$ for each $f\in\mathcal{D}(\boldsymbol{R}^n)$ which is non-negative at every point. Then there exists a unique positive Radon measure ν on $\boldsymbol{R}^n$ such that $(T,f)=\int f(x)\,\nu(dx)$ for each $f\in\mathcal{D}(\boldsymbol{R}^n)$ ([15]). In this section, we will claim that an analogous theorem holds replacing T by a GWF Φ, positive in the sense of Definition 4.1 below, and $\boldsymbol{R}^n$ by the Banach space W. Then the corresponding measure will be a Borel measure on W. After proving it, the relations between the capacities and measures corresponding to positive GWF's will be revealed.

Before entering into the subject, let us fix one more notation; As in the previous section, we assume that the indices p and r are given and fixed such that $1<p<\infty$ and $r>0$ respectively. *In addition, we will fix an index* q, $1<q<\infty$, *to denote the dual index of* p, *i.e., such that* $1/p+1/q=1$ *from now on.*

DEFINITION 4.1. Let Φ be a GWF (i.e., an element of $\boldsymbol{D}^{-\infty}$). We say Φ is positive, if it holds that $(\Phi,F)\geqq 0$ for each $F\in\boldsymbol{D}^{+\infty}$ such that $F(w)\geqq 0$ μ-a.e.$w\in W$.

Positive GWF is abbreviated to PGWF, and is denoted by $\Phi\geqq 0$. We remark that if $\Phi\in\boldsymbol{D}_p^{-r}$, we have "$\Phi\geqq 0$ *if and only if* $(\Phi,F)\geqq 0$ *for each* $F\in\boldsymbol{D}_p^r$ *such that* $F(w)\geqq 0$ μ-*a.e.*$w\in W$." In particular, if $\Phi\in L_q(\mu)$, then we have "$\Phi\geqq 0$ *if and only if* $\Phi(w)\geqq 0$ μ-*a.e.*$w\in W$." For the proof, use Lemma 2.4.

Theorem 4.1. *For each PGWF* Φ, *there exists a unique finite positive measure* ν_Φ *on* $(W,\mathcal{B}(W))$ *such that*

$$(\Phi,F)=\int F(w)\,\nu_\Phi(dw) \tag{4.1}$$

for all $F\in\mathcal{F}C_b^\infty$. *Here the space* $\mathcal{F}C_b^\infty$ *is defined by*

$$\mathcal{F}C_b^\infty:=\{F\in\boldsymbol{D}^{+\infty};\ F(w)=f((l_1,w),\cdots,(l_n,w)),\ w\in W,\ \text{for some } n\in\boldsymbol{N},\ l_1,\cdots,l_n\in W^*\ \text{and } f:\boldsymbol{R}^n\to\boldsymbol{R}\ \text{which is bounded and of } C^\infty\text{-class}\}\ .$$

REMARK. $\mathcal{F}C_b^\infty$ is a dense subspace of $\boldsymbol{D}^{+\infty}$.

Proof. Without a loss of generality, we may assume that $(\Phi, 1)=1$ and then we can use the probabilistic terminology. (Note that $1\in \boldsymbol{D}^{+\infty}$ and that the condition $(\Phi, 1)=1$ assures the total mass of ν_Φ to be equal to one, if it exists.)

Let $l_1, \cdots, l_n \in W^*$ be arbitrarily chosen. For $f, f_j \in \mathcal{D}(\boldsymbol{R}^n)$, $j=1, 2, \cdots$, we put

$$F(w) := f((l_1, w), \cdots, (l_n, w)), \quad w\in W,$$
$$F_j(w) := f_j((l_1, w), \cdots, (l_n, w)), \quad w\in W, \quad j=1, 2, \cdots.$$

It is clear that $F, F_j \in \boldsymbol{D}^{+\infty}$ and that if f_j converges to f in $\mathcal{D}(\boldsymbol{R}^n)$ as $j\to\infty$, F_j converges to F in $\boldsymbol{D}^{+\infty}$. Then since Φ is a GWF, it follows that (Φ, F_j) converges to (Φ, F), which in turn implies the continuity of a mapping $\overline{\Phi}: \mathcal{D}(\boldsymbol{R}^n) \ni f \mapsto (\Phi, F)\in \boldsymbol{R}$. Of course $(\Phi, F)\geqq 0$ for $f\geqq 0$ and hence $\overline{\Phi}$ is a positive distribution. Therefore it follows from the Schwartz theorem that there exists a unique Radon measure $\nu_{\Phi; l_1, \cdots, l_n}$ on $\boldsymbol{R}^n$ such that

$$(\Phi, G) = \int_{\boldsymbol{R}^n} g(x_1, \cdots, x_n)\, \nu_{\Phi; l_1, \cdots, l_n}(dx_1 \cdots dx_n), \tag{4.2}$$

for all $G(w)=g((l_1, w), \cdots, (l_n, w))$, $g\in \mathcal{D}(\boldsymbol{R}^n)$. It is easy to see that (4.2) holds for all $G(w)=g((l_1, w), \cdots, (l_n, w))\in \mathcal{F}C_b^\infty$, by means of a smooth partition of unity over $\boldsymbol{R}^n$.

Thus we obtained a family of finite dimensional probability distributions $\{\nu_{\Phi; l_1, \cdots, l_n}; l_1, \cdots, l_n \in W^*, n\in \boldsymbol{N}\}$, which is obviously consistent. That is to say, ν_Φ is realized as a *cylindrical measure* on W. In order for ν_Φ to be countably additive on $\mathcal{B}(W)$, it is sufficient to show the following ([6] [8]).

$$\begin{cases} \text{For an arbitrary } \varepsilon>0, \text{ there exists a compact set } K_\varepsilon \\ \text{such that } \nu_\Phi(C)<\varepsilon \text{ for any cylinder set } C \text{ with } C\cap K_\varepsilon = \phi. \end{cases} \tag{4.3}$$

Now let us assume $\Phi\in \boldsymbol{D}_q^{-k}$ for a positive integer k. Let F and F_R be the functions defined in the proof of Lemma 3.3. Then take R sufficiently large so that

$$||\Phi||_{q,-k}||F_R||_{p,k}<\varepsilon. \tag{4.4}$$

This is possible by virtue of (3.6). We know from (3.3) and (3.4) that taking sufficiently large $R'(R'>(1+\delta_2)(R+\delta_2+\delta_3)/(1-\delta_1)+\delta_2$ will do), the two sets $\{w\in W; F(w)\leqq R+\delta_3\}$ and $\{w\in W; F(w)>R'\}$ are separated by two W_1-balls both centered at the origin but with distinct radii. Namely, the former lies inside the smaller W_1-ball, say B_1, and the latter lies outside the bigger W_1-ball, say B_2.

Let K_ε be the closure of the set $\{w\in W; F(w)\leqq R'\}$. We already know that K_ε is compact. Suppose that a cylinder set C with an expression $C=\{w\in W; ((l_1, w), \cdots, (l_n, w))\in E_n\}$, $E_n\in \mathcal{B}(\boldsymbol{R}^n)$, $l_1, \cdots, l_n\in W^*$, does not intersect K_ε. Let

$C^\rho=\{w\in W;\ ((l_1, w), \cdots, (l_n, w))\in E_n^\rho\}$, where $E_n^\rho\in\mathcal{B}(\boldsymbol{R}^n)$ is the ρ-neighborhood of E_n, $\rho>0$. Then we have $C^\rho\cap\{w\in W;\ F(w)\leqq R+\delta_3\}=\phi$ for sufficiently small ρ. (This follows from the fact that the interior of the ball B_1 and the exterior of the ball B_2 are separated from each other with a positive distance when restricted to a finite dimensional subspace.) Now taking a function $\psi\in C^\infty(\boldsymbol{R}^n\to\boldsymbol{R})$ with properties

$$\begin{cases} 0\leqq\psi(x)\leqq 1,\ x\in\boldsymbol{R}^n, \\ \text{all its derivatives are bounded}, \\ \psi(x)=1 \quad \text{if} \quad x\in E_n, \\ \psi(x)=0 \quad \text{if} \quad x\in\boldsymbol{R}^n\backslash E_n^\rho, \end{cases}$$

we put $G(w):=\psi((l_1, w), \cdots, (l_n, w))$. Obviously $G\in\mathcal{F}C_b^\infty$. Therefore we see

$$\nu_\Phi(C)\leqq\int_{\boldsymbol{R}^n}\psi(x_1, \cdots, x_n)\,\nu_{\Phi;l,\cdots,l_n}(dx_1\cdots dx_n)=(\Phi, G)\,.$$

On the other hand, that $F_R(W)\geqq G(w)$, $w\in W$, and the positivity of Φ imply that $(\Phi, G)\leqq(\Phi, F_R)\leqq\|\Phi\|_{q,-k}\|F_R\|_{p,k}$. But the last term is smaller than ε, hence we have $\nu_\Phi(C)<\varepsilon$. Q.E.D.

The measure corresponding to a PGWF Φ by this theorem will be denoted by ν_Φ in the sequel.

REMARK (i). In the case of the classical Wiener space (Example 1.1), the countable additivity of ν_Φ on $\mathcal{B}(W)$ can be directly proved by the following inequality.

$$\int_{\boldsymbol{R}^d\times\boldsymbol{R}^d}|x_1-x_2|^4\,\nu_{\Phi;t_1,t_2}(dx_1\,dx_2)\leqq c|t_1-t_2|^2,\quad 0\leqq t_1, t_2\leqq T\,, \tag{4.5}$$

where $\nu_{\Phi;t_1,t_2}$ denotes the joint distribution of $(w(t_1), w(t_2))\in\boldsymbol{R}^d\times\boldsymbol{R}^d$ under the cylindrical measure ν_Φ, and $c>0$ is a constant independent of t_1 and t_2. To show (4.5), we first claim that the left-hand side of (4.5) is equal to $(\Phi, |w(t_1)-w(t_2)|^4)$. Then noting that $|w(t_1)-w(t_2)|^4\in\boldsymbol{P}_4$, we have

$$\begin{aligned}(\Phi, |w(t_1)-w(t_2)|^4)&\leqq\|\Phi\|_{q,-r}\||w(t_1)-w(t_2)|^4\|_{p,r}\\ &\leqq\|\Phi\|_{q,-r}\,c_1\||w(t_1)-w(t_2)|^4\|_{L_p(\mu)}\\ &\leqq\|\Phi\|_{q,-r}\,c_2\||w(t_1)-w(t_2)|^4\|_{L_2(\mu)}\\ &=\|\Phi\|_{q,-}, c_3|t_1-t_2|^2\,.\end{aligned}$$

Here the constants c_1, c_2 and c_3 are all positive and independent of t_1 and t_2. (For the proof of these inequalities, see [17] Lem. 1.1, Th. 2.3.)

REMARK (ii). If a PGWF Φ is an element of $\boldsymbol{D}_q^{-1}$, we need not take the regularization $T_t\|\cdot\|_{W_1}$ of $\|\cdot\|_{W_1}$, because $\|\cdot\|_{W_1}$ itself belongs to $\boldsymbol{D}_p^1$ ([9][18]).

Therefore we can proceed the above proof replacing (3.5) by

$$(3.5)' \qquad F_R(w) := \begin{cases} \{0 \vee (\delta_3^{-1}(\|w\|_{W_1} - R))\} \wedge 1, & w \in W_1 \\ \infty, & w \in W \setminus W_1. \end{cases}$$

Now, note that μ need not be Gaussian in this case. Namely, only Lemma 1.1 (i) (ii) and (3.5)′ are needed to prove (4.3). Particularly, in the case of Dirichlet spaces over infinite dimensional spaces discussed by Kusuoka [9], we obtain the following.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet space over a Banach space B with a probability measure ν on it (not necessarily Gaussian) satisfying all the conditions of Theorem 2 of Kusuoka [9]. *Then, for each positive bounded linear functional Φ on $\mathcal{F}$ with respect to a bilinear form $\mathcal{E}_1(=\mathcal{E}+$the inner product of $L_2(\nu; B \to \boldsymbol{R}))$, there exists a unique positive finite Borel measure ν_Φ on B such that an analogy of Theorem 4.1 holds.*

EXAMPLE 4.1. Let $F=(F^1, \cdots, F^d) \in \boldsymbol{D}^{+\infty}(\boldsymbol{R}^d)$. If F is *non-degenerate in Malliavin's sense,* i.e,, $\det(\langle DF^i, DF^j \rangle_H)^{-1} \in L_{\infty-}(\mu)$, we can give a rigorous meaning to the *pull back* $T \circ F \in \boldsymbol{D}^{-\infty}$ of any d-dimensional Schwartz tempered distribution T under the mapping F ([20] [21]). *Then if T is positive, and hence a positive measure, the pull back $T \circ F$ is a PGWF. If, in particular, $\delta_y \circ F \not\equiv 0$, the probability measure corresponding to a PGWF $\Phi_{F,y}$, $y \in \boldsymbol{R}^d$, defined by*

$$(4.6) \qquad \Phi_{F,y} := \delta_y \circ F / (\delta_y \circ F, 1),$$

where δ_y is the Dirac measure concentrated at y, is nothing but the conditional probability measure $\mu(\cdot \mid F=y)$ given $F=y$. Of course, this measure is singular with respect to μ.

Lemma 4.1. *Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of PGWF's belonging to $\boldsymbol{D}_q^{-r}$. We assume $(\Phi_n, 1)=1$, $n=1, 2, \cdots$.*

(i) *If $\sup_n \|\Phi_n\|_{q,-r} < \infty$, then the family of probability measures $\{\nu_{\Phi_n}\}_{n=1}^\infty$ is uniformly tight.*

(ii) *If Φ_n converges to some $\Phi \in \boldsymbol{D}_q^{-r}$ weakly in $\boldsymbol{D}_q^{-r}$ (i.e., (Φ_n, F) converges to (Φ, F) for each $F \in \boldsymbol{D}_p'$), then Φ is positive and ν_{Φ_n} converges to ν_Φ weakly.*

(iii) *If $\sup_n \|\Phi_n\|_{q,-r} < \infty$ and ν_{Φ_n} converges to some ν weakly, then there exists a unique PGWF $\Phi \in \boldsymbol{D}_q^{-r}$ such that $\nu = \nu_\Phi$ and Φ_n converges to Φ in $\boldsymbol{D}_q^{-r}$ weakly.*

Proof. (i) Taking an integer $k \geqq r$, we have $\sup_n \|\Phi_n\|_{q,-k} \leqq \sup_n \|\Phi_n\|_{q,-r} < \infty$. For an arbitrary $\varepsilon > 0$, take $R > 0$ so large that $\sup_n \|\Phi_n\|_{q,-k} \|F_R\|_{p,k} < \varepsilon$, where F_R is given by (3.2) and let K_ε be the compact set defincd in the proof of Theorem 4.1. Then we have $\nu_{\Phi_n}(C) < \varepsilon$ in the same way as Theorem 4.1 for

any $n\in\boldsymbol{N}$ and any cilynder set C with $C\cap K_\varepsilon=\phi$. Thus the family of the probability measures $\{\nu_{\Phi_n}\}_{n=1}^\infty$ is uniformly tight .(ii) $\Phi\geqq 0$ is obvious. Since $\sup_n ||\Phi_n||_{q,-r}<\infty$ follows from the weak convergence of Φ_n in $\boldsymbol{D}_q^{-r}$, we see $\{\nu_{\Phi_n}\}_{n=1}^\infty$ is uniformly tight by (i). Therefore we have only to check the convergence of each finite dimensional distribution of $\{\nu_{\Phi_n}\}_{n=1}^\infty$. But it is easy, because we can approximate any bounded continuous function defined on a finite dimensional space by C^∞-functions which are bounded together with their all derivatives. (iii) Since $\boldsymbol{D}_q^{-r}$ is a reflexive Banach space, every bounded set of it is weakly compact. Consequently, we can take an appropriate subsequence of $\{\Phi_n\}_{n=1}^\infty$ which converges to some $\Phi\in\boldsymbol{D}_q^{-r}$ weakly. Then $\nu=\nu_\Phi$ follows from (ii). Now the weak convergence of Φ_n to Φ in $\boldsymbol{D}_q^{-r}$ is clear.

Q.E.D.

Lemma 4.2. *Let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive. Putting $\Phi_n:=T_{1/n}\Phi$, $n=1, 2, \cdots$, we have*

(i) $\Phi_n\in L_q(\mu)$, $\Phi_n(w)\geqq 0$ μ-*a.e.*$w\in W$, $n=1, 2, \cdots$,

(ii) Φ_n *converges to* Φ *in* $\boldsymbol{D}_q^{-r}$ (*i.e., in the norm* $||\cdot||_{q,-r}$),

(iii) $(\Phi_n, 1)=(\Phi, 1)$, $n=1, 2, \cdots$.

Proof. That $\Phi_n\in L_q(\mu)$ and (ii) were already proved in Lemma 2.2 and Lemma 2.3 respectively. So we have to prove the positivity of Φ_n and (iii) only. We first prove $\Phi_n\geqq 0$. Let $F\in\boldsymbol{D}^{+\infty}$ be non-negative μ-a.e. Since $T_{1/n}$ is a symmetric operator on $L_2(\mu)$, it holds that $(\Phi_n, F)=(\Phi, T_{1/n}F)$. In addition, the positivity of $T_{1/n}$ leads us to conclude that $(\Phi, T_{1/n}F)\geqq 0$. Therefore Φ_n is a *PGWF*, which means $\Phi_n(w)\geqq 0$ μ-a.e. (iii) is also proved using the symmetry of $T_{1/n}$, i.e., $(\Phi_n, 1)=(T_{1/n}\Phi, 1)=(\Phi, T_{1/n}1)=(\Phi, 1)$. Q.E.D.

Lemma 4.1 and Lemma 4.2 give us a method to approximate the measure ν_Φ corresponding to a PGWF Φ by a sequence of measures which are absolutely continuous relative to the measure μ. Namely, let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive and $\{\Phi_n\}_{n=1}^\infty$ be as in Lemma 4.2. Then we have the followings.

(4.7) $\Phi_n\in L_q(\mu)$ are non-negative μ-a.e. and $\nu_{\Phi_n}(dw)=\Phi_n(w)\,\mu(dw)$.

(4.8) Φ_n converges to Φ in $\boldsymbol{D}_q^{-r}$.

(4.9) ν_{Φ_n} converges to ν_Φ weakly.

As an easy consequence of this, we can show that if $F\in\boldsymbol{D}_p^r$ is continuous and bounded, we have $(\Phi, F)=\int F(w)\,\nu_\Phi(dw)$. Indeed, it is sufficient to note that $(\Phi_n, F)\to(\Phi, F)$, $\int F(w)\,\nu_{\Phi_n}(dw)\to\int F(w)\,\nu_\Phi(dw)$ and that $(\Phi_n, F)=\int F(w)\,\Phi_n(w)\,\mu(dw)=\int F(w)\,\nu_{\Phi_n}(dw)$. This assertion is an improvement of Theorem 4.1, but it will be fully improved in the coming Theorem 4.3.

The following theorem is another consequence of the above argument, which directly connects PGWF's with the potential theory.

Theorem 4.2. *Let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive. Then its corresponding measure ν_Φ has no mass in any set of (p,r)-capacity zero. More precisely, we have*

$$\bar{\nu}_\Phi(A)\leqq\|\Phi\|_{q,-r}(C_p^r(A))^{1/p} \tag{4.10}$$

for any set A of W, where $\bar{\nu}_\Phi$ denotes the outer measure induced by ν_Φ. In particular, a measure corresponding to a PGWF never has its mass in slim sets.

Proof. Let $\Phi_n:=T_{1/n}\Phi$, $n=1, 2, \cdots$. We already know (4.7)$\sim$(4.9) hold. Let O and U_O be an arbitrary open set and its (p,r)-equilibrium potential respectively. By (4.7) and Lemma 3.2 (ii) (iii), we have

$$(\Phi_n, U_O)=\int_W \Phi_n U_O d\mu\geqq\int_O \Phi_n U_O d\mu\geqq\int_O \Phi_n d\mu=\nu_{\Phi_n}(O)\,.$$

On account of (4.9) and since O is open, we see $\varliminf_{n\to\infty}\nu_{\Phi_n}(O)\geqq\nu_\Phi(O)$. On the other hand, (4.8) implies that

$$\lim_{n\to\infty}(\Phi_n, U_O)=(\Phi, U_O)\leqq\|\Phi\|_{q,-r}\|U_O\|_{p,r}=\|\Phi\|_{q,-r}(C_p^r(O))^{1/p}\,.$$

Thus we have proved (4.10) for an open O. And hence (4.10) holds for any set A by Definition 3.1. Q.E.D.

Recall that $W\setminus W_1$ is a slim set (Corollary to Lemma 3.3). This fact together with Theorem 4.2 implies that $\nu_\Phi(W\setminus W_1)=0$, i.e., ν_Φ is actually supported by W_1. But of course it is quite trivial, because (W_1, H, μ) is again an abstract Wiener space. Thus ν_Φ is always supported by a Banach space contained in W which supports μ. So we can say ν_Φ is very close to μ in a sense, although it may be singular with respect to μ. In fact, in the case of the classical 1-dimensional Wiener space, Takeda [19] proved that ∞-quasi-all paths have nowhere differentiability, Lévy's Hölder continuity and they obey the law of iterated logarithm at $t=0$. Consequently, ν_Φ-almost all paths should possess all these properties as almost all Brownian paths do. Therefore, the class of measures corresponding to PGWF's is a rather small class in the totality of finite Borel measures on the path space W_0^1.

Let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive. Since ν_Φ may fail to be absolutely continuous with respect to μ, a Wiener function F need not be ν_Φ-measurable. However the pairing (Φ, F) for $F\in\boldsymbol{D}_p^r$ has a definite value, hence it is natural to guess that the integration of F by ν_Φ should be rigorously defined in a certain manner, and that its value should be equal to (Φ, F). In fact, this idea is realized in the following theorem.

Theorem 4.3.

(A) *Let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive. Then for each $F\in\boldsymbol{D}_p^r$, any (p,r)-quasi-continuous version $\widetilde{F}$ of F is ν_Φ-measurable and integrable. In addition, it holds that*

$$(\Phi, F) = \int \widetilde{F}(w)\,\nu_\Phi(dw)\,. \tag{4.11}$$

(B) *Let Φ be a PGWF. Then for each $F\in\boldsymbol{D}^{+\infty}$, any ∞-quasi-continuous version $\widetilde{F}$ of F is ν_Φ-measurable and integrable, and in addition,* (4.11) *holds.*

Proof. It is enough to prove (A) only.

Step 1: *The ν_Φ-measurability of $\widetilde{F}$*; By the definition of (p,r)-quasi-continuity, there exists a decreasing sequence of open sets $\{O_n\}_{n=1}^\infty$ such that $C_p^r(O_n)$ converges to zero and that $\widetilde{F}$ is continuous on $W\backslash O_n$, $n=1,2,\cdots$. Putting $F_n(w):=\widetilde{F}(w)\,I_{W\backslash O_n}(w)$, we see F_n is $\mathcal{B}(W)$-measurable. Because the (p,r)-capacity is a monotonous set function, we have $C_p^r(\cap O_n)=0$ and hence $\nu_\Phi(\cap O_n)=0$ by Theorem 4.2. Therefore the convergence of F_n to $\widetilde{F}$ on $W\backslash(\cap O_n)$ is a ν_Φ-a.e. convergence. Thus $\widetilde{F}$ is ν_Φ-measurable.

Step 2: *For $F\in\boldsymbol{D}^{+\infty}$ which is bounded μ-a.e.*; We first note that $|F|<M$ μ-a.e. implies $|\widetilde{F}|<M$ ∞-q.e. (Lemma 3.1 (B) (iii)), and hence $|\widetilde{F}|<M$ ν_Φ-a.e. Let $\{l_i\}_{i=1}^\infty$ be a CONS of H each l_i of which is taken from W^* and $\mathcal{B}_n$ be a sub σ-algebra of $\mathcal{B}(W)$ generated by linear functions $\{(l_i, w); i=1,2,\cdots,n\}$. Now we put $F_n:=E[F\,|\,\overline{\mathcal{B}}_n^\mu]$, $n=1,2,\cdots$, i.e., F_n is the conditional expectation of F with respect to $\overline{\mathcal{B}}_n^\mu$ under the probability μ. Then each F_n has a version $\widetilde{F}_n$ which belongs to $\mathcal{F}C_b^\infty$. (This is due to the Sobolev imbedding theorem.) Hence we have $\int \widetilde{F}_n d\nu_\Phi=(\Phi, F_n)$. On the other hand, we can prove that F_n converges to F in $\boldsymbol{D}^{+\infty}$ by the convergence theorem of martingales and the fact that $LF_n=E[LF\,|\,\overline{\mathcal{B}}_n^\mu]$ (see [10]). This implies that (Φ, F_n) converges to (Φ, F) and also that some subsequence $\widetilde{F}_{n_j}$ of $\widetilde{F}_n$ converges to $\widetilde{F}$ ∞-q.e. (Lemma 3.1 (B) (ii)). Then it holds that $\int \widetilde{F}_{n_j} d\nu_\Phi$ converges to $\int \widetilde{F} d\nu_\Phi$ by the bounded convergence theorem. Thus, we have (4.11) for $F\in\boldsymbol{D}^{+\infty}$ which is bounded μ-a.e.

Step 3: *For $F\in\boldsymbol{D}_p^r$ which is bounded μ-a.e.*; Let $F_n:=T_{1/n}F$, $n=1,2,\cdots$. Since F is bounded, each F_n is an element of $\boldsymbol{D}^{+\infty}$ (Lemma 2.2 (ii)) and if $|F|\leqq M$ μ-a.e., then $|F_n|\leqq M$ μ-a.e. (Lemma 2.1 (iv)) and hence $|\widetilde{F}_n|\leqq M$ (p,r)-q.e. (Lemma 3.1 (A) (iii)). Consequently, by the bounded convergence theorem, Lemma 3.1 (A) (ii), Lemma 2.3 and Step 2 above, (4.11) is valid for $F\in\boldsymbol{D}_p^r$ which is bounded μ-a.e.

Step 4: *For general $F\in\boldsymbol{D}_p^r$*; We put

$$F^+ := (I-L)^{-r/2}\left((I-L)^{r/2}F\vee 0\right),\quad F^- := (I-L)^{-r/2}\left((I-L)^{r/2}F\wedge 0\right).$$

The positivity of the operator $(I-L)^{-r/2}$ implies that $F^+\geqq 0$ and $F^-\leqq 0$ μ-a.e. Of course we have $F=F^+ + F^-$, hence $|F|\leqq F^+ - F^-$. Now let us show ν_Φ-

integrability of F^+ and (4.11) for F^+. To this end, we put $F_n^+:=(I-L)^{-r/2}\{((I-L)^{r/2}F\vee 0)\wedge n\}$, $n=1, 2, \cdots$. It is easy to see that F_n^+ converges to F^+ in $\boldsymbol{D}_p^r$. Moreover, the positivity of $(I-L)^{-r/2}$ again implies that $0\leqq F_n^+\leqq n$, $F_n^+\leqq F_{n+1}^+\leqq F^+$, μ-a.e., $n=1, 2, \cdots$. Therefore $\widetilde{F}_n^+$ converges non-decreasingly to $\widetilde{F}^+$ ν_Φ-a.e., and hence $\int \widetilde{F}_n^+ d\nu_\Phi$ converges to $\int \widetilde{F}^+ d\nu_\Phi$ (=possibly ∞). On the other hand, because F_n^+ is bounded, it follows from Step 3 that (4.11) holds for F_n^+. Combining these two with the convergence of (Φ, F_n^+) to (Φ, F^+), we conclude that $\widetilde{F}^+$ is ν_Φ-integrable and (4.11) holds for F^+. Getting the same result for F^-, we finish the proof. Q.E.D.

EXAMPLE 4.2. Let $\boldsymbol{C}:=C([0, \infty)\to\boldsymbol{R}^n)$ be the space of all n-dimensional continuous paths with the usual topology of uniform convergence on bounded time intervals. A continuous n-dimensional stochastic process X, defined on an abstract Wiener space (W, H, μ), is nothing but a $\boldsymbol{C}$-valued μ-measurable function. X is said to be *∞-quasi-continuous* if for every $1<p<\infty$, $r>0$ and $\varepsilon>0$, there exists an open set O in W such that $C_p^r(O)<\varepsilon$ and $W\backslash O\ni w\mapsto X(w)\in\boldsymbol{C}$ is continuous.

Let (W_0^d, H^1, P_0) be the classical d-dimensional Wiener space (Example 1.1) and $V_0, V_1, \cdots, V_d$ be vector fields on $\boldsymbol{R}^n$; $V_\alpha(x)=\sum_{i=1}^n V_\alpha^i(x)\dfrac{\partial}{\partial x_i}$, $\alpha=0, 1, \cdots, d$, where the coefficients $V_\alpha^i(x)$ are C^∞-functions whose derivatives of orders $\geqq 1$ are all bounded. Given $x\in\boldsymbol{R}^n$, an n-dimensional continuous process X_t is defined by the solution of the following stochastic differential equation (SDE);

$$\begin{cases} dX_t = \sum_{\alpha=1}^d V_\alpha(X_t)\, dw_t^\alpha + V_0(X_t)\, dt\,, \quad w=(w_t^1, \cdots, w_t^d)\in W_0^d\,, \\ X_0 = x\,. \end{cases}$$

Then the following assertion holds.

$X=(X_t)$ has an ∞-quasi-continuous version $\tilde{X}$ as a $\boldsymbol{C}$-valued μ-measurable function. $\tilde{X}$ is uniquely defined ∞-q.e.

To show this assertion, let $T>0$ be arbitrarily fixed. We define $\boldsymbol{C}_T:=C([0, T]\to\boldsymbol{R}^n)$ and $\boldsymbol{B}_T:=W^{\alpha,2m}(W^{\alpha,2m}$ and $\|\cdot\|_{\alpha,2m}$ below are those defined in Example 1.1. We also assume $0<\alpha<\dfrac{1}{2}$, $m\in\boldsymbol{N}$ and $2m\alpha>1$). $\boldsymbol{C}_T$ with the maximum norm and $\boldsymbol{B}_T$ with a norm $\|\cdot\|_{\boldsymbol{B}_T}:=[\|\cdot\|_{L_{2m}}^{2m}+\|\cdot\|_{\alpha,2m}^{2m}]^{1/2m}$ are Banach spaces. As we mentioned in Example 1.1, we can regard $\boldsymbol{B}_T$ as a continuously imbedded dense subspace of $\boldsymbol{C}_T$. Then we can verify that X is a $\boldsymbol{B}_T$-valued μ-measurable function and moreover, that the function $\|X(\cdot)\|_{\boldsymbol{B}_T}^{2m}: W\to\boldsymbol{R}$ is an element of $\boldsymbol{D}^{+\infty}$. Let $\{X_n\}=\{(X_{n,t})\}$ be a sequence of $\boldsymbol{B}_T$-valued continuous functions such that $\|X(\cdot)-X_n(\cdot)\|_{\boldsymbol{B}_T}^{2m}$ converges to 0 in $\boldsymbol{D}^{+\infty}$ (for example, take polygonal approximations. cf. [7] [21]). Then, for any pair of integers $m>n$, X_m-X_n is continuous and hence we have the following Chebyshev-type ine-

quality;

$$C_p^r(\{w\in W;\ ||X_m(w)-X_n(w)||_{B_T}^{2m}>\lambda\})\leqq\lambda^{-p}||\ ||X_m(\cdot)-X_n(\cdot)||_{B_T}^{2m}||_{p,r}^p.$$

Hence applying the similar argument of Fukushima [2], we can show that X has an ∞-quasi-continuous version $\tilde{X}$ as a $\boldsymbol{B}_T$-valued function, and therefore as a $\boldsymbol{C}_T$-valued function.

Now assume further that, for a fixed $T>0$, X_T is non-degenerate in the sense of Malliavin (cf. [21] for a sufficient condition in terms of the vector fields V_α). Then a PGWF $\delta_y\circ X_T$, $y\in\boldsymbol{R}^d$, is well-defined and, if furthermore $\delta_y\circ X_T\neq 0$, we have a Borel probability measure ν_Φ on W_0^d corresponding to $\Phi=\delta_y\circ X_T/(\delta_y\circ X_T, 1)$. Clearly, every $\boldsymbol{C}$-valued ∞-quasi-continuous function is ν_Φ-measurable, that is, it is a continuous process on the probability space (W_0^d, ν_Φ). Hence, for any solution Y of an SDE with regular coefficients as above, the process $X=(X_t)$ in particular, its ∞-quasi-continuous version $\tilde{Y}$ is a continuous process on the probability space (W_0^d, ν_Φ). This remark will give a new approach to pinned processes. Indeed, the process $\tilde{X}$ on (W_0^d, ν_Φ) is just the pinned process of X conditioned by $X_T=y$.

5. Equilibrium measures

Let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive. A function $U\in\boldsymbol{D}_p^r$ defined by (5.1) below is called the *(p, r)-potential of Φ or of ν_Φ*.

$$U=(I-L)^{-r/2}\{(I-L)^{-r/2}\Phi\}^{q-1}. \tag{5.1}$$

Observing that $(I-L)^{-r/2}\Phi\in L_q(\mu)$ and it is non-negative μ-a.e., we know U is well-defined as a non-negative element of $\boldsymbol{D}_p^r$. If $p=q=2$ and $r=1$, (5.1) becomes $U=(I-L)^{-1}\Phi$, which is well-known as a 1-potential in the usual potential theory ([2]). If $p=q=2$ but $r\neq 1$, the potential U is expressed as $U=(I-L)^{-r/2}\Phi$, which is an infinite dimensional analogue of the Riesz potential. In these cases, the equation (5.1) is linear and hence the potential theory is said to be *linear*, otherwise it is said to be *non-linear* ([11]).

Let $\Phi\in\boldsymbol{D}_q^{-r}$ be positive and U be its (p, r)-potential. We readily see that $\int\tilde{U}d\nu_\Phi=||U||_{p,r}^p=||\Phi||_{q,-r}^q$, the common value of which is called the *(p, r)-energy of U or Φ or ν_Φ*. In this context, ν_Φ is called a *measure of finite (p, r)-energy* ([11]).

We first claim that a (p, r)-equilibrium potential introduced in Definition 3.2 is actually a (p, r)-potential. Namely, we have the following.

Theorem 5.1. *Let $U_A=U_{A;p,r}$ be the (p, r)-equilibrium potential of a set A. Then there exists a unique PGWF $\Phi_A=\Phi_{A;p,r}\in\boldsymbol{D}_q^{-r}$ such that U_A is the (p, r)-potential of Φ_A or of its corresponding measure $\nu_A=\nu_{A;p,r}=\nu_{\Phi_A}$. Furthermore, the*

topological support of ν_A is contained in the topological closure $\bar{A}$ of A.

Proof. Although the proof is merely a paraphrase of [11] Lem. 4.1, we will present it for completeness.

First we extend (5.1) for all $\Phi\in\boldsymbol{D}_q^{-r}$ that are not necessarily positive as follows.

$$U=(I-L)^{-r/2}\{|(I-L)^{-r/2}\Phi|^{q-2}(I-L)^{-r/2}\Phi\} \tag{5.2}$$

Then (5.2) turns out to be a one-to-one mapping from $\boldsymbol{D}_q^{-r}$ onto $\boldsymbol{D}_p^r$, and its inverse is written down explicitly as

$$\Phi=(I-L)^{r/2}\{|(I-L)^{r/2}U|^{p-2}(I-L)^{r/2}U\}\ . \tag{5.3}$$

Therefore what we must show is the positivity of $\Phi_A\in\boldsymbol{D}_q^{-r}$, where Φ_A is defined by (5.3) for $U=U_A$.

Take an arbitrary $F\in\boldsymbol{D}_p^r$ such that $F\geqq 0$ μ-a.e. Recall that $\tilde{F}\geqq 0$ (p,r)-q.e. by virtue of Lemma 3.1 (A) (iii). A parameter λ being assumed to be non-negative, Lemma 3.2 implies that the quantity $\|U_A+\lambda F\|_{p,r}^p$ takes its minimum at $\lambda=0$. Therefore if it is differentiable in λ, we must have $\frac{d}{d\lambda}\|U_A+\lambda F\|_{p,r}^p|_{\lambda=0}\geqq 0$. In fact, it is differentiable, because we can easily justify the commutation of $\int$ and $\frac{d}{d\lambda}$ in the following calculation.

$$\begin{aligned}
&\frac{d}{d\lambda}\|U_A+\lambda F\|_{p,r}^p\\
&\quad=\frac{d}{d\lambda}\int|(I-L)^{r/2}U_A+\lambda(I-L)^{r/2}F|^p\,d\mu\\
&\quad=\int\frac{d}{d\lambda}|(I-L)^{r/2}U_A+\lambda(I-L)^{r/2}F|^p\,d\mu\\
&\quad=p\int(I-L)^{r/2}F\,|(I-L)^{r/2}U_A+\lambda(I-L)^{r/2}F|^{p-2}\{(I-L)^{r/2}U_A+\\
&\qquad+\lambda(I-L)^{r/2}F\}d\mu\ .
\end{aligned}$$

Here we used $\frac{d}{dx}|x|^p=p|x|^{p-2}x$. Substituting $\lambda=0$, we get

$$\begin{aligned}
\frac{d}{d\lambda}\|U_A+\lambda F\|_{p,r}^p|_{\lambda=0}&=p\int(I-L)^{r/2}F\,|(I-L)^{r/2}U_A|^{p-2}(I-L)^{r/2}U_A\,d\mu\\
&=p\int(I-L)^{r/2}F(I-L)^{-r/2}\Phi_A\,d\mu\\
&=p(\Phi_A,F)\ .
\end{aligned}$$

Consequently, we have $(\Phi_A,F)\geqq 0$, which shows the positivity of Φ_A.

Clearly, the same reasoning applies to $F\in\boldsymbol{D}_p^r$ satisfying $\tilde{F}\geqq 0$, (p,r)-q.e.

only on A: We can still conclude that $\int \tilde{F} d\nu_\Phi = (\Phi_A, F) \geqq 0$. In particular, if $\tilde{F}=0$ (p,r)-q.e. on A, then $\int \tilde{F} d\nu_A = 0$ holds, because both $(\Phi_A, F) \geqq 0$ and $(\Phi_A, -F) \geqq 0$ should hold. Take an arbitrary $w_0 \in W \setminus \bar{A}$. Put $\varepsilon := \frac{1}{2}$ dist (w_0, A), which is strictly positive, and $B := \{w \in W;\ \|w - w_0\|_W \leqq \varepsilon\}$. By virtue of Lemma 2.5, we can take a function $F \in \boldsymbol{D}^{+\infty}$ such that F is non-negative, continuous, $F(w)=0$ for $w \in \bar{A}$ and $F(w)=1$ for $w \in B$. Then $\int F d\nu_A = 0$ and this implies $\nu_A(B)=0$. Thus we conclude that the topological support of ν_A is contained in $\bar{A}$. Q.E.D.

DEFINITION 5.1. The measure $\nu_A = \nu_{A;p,r}$ in the above theorem is called the (p,r)-*equilibrium measure* or the (p,r)-*capacity measure* of the set A.

Combining Theorem 3.1 with this theorem, we obtain the following theorem, which characterizes Borel slim sets by means of PGWF's.

Theorem 5.2. *Let $B \in \mathcal{B}(W)$.*

(A) *B is of (p,r)-capacity zero, if and only if B is of ν_Φ-measure zero for each PGWF $\Phi \in \boldsymbol{D}_q^{-r}$.*

(B) *B is slim, if and only if B is of ν_Φ-measure zero for each PGWF Φ.*

(B) is an immediate consequence of (A), while (A) is proved in the same way as [2] Th. 3.3.2.

EXAMPLE 5.1. Let $l_1, \cdots, l_d \in W^*$ be such that the matrix $V = (\langle l_i, l_j \rangle_H)$ is non-singular. Put $F(w) := ((l_1, w), \cdots, (l_d, w)) \in \boldsymbol{R}^d$, $w \in W$, and define a closed set $A_{F,y}, y \in \boldsymbol{R}^d$, by $A_{F,y} := F^{-1}(y)$. Then provided that $r/2 \geqq [d/2]+1$ ($[d/2]$ denotes the smallest integer not exceeding $d/2$), the $(2,r)$-equilibrium measure of $A_{F,y}$ is a constant times the measure corresponding to a PGWF $\delta_y \circ F$, or equivalently, a constant times the Gaussian probability measure $\mu(\cdot \mid F=y)$. More precisely, we have

$$d\nu_{A_{F,y}} = \left[\frac{1}{\Gamma(r)} \int_0^{+\infty} s^{r-1} \exp\left(-s - \frac{e^{-2s} - e^{-s}}{1-e^{-2s}} \langle V^{-1} y, y \rangle\right) ds \right]^{-1} d\mu(\cdot \mid F=y) .$$

In particular, when $y=0$ (then $A_{F,0}$ is a closed linear subspace of W), $\nu_{A_{F,0}}$ is nothing but $\mu(\cdot \mid F=0)$.

In the remainder of the section, we will verify the above example. For this, we need the following lemma which presents a sufficient condition for a $(2,r)$-potential to be equilibrium one.

Lemma 5.1. *Let $\Phi \in \boldsymbol{D}_2^{-r}$ be positive and $U \in \boldsymbol{D}_2^r$ be the $(2,r)$-potential of Φ. Suppose that a ν_Φ-measurable set A is a support of ν_Φ, i.e., $\nu_\Phi(W \setminus A)=0$, and*

that $\tilde{U}=1$, $(2,r)$-q.e. on A. Then ν_Φ and U are the $(2,r)$-equilibrium measure and the $(2,r)$-equilibrium potential of A, respectively.

Proof. Take an arbitrary $F\in\boldsymbol{D}_2^r$ whose $(2,r)$-quasi-continuous version $\tilde{F}$ is non-negative $(2,r)$-q.e. on A. We have $\frac{d}{d\lambda}\|U+\lambda F\|_{2,r}^2|_{\lambda=0}=2(\Phi,F)$ as before, and this is non-negative because it coincides with $2\int\tilde{F}d\nu_\Phi$. Thus we have $\frac{d}{d\lambda}\|U+\lambda F\|_{2,r}^2|_{\lambda=0}\geqq 0$. It follows from this that the quantity $\|U+\lambda F\|_{2,r}^2$ decreases as $\lambda\downarrow 0$ in some neighborhood of $\lambda=0$. However since it is a polynomial in λ of degree two (provided F is non-zero), it holds that $\|U\|_{2,r}^2\leqq\|U+\lambda F\|_{2,r}^2$ for all $\lambda>0$, particularly, $\|U\|_{2,r}^2\leqq\|U+F\|_{2,r}^2$. Therefore we conclude that U is the $(2,r)$-equilibrium potential of A by referring to Lemma 3.2. Q.E.D.

Now let us verify Example 5.1. We define

$$A_n:=\{w\in W;\ y^i-\frac{1}{2n}<(l_i,w)<y^i+\frac{1}{2n},\ i=1,\cdots,d\},\quad n=1,2,\cdots.$$

Here y^i denotes the i-th component of the vector $y\in\boldsymbol{R}^d$. We put

$$\Phi_n(w):=I_{A_n}(w)/\mu(A_n),\quad w\in W,\quad n=1,2,\cdots.$$

First we note that

(5.5) $$d\nu_{\Phi_n}=\Phi_n d\mu \text{ converges to } \mu(\cdot\,|\,F=y) \text{ weakly.}$$

Next, we calculate the following (cf. (2.4)).

$$\begin{aligned}(I-L)^{-r/2}\Phi_n(w)&=\frac{1}{\Gamma(r/2)\,\mu(A_n)}\int_0^\infty s^{r/2-1}e^{-s}\,T_s\,I_{A_n}(w)\,ds\\&=\frac{1}{\Gamma(r/2)\,\mu(A_n)}\int_0^\infty s^{r/2-1}e^{-s}\int_W I_{A_n}(e^{-s}w+\sqrt{1-e^{-2s}}\,v)\,\mu(dv)\,ds\end{aligned}$$

For $s>0$, putting $y_\pm^i:=\{y^i-e^{-s}(l_i,v)\pm\frac{1}{2n}\}/(1-e^{-2s})^{1/2}$, we have

$$\begin{aligned}&\int_W I_{A_n}(e^{-s}w+\sqrt{1-e^{-2s}}\,v)\,\mu(dv)\\&=\mu(\{v\in W;\ y_-^i<(l_i,v)<y_+^i,\ i=1,\cdots,d\})\\&=(2\pi\det V)^{-d/2}\int_{y_-^1}^{y_+^1}\cdots\int_{y_-^d}^{y_+^d}\exp\left(-\frac{1}{2}\langle V^{-1}\xi,\xi\rangle\right)d\xi\\&\leqq(2\pi\det V)^{-d/2}n^{-d}(1-e^{-2s})^{-d/2}.\end{aligned}$$

Therefore it holds that

$$0\leqq(I-L)^{-r/2}\Phi_n(w)\leqq\frac{(2\pi\det V)^{-d/2}n^{-d}}{\Gamma(r/2)\,\mu(A_n)}\int_0^\infty s^{r/2-1}e^{-s}(1-e^{-2s})^{-d/2}\,ds,$$

for all $w\in W$. The integral of the last term is finite if $r/2-1\geqq[d/2]$. Since $\frac{1}{\mu(A_n)}(2\pi \det V)^{-d/2}n^{-d}$ converges to $\exp(\frac{1}{2}\langle V^{-1}y,y\rangle)$ as $n\to\infty$, we see that $\sup_{n\in N}\sup_{w\in W}|(I-L)^{-r/2}\Phi_n(w)|<\infty$ and hence $\sup_{n\in N}\|\Phi_n\|_{2,-r}<\infty$, if $r/2\geqq[d/2]+1$. Then according to Lemma 4.1 (iii) and (5.5), we know that there exists a unique $\Phi\in \boldsymbol{D}_2^{-r}$ such that $\Phi\geqq 0$, $\mu(\cdot\,|F=y)=\nu_\Phi$ and that Φ_n converges to Φ in $\boldsymbol{D}_2^{-r}$ weakly. Obviously, we have $\Phi=\Phi_{F,y}$ (Example 4.1 (4.6)). Furthermore Φ_n actually converges to Φ in the norm $\|\cdot\|_{2,-r}$. Indeed, it is not hard to see that $(I-L)^{-r/2}\Phi_n(w)$ converges to

$$(5.6)\quad \frac{1}{\Gamma(r/2)}\exp(\frac{1}{2}\langle V^{-1}y,y\rangle)\times \times\int_0^\infty s^{r/2-1}\exp\left[-s-\frac{1}{2}\langle V^{-1}(y-e^{-s}F(w)),y-e^{-s}F(w)\rangle/(1-e^{-2s})\right]ds\,,$$

for all $w\in W$. Since $\sup_n\sup_w|(I-L)^{-r/2}\Phi_n(w)|<\infty$, we can regard this convergence as the convergence in $L_2(\mu)$. Therefore Φ_n is convergent in $\|\cdot\|_{2,-r}$, and of course the limit is nothing but $\Phi_{F,y}$. Namely, (5.6) is equal to $(I-L)^{-r/2}\Phi_{F,y}(w)$. Replacing $r/2$ by r, we explicitly have the $(2,r)$-potential $U_{F,y}$ corresponding to $\Phi_{F,y}$ as follows.

$$U_{F,y}(w)=\frac{1}{\Gamma(r)}\exp(\frac{1}{2}\langle V^{-1}y,y\rangle)\times \times\int_0^\infty s^{r-1}\exp\left[-s-\frac{1}{2}\langle V^{-1}(y-e^{-s}F(w)),y-e^{-s}F(w)\rangle/(1-e^{-2s})\right]ds\,,$$

$w\in W$. Now suppose $w\in A_{F,y}$, i.e., $F(w)=y$, then we have

$$U_{F,y}=\frac{1}{\Gamma(r)}\int_0^\infty s^{r-1}\exp(-s-\frac{e^{-2s}-e^{-s}}{1-e^{-2s}}\langle V^{-1}y,y\rangle)\,ds\,.$$

Hence the assertion of Example 5.1 follows from Lemma 5.1.

6. Other properties of PGWF's

In this section, we will survey properties of PGWF's and their corresponding measures from viewpoints other than the potential theory.

Since the space of the polynomials $\boldsymbol{P}$ is dense in each Sobolev space $\boldsymbol{D}_p^r$ by the definition, a linear functional over $\boldsymbol{P}$ which is continuous in the norm $\|\cdot\|_{p,r}$ will be uniquely extended to an element of $(\boldsymbol{D}_p^r)^*$, or equivalently of $\boldsymbol{D}_q^{-r}$. Therefore we can characterize the measures corresponding to PGWF's as follows.

Theorem 6.1. *Let ν be a positive finite measure over $(W,\mathcal{B}(W))$. Then* (A) (B) *and* (C) *below are equivalent to each other.*

(A) *There exists a PGWF* Φ *such that* $\nu=\nu_\Phi$.

(B) $P\subset L_1(\nu)$ *and there exist* $1<p<\infty, r>0$ *and* $c<0$ *such that*

$$|\int F(w)\,\nu(dw)|\leqq c\|F\|_{p,r} \quad \textit{for each } F\in\boldsymbol{P}.$$

(C) (i) ν *has no mass in any slim set,*

(ii) *for each* $F\in\boldsymbol{D}^{+\infty}$, *any* ∞*-quasi-continuous version* $\tilde{F}$ *of* F *is an element of* $L_1(\nu)$,

(iii) *there exist* $1<p<\infty, r>0$ *and* $c>0$ *such that*

$$|\int \tilde{F}(w)\,\nu(dw)|\leqq c\,\|F\|_{p,r} \quad \textit{for each } F\in\boldsymbol{D}^{+\infty}.$$

Proof. (A)$\leftrightarrow$(B) is just mentioned. (A)$\leftrightarrow$(C) is clear by the previous section. Q.E.D.

As we consider the product of elements of $\mathcal{D}(\boldsymbol{R}^n)$ and $\mathcal{D}'(\boldsymbol{R}^n)$, we can define the product $G\Phi\in\boldsymbol{D}^{-\infty}$ of $G\in\boldsymbol{D}^{+\infty}$ and $\Phi\in\boldsymbol{D}^{-\infty}$ by $(G\Phi, F):=(\Phi, GF)$, $F\in\boldsymbol{D}^{+\infty}$. But in case Φ is positive, we are allowed to define the product $G\Phi\in\boldsymbol{D}^{-\infty}$ when G belongs to a certain space much wider than $\boldsymbol{D}^{+\infty}$. Namely;

Theorem 6.2. *Let* Φ *be a PGWF and* $G\in L_{1+}(\nu_\Phi)$. *Then the product* $G\Phi\in\boldsymbol{D}^{-\infty}$ *is well-defined by* $(G\Phi, F):=\int \tilde{F}(w)\,G(w)\,\nu_\Phi(dw)$, $F\in\boldsymbol{D}^{+\infty}$.

Proof. It is obvious that the measure $G(w)\,\nu_\Phi(dw)$ (in general, a signed measure) has no mass in any slim set and that $\tilde{F}$ is $G\nu_\Phi$-measurable for each $F\in\boldsymbol{D}^{+\infty}$. Since $\boldsymbol{D}^{+\infty}$ is an algbera, $\tilde{F}$ belongs to $L_{\infty-}(\nu_\Phi)$, and hence it is $|G|\nu_\Phi$-integrable. So let us verify the condition of Theorem 6.1 (C) (iii) for the measure $|G|\nu_\Phi$.

We may assume $G\in L_{1+\varepsilon}(\nu_\Phi)$ for some $0<\varepsilon\leqq 1$. By the Hölder inequality, we have $|\int \tilde{F}G\,d\nu_\Phi|\leqq\|\tilde{F}\|_{L_m(\nu\Phi)}\|G\|_{L_{m'}(\nu\Phi)}$, where m is an even integer not less than $(1+\varepsilon)/\varepsilon$ and $m':=m/(m-1)$ $(\leqq 1+\varepsilon)$. Similarly we get $\|\tilde{F}\|_{L_m(\nu\Phi)}=\left[\int \tilde{F}^m\,d\nu_\Phi\right]^{1/m}=(\Phi, F^m)^{1/m}\leqq\|\Phi\|_{q,-k}^{1/m}\|F^m\|_{p,k}^{1/m}$. Here indices q, $-k$ are chosen so that $\Phi\in\boldsymbol{D}_q^{-k}$ and $k\in\boldsymbol{N}$. Recalling that $\|F^m\|_{p,k}\leqq c\,\|F\|_{mp,k}^m$ for a suitable constant $c>0$ independent of F ([17] Th.3.1), we see, with the help of the above inequalities, that $|\int \tilde{F}G\,d\nu_\Phi|\leqq c^{1/m}\|G\|_{L_{m'}(\nu\Phi)}\|\Phi\|_{q,-k}^{1/m}\|F\|_{mp,k}$, which completes the proof. Q.E.D.

EXAMPLE 6.1. Let $X(w)=(X_t(w))$ be the solution of the SDE discussed in Example 4.2. Here we also assume that X_T is non-degenerate in Malliavin's sense for a fixed $T>0$. We define $\sigma_D(w):=\inf\{s>0;\tilde{X}_s(w)\in D\}$, $D\in\mathcal{B}(\boldsymbol{R}^d)$, where $\tilde{X}=(\tilde{X}_t)$ is an ∞-quasi-continuous version of $X=(X_t)$ in the sense of

Example 4.2. Then the product $I_{\{\sigma_D>T\}}(w)\,(\delta_y\circ X_T)$, $y\in\boldsymbol{R}^d$, is well-defined as a PGWF by Theorem 6.2. This is because $I_{\{\sigma_D>T\}}(w)$ is measurable with respect to the measure corresponding to the PGWF $\delta_y\circ X_T$.

We mentioned after the proof of Theorem 4.2 that the measures corresponding to PGWF's are very close to μ in a sense, though they may be singular relative to μ. The following theorem shows an example of this similarity concerning the integrability or, equivalently, the order of decay of the tail.

Theorem 6.3 (*Fernique-type theorem*). *Let Φ be a PGWF. Then there exists $\beta>0$ such that a function* $\exp(\beta\|w\|_W^2)$, $w\in W$, *is ν_Φ-integrable.*

Proof. It is enough to show the following.

$$\text{(6.1)}\qquad \begin{cases} \text{There exist } c>0,\ \alpha>0 \text{ and } a_0>0 \text{ such that} \\ \nu_\Phi(\{w\in W;\ \|w\|_W>a\})\leqq c\,e^{-\alpha a^2} \quad \text{if} \quad a>a_0\,. \end{cases}$$

Let $t>0$ and a family of functions $\{\phi_R\}_{R>0}$ satisfy the conditions of (3.4). As we saw in the proof of Lemma 3.3, we have

$$\text{(6.2)}\qquad \begin{cases} \{w\in W;\ \|w\|_W>a\}\subset\{w\in W;\ T_t\|w\|_W>(1-\delta_1)\,a-\delta_2\}\,, \\ \{w\in W;\ T_t\|w\|_W>a\}\subset\{w\in W;\ \|w\|_W>(a-\delta_2)/(1+\delta_1)\}\,. \end{cases}$$

where $\delta_1:=1-e^{-t}$ and $\delta_2:=\sqrt{1-e^{-2t}}\int\|v\|_W\,\mu(dv)$. Putting $F_R(w):=\phi_R(T_t\|w\|_W)$, we have by (3.6) that

$$\text{(6.3)}\qquad \|F_R\|_{p,k}\leqq c'_{p,k}\,\mu(\{w\in W;\ T_t\|w\|_W>R\}),\quad 1<p<\infty,\ k\in\boldsymbol{N}\,,$$

where $c'_{p,k}>0$ is a certain constant independent of R. Then it follows from (6.2) and (6.3) that

$$\begin{aligned}
&\nu_\Phi(\{w\in W;\ \|w\|_W>a\}) \\
&\quad\leqq\nu_\Phi(\{w\in W;\ T_t\|w\|_W>(1-\delta_1)\,a-\delta_2)\} \\
&\quad\leqq\int F_{(1-\delta_1)a-\delta_2-\delta_3}(w)\,\nu_\Phi(dw) \\
&\quad=(\Phi,\,F_{(1-\delta_1)a-\delta_2-\delta_3}) \\
&\quad\leqq\|\Phi\|_{q,-k}\,\|F_{(1-\delta_1)a-\delta_2-\delta_3}\|_{p,k} \\
&\quad\leqq\|\Phi\|_{q,-k}\,c'_{p,k}\,\mu(\{w\in W;\ T_t\|w\|_W>(1-\delta_1)\,a-\delta_2-\delta_3\}) \\
&\quad\leqq\|\Phi\|_{q,-k}\,c'_{p,k}\,\mu(\{w\in W;\ \|w\|_W>\{(1-\delta_1)\,a-2\delta_2-\delta_3\}/(1+\delta_1)\})
\end{aligned}$$

Here we assumed $\Phi\in\boldsymbol{D}_q^{-k}$ and $k\in\boldsymbol{N}$.

On the other hand, Fernique's theorem [1] says that there exist $c'>0$ and $\alpha'>0$ such that

$$\text{(6.4)}\qquad \mu(\{w\in W;\ \|w\|_W>a\})\leqq c'\,e^{-\alpha' a^2},\quad a>0\,.$$

Therefore we have

$$\nu_\Phi(\{w\in W;\ \|w\|_W>a\})\leqq\|\Phi\|_{q,-k}\, c'_{p,k}\, c'\exp\left[-\alpha'\left\{\frac{(1-\delta_1)\,a-2\delta_2-\delta_3)}{1+\delta_1}\right\}^2\right],$$

from which (6.1) easily follows. Q.E.D.

REMARK. By taking $t>0$ and $\delta_3>0$ sufficiently small, α of (6.1) can be taken arbitrarily close to α' of (6.4).

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