Doctoral Thesis

Weighted norm inequalities for multilinear Fourier multiplier operators

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Chapter 1

Introduction and results

In recent years, multilinear operators in harmonic analysis have been well studied by many mathematicians. In this thesis, we consider weighted norm inequalities for multilinear Fourier multiplier operators.

In this chapter, we shall describe an introduction of the thesis. The definitions and notations will be given in Chapter 2.

1.1 Fourier multiplier operators in the linear setting

In this section, we recall Fourier multiplier operators in the linear setting. For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$\begin{aligned} m(D)f(x) &= \mathcal{F}^{-1}\left[m(\xi)\widehat{f}(\xi)\right](x) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi)\widehat{f}(\xi)\,d\xi \end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $x, \xi \in \mathbb{R}^n$. The purpose of the thesis is to study "smoothness" of multipliers m which is measured by several function spaces. In particular, we are interested in a relationship with weight classes in the multilinear setting. Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\begin{aligned} &\operatorname{supp}\Psi \subset \left\{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\right\}, \\ &\sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1 \quad (\xi \in \mathbb{R}^d \setminus \{0\}). \end{aligned} \tag{1.1.1}$$

For this cut-off function Ψ with $d = n$ and $m \in L^\infty(\mathbb{R}^n)$, we set

$$m_j(\xi) = m(2^j\xi)\Psi(\xi),$$

where $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. The Hörmander multiplier theorem [15] is very well known as the theorem which describes the L^p-boundedness of $m(D)$ and we recall it (see also [6, Theorem 8.3]).

Theorem A ([15]). *Let $1 < p < \infty$ and $s > n/2$. Then*

$$\|m(D)\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^s(\mathbb{R}^n)},$$

where $\|\cdot\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$ is the operator norm and $H^s(\mathbb{R}^n)$ is the Sobolev space.

If $m \in C^{[n/2]+1}(\mathbb{R}^n\backslash\{0\})$ satisfies

$$|\partial^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{1.1.2}$$

for all $\alpha \in \mathbb{Z}^n_+$, $|\alpha| \le [n/2] + 1$, then Theorem A also holds, where $[n/2]$ is the integer part of $n/2$. For example, the multiplier m of the Hilbert transform on $\mathbb{R}$, namely, $m(\xi) = -i\mathrm{sgn}(\xi)$ satisfies (1.1.2). If $m \in H^s(\mathbb{R}^n)$, $s > n/2$, then the L^p-boundedness of $m(D)$ follows since $\mathcal{F}^{-1}[m] \in L^1$ and Young's inequality. In contrast to this fact, Theorem A says that by the smoothness of each part of the multiplier m, namely, $m_j \in H^s(\mathbb{R}^n)$, the L^p-boundedness of $m(D)$ follows.

Weighted norm inequalities for Fourier multiplier operators in the linear setting are known as the result of Kurtz-Wheeden [18] and we recall it.

Theorem B ([18, Theorem 1]). *Let $1 < p < \infty$ and $n/2 < s \le n$. Assume*

$$p > n/s \quad \text{and} \quad w \in A_{p/(n/s)}.$$

Then

$$\|m(D)\|_{L^p(w)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^s(\mathbb{R}^n)},$$

where w is a weight, $L^p(w)$ is the L^p space with the Lebesgue measure dx replaced by wdx and A_p is the Muckenhoupt class.

Theorem 1.4.1 in the thesis is a multilinear version of Theorem B. Theorem B gives us a consideration as follows: exponent p and weight w to assure the $L^p(w)$-boundedness of $m(D)$ depend on "smoothness s" of multipliers m. That is to say if the condition on m is strong, namely, m is smooth, then the conditions on p and w should be weak and vice versa. Theorem 1.4.3 in the thesis corresponds to this consideration in the multilinear setting.

1.2 Fourier multiplier operators in the multilinear setting

In this section, we recall Fourier multiplier operators in the multilinear setting. Let N be a natural number, $N \ge 2$, $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$T_m(f_1, \dots, f_N)(x)$$

$$= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N)\, d\xi$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \dots d\xi_N$. Coifman and Meyer [5] proved that if $m \in C^L(\mathbb{R}^{Nn} \backslash \{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \cdots, \xi_N)| \leq C_{\alpha_1,\dots,\alpha_N} (|\xi_1| + \cdots + |\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}$$

for all $|\alpha_1| + \cdots + |\alpha_N| \leq L$, where L is a sufficiently large natural number, then T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. For the cut-off function Ψ in (1.1.1) with $d = Nn$ and $m \in L^\infty(\mathbb{R}^{Nn})$, we set

$$m_j(\xi) = m(2^j \xi)\Psi(\xi),$$

where $j \in \mathbb{Z}$ and $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$. The starting point of the thesis is the Hörmander multiplier theorem in the multilinear setting given by Tomita [31] and we recall it.

Theorem C ([31, Theorem 1.1])**.** *Let $1 < p_1, \dots, p_N, p < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $s > Nn/2$. Then*

$$\|T_m\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^s(\mathbb{R}^{Nn})},$$

where $\|\cdot\|_{L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}$ is the operator norm.

Grafakos-Si [13] treated the case $p \leq 1$ by using the L^r-based Sobolev spaces, $1 < r \leq 2$. As a corollary of Theorem C, we can reduce the number of derivatives of m to assure the boundedness of T_m ([31, Corollary 1.2]). After Theorem C, problems to find minimal smoothness conditions on m to assure the boundedness of T_m were considered by Grafakos-Miyachi-Tomita [12], Miyachi-Tomita [22] and Miyachi-Tomita [23]. Theorem 1.4.1 in the thesis is a weighted version of Theorem C.

1.3 Weighted norm inequalities in the multilinear setting

In this section, we recall weighted norm inequalities in the multilinear setting. Let $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. For $w_1, \dots, w_N$ are weights, we set $w = w_1^{p/p_1} \dots w_N^{p/p_N}$. We first consider the case multiple A_p weights of direct product type, namely, $A_{p_1} \times \cdots \times A_{p_N}$. We define the multi (sub) linear maximal operator $\widetilde{\mathcal{M}}$ by

$$\widetilde{\mathcal{M}}(f_1, \dots, f_N)(x) = \prod_{i=1}^{N} Mf_i(x)$$

for $f_1, \dots, f_N \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator. For $(w_1, \dots, w_N) \in A_{p_1} \times \cdots \times A_{p_N}$, by Hölder's inequality, it is easy to see that

$$\left\| \widetilde{\mathcal{M}}(f_1, \dots, f_N) \right\|_{L^p(w)} \lesssim \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i)}.$$

Moreover, it is known that $\widetilde{\mathcal{M}}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_N}(w_N)$ to $L^p(w)$ if and only if $(w_1, \dots, w_N)$ belongs to the class $A_{p_1} \times \cdots \times A_{p_N}$ (see [29]). In [14], it was proposed to develop a more suitable class of multiple A_p weights in the multilinear setting.

We next consider the case multiple A_p weights of vector type, namely, $A_{(p_1,\dots,p_N)}$ introduced by Lerner, Ombrosi, Pérez, Torres and Trujillo-Gonzĺez in [19]. The new multi (sub) linear maximal operator $\mathcal{M}$ ([19]) is defined by

$$\mathcal{M}(f_1, \dots, f_N)(x) = \sup_{Q \ni x} \prod_{i=1}^{N} \frac{1}{|Q|} \int_Q |f_i(y_i)|\, dy_i$$

for $f_1, \dots, f_N \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. In [19, Theorem 3.7], it was proved that $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_N}(w_N)$ to $L^p(w)$ if and only if $(w_1, \dots, w_N)$ belongs to the class $A_{(p_1,\dots,p_N)}$. It should be remarked that

$$\mathcal{M}(f_1, \dots, f_N)(x) \le \widetilde{\mathcal{M}}(f_1, \dots, f_N)(x), \quad x \in \mathbb{R}^n,$$

and

$$A_{p_1} \times \cdots \times A_{p_N} \subsetneq A_{(p_1,\dots,p_N)}. \tag{1.3.1}$$

For the strictness of the above inclusion, see [19, Remark 7.2]. By these remarks, it can be thought that $\mathcal{M}$ and $A_{(p_1,\dots,p_N)}$ are more suitable in the multilinear setting.

Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $n/2 < s_i \le n$ for $i = 1, \dots, N$. Assume $p_i > n/s_i$ for $i = 1, \dots N$. We shall prove in Theorem 1.4.1 that if $(w_1, \dots, w_N) \in A_{p_1/(n/s_1)} \times \cdots \times A_{p_N/(n/s_N)}$, then

$$\|T_m\|_{L^{p_1}(w_1) \times \cdots \times L^{p_N}(w_N) \to L^p(w)} \lesssim \sup_{j \in Z} \|m_j\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)}, \tag{1.3.2}$$

where $H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ is the Sobolev space of product type. This result can also be obtained from another approach of [16]. See [21, 2] for the endpoint cases. In particular, for $Nn/2 < s \le Nn$ and $p_i > Nn/s$, $i = 1, \dots, N$, taking $s_1 = \cdots = s_N = s/N$, we have that if $(w_1, \dots, w_N) \in A_{p_1 s/(Nn)} \times \cdots \times A_{p_N s/(Nn)}$, then

$$\|T_m\|_{L^{p_1}(w_1) \times \cdots \times L^{p_N}(w_N) \to L^p(w)} \lesssim \sup_{j \in Z} \|m_j\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)}. \tag{1.3.3}$$

On the other hand, Bui-Duong [4] and Li-Sun [20] proved that if $(w_1, \dots, w_N) \in A_{(p_1 s/(Nn), \dots, p_N s/(Nn))}$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in Z} \|m_j\|_{H^s(\mathbb{R}^{Nn})}.$$

It should be remarked that

$$H^{s_1+\cdots+s_N}(\mathbb{R}^{Nn}) \hookrightarrow H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N), \quad s_1, \dots, s_N \geq 0.$$

By this remark and (1.3.1), it is natural to consider a question whether (1.3.3) holds under the condition $(w_1, \dots, w_N) \in A_{(p_1 s/(Nn), \dots, p_N s/(Nn))}$, and we shall answer this question negatively in Theorem 1.4.3. It means that both conditions on weights and multipliers cannot be weaken at the same time. It corresponds to the consideration which we thought in the linear setting.

In Theorem 1.4.2, we consider weighted norm inequalities for multilinear Fourier multipliers with the L^r-based Sobolev regularity, $1 < r \leq 2$ with mixed norm. In [27], weighted norm inequalities for multilinear Fourier multipliers with the L^r-based Sobolev regularity were obtained.

In Theorem 1.4.4, we study a critical case of Theorem 1.4.1, namely, $s_i = n/2$, $i = 1, \dots, N$ in (1.3.2) and measure the smoothness of multipliers m by the Besov spaces. It should be remarked that

$$H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N) \hookrightarrow B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N) \hookrightarrow L^\infty(\mathbb{R}^{Nn}),$$

where $s_1, \dots, s_N > n/2$. In this sense, it can be thought that Theorem 1.4.4 is a critical case of Theorem 1.4.1. As a Corollary of Theorem 1.4.4, we obtain a critical case of Theorem C ([10, Corollary 1.2]). In the linear setting, Seeger [26] considered Fourier multiplier operators with Besov regularity.

1.4 Results

In the thesis, we consider the following four results.

Theorem 1.4.1 ([7, Theorem 6.2])**.** *Let $1 < p_1, \dots, p_N < \infty, 1/p_1 + \cdots + 1/p_N = 1/p$ and $n/2 < s_i \leq n$ for $i = 1, \cdots, N$. Assume*

$$p_i > n/s_i \quad \text{and} \quad w_i \in A_{p_i/(n/s_i)} \quad \text{for} \quad i = 1, \cdots, N.$$

Then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)},$$

where $w = w_1^{p/p_1} \cdots w_N^{p/p_N}$.

Theorem 1.4.2 ([8, Result 1])**.** *Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$, $1 < r_N \leq r_{N-1} \leq \cdots \leq r_2 \leq r_1 \leq 2$ and $n/r_i < s_i \leq n$ for $i = 1, \dots, N$. Assume*

$$p_i > n/s_i \quad \text{and} \quad w_i \in A_{p_i/(n/s_i)} \quad \text{for} \quad i = 1, \dots, N.$$

Then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\ldots,s_N)}_{(r_1,\ldots,r_N)}((\mathbb{R}^n)^N)},$$

where $w = w_1^{p/p_1}\cdots w_N^{p/p_N}$.

Theorem 1.4.3 ([9, Theorem 1.1])**.** *Let* $N \geq 2$, $1 < p_1, \ldots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ *and* $Nn/2 < s \leq Nn$. *Assume*

$$p_i > Nn/s \quad \text{for} \quad i = 1, \ldots N.$$

Then there exists $\vec{w_0} = (w_1, \ldots, w_N) \in A_{(p_1 s/(Nn),\ldots,p_N s/(Nn))}$ *such that the estimate*

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w_0}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s/N,\ldots,s/N)}((\mathbb{R}^n)^N)}$$

does not hold, where $\nu_{\vec{w_0}} = w_1^{p/p_1}\cdots w_N^{p/p_N}$.

Theorem 1.4.4 ([10, Theorem 1.1])**.** *Let* $2 < p_1, \ldots, p_N < \infty$ *and* $1/p_1 + \cdots + 1/p_N = 1/p$. *Assume*

$$w_i \in A_{p_i/2} \quad \text{for} \quad i = 1, \cdots, N.$$

Then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{(n/2,\ldots,n/2)}_{2,1}((\mathbb{R}^n)^N)},$$

where $w = w_1^{p/p_1}\cdots w_N^{p/p_N}$.

Chapter 2

Preliminaries

In this chapter, we collect notations and lemmas which will be used later on, and recall definitions of several functions spaces.

2.1 Notations

Let $n \in \mathbb{N}$ be the dimension of the Euclidean space and $\mathbb{Z}_+^n$ is defied by $\{0, 1, 2, \dots\}^n$. Lebesgue measure in $\mathbb{R}^n$ is denoted by dx (See, for example, [24, Chapter 1, 2]). For two non-negative quantities A and B, the notation $A \lesssim B$ means that $A \le CB$ for some unspecified constant $C > 0$ independent of A and B , and the notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively (See, for example, [6, Chapter 1, Section 7]). We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx,$$
$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

The Laplacian Δ is defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We say that a function w is a weight, if w is a non-negative almost everywhere and locally integrable function. Let $0 < p < \infty$ and $w \ge 0$. The weighted Lebesgue space $L^p(w)$ consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |f(y)|\, dy$$

for locally integrable functions f on $\mathbb{R}^n$. We say that a weight $w \geq 0$ belongs to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of B and p' is the conjugate exponent of p, namely, $1/p + 1/p' = 1$. It is well known that M is bounded on $L^p(w)$ if and only if $w \in A_p$([6, Theorem 7.3]). We also say that $(w_1, \dots, w_N)$ belongs to the class $A_{(p_1,\dots,p_N)}$, if

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)\, dx \right)^{1/p} \prod_{i=1}^{N} \left(\frac{1}{|B|} \int_B w_i(x)^{1-p_i'}\, dx \right)^{1/p_i'} < \infty,$$

where $w = w_1^{p/p_1} \cdots w_N^{p/p_N}$ ([19]).

For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$\begin{aligned} m(D)f(x) &= \mathcal{F}^{-1} \left[m(\xi) \widehat{f}(\xi) \right] (x) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} m(\xi) \widehat{f}(\xi)\, d\xi \end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $x, \xi \in \mathbb{R}^n$. Let N be a natural number, $N \geq 2$, $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. For $m \in L^\infty(\mathbb{R}^{Nn})$, the N-linear Fourier multiplier operator T_m is defined by

$$\begin{aligned} &T_m(f_1, \dots, f_N)(x) \\ &= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \cdots \widehat{f_N}(\xi_N)\, d\xi \end{aligned}$$

for $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $d\xi = d\xi_1 \dots d\xi_N$. We denote by $\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(w)}$ the smallest constant C satisfying

$$\|T_m(f_1, \dots, f_N)\|_{L^p(w)} \leq C \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i)}, \quad f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n).$$

2.2 Cut-off functions

In this section, we collect cut-off functions which will be used later on. Let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\operatorname{supp} \Psi \subset \left\{ \xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2 \right\}, \tag{2.2.1}$$

$$\sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1 \quad (\xi \in \mathbb{R}^d \setminus \{0\}).$$

For a method for the construction of such cut-off functions, see, for example, [6, p.162]. For this cut-off function Ψ with $d = Nn$ and $m \in L^\infty(\mathbb{R}^{Nn})$, we set

$$m_j(\xi) = m(2^j\xi)\Psi(\xi), \tag{2.2.2}$$

where $j \in \mathbb{Z}$ and $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$.

Let ϕ_0 be a C^∞-function on $[0, \infty)$ satisfying

$$\phi_0(t) = 1 \text{ on } [0, 1/(4N)]\ ,\ \operatorname{supp}\phi_0 \subset [0, 1/(2N)].$$

We set $\phi_1(t) = 1 - \phi_0(t)$. For $(i_1, \cdots, i_N) \in \{0, 1\}^N$, we define the function $\Phi_{(i_1,\cdots,i_N)}$ on $\mathbb{R}^{Nn}\setminus\{0\}$ by

$$\Phi_{(i_1,\cdots,i_N)}(\xi) = \phi_{i_1}(|\xi_1|/|\xi|)\cdots\phi_{i_N}(|\xi_N|/|\xi|), \tag{2.2.3}$$

where $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $|\xi| = \sqrt{|\xi_1|^2 + \cdots + |\xi_N|^2}$. Note that

$$\Phi_{(0,\dots,0)}(\xi) = 0.$$

(See [7, p.6339]).

According to the notation of [12, p.8] or [22, p.17], we also set $\mathcal{A}_0$, $\mathcal{A}_1$: $\mathcal{A}_0$ denotes the set of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\varphi$ is compact and $\varphi = 1$ on some neighborhood of the origin; $\mathcal{A}_1$ denotes the set of $\widetilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\widetilde{\psi}$ is a compact subset of $\mathbb{R}^n\setminus\{0\}$.

2.3 Function spaces

In this section, we recall definitions of several function spaces. To distinguish spaces of usual type and product type, we use $\mathbb{R}^{Nn}$ and $(\mathbb{R}^n)^N$, respectively.

Definition 2.3.1 (The Sobolev space of usual type)**.** *For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{H^s(\mathbb{R}^n)} = \left\|(1 + |\xi|^2)^{s/2}\widehat{f}\right\|_{L^2(\mathbb{R}^n)} < \infty.$$

Definition 2.3.2 (The Sobolev space of product type)**.** *For $(s_1, \dots, s_N) \in \mathbb{R}^N$, the norm of the Sobolev space of product type $H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ for $F \in \mathcal{S}'(\mathbb{R}^{Nn})$ is defined by*

$$\begin{aligned}&\|F\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)}\\ &= \left\|(1 + |\xi_1|^2)^{s_1/2}\dots(1 + |\xi_N|^2)^{s_N/2}\widehat{F}(\xi_1, \dots, \xi_N)\right\|_{L^2((\mathbb{R}^n)^N)}.\end{aligned}$$

Definition 2.3.3 (The Sobolev space of product type with mixed norm)**.** *For $(s_1, \dots, s_N) \in \mathbb{R}^N$ and $(r_1, \dots, r_N) \in (1, \infty)^N$, the norm of the Sobolev space of product type with mixed norm $H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)$ is defined by*

(2.3.1)

$$\|F\|_{H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)} = \left\| \left\| \mathcal{F}^{-1}\left[\langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \left| \widehat{F}(\xi) \right| \right](x_1, \dots, x_N) \right\|_{L^{r_1}(\mathbb{R}^n_{x_1})} \cdots \right\|_{L^{r_N}(\mathbb{R}^n_{x_N})},$$

where $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$ and $\langle \xi_i \rangle = (1 + |\xi_i|^2)^{1/2}, i = 1, \dots, N$.

For the L^p space with mixed norm, see [3].

Definition 2.3.4 (The weighted Lebesgue space with mixed norm)**.** *For $(s_1, \dots, s_N) \in \mathbb{R}^N$ and $(q_1, \dots, q_N) \in [1, \infty)^N$, the norm of the weighted Lebesgue space with mixed norm $L^{(q_1,\dots,q_N)}_{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ is defined by*

$$\|F\|_{L^{(q_1,\dots,q_N)}_{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} = \left\| \|F(x_1, \dots, x_N)\|_{L^{q_1}(\langle x_1 \rangle^{s_1})} \cdots \right\|_{L^{q_N}(\langle x_N \rangle^{s_N})},$$

where $x = (x_1, \dots, x_N) \in (\mathbb{R}^n)^N$ and $\langle x_i \rangle^{s_i} = (1 + |x_i|^2)^{s_i/2}$, $i = 1, \dots, N$.

We recall the definition of the Besov spaces of usual type and product type, respectively. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (2.2.1) with $d = n$, and set

$$\psi_k(\eta) = \psi(\eta/2^k), \quad k \geq 1,$$
$$\psi_0(\eta) = 1 - \sum_{k=1}^{\infty} \psi_k(\eta),$$

where $\eta \in \mathbb{R}^n$. Note that

$$\operatorname{supp} \psi_k \subset \{\eta \in \mathbb{R}^n : 2^{k-1} \leq |\eta| \leq 2^{k+1}\}, \quad k \geq 1,$$
$$\operatorname{supp} \psi_0 \subset \{\eta \in \mathbb{R}^n : |\eta| \leq 2\}, \quad \sum_{k=0}^{\infty} \psi_k(\eta) = 1.$$

Definition 2.3.5 (The Besov space of usual type)**.** *For $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|\psi_k(D) f\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q} < \infty,$$

where $\psi_k(D) f = \mathcal{F}^{-1}[\psi_k \widehat{f}]$.

We refer to [32], [25] for details on Besov spaces. We also recall the definition of Besov spaces of product type (See [28]). Let $\{\psi_{k_1}\}_{k_1=0}^{\infty}, \dots, \{\psi_{k_N}\}_{k_N=0}^{\infty}$ be as above and set

$$\begin{aligned}\Psi_{(k_1,\dots,k_N)}(\xi) &= (\psi_{k_1} \otimes \cdots \otimes \psi_{k_N})(\xi)\\ &= \psi_{k_1}(\xi_1) \times \cdots \times \psi_{k_N}(\xi_N),\end{aligned}$$

where $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$.

Definition 2.3.6 (The Besov space of product type)**.** *For $1 \le p, q \le \infty$ and $(s_1, \dots, s_N) \in \mathbb{R}^N$, the norm of the Besov space of product type $B_{p,q}^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ for $F \in \mathcal{S}'(\mathbb{R}^{Nn})$ is defined by*

$$\begin{aligned}&\|F\|_{B_{p,q}^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)}\\ &= \left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1 s_1 + \cdots + k_N s_N)q} \|\Psi_{(k_1,\dots,k_N)}(D)F\|_{L^p((\mathbb{R}^n)^N)}^q \right)^{1/q}.\end{aligned}$$

2.4 Lemmas

In this section, we collect lemmas which will be used in the thesis.

Lemma 2.4.1 ([7, Lemma 3.1])**.** *Let $\Phi_{(i_i,\dots,i_N)}$ be the same as in* (2.2.3)*. Then the following are true:*
(1) *For $(\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$,*

$$\sum_{\substack{(i_1,\dots,\imath_N)\in\{0,1\}^N,\\ (i_1,\dots,\imath_N)\neq(0,\dots,0)}} \Phi_{(i_1,\dots,i_N)}(\xi_1, \dots, \xi_N) = 1.$$

(2) *For $(i_1, \dots, i_N) \in \{0,1\}^N$ and $(\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^n \times \cdots \times \mathbb{Z}_+^n$, then there exists a constant $C_{(i_1,\dots,i_N)}^{(\alpha_1,\dots,\alpha_N)} > 0$ such that*

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N} \Phi_{(i_1,\dots,i_N)}(\xi)| \le C_{(i_1,\dots,i_N)}^{(\alpha_1,\dots,\alpha_N)} (|\xi_1| + \cdots + |\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}$$

for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \setminus \{(0, \dots, 0)\}$.
(3) *If $i_j = 1$ for some $j = 1, \cdots, N$ and $i_k = 0$ for all $k = 1, \cdots, N$ with $j \neq k$, then $\operatorname{supp} \Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1, \dots, \xi_N) : |\xi_k| \le |\xi_j|/N \text{ for } k \neq j\}$. If $i_j = i_{j'} = 1$ for some $j, j' = 1, \dots, N$ with $j \neq j'$, then $\operatorname{supp} \Phi_{(i_1,\dots,i_N)} \subset \{(\xi_1, \dots, \xi_N) : |\xi_j|/(4N) \le |\xi_{j'}| \le 4N|\xi_j|,\ |\xi_k| \le 4N|\xi_j| \text{ for } k \neq j, j'\}$.*

Lemma 2.4.2 ([6, Chapter 7])**.** *Let $1 < p < \infty$ and $w \in A_p$. Then*
(1) $w^{1-p'} \in A_{p'}$ (2) *there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.*

Lemma 2.4.3 ([17]). *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that* $\operatorname{supp}\psi \subset \{\eta \in \mathbb{R}^n : 1/r \le |\eta| \le r\}$ *for some $r > 1$. If $1 < p < \infty$ and $w \in A_p$, then*

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} |\psi(D/2^j)f|^2 \right\}^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p},$$

where $\psi(D/2^j)f = \mathcal{F}^{-1}[\psi(\cdot/2^j)\widehat{f}]$.

Lemma 2.4.4 ([1]). *Let $1 < p, q < \infty$ and $w \in A_p$. Then*

$$\left\| \left\{ \sum_{k\in Z} (Mf_k)^q \right\}^{1/q} \right\|_{L^p(w)} \lesssim \left\| \left\{ \sum_{k\in\mathbb{Z}} |f_k|^q \right\}^{1/q} \right\|_{L^p(w)}$$

Lemma 2.4.5 ([6, Proposition 2.7]). *Let ϕ be a function which is positive, radial, decreasing* (*as a function on* $(0,\infty)$) *and integrable. Set $\phi_t(x) = 1/t^n \phi(x/t)$ for $t > 0$. Then*

$$\sup_{t>0} |\phi_t * f(x)| \lesssim Mf(x)$$

for $x \in \mathbb{R}^n$.

Lemma 2.4.6 ([31, Lemma 3.2]). *Let $L \in \mathbb{Z}_+$. Assume that $m \in C^L(\mathbb{R}^{Nn} \setminus \{0,\dots,0\})$ satisfies*

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N} m(\xi)| \le C_{(\alpha_1,\dots,\alpha_N)}(|\xi_1| + \cdots + |\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}$$

for all $|\alpha_1| + \cdots + |\alpha_N| \le L$, $\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \setminus \{(0,\dots,0)\}$. Let $\Phi \in \mathcal{S}(\mathbb{R}^{Nn})$ be such that $\operatorname{supp}\Phi \not\ni (0,\dots,0)$. *Then*

$$\sup_{j\in\mathbb{Z}} \left\| m(2^j\cdot)\Phi(\cdot) \right\|_{H^L} < \infty.$$

Lemma 2.4.7 ([3, Section10, Theorem 1]). *Let $1 \le p_1,\dots,p_N \le \infty$, then*

$$\begin{aligned} &\left\| \|f * g(x_1,\dots,x_N)\|_{L^{p_1}(\mathbb{R}^n_{x_1})} \cdots \right\|_{L^{p_N}(\mathbb{R}^n_{x_N})} \\ &\le \left\| \|f(x_1,\dots,x_N)\|_{L^{p_1}(\mathbb{R}^n_{x_1})} \cdots \right\|_{L^{p_N}(\mathbb{R}^n_{x_N})} \times \left\| \|g(x_1,\dots,x_N)\|_{L^1(\mathbb{R}^n_{x_1})} \cdots \right\|_{L^1(\mathbb{R}^n_{x_N})}. \end{aligned}$$

Lemma 2.4.8 ([3, Section12. Theorem 1]). *Let $1 \le p_N \le p_{N-1} \le \cdots \le p_1 \le 2$, then*

$$\begin{aligned} &\left\| \|\mathcal{F}[f](\xi_1,\dots,\xi_N)\|_{L^{p'_1}(\mathbb{R}^n_{\xi_1})} \cdots \right\|_{L^{p'_N}(\mathbb{R}^n_{\xi_N})} \\ &\le \left\| \|f(\xi_1,\dots,\xi_N)\|_{L^{p_1}(\mathbb{R}^n_{\xi_1})} \cdots \right\|_{L^{p_N}(\mathbb{R}^n_{\xi_N})}. \end{aligned}$$

Lemma 2.4.9 ([10, Lemma 2.3]). *Let $s > 0$ and $\ell \in \mathbb{Z}$. Then the estimates*

(1) $\|FG\|_{B_{2,1}^{(s,\dots,s)}((\mathbb{R}^n)^N)} \lesssim \|F\|_{B_{2,1}^{(s,\dots,s)}((\mathbb{R}^n)^N)} \|G\|_{B_{\infty,1}^{(s,\dots,s)}((\mathbb{R}^n)^N)}$,

(2) $\|F(2^\ell \cdot)\|_{B_{2,1}^{(s,\dots,s)}((\mathbb{R}^n)^N)} \lesssim \left(\max\{1, 2^{\ell s}\} 2^{-\ell n/2}\right)^N \|F\|_{B_{2,1}^{(s,\dots,s)}((\mathbb{R}^n)^N)}$

holds.

Chapter 3

The proof of Theorem 1.4.1

In this chapter, we consider weighted norm inequalities for multilinear Fourier multipliers with Sobolev regularity. We first prove the following three lemmas which will be used in the proof of Theorem 1.4.1.

3.1 Lemmas

The proof of the following lemma is based on the argument of [32, Proposition 1.3.2] or [31, Lemma 3.3].

Lemma 3.1.1 ([7, Lemma A.1]). *Let $r > 0$, $2 \le q_i < \infty$ and $s_i \ge 0$, $i = 1, \dots, N$. Then,*

$$\|\widehat{F}\|_{L^{(q_1,\dots,q_N)}_{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} \lesssim \|F\|_{H^{(s_1/q_1,\dots,s_N/q_N)}((\mathbb{R}^n)^N)},$$

for all $F \in H^{(s_1/q_1,\dots,s_N/q_N)}((\mathbb{R}^n)^N)$ with $\operatorname{supp} F \subset \{x = (x_1, \dots, x_N) \in (\mathbb{R}^n)^N : |x| \le r\}$, where $L^{(q_1,\dots,q_N)}_{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ and $H^{(s_1/q_1,\dots,s_N/q_N)}((\mathbb{R}^n)^N)$ are the weighted Lebesgue space with mixed norm and the Sobolev space of product type, respectively.

Proof. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\phi(y) = 1 \text{ on } \{y \in \mathbb{R}^n : |y| \le r\}, \ \operatorname{supp} \phi \subset \{y \in \mathbb{R}^n : |y| \le 2r\}.$$

Since $\operatorname{supp} F \subset \{x = (x_1, \dots, x_N) \in (\mathbb{R}^n)^N : |x_i| \le r, i = 1, \dots, N\}$, we have $F(x_1, \dots, x_N) = \phi(x_1) \dots \phi(x_N) F(x_1, \dots, x_N)$. Then, it follows that

$$\mathcal{F}[F](\xi_1, \dots, \xi_N) = \left(\left[\widehat{\phi} \otimes \cdots \otimes \widehat{\phi} \right] * \widehat{F} \right) (\xi_1, \dots, \xi_N).$$

For $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$, we see that

$$\left\| \widehat{F}(\xi_1, \dots, \xi_N) \right\|^{q_1}_{L^{q_1}(\langle \xi_1 \rangle^{s_1})}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n_{\xi_1}} |\widehat{F}(\xi)|^{q_1} (1+|\xi_1|^2)^{s_1/2}\, d\xi_1 \\
&= \int_{\mathbb{R}^n_{\xi_1}} \left| \left(\widehat{\phi}\otimes\cdots\otimes\widehat{\phi}\right) * \widehat{F}(\xi)\right|^{q_1} (1+|\xi_1|^2)^{s_1/2}\, d\xi_1 \\
&= \int_{\mathbb{R}^n_{\xi_1}} \left| \int_{(\mathbb{R}^n)^N} \left(\widehat{\phi}\otimes\cdots\otimes\widehat{\phi}\right)(\xi_1-\eta_1,\dots\xi_N-\eta_N)\widehat{F}(\eta)\, d\eta\right|^{q_1} (1+|\xi_1|^2)^{s_1/2}\, d\xi_1 \\
&\le \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{(\mathbb{R}^n)^N} |\widehat{\phi}(\xi_1-\eta_1)|\dots|\widehat{\phi}(\xi_N-\eta_N)||\widehat{F}(\eta)|\, d\eta\right)^{q_1} (1+|\xi_1|^2)^{s_1/2}\, d\xi_1,
\end{aligned}
$$

where $\eta = (\eta_1, \dots, \eta_N) \in (\mathbb{R}^n)^N$ and $d\eta = d\eta_1 \dots d\eta_N$. By Minkowski's inequality for integrals, we obtain

(3.1.1)

$$
\begin{aligned}
&\left\|\widehat{F}(\xi_1,\dots,\xi_N)\right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \\
&\le \left\| \int_{(\mathbb{R}^n)^N} |\widehat{\phi}(\xi_1-\eta_1)|\dots|\widehat{\phi}(\xi_N-\eta_N)||\widehat{F}(\eta)|\, d\eta \right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \\
&\le \int_{(\mathbb{R}^n)^{N-1}_{\eta_2,\dots,\eta_N}} |\widehat{\phi}(\xi_N-\eta_N)|\dots|\widehat{\phi}(\xi_2-\eta_2)| \left\| \int_{\mathbb{R}^n_{\eta_1}} |\widehat{\phi}(\xi_1-\eta_1)||\widehat{F}(\eta)|\, d\eta_1 \right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \\
&\quad d\eta_2 \dots d\eta_N.
\end{aligned}
$$

Since $\langle\xi_1\rangle^{s_1} \lesssim \langle\xi_1-\eta_1\rangle^{s_1}\langle\eta_1\rangle^{s_1}$, we have

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}^n_{\eta_1}} |\widehat{\phi}(\xi_1-\eta_1)||\widehat{F}(\eta)|\, d\eta_1 \right\|^{q_1}_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \\
&= \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} \left|\widehat{\phi}(\xi_1-\eta_1)\right| \left|\widehat{F}(\eta)\right| \, d\eta_1 \right)^{q_1} (1+|\xi_1|^2)^{s_1/2}\, d\xi_1 \\
&= \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} \frac{(1+|\xi_1|^2)^{s_1/2q_1}}{(1+|\eta_1|^2)^{s_1/2q_1}} |\widehat{\phi}(\xi_1-\eta_1)|(1+|\eta_1|^2)^{s_1/2q_1}|\widehat{F}(\eta)|\, d\eta_1 \right)^{q_1} d\xi_1 \\
&\lesssim \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)|(1+|\eta_1|^2)^{s_1/2q_1}|\widehat{F}(\eta)|\, d\eta_1 \right)^{q_1-2+2} d\xi_1.
\end{aligned}
$$

Hence, we see that

(3.1.2)

$$
\left\| \int_{\mathbb{R}^n_{\eta_1}} |\widehat{\phi}(\xi_1-\eta_1)||\widehat{F}(\eta)|\, d\eta_1 \right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})}
$$

$$\leq \sup_{\xi_1 \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)| (1+|\eta_1|^2)^{s_1/2q_1} |\widehat{F}(\eta)| \, d\eta_1 \right)^{(q_1-2)/q_1}$$
$$\times \left\{ \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)| (1+|\eta_1|^2)^{s_1/2q_1} |\widehat{F}(\eta)| \, d\eta_1 \right)^2 d\xi_1 \right\}^{1/q_1}.$$

For the first term on the right hand side of (3.1.2), by Schwarz's inequality and a change of variables, we have

$$\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)| (1+|\eta_1|^2)^{s_1/2q_1} |\widehat{F}(\eta)| \, d\eta_1$$
$$\leq \left\| (1+|\xi_1-\cdot|^2)^{s_1/2q_1} \widehat{\phi}(\xi_1-\cdot) \right\|_{L^2(\mathbb{R}^n_{\eta_1})} \| (1+|\cdot|^2)^{s_1/2q_1} \widehat{F}(\cdot,\eta_2,\ldots,\eta_N) \|_{L^2(\mathbb{R}^n_{\eta_1})}$$
$$= \left\| (1+|\cdot|^2)^{s_1/2q_1} \widehat{\phi} \right\|_{L^2(\mathbb{R}^n_{\eta_1})} \left\| (1+|\cdot|^2)^{s_1/2q_1} \widehat{F}(\cdot,\eta_2,\ldots,\eta_N) \right\|_{L^2(\mathbb{R}^n_{\eta_1})},$$

where $\xi_1 \in \mathbb{R}^n$. Thus, we obtain

(3.1.3)
$$\sup_{\xi_1 \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)| (1+|\eta_1|^2)^{s_1/2q_1} |\widehat{F}(\eta)| \, d\eta_1 \right)^{(q_1-2)/q_1}$$
$$\leq \left\| (1+|\cdot|^2)^{s_1/2q_1} \widehat{\phi} \right\|^{(q_1-2)/q_1}_{L^2(\mathbb{R}^n_{\eta_1})} \left\| (1+|\cdot|^2)^{s_1/2q_1} \widehat{F}(\cdot,\eta_2,\ldots,\eta_N) \right\|^{(q_1-2)/q_1}_{L^2(\mathbb{R}^n_{\eta_1})}.$$

For the second term on the right hand side of (3.1.2), by Young's inequality, we have

$$\int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)| (1+|\eta_1|^2)^{s_1/2q_1} |\widehat{F}(\eta)| \, d\eta_1 \right)^2 d\xi_1$$
$$= \int_{\mathbb{R}^n_{\xi_1}} \left[\left\{ (1+|\cdot|^2)^{s_1/2q_1} |\widehat{\phi}(\cdot)| \right\} * \left\{ (1+|\cdot|^2)^{s_1/2q_1} |\widehat{F}(\cdot,\eta_2,\ldots,\eta_N)| \right\} (\xi_1) \right]^2 d\xi_1$$
$$= \left\| \left\{ (1+|\cdot|^2)^{s_1/2q_1} |\widehat{\phi}(\cdot)| \right\} * \left\{ (1+|\cdot|^2)^{s_1/2q_1} |\widehat{F}(\cdot,\eta_2,\ldots,\eta_N)| \right\} (\xi_1) \right\|^2_{L^2(\mathbb{R}^n_{\xi_1})}$$
$$\leq \left\| (1+|\cdot|^2)^{s_1/2q_1} \widehat{\phi} \right\|^2_{L^1(\mathbb{R}^n_{\xi_1})} \left\| (1+|\cdot|^2)^{s_1/2q_1} |\widehat{F}(\cdot,\eta_2,\ldots,\eta_N)| \right\|^2_{L^2(\mathbb{R}^n_{\xi_1})}.$$

Thus, we obtain

(3.1.4)
$$\left\{ \int_{\mathbb{R}^n_{\xi_1}} \left(\int_{\mathbb{R}^n_{\eta_1}} (1+|\xi_1-\eta_1|^2)^{s_1/2q_1} |\widehat{\phi}(\xi_1-\eta_1)| (1+|\eta_1|^2)^{s_1/2q_1} |\widehat{F}(\eta)| \, d\eta_1 \right)^2 d\xi_1 \right\}^{1/q_1}$$

$$\leq \left\|(1+|\cdot|^2)^{s_1/2q_1}\widehat{\phi}\right\|_{L^1(\mathbb{R}^n_{\xi_1})}^{2/q_1} \left\|(1+|\cdot|^2)^{s_1/2q_1}|\widehat{F}(\cdot,\eta_2,\dots,\eta_N)|\right\|_{L^2(\mathbb{R}^n_{\xi_1})}^{2/q_1}.$$

By (3.1.2), (3.1.3) and (3.1.4), we see that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}^n_{\eta_1}} |\widehat{\phi}(\xi_1-\eta_1)||\widehat{F}(\eta)|\,d\eta_1\right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \\
&\leq \left\|(1+|\cdot|^2)^{s_1/2q_1}\widehat{\phi}\right\|_{L^2(\mathbb{R}^n_{\eta_1})}^{(q_1-2)/q_1} \left\|(1+|\cdot|^2)^{s_1/2q_1}\widehat{F}(\cdot,\eta_2,\dots,\eta_N)\right\|_{L^2(\mathbb{R}^n_{\eta_1})}^{(q_1-2)/q_1} \\
&\quad \times \left\|(1+|\cdot|^2)^{s_1/2q_1}\widehat{\phi}\right\|_{L^1(\mathbb{R}^n_{\xi_1})}^{2/q_1} \left\|(1+|\cdot|^2)^{s_1/2q_1}|\widehat{F}(\cdot,\eta_2,\dots,\eta_N)|\right\|_{L^2(\mathbb{R}^n_{\xi_1})}^{2/q_1} \\
&\lesssim \left\|(1+|\cdot|^2)^{s_1/2q_1}|\widehat{F}(\cdot,\eta_2,\dots,\eta_N)|\right\|_{L^2(\mathbb{R}^n_{\xi_1})}.
\end{aligned}$$

Therefore, it follows that

$$\left\|\int_{\mathbb{R}^n_{\eta_1}} \left|\widehat{\phi}(\xi_1-\eta_1)\right| \left|\widehat{F}(\eta)\right|\,d\eta_1\right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})} \lesssim \left\|\widehat{F}(\xi_1,\eta_2,\dots,\eta_N)\right\|_{L^2(\langle\xi_1\rangle^{s_1/q_1})}.$$

By the same way for $\xi_2 \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
(3.1.5)\qquad &\left\|\int_{\mathbb{R}^n_{\eta_2}} |\widehat{\phi}(\xi_2-\eta_2)| \left\|\widehat{F}(\xi_1,\eta_2,\dots,\eta_N)\right\|_{L^2(\langle\xi_1\rangle^{s_1/q_1})}\,d\eta_2\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})} \\
&\lesssim \left\|\left\|\widehat{F}(\xi_1,\xi_2,\eta_3,\dots,\eta_N)\right\|_{L^2(\langle\xi_1\rangle^{s_1/q_1})}\right\|_{L^2(\langle\xi_2\rangle^{s_2/q_2})}.
\end{aligned}$$

By (3.1.1) and (3.1.5), we have

$$\begin{aligned}
&\left\|\left\|\widehat{F}(\xi_1,\xi_2,\dots,\xi_N)\right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})}\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})} \\
&\leq \left\|\int_{(\mathbb{R}^n)^{N-1}_{\eta_2,\dots,\eta_N}} |\widehat{\phi}(\xi_N-\eta_N)|\dots|\widehat{\phi}(\xi_2-\eta_2)| \left\|\int_{\mathbb{R}^n_{\eta_1}} |\widehat{\phi}(\xi_1-\eta_1)||\widehat{F}(\eta)|\,d\eta_1\right\|_{L^{q_1}(\langle\xi_1\rangle^{s_1})}\right. \\
&\quad \left.\times\ d\eta_2\dots d\eta_N\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})} \\
&\lesssim \left\|\int_{(\mathbb{R}^n)^{N-1}_{\eta_2,\dots,\eta_N}} |\widehat{\phi}(\xi_N-\eta_N)|\dots|\widehat{\phi}(\xi_2-\eta_2)| \left\|\widehat{F}(\xi_1,\eta_2,\dots,\eta_N)\right\|_{L^2(\langle\xi_1\rangle^{s_1/q_1})}\right. \\
&\quad \left.\times\ d\eta_2\dots d\eta_N\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})} \\
&\leq \int_{(\mathbb{R}^n)^{N-2}_{\eta_3,\dots,\eta_N}} |\widehat{\phi}(\xi_N-\eta_N)|\dots|\widehat{\phi}(\xi_3-\eta_3)| \\
&\quad \times \left\|\int_{\mathbb{R}^n_{\eta_2}} |\widehat{\phi}(\xi_2-\eta_2)| \left\|\widehat{F}(\xi_1,\eta_2,\dots,\eta_N)\right\|_{L^2(\langle\xi_1\rangle^{s_1/q_1})}\,d\eta_2\right\|_{L^{q_2}(\langle\xi_2\rangle^{s_2})} d\eta_3\dots d\eta_N
\end{aligned}$$

$$\lesssim \int_{(\mathbb{R}^n)^{N-2}_{\eta_3,\ldots,\eta_N}} |\widehat{\phi}(\xi_N - \eta_N)| \ldots |\widehat{\phi}(\xi_3 - \eta_3)|$$

$$\times \left\| \left\| \widehat{F}(\xi_1, \xi_2, \eta_3, \ldots, \eta_N) \right\|_{L^2(\langle \xi_1 \rangle^{s_1/q_1})} \right\|_{L^2(\langle \xi_2 \rangle^{s_2/q_2})} d\eta_3 \ldots d\eta_N.$$

By the same way for $i = 3, \ldots, N$, we have the desired estimate.

□

The following is a key lemma in the proof of Theorem 1.4.1.

Lemma 3.1.2 ([7, Lemma 6.1])**.** *Let $r > 0$, $n/2 < s_i$ and $\max\{1, n/s_i\} < q_i < 2$, $i = 1, \ldots, N$. Then, the estimate*

$$\left| T_{m(\cdot/2^j)}(f_1, \ldots, f_N)(x) \right| \lesssim \|m\|_{H^{(s_1,\ldots,s_N)}((\mathbb{R}^n)^N)} \prod_{i=1}^{N} M(|f_i|^{q_i})(x)^{1/q_i} \tag{3.1.6}$$

holds, for all $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$ and $m \in H^{(s_1,\ldots,s_N)}((\mathbb{R}^n)^N)$ with $\operatorname{supp} m \subset \{\xi = (\xi_1, \ldots, \xi_N) \in (\mathbb{R}^n)^N : |\xi| \le r\}$, where $H^{(s_1,\ldots,s_N)}((\mathbb{R}^n)^N)$ is the Sobolev space of product type.

Proof. For $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, by Fubini's Theorem, we obtain

(3.1.7)

$$\begin{aligned}
&T_{m(\cdot/2^j)}(f_1, \ldots, f_N)(x) \\
&= \int_{(\mathbb{R}^n)^N} \mathcal{F}^{-1}[m(\cdot/2^j, \ldots, \cdot/2^j)](x - y_1, \ldots, x - y_N) f_1(y_1) \ldots f_N(y_N)\, dy_1 \ldots dy_N \\
&= (2^{jn})^N \int_{(\mathbb{R}^n)^N} \mathcal{F}^{-1}[m] \left(2^j(x - y_1), \ldots, 2^j(x - y_N)\right) f_1(y_1) \ldots f_N(y_N)\, dy_1 \ldots dy_N \\
&= (2^{jn})^N \int_{(\mathbb{R}^n)^N} (1 + 2^j|x - y_1|)^{s_1 - s_1} \ldots (1 + 2^j|x - y_N|)^{s_N - s_N} \\
&\quad \times \mathcal{F}^{-1}[m] \left(2^j(x - y_1), \ldots, 2^j(x - y_N)\right) f_1(y_1) \ldots f_N(y_N)\, dy_1 \ldots dy_N \\
&= (2^{jn})^N \int_{(\mathbb{R}^n)^{N-1}_{y_2,\ldots,y_N}} (1 + 2^j|x - y_N|)^{s_N - s_N} f_N(y_N) \ldots (1 + 2^j|x - y_2|)^{s_2 - s_2} f_2(y_2) \\
&\quad \times \Bigg(\int_{\mathbb{R}^n_{y_1}} (1 + 2^j|x - y_1|)^{s_1} \mathcal{F}^{-1}[m] \left(2^j(x - y_1), \ldots, 2^j(x - y_N)\right) \\
&\qquad \times (1 + 2^j|x - y_1|)^{-s_1} f_1(y_1)\, dy_1 \Bigg) \\
&\quad \times\, dy_2 \ldots dy_N.
\end{aligned}$$

By Hölder's inequality and Lemma 2.4.5 with $\phi(x) = (1 + |x|)^{-s_1 q_1}$, we see that

(3.1.8)

$$\left| \int_{\mathbb{R}^n_{y_1}} (1 + 2^j|x - y_1|)^{s_1} \mathcal{F}^{-1}[m] \left(2^j(x - y_1), \ldots, 2^j(x - y_N)\right) \right.$$

$$\times (1+2^j|x-y_1|)^{-s_1} f_1(y_1)\, dy_1 \Big|$$

$$\le \left\{ \int_{\mathbb{R}^n_{y_1}} (1+2^j|x-y_1|)^{s_1 q_1'} \left|\mathcal{F}^{-1}[m]\left(2^j(x-y_1),\dots,2^j(x-y_N)\right)\right|^{q_1'}\, dy_1 \right\}^{1/q_1'}$$

$$\times \left\{ \int_{\mathbb{R}^n_{y_1}} \frac{|f_1(y_1)|^{q_1}}{(1+2^j|x-y_1|)^{s_1 q_1}}\, dy_1 \right\}^{1/q_1}$$

$$\lesssim \left\{ \int_{\mathbb{R}^n_{y_1}} (1+2^j|x-y_1|)^{s_1 q_1'} \left|\mathcal{F}^{-1}[m](2^j(x-y_1),\dots,2^j(x-y_N))\right|^{q_1'}\, dy_1 \right\}^{1/q_1'}$$

$$\times M(|f_1|^{q_1})(x)^{1/q_1},$$

where we have used the fact that $s_1 q_1 > n$. Thus, we have

$$\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right|$$

$$\lesssim (2^{jn})^N M(|f_1|^{q_1})(x)^{1/q_1}$$

$$\times \int_{(\mathbb{R}^n)^{N-2}_{y_3,\dots,y_N}} (1+2^j|x-y_N|^{s_N-s_N}|f_N(y_N)|\dots(1+2^j|x-y_3|^{s_3-s_3}|f_3(y_3)|$$

$$\times \Bigg(\int_{\mathbb{R}^n_{y_2}} (1+2^j|x-y_2|)^{s_2}$$

$$\times \left\{ \int_{\mathbb{R}^n_{y_1}} (1+2^j|x-y_1|)^{s_1 q_1'} \left|\mathcal{F}^{-1}[m]\left(2^j(x-y_1),\dots,2^j(x-y_N)\right)\right|^{q_1'}\, dy_1 \right\}^{1/q_1'}$$

$$\times (1+2^j|x-y_2|)^{-s_2}|f_2(y_2)|\, dy_2 \Bigg)$$

$$\times\, dy_3\dots dy_N.$$

By the same way, we see that

$$\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right|$$

$$\lesssim (2^{jn})^N \prod_{i=1}^{2} M\left(|f_i|^{q_i}\right)(x)^{1/q_i}$$

$$\times \int_{(\mathbb{R}^n)^{N-2}_{y_3,\dots,y_N}} (1+2^j|x-y_N|^{s_N-s_N}|f_N(y_N)|\dots(1+2^j|x-y_3|^{s_3-s_3}|f_3(y_3)|$$

$$\times \Bigg\{ \int_{\mathbb{R}^n_{y_2}} (1+2^j|x-y_2|)^{s_2 q_2'}$$

$$\times \left(\int_{\mathbb{R}^n_{y_1}} (1+2^j|x-y|)^{s_1 q_1'} \left|\mathcal{F}^{-1}[m]\left(2^j(x-y_1),\dots,2^j(x-y_N)\right)\right|^{q_1'}\, dy_1 \right)^{q_2'/q_1'}$$

$$\times\, dy_2 \Bigg\}^{1/q_2'} dy_3\dots dy_N.$$

By the same way, we obtain

(3.1.9)

$$\begin{aligned}
&\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right| \\
&\lesssim (2^{jn})^N \prod_{i=1}^N M\left(|f_i|^{q_i}\right)(x)^{1/q_i} \\
&\quad \times \Bigg\{ \int_{\mathbb{R}^n_{y_N}} (1+2^j|x-y_N|)^{s_N q_N'} \dots \\
&\qquad \times \left\{ \int_{\mathbb{R}^n_{y_1}} (1+2^j|x-y_1|)^{s_1 q_1'} \left|\mathcal{F}^{-1}[m]\left(2^j(x-y_1),\dots,2^j(x-y_N)\right)\right|^{q_1'} dy_1 \right\}^{q_2'/q_1'} \\
&\quad \times \dots dy_N \Bigg\}^{1/q_N'}.
\end{aligned}$$

By the change of variables, we see that

$$\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right| \lesssim \|\widehat{m}\|_{L^{(q_1',\dots,q_N')}_{(s_1q_1',\dots,s_Nq_N')}} \prod_{i=1}^N M\left(|f_i|^{q_i}\right)(x)^{1/q_i}. \tag{3.1.10}$$

By Lemma 3.1.1, we have the desired estimate. □

Lemma 3.1.3 ([7, Proposition A.2])**.** *If $s_i > n/2$, $i=1,\dots,N$, then $H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)$ is a multiplication algebra.*

Proof. We consider only the case $N=2$. Note that for all $\eta_i \in \mathbb{R}^n$,

$$\langle \xi_i\rangle^{s_i} \lesssim \langle \xi_i-\eta_i\rangle^{s_i} + \langle \eta_i\rangle^{s_i}, \quad i=1,2.$$

Hence, we obtain

(3.1.11)

$$\begin{aligned}
&\|FG\|_{H^{(s_1,s_2)}((\mathbb{R}^n)^2)} \\
&= \left\|\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\widehat{FG}(\xi_1,\xi_2)\right\|_{L^2_{(\xi_1,\xi_2)}} \\
&= \left\|\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}\left(\widehat{F}*\widehat{G}\right)(\xi_1,\xi_2)\right\|_{L^2_{(\xi_1,\xi_2)}} \\
&\lesssim \Bigg\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \left(\langle\xi_1-\eta_1\rangle^{s_1} + \langle\eta_1\rangle^{s_1}\right)\left(\langle\xi_2-\eta_2\rangle^{s_2} + \langle\eta_2\rangle^{s_2}\right) \\
&\quad \times \left|\widehat{F}(\xi_1-\eta_1,\xi_2-\eta_2)\right| \left|\widehat{G}(\eta_1,\eta_2)\right| d\eta_1 d\eta_2 \Bigg\|_{L^2_{(\xi_1,\xi_2)}}
\end{aligned}$$

$$\le \left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \xi_1 - \eta_1 \rangle^{s_1} \langle \xi_2 - \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}}$$

$$+ \left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \xi_1 - \eta_1 \rangle^{s_1} \langle \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}}$$

$$+ \left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \eta_1 \rangle^{s_1} \langle \xi_2 - \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}}$$

$$+ \left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \eta_1 \rangle^{s_1} \langle \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}}.$$

For the first term on the right hand side of (3.1.11), by Young's inequality, we see that

(3.1.12)

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \xi_1 - \eta_1 \rangle^{s_1} \langle \xi_2 - \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}} \\
&= \left\| \left(\langle \cdot \rangle^{s_1} \langle \cdot \rangle^{s_2} \widehat{F} * \widehat{G} \right) (\xi_1, \xi_2) \right\|_{L^2_{(\xi_1,\xi_2)}} \\
&\lesssim \left\| \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \widehat{F}(\xi_1, \xi_2) \right\|_{L^2_{(\xi_1,\xi_2)}} \left\| \widehat{G}(\xi_1, \xi_2) \right\|_{L^1_{(\xi_1,\xi_2)}} \\
&\lesssim \|F\|_{H^{(s_1,s_2)}} \|G\|_{H^{(s_1,s_2)}},
\end{aligned}$$

where we have used the fact that $s_i > n/2$, $i = 1, 2$. For the forth term on the right hand side of (3.1.11), by the same way, we also have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \eta_1 \rangle^{s_1} \langle \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}} \\
&= \left\| \left(\widehat{F} * \langle \cdot \rangle^{s_1} \langle \cdot \rangle^{s_2} \widehat{G} \right) (\xi_1, \xi_2) \right\|_{L^2_{(\xi_1,\xi_2)}} \\
&\lesssim \left\| \widehat{F}(\xi_1, \xi_2) \right\|_{L^1_{(\xi_1,\xi_2)}} \left\| \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \widehat{G}(\xi_1, \xi_2) \right\|_{L^2_{(\xi_1,\xi_2)}} \\
&\lesssim \|F\|_{H^{(s_1,s_2)}} \|G\|_{H^{(s_1,s_2)}},
\end{aligned} \tag{3.1.13}$$

where we have used the fact that $s_i > n/2$, $i = 1, 2$.

For the second term on the right hand side of (3.1.11), by Minkowski's inequality for integrals and Young's inequality, we see that

(3.1.14)

$$\left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \xi_1 - \eta_1 \rangle^{s_1} \langle \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| \, d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}}$$

$$\lesssim \left\| \left\| \langle \xi_1 \rangle^{s_1} \widehat{F}(\xi_1, \xi_2) \right\|_{L^2(\xi_1)} \right\|_{L^1(\xi_2)} \left\| \left\| \langle \xi_2 \rangle^{s_2} \widehat{G}(\xi_1, \xi_2) \right\|_{L^1(\xi_1)} \right\|_{L^2(\xi_2)}.$$

For the first term on the right hand side of (3.1.14), by Schwartz's inequality, we obtain

$$\left\| \left\| \langle \xi_1 \rangle^{s_1} \widehat{F}(\xi_1, \xi_2) \right\|_{L^2(\xi_1)} \right\|_{L^1(\xi_2)} \lesssim \|F(\xi_1, \xi_2)\|_{H^{(s_1,s_2)}}, \tag{3.1.15}$$

where we have used the fact that $s_2 > n/2$. For the second term of (3.1.14), by Schwartz's inequality, we also have

$$\left\| \left\| \langle \xi_2 \rangle^{s_2} \widehat{G}(\xi_1, \xi_2) \right\|_{L^1(\xi_1)} \right\|_{L^2(\xi_2)} \lesssim \|G(\xi_1, \xi_2)\|_{H^{(s_1,s_2)}}, \tag{3.1.16}$$

where we have used the fact that $s_1 > n/2$. By (3.1.14), (3.1.15) and (3.1.16), we have

(3.1.17)

$$\left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \xi_1 - \eta_1 \rangle^{s_1} \langle \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}} \lesssim \|F\|_{H^{(s_1,s_2)}} \|G\|_{H^{(s_1,s_2)}}.$$

For the third term on the right hand side of (3.1.11), by the same way, we see that

(3.1.18)

$$\left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \eta_1 \rangle^{s_1} \langle \xi_2 - \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}} \lesssim \left\| \left\| \langle \xi_2 \rangle^{s_2} \widehat{F}(\xi_1, \xi_2) \right\|_{L^2(\xi_1)} \right\|_{L^1(\xi_2)} \left\| \left\| \langle \xi_1 \rangle^{s_1} \widehat{G}(\xi_1, \xi_2) \right\|_{L^1(\xi_1)} \right\|_{L^2(\xi_2)}.$$

For the first term on the right hand side of (3.1.18), by Schwartz's inequality, we obtain

$$\left\| \left\| \langle \xi_2 \rangle^{s_2} \widehat{F}(\xi_1, \xi_2) \right\|_{L^2(\xi_1)} \right\|_{L^1(\xi_2)} \lesssim \|F(\xi_1, \xi_2)\|_{H^{(s_1,s_2)}}, \tag{3.1.19}$$

where we have used the fact that $s_1 > n/2$. For the second term on the right hand side of (3.1.18), by Schwartz's inequality, we have

$$\left\| \left\| \langle \xi_1 \rangle^{s_1} \widehat{G}(\xi_1, \xi_2) \right\|_{L^1(\xi_1)} \right\|_{L^2(\xi_2)} \lesssim \|G(\xi_1, \xi_2)\|_{H^{(s_1,s_2)}}, \tag{3.1.20}$$

where we have used the fact that $s_2 > n/2$. By (3.1.18), (3.1.19) and (3.1.20), we see that

(3.1.21)

$$\left\| \int_{\mathbb{R}^2_{\eta_1,\eta_2}} \langle \eta_1 \rangle^{s_1} \langle \xi_2 - \eta_2 \rangle^{s_2} \left| \widehat{F}(\xi_1 - \eta_1, \xi_2 - \eta_2) \right| \left| \widehat{G}(\eta_1, \eta_2) \right| d\eta_1 d\eta_2 \right\|_{L^2_{(\xi_1,\xi_2)}}$$

$$\lesssim \|F\|_{H^{(s_1,s_2)}} \|G\|_{H^{(s_1,s_2)}}.$$

By (3.1.11), (3.1.12), (3.1.13), (3.1.17) and (3.1.21), we have the desired estimate. □

Now, we prove Theorem 1.4.1. Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $n/2 < s_i \le n$ for $i = 1, \dots, N$. Assume $p_i > n/s_i$ and $w_i \in A_{p_i/(n/s_i)}$ for $i = 1, \cdots, N$ and set $w = w_1^{p/p_1} \dots w_N^{p/p_N}$. We also assume that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies $\sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} < \infty$, where m_j is defined by (2.2.2). Since $n/s_i < \min\{2, p_i\}$ and $w_i \in A_{p_i/(n/s_i)}$ for $i = 1, \dots, N$, by Lemma 2.4.2 (2), we can take $n/s_i < q_i < \min\{2, p_i\}$ satisfying $w_i \in A_{p_i/q_i}$ for $i = 1, \dots, N$. By Lemma 2.4.1 (1), we decompose m as follows:

$$\begin{aligned} m(\xi) &= \sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N, \\ (i_1,\dots,i_N)\neq(0,\dots,0)}} \Phi_{(i_1,\dots,i_N)}(\xi) m(\xi) \\ &= \sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N, \\ (i_1,\dots,i_N)\neq(0,\dots,0)}} m_{(i_1,\dots,i_N)}(\xi). \end{aligned}$$

3.2 Estimate for $m_{(1,0,\dots,0)}$ type

We first consider the case where $(i_1, \dots, i_N)$ satisfies $\sharp\{j : i_j = 1\} = 1$. Without loss of generality, we may assume that $i_1 = 1$. We simply write m instead of $m_{(1,0,\dots,0)}$. Note that by Lemma 2.4.1 (3),

$$\text{(3.2.1)} \qquad \operatorname{supp} m \subset \{\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N : |\xi_i| \le |\xi_1|/N, i = 2, \dots, N\}.$$

It is easy to see that if $\xi = (\xi_1, \dots, \xi_N) \in \operatorname{supp} m$, then $|\xi_1 + \cdots + \xi_N| \approx |\xi_1|$.

Proof. In this case, we shall prove the estimate

$$\text{(3.2.2)} \qquad \begin{aligned} &\|T_m(f_1, \dots, f_N)\|_{L^p(w)} \\ &\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} \right) \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)} \end{aligned}$$

holds, where $m^{(j)}$ will be defined later on. In Section 3.4, we shall complete the proof.

Let ψ be as in (2.2.1) with $d = n$. Since $w \in A_{Np} \subset A_\infty$ ([19, p.1232]), we can use the way of Grafakos-Si ([13, Lemma 2.4] or [7, Remark 2.6]),

$$\text{(3.2.3)} \qquad \|T_m(f_1, \dots, f_N)\|_{L^p(w)} \lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} |\Delta_j T_m(f_1, \dots, f_N)|^2 \right\}^{1/2} \right\|_{L^p(w)},$$

where $\Delta_j g = \psi(D/2^j) g$.

By Fubini's Theorem and the Fourier inversion formula, we see that

$$\begin{aligned}
&\Delta_j T_m(f_1, \dots, f_N)(x) \\
&= \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j) \\
&\quad \times \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi.
\end{aligned}$$

We shall prove that for $\xi = (\xi_1, \dots, \xi_N) \in \operatorname{supp} m$ satisfying $2^{j-1} \le |\xi_1 + \cdots + \xi_N| \le 2^{j+1}$, we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$\begin{aligned}
(3.2.4) \qquad & m(\xi)\psi\left(\frac{\xi_1 + \cdots + \xi_N}{2^j}\right) \\
&= m(\xi)\psi\left(\frac{\xi_1 + \cdots + \xi_N}{2^j}\right) \widetilde{\psi}(\xi_1/2^j)^2 \varphi(\xi_2/2^j) \cdots \varphi(\xi_N/2^j).
\end{aligned}$$

Once this is proved, setting

$$(3.2.5) \qquad m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1 + \cdots + \xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2) \cdots \varphi(\xi_N),$$

we have

$$(3.2.6) \qquad \Delta_j T_m(f_1, \dots, f_N)(x) = T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, f_2, \dots, f_N)(x),$$

where $\tilde{\Delta}_j f_1 = \widetilde{\psi}(D/2^j) f_1$. Let $\xi = (\xi_1, \dots, \xi_N) \in \operatorname{supp} m$ satisfying $2^{j-1} \le |\xi_1 + \cdots + \xi_N| \le 2^{j+1}$. We can find a function $\widetilde{\psi} \in \mathcal{A}_1$ such that $\widetilde{\psi}(\xi_1) = 1$ on $\{\xi_1 \in \mathbb{R}^n : N/2(2N-1) \le |\xi_1| \le 2N\}$, and we obtain

$$m(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j) = m(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j)\widetilde{\psi}(\xi_1/2^j)^2,$$

where we have used the fact that by (3.2.1), in this area, $N/2(2N-1) \le |\xi_1|/2^j \le 2N$ holds. Moreover we can find a function $\varphi \in \mathcal{A}_0$ such that $\varphi(\xi_i) = 1$ on $\{\xi_i \in \mathbb{R}^n : |\xi_i| \le 2\}$, $i = 2, \dots, N$, and we obtain

$$\begin{aligned}
& m(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j)\widetilde{\psi}(\xi_1/2^j)^2 \\
&= m(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j)\widetilde{\psi}(\xi_1/2^j)^2\varphi(\xi_2/2^j) \dots \varphi(\xi_N/2^j)
\end{aligned}$$

where we have the fact that by (3.2.1), in this area, $|\xi_i|/2^j \le 2$, $i = 2, \dots, N$ holds. Combining these, we have (3.2.4).

By Lemma 3.1.2, we see that

$$\begin{aligned}
& \sum_{j\in\mathbb{Z}} \left| T_{m^{(j)}(\cdot/2^j)}(\widetilde{\Delta}_j f_1, f_2, \dots, f_N)(x) \right|^2 \\
& \lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}} \right)^2
\end{aligned}$$

$$\times \sum_{j\in\mathbb{Z}} M(|\widetilde{\Delta}_j f_1|^{q_1})(x)^{2/q_1} \times \prod_{i=2}^N M(|f_i|^{q_i})(x)^{2/q_i}.$$

By Hölder's inequality, we have

$$\begin{aligned}
(3.2.7)\qquad & \left\| \left\{ \sum_{j\in\mathbb{Z}} \left| T_{m^{(j)}(\cdot/2^j)}(\widetilde{\Delta}_j f_1, f_2, \dots, f_N) \right|^2 \right\}^{1/2} \right\|_{L^p(w)} \\
& \lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}} \right) \\
& \quad \times \left\| \left\{ \sum_{j\in\mathbb{Z}} M(|\widetilde{\Delta}_j f_1|^{q_1})^{2/q_1} \right\}^{1/2} \right\|_{L^{p_1}(w_1)} \prod_{i=2}^N \left\| M(|f_i|^{q_i})^{1/q_i} \right\|_{L^{p_i}(w_i)}.
\end{aligned}$$

For the second term on the right hand side of (3.2.7), since $1 < 2/q_1, p_1/q_1$ and $w_1 \in A_{p_1/q_1}$, it follows from Lemmas 2.4.4 and 2.4.3 that

(3.2.8)

$$\begin{aligned}
\left\| \left\{ \sum_{j\in\mathbb{Z}} M(|\widetilde{\Delta}_j f_1|^{q_1})^{2/q_1} \right\}^{1/2} \right\|_{L^{p_1}(w_1)} &= \left\| \left\{ \sum_{j\in\mathbb{Z}} M(|\widetilde{\Delta}_j f_1|^{q_1})^{2/q_1} \right\}^{q_1/2} \right\|_{L^{p_1/q_1}(w_1)}^{1/q_1} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f_1|^2 \right)^{q_1/2} \right\|_{L^{p_1/q_1}(w_1)}^{1/q_1} \\
&= \left\| \left(\sum_{j\in\mathbb{Z}} |\widetilde{\Delta}_j f_1|^2 \right)^{1/2} \right\|_{L^{p_1}(w_1)} \\
&\lesssim \|f\|_{L^{p_1}(w_1)}.
\end{aligned}$$

For the third term on the right hand side of (3.2.7), since $p_i > q_i$ and $w_i \in A_{p_i/q_i}$, $i = 2, \dots, N$, we see that

$$\begin{aligned}
(3.2.9)\qquad \prod_{i=2}^N \left\| M\left(|f_i|^{q_i}\right)^{1/q_i} \right\|_{L^{p_i}(w_i)} &= \prod_{i=2}^N \|M\left(|f_i|^{q_i}\right)\|_{L^{p_i/q_i}(w_i)}^{1/q_i} \\
&\lesssim \prod_{i=2}^N \||f_i|^{q_i}\|_{L^{p_i/q_i}(w_i)}^{1/q_i} \\
&= \prod_{i=2}^N \|f_i\|_{L^{p_i}(w_i)},
\end{aligned}$$

where we have used the boundedness of M on $L^{p_i/q_i}(w_i)$, $i = 2, \dots, N$. By (3.2.3), (3.2.6), (3.2.7), (3.2.8) and (3.2.9), we obtain (3.2.2). □

3.3 Estimate for $m_{(1,1,i_3,\dots,i_N)}$ type

Next, we consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j = 1\} \geq 2$. Without loss of generality, we may assume that $i_1 = i_2 = 1$. We simply write m instead of $m_{(1,1,i_3,\dots,i_N)}$, where $i_3,\dots,i_N \in \{0,1\}$. Note that by Lemma 2.4.1 (3),

$$\text{(3.3.1)} \qquad \operatorname{supp} m \subset \{|\xi_1|/(4N) \leq |\xi_2| \leq 4N|\xi_1|, |\xi_i| \leq 4N|\xi_1|, i = 3,\dots,N\}.$$

Proof. In this case, we shall prove the estimate

$$\text{(3.3.2)} \qquad \|T_m(f_1,\dots,f_N)\|_{L^p(w)} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)}\right) \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}$$

holds, where $m^{(j)}$ will be defined later on. In Section 3.4, we shall complete the proof.

Since ψ is in (2.2.1) with $d = n$, we see that

$$T_m(f_1,\dots,f_N)(x) = \sum_{j\in\mathbb{Z}} \frac{1}{(2\pi)^{Nn}} \int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\dots+\xi_N)} m(\xi)\psi(\xi_1/2^j)\widehat{f_1}(\xi_1)\dots\widehat{f_N}(\xi_N)\,d\xi.$$

We shall prove that for $\xi = (\xi_1,\dots,\xi_N) \in \operatorname{supp} m$ and $\xi_1 \in \operatorname{supp}\psi(\cdot/2^j)$, we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$\text{(3.3.3)} \qquad m(\xi)\psi(\xi_1/2^j) = m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2\varphi(\xi_3/2^j)\dots\varphi(\xi_N/2^j).$$

Once this is proved, setting

$$\text{(3.3.4)} \qquad m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1)\widetilde{\psi}(\xi_2)\varphi(\xi_3)\dots\varphi(\xi_N),$$

we have

$$\text{(3.3.5)} \qquad T_m(f_1,\dots,f_N)(x) = \sum_{j\in\mathbb{Z}} T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, \tilde{\Delta}_j f_2, f_3,\dots,f_N)(x),$$

where $\tilde{\Delta}_j f_i = \widetilde{\psi}(D/2^j)f_i, i = 1,2$. Let $\xi = (\xi_1,\dots,\xi_N) \in \operatorname{supp} m$ and $\xi_1 \in \operatorname{supp}\psi(\cdot/2^j)$. We can find a function $\widetilde{\psi} \in \mathcal{A}_1$ such that $\widetilde{\psi}(\xi_1) = 1$ on $\{\xi_1 \in \mathbb{R}^n : 1/2 \leq |\xi_1| \leq 2\}$, and we obtain

$$m(\xi)\psi(\xi_1/2^j) = m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)$$

where we have used the fact that by $\xi_1 \in \operatorname{supp}\psi(\cdot/2^j)$, in this area, $1/2 \leq |\xi_1|/2^j \leq 2$ holds. Moreover, we can find a function $\widetilde{\psi} \in \mathcal{A}_1$ such that $\widetilde{\psi}(\xi_2) = 1$ on $\{\xi_2 \in \mathbb{R}^n : 1/(8N) \leq |\xi_2| \leq 8N\}$, and we obtain

$$m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j) = m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2$$

where we have used the fact that by (3.3.1), in this area, $1/(8N) \le |\xi_2|/2^j \le 8N$ holds. Moreover, we can find a function $\varphi \in \mathcal{A}_0$ such that $\varphi(\xi_i) = 1$ on $\{\xi_i \in \mathbb{R}^n : |\xi_i| \le 8N\}$, $i = 3, \dots, N$, and we see that

$$\begin{aligned}
&m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2 \\
&= m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2\varphi(\xi_3/2^j)\dots\varphi(\xi_N/2^j)
\end{aligned}$$

where we have used the fact that by (3.3.1) in this area, $|\xi_i|/2^j \le 8N$, $i = 3, \dots, N$ holds. Combining these, we have (3.3.3).

By (3.3.5) and Lemma 3.1.2, it follows that

$$\begin{aligned}
|T_m(f_1, \dots, f_N)(x)| &\le \sum_{j\in\mathbb{Z}} \left| T_{m^{(j)}}\left(\widetilde{\Delta}_j f_1, \widetilde{\Delta}_j f_2, f_3, \dots, f_N\right)(x) \right| \\
&\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} \right) \\
&\quad \times \sum_{j\in\mathbb{Z}} \prod_{i=1}^{2} M\left(|\widetilde{\Delta}_j f_i|^{q_i}\right)(x)^{1/q_1} \prod_{i=3}^{N} M\left(|f_i|^{q_i}\right)(x)^{1/q_i}.
\end{aligned}$$

By Schwarz's inequality and Hölder's inequality, we see that

$$\begin{aligned}
&\|T_m(f_1, \dots, f_N)\|_{L^p(w)} \\
&\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} \right) \\
&\quad \times \prod_{i=1}^{2} \left\| \left\{ \sum_{j\in\mathbb{Z}} M\left(|\widetilde{\Delta}_j f_i|^{q_i}\right)(x)^{2/q_i} \right\}^{1/2} \right\|_{L^{p_i}(w_i)} \prod_{i=3}^{N} \left\| M\left(|f_i|^{q_i}\right)(x)^{1/q_i} \right\|_{L^{p_i}(w_i)}.
\end{aligned}$$

By the same way as for $m_{(1,0,\dots,0)}$, we obtain (3.3.2). □

3.4 Completion of the proof of Theorem 1.4.1

In this section, we shall prove the estimate

$$\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}((\mathbb{R}^n)^N)}$$

holds, where $m^{(j)}$ is defined by (3.2.5) or (3.3.4), and m_j is defined by (2.2.2).

Proof. We first consider the case where $m^{(j)}$ is the same as in (3.2.5). Since $\operatorname{supp}\Psi(\cdot/2^\ell) \subset \{\xi \in (\mathbb{R}^n)^N : 2^{\ell-1} \le |\xi| \le 2^{\ell+1}\}$ and

$$\operatorname{supp}\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N) \subset \{\xi \in (\mathbb{R}^n)^N : 2^{-j_0} \le |\xi| \le 2^{j_0}\} \tag{3.4.1}$$

for some $j_0 \in \mathbb{N}$, for example, $j_0 = [\log_2 2(N+1)]$, where $[s]$ is the integer part of $s \in \mathbb{R}$. It follows from Lemma 3.1.3 that

(3.4.2)

$$\begin{aligned}
&\|m^{(j)}\|_{H^{(s_1,\dots,s_N)}} \\
&= \left\| m(2^j\xi)\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N) \right\|_{H^{(s_1,\dots,s_N)}} \\
&\lesssim \sum_{\ell=-j_0}^{j_0} \big\| m(2^j\xi)\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N) \\
&\quad \times \widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\Psi(\xi/2^\ell)\big\|_{H^{(s_1,\dots,s_N)}} \\
&\lesssim \sum_{\ell=-j_0}^{j_0} \|m(2^j\xi)\Psi(\xi/2^\ell)\|_{H^{(s_1,\dots,s_N)}} \\
&\quad \times \|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{H^{(s_1,\dots,s_N)}}.
\end{aligned}$$

For the first term on the right hand side of (3.4.2), by a change of variables, we have

$$\begin{aligned}
\text{(3.4.3)} \qquad \|m(2^j\xi)\Psi(\xi/2^\ell)\|_{H^{(s_1,\dots,s_N)}} &= \|m\left(2^{j+\ell}(\xi/2^\ell)\right)\Psi(\xi/2^\ell)\|_{H^{(s_1,\dots,s_N)}} \\
&= \|m_{j+\ell}(\xi/2^\ell)\|_{H^{(s_1,\dots,s_N)}} \\
&\lesssim \|m_{j+\ell}\|_{H^{(s_1,\dots,s_N)}} \\
&\lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}}.
\end{aligned}$$

For the second term on the right hand side of (3.4.2), by Lemma 2.4.1 (2), (3.4.1) and Lemma 2.4.6, we see that

(3.4.4)

$$\begin{aligned}
&\left\|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right\|_{H^{(s_1,\dots,s_N)}} \\
&\lesssim \left\|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right\|_{H^{s_1+\cdots+s_N}} \\
&\lesssim \left\|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right\|_{H^{[s_1+\cdots+s_N]+1}} \\
&\le \sup_{j\in\mathbb{Z}} \left\|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right\|_{H^{[s_1+\cdots+s_N]+1}} \\
&< \infty.
\end{aligned}$$

By (3.4.2), (3.4.3) and (3.4.4), we have the desired estimate. In the case where $m^{(j)}$ is the same as in (3.3.4), the proof is similar to that of $m_{(1,0,\dots,0)}$, and we omit it. □

Chapter 4

The proof of Theorem 1.4.2

In this chapter, we study weighted norm inequalities for multilinear Fourier multipliers with the L^r-based Sobolev regularity, $1 < r \le 2$. In [30], the boundedness of T_m under the condition

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{B^{(n/r,\dots,n/r)}_{r,1}((\mathbb{R}^n)^N)} < \infty, \quad 1 \le r < 2$$

was discussed. And weighted norm inequalities for multilinear Fourier multipliers with Besov regularity will be considered in Chapter 6.

4.1 A Lemma

Lemma 4.1.1. *Let $r > 0$ and $p_i \le q_i$, $i = 1, \dots, N$. Then, the estimate*

$$\left\| \left\| \langle \xi_1 \rangle^{s_1} \widehat{F}(\xi_1, \dots, \xi_N) \right\|_{L^{q_1}(\mathbb{R}^n_{\xi_1})} \cdots \langle \xi_N \rangle^{s_N} \right\|_{L^{q_N}(\mathbb{R}^n_{\xi_N})} \lesssim \left\| \left\| \langle \xi_1 \rangle^{s_1} \widehat{F}(\xi_1, \dots, \xi_N) \right\|_{L^{p_1}(\mathbb{R}^n_{\xi_1})} \cdots \langle \xi_N \rangle^{s_N} \right\|_{L^{p_N}(\mathbb{R}^n_{\xi_N})} \tag{4.1.1}$$

holds, where $\operatorname{supp} F \subset \{\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N : |\xi| \le r\}$.

Proof. By the same way in Lemma 3.1.1, we obtain the desired estimate. □

Now, we prove Theorem 1.4.2. Let $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $1 < r_N \le r_{N-1} \le \cdots \le r_2 \le r_1 \le 2$, $n/r_i < s_i \le n$ for $i = 1, \dots, N$. Assume $p_i > n/s_i$ and $w_i \in A_{p_i/(n/s_i)}$ for $i = 1, \cdots, N$ and set $w = w_1^{p/p_1} \dots w_N^{p/p_N}$. We also assume that $m \in L^\infty(\mathbb{R}^{Nn})$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)} < \infty,$$

where m_j is defined by (2.2.2) and $H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)$ is defined by (2.3.1). Since $n/s_i < \min\{r_i, p_i\}$ and $w_i \in A_{p_i/(n/s_i)}$ for $i = 1, \dots, N$, by Lemma 2.4.2 (2), we can take $n/s_i < q_i < r_i$ satisfying $w_i \in A_{p_i/q_i}$, $i = 1, \dots, N$. By Lemma 2.4.1 (1), we decompose m as follows:

$$\begin{aligned} m(\xi) &= \sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N,\\ (i_1,\dots,i_N)\neq(0,\dots,0)}} \Phi_{(i_1,\dots,i_N)}(\xi) m(\xi) \\ &= \sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N,\\ (i_1,\dots,i_N)\neq(0,\dots,0)}} m_{(i_1,\dots,i_N)}(\xi). \end{aligned}$$

4.2 Estimate for $m_{(1,0,\dots,0)}$ type

We first consider the case where $(i_1, \dots, i_N)$ satisfies $\sharp\{j : i_j = 1\} = 1$. Without loss of generality, we may assume that $i_1 = 1$. We simply write m instead of $m_{(1,0,\dots,0)}$.

As in Section 3.2, we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j satisfying (3.2.4). Set $B_j(\xi) = \Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)$. The following is a key lemma in the proof of Theorem 1.4.2.

Lemma 4.2.1. *Let $r > 0$, $r_N \le r_{N-1} \le \dots \le r_2 \le r_1$, $n/r_i < s_i \le n$ and $n/s_i < q_i < r_i$, $i = 1, \dots, N$. Then, the estimate*

$$\begin{aligned} &\left|T_{m^{(j)}(\cdot/2^j)}(f_1, \dots, f_N)(x)\right| \\ &\lesssim \|m_j\|_{H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)} \prod_{i=1}^N M(|f_i|^{q_i})(x)^{1/q_i} \end{aligned} \tag{4.2.1}$$

holds for all $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$ and $m \in H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)$ with $\operatorname{supp} m \subset \{\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N : |\xi| \le r\}$, where $m^{(j)}(\xi) = m_j(\xi)B_j(\xi)$.

Proof. By (3.1.10), we obtain

$$\left|T_{m^{(j)}(\cdot/2^j)}(f_1, \dots, f_N)(x)\right| \lesssim \left\|\widehat{m^{(j)}}\right\|_{L^{(q_1',\dots,q_N')}_{(s_1q_1',\dots,s_Nq_N')}} \prod_{i=1}^N M\left(|f_i|^{q_i}\right)(x)^{1/q_i}.$$

We shall prove that the estimate

$$\left\|\widehat{m^{(j)}}\right\|_{L^{(q_1',\dots,q_N')}_{(s_1q_1',\dots,s_Nq_N')}} \lesssim \|m_j\|_{H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)} \tag{4.2.2}$$

holds. Once this is proved, we have the desired estimate. The left hand side of (4.2.2) is as follows:

$$\left\|\widehat{m^{(j)}}\right\|_{L^{(q_1',\dots,q_N')}_{(s_1q_1',\dots,s_Nq_N')}}$$

$$= \left\{ \int_{\mathbb{R}^n_{\xi_N}} \cdots \right.$$
$$\times \left\{ \int_{\mathbb{R}^n_{\xi_2}} \left(\int_{\mathbb{R}^n_{\xi_1}} \langle \xi_1 \rangle^{s_1 q_1'} \langle \xi_2 \rangle^{s_2 q_1'} \dots \langle \xi_N \rangle^{s_N q_1'} |\widehat{m^{(j)}}(\xi_1, \xi_2, \dots, \xi_N)|^{q_1'} \, d\xi_1 \right)^{q_2'/q_1'} d\xi_2 \right\}^{q_3'/q_2'}$$
$$\left. \times \dots \, d\xi_N \right\}^{1/q_N'}.$$

Since $\langle \xi_i \rangle^{s_i} \lesssim \langle \xi_i - \eta_i \rangle^{s_i} \langle \eta_i \rangle^{s_i}$, $i = 1, \dots, N$, we have

$$\langle \xi_1 \rangle^{s_1 q_1'} \langle \xi_2 \rangle^{s_2 q_1'} \dots \langle \xi_N \rangle^{s_N q_1'} |\widehat{m^{(j)}}(\xi_1, \xi_2, \dots, \xi_N)|^{q_1'}$$
$$= \left| \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \dots \langle \xi_N \rangle^{s_N} \left[\widehat{m_j} * \widehat{B_j} \right] (\xi_1, \xi_2, \dots, \xi_N) \right|^{q_1'}$$
$$\le |\langle \xi_1 - \eta_1 \rangle^{s_1} \dots \langle \xi_N - \eta_N \rangle^{s_N} \langle \eta_1 \rangle^{s_1} \dots \langle \eta_N \rangle^{s_N}$$
$$\times \int_\eta \widehat{m_j}(\xi_1 - \eta_1, \dots, \xi_N - \eta_N) \widehat{B_j}(\eta_1, \dots, \eta_N) \, d\eta_1 \dots d\eta_N \Big|^{q_1'}$$
$$= \Big| \int_\eta \langle \xi_1 - \eta_1 \rangle^{s_1} \dots \langle \xi_N - \eta_N \rangle^{s_N} \widehat{m_j}(\xi_1 - \eta_1, \dots, \xi_N - \eta_N)$$
$$\times \langle \eta_1 \rangle^{s_1} \dots \langle \eta_N \rangle^{s_N} \widehat{B_j}(\eta_1, \dots, \eta_N) \, d\eta_1 \dots d\eta_N \Big|^{q_1'}$$
$$= \left| \left\{ \langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \widehat{m_j}(\xi_1, \dots, \xi_N) * \langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \widehat{B_j}(\xi_1, \dots, \xi_N) \right\} (\xi_1, \dots, \xi_N) \right|^{q_1'},$$

where $\eta = (\eta_1, \dots, \eta_N) \in (\mathbb{R}^n)^N$. By Young's inequality with mixed type (Lemma 2.4.7), we see that

$$\left\| \widehat{m^{(j)}} \right\|_{L^{(q_1', \dots, q_N')}_{(s_1 q_1', \dots, s_N q_N')}} \tag{4.2.3}$$
$$\le \left\| \| \langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \widehat{m_j}(\xi_1, \dots, \xi_N) \|_{L^{q_1'}(\mathbb{R}^n_{\xi_1})} \cdots \right\|_{L^{q_N'}(\mathbb{R}^n_{\xi_N})}$$
$$\times \left\| \left\| \langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \widehat{B_j}(\xi_1, \dots, \xi_N) \right\|_{L^1(\mathbb{R}^n_{\xi_1})} \cdots \right\|_{L^1(\mathbb{R}^n_{\xi_N})}.$$

For the first term of the right hand side of (4.2.3), by Lemma 4.1.1 and Hausdorff-Young's inequality with mixed type (Lemma 2.4.8), it follows that

$$\left\| \| \langle \xi_1 \rangle^{s_1} \widehat{m_j}(\xi_1, \dots, \xi_N) \|_{L^{q_1'}(\mathbb{R}^n_{\xi_1})} \cdots \langle \xi_N \rangle^{s_N} \right\|_{L^{q_N'}(\mathbb{R}^n_{\xi_N})}$$
$$\le \left\| \| \langle \xi_1 \rangle^{s_1} \widehat{m_j}(\xi_1, \dots, \xi_N) \|_{L^{r_1'}(\mathbb{R}^n_{\xi_1})} \cdots \langle \xi_N \rangle^{s_N} \right\|_{L^{r_N'}(\mathbb{R}^n_{\xi_N})}$$
$$\le \left\| \left\| \mathcal{F}^{-1} \left[\langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \widehat{m_j}(\xi_1, \dots, \xi_N) \right] (y_1, \dots, y_N) \right\|_{L^{r_1}(\mathbb{R}^n_{y_1})} \cdots \right\|_{L^{r_N}(\mathbb{R}^n_{y_N})}$$

$$= \|m_j\|_{H^{(s_1,\dots,s_N)}_{(r_1,\dots,r_N)}((\mathbb{R}^n)^N)}.$$

For the second term of the right hand side of (4.2.3), by B_j is homogeneous of degree 0, we obtain

$$\left\| \left\| \langle \xi_1 \rangle^{s_1} \dots \langle \xi_N \rangle^{s_N} \widehat{B_j}(\xi_1, \dots, \xi_N) \right\|_{L^1(\mathbb{R}^n_{\xi_1})} \dots \right\|_{L^1(\mathbb{R}^n_{\xi_N})} < \infty.$$

Therefore, we have (4.2.2). □

4.3 Estimate for $m_{(1,1,i_3,\dots,i_N)}$ type

In the case where $(i_1, \dots, i_N)$ satisfies $\sharp\{j : i_j = 1\} \geq 2$, the proof is similar to and we omit it.

Chapter 5

The proof of Theorem 1.4.3

In this chapter, we consider the problem whether weighted norm inequalities for multilinear Fourier multipliers with Sobolev regularity hold under the weak condition on weights. We first prove the following lemmas which will be used in the proof of Theorem 1.4.3.

5.1 Lemmas

The proof of the following lemma is based on the argument of [11, Example 9.1.7].

Lemma 5.1.1 ([9, Lemma 2.1])**.** *Let N be a natural number and $N \geq 2$, $1 < p_1, \ldots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. If α_1, α_2 satisfy*

$$\alpha_1/p_1 + \alpha_2/p_2 > -n/p \quad \textit{and} \quad \alpha_i < n(p_i - 1) \quad \textit{for} \quad i = 1, 2,$$

then the conclusion

$$(|x|^{\alpha_1}, |x|^{\alpha_2}, 1, \ldots, 1) \in A_{(p_1, p_2, p_3, \ldots, p_N)}$$

holds.

Proof. Since $w_i = 1$, $i = 3, \ldots, N$, the desired conclusion is the following.

$$\sup_B \left(\frac{1}{|B|} \int_B |x|^{(\alpha_1/p_1 + \alpha_2/p_2)p} \, dx \right)^{1/p} \prod_{i=1}^{2} \left(\frac{1}{|B|} \int_B |x|^{\alpha_i(1-p_i')} \, dx \right)^{1/p_i'} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ (instead of cubes). Let B be the ball with center x_0 and radius r.

We first consider the case $|x_0| \geq 2r$. Note that for all $x \in B$,

$$|x| \leq |x_0| + |x - x_0| \leq |x_o| + r \leq |x_0| + 1/2|x_0| = 3/2|x_0|,$$

and

$$|x_0| \le |x| + |x - x_0| \le |x| + r \le |x| + 1/2|x_0|,$$

so we have $|x_0| \le 2|x|$. Hence, for all $x \in B$ we obtain $|x| \approx |x_0|$. Then

$$\begin{aligned}
(5.1.1)\quad & \left(\frac{1}{|B|}\int_B |x|^{(\alpha_1/p_1+\alpha_2/p_2)p}\,dx\right)^{1/p} \prod_{i=1}^{2}\left(\frac{1}{|B|}\int_B |x|^{\alpha_i(1-p_i')}\,dx\right)^{1/p_i'} \\
& \lesssim \left(\frac{1}{|B|}\int_B |x_0|^{(\alpha_1/p_1+\alpha_2/p_2)p}\,dx\right)^{1/p} \prod_{i=1}^{2}\left(\frac{1}{|B|}\int_B |x_0|^{\alpha_i(1-p_i')}\,dx\right)^{1/p_i'} \\
& = |x_0|^{\alpha_1/p_1+\alpha_2/p_2} \prod_{i=1}^{2} |x_0|^{\alpha_i(1-p_i')1/p_i'} \\
& = 1.
\end{aligned}$$

We next consider the case $|x_0| < 2r$. For all $x \in B$,

$$|x| \le |x_0| + |x - x_0| < 2r + r = 3r.$$

Hence, we have $B \subset \{x \in \mathbb{R}^n : |x| < 3r\}$. Thus, we obtain

$$\begin{aligned}
(5.1.2)\quad \frac{1}{|B|}\int_B |x|^{(\alpha_1/p_1+\alpha_2/p_2)p}\,dx &\le \frac{1}{|B|}\int_{\{x\in\mathbb{R}^n:|x|<3r\}} |x|^{(\alpha_1/p_1+\alpha_2/p_2)p}\,dx \\
&< \frac{1}{|B|}(3r)^{(\alpha_1/p_1+\alpha_2/p_2)p}\int_{\{|x|<3r\}} 1\,dx \\
&\lesssim r^{(\alpha_1/p_1+\alpha_2/p_2)p}.
\end{aligned}$$

For $i = 1, 2$, we see that

$$\begin{aligned}
(5.1.3)\quad \frac{1}{|B|}\int_B |x|^{\alpha_i(1-p_i')}\,dx &\le \frac{1}{|B|}\int_{\{x\in\mathbb{R}^n:|x|<3r\}} |x|^{\alpha_i(1-p_i')}\,dx \\
&< \frac{1}{|B|}(3r)^{\alpha_i(1-p_i')}\int_{\{x\in\mathbb{R}^n:|x|<3r\}} 1\,dx \\
&\lesssim r^{\alpha_i(1-p_i')}.
\end{aligned}$$

By (5.1.2) and (5.1.3), we obtain

$$\begin{aligned}
(5.1.4)\quad & \left(\frac{1}{|B|}\int_B |x|^{(\alpha_1/p_1+\alpha_2/p_2)p}\,dx\right)^{1/p} \prod_{i=1}^{2}\left(\frac{1}{|B|}\int_B |x|^{\alpha_i(1-p_i')}\,dx\right)^{1/p_i'} \\
& \lesssim r^{\alpha_1/p_1+\alpha_2/p_2} r^{\alpha_i(1-p_i')/p_i'} \\
& = 1.
\end{aligned}$$

By (5.1.1) and (5.1.4), we have the desired conclusion. □

Lemma 5.1.2 ([9, Lemma 2.2])**.** *Let $r > 0$ and $\ell \in \mathbb{Z}_+$. Then there is a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that* $\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n : |x| \le r\}$*,*

$$\int_{\mathbb{R}^n} \varphi(x)^2 \, dx \neq 0, \tag{5.1.5}$$

and

$$\int_{\mathbb{R}^n} x^\beta \varphi(x) \, dx = 0, \quad \beta \in \mathbb{Z}_+^n, \ |\beta| \le \ell. \tag{5.1.6}$$

Proof. We can take a real-valued function $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ satisfying $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le r\}$. We set

$$\varphi(x) = (-\Delta)^{\ell+1}\psi(x).$$

Since $\psi \neq 0$, we have $\widehat{\psi} \neq 0$. Hence, we can take $\xi_0 \in \mathbb{R}^n$ and $r_0 > 0$ satisfying

$$\xi \in \mathbb{R}^n, |\xi - \xi_0| \le r_0 \Longrightarrow \widehat{\psi}(\xi) \neq 0.$$

Since φ is a real-valued function, by Plancherel's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi(x)^2 \, dx &= \int_{\mathbb{R}^n} (-\Delta)^{2(\ell+1)}\psi(x)^2 \, dx \\
&= \int_{\mathbb{R}^n} \left|(-\Delta)^{\ell+1}\psi(x)\right|^2 \, dx \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left||\xi|^{2(\ell+1)}\widehat{\psi}(\xi)\right|^2 \, d\xi \\
&\ge \frac{1}{(2\pi)^n} \int_{|\xi-\xi_0|\le r_0} \left||\xi|^{2(\ell+1)}\widehat{\psi}(\xi)\right|^2 \, d\xi \\
&\ge \frac{1}{(2\pi)^n} \inf_{|\xi-\xi_0|\le r_0} |\widehat{\psi}(\xi)|^2 \int_{|\xi-\xi_0|\le r_0} |\xi|^{4(\ell+1)} \, d\xi \neq 0.
\end{aligned}$$

Thus, we have (5.1.5). For $|\beta| \le \ell$, we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} (-ix)^\beta \varphi(x) \, dx &= \int_{\mathbb{R}^n} e^{-i\langle x,0\rangle} (-ix)^\beta \varphi(x) \, dx \\
&= \mathcal{F}\left[(-ix)^\beta \varphi(x)\right](0) \\
&= \left(\partial_\xi^\beta \mathcal{F}[\varphi](\xi)\right)(0).
\end{aligned} \tag{5.1.7}$$

By Leibniz's formula, we obtain

$$\begin{aligned}
\left(\partial_\xi^\beta \mathcal{F}[\varphi]\right)(\xi) &= \left(\partial_\xi^\beta \mathcal{F}\left[(-\Delta)^{\ell+1}\psi(x)\right]\right)(\xi) \\
&= \left(\partial_\xi^\beta \left\{|\xi|^{2(\ell+1)}\widehat{\psi}(\xi)\right\}\right)(\xi) \\
&= \sum_{\alpha\le\beta} {}_\beta C_\alpha \partial^\alpha(|\xi|^{2(\ell+1)})\partial^{\beta-\alpha}\widehat{\psi}(\xi).
\end{aligned}$$

By $|\beta| \le \ell$, we see that

$$\left(\partial_\xi^\beta \mathcal{F}[\varphi](\xi)\right)(0) = 0. \tag{5.1.8}$$

By (5.1.7) and (5.1.8), we have (5.1.6). □

Lemma 5.1.3. *Let $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ be a function as in* Lemma 5.1.2 *with* $\operatorname{supp} \widehat{\varphi} \subset \{\eta \in \mathbb{R}^n : |\eta| \le 1/(10N)\}$ *and ℓ satisfying $p_1(\ell+1)+\alpha_1 > -n$. For sufficiently small $\varepsilon > 0$, we set*

$$m^{(\varepsilon)}(\xi) = \widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right)\widehat{\varphi}(\xi_2)\dots\widehat{\varphi}(\xi_N),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ and $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$. Then the estimate

$$\sup_{j \in \mathbb{Z}} \|(m^{(\varepsilon)})_j\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)} \lesssim \varepsilon^{-s/N+n/2}$$

holds, where $(m^{(\varepsilon)})_j$ is defined by (2.2.2) *with m replaced by $m^{(\varepsilon)}$.*

Proof. For sufficiently small $\alpha > 0$, we can take $\Psi \in \mathcal{S}(\mathbb{R}^{Nn})$ appearing in the definition of $(m^{(\varepsilon)})_j$ satisfying

$$\begin{aligned} \operatorname{supp}\Psi \subset& \left\{\xi \in \mathbb{R}^{Nn} : 2^{-1/2-\alpha} \le |\xi| \le 2^{1/2+\alpha}\right\}, \\ \Psi(\xi) =& 1 \text{ on } \left\{\xi \in \mathbb{R}^{Nn} : 2^{-1/2+\alpha} \le |\xi| \le 2^{1/2-\alpha}\right\}. \end{aligned}$$

If $\varepsilon > 0$ is sufficiently small, then we see that

$$\begin{aligned} &\operatorname{supp} m^{(\varepsilon)} \\ &\subset \left\{\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N : |\xi_1 - e_1| \le \varepsilon/(10N), |\xi_i| \le 1/(10N), i = 2, \dots, N\right\} \\ &\subset \left\{\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N : 2^{-1/2+\alpha} \le |\xi| \le 2^{1/2-\alpha}\right\}. \end{aligned}$$

Hence, we have

$$\left(m^{(\varepsilon)}\right)_j(\xi) = m^{(\varepsilon)}(2^j\xi)\Psi(\xi) = \begin{cases} m^{(\varepsilon)}(\xi) & (j = 0) \\ 0 & (j \ne 0), \end{cases}$$

where $\xi = (\xi_1, \dots, \xi_N) \in (\mathbb{R}^n)^N$. Thus, we see that

$$\begin{aligned} &\sup_{j \in \mathbb{Z}} \left\|\left(m^{(\varepsilon)}\right)_j\right\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)} \\ &= \left\|m^{(\varepsilon)}\right\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)} \\ &= \left\|\widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right)\widehat{\varphi}(\xi_2)\dots\widehat{\varphi}(\xi_N)\right\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)} \end{aligned} \tag{5.1.9}$$

$$= \left\| \widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right) \right\|_{H^{s/N}(\mathbb{R}^n_{\xi_1})} \|\widehat{\varphi}\|^{N-1}_{H^{s/N}(\mathbb{R}^n)} .$$

For the first term of the right hand side of (5.1.9), by a change of variables and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, for sufficiently large $M > 0$, we obtain

(5.1.10)

$$\begin{aligned}
&\left\| \widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right) \right\|_{H^{s/N}(\mathbb{R}^n_{\xi_1})} \\
&= \left\| \varepsilon^n \varphi(\varepsilon x)(1+|x|^2)^{s/2N} \right\|_{L^2} \\
&= \left\{ \int_{\mathbb{R}^n} \varepsilon^{2n} |\varphi(\varepsilon x)|^2 (1+|x|^2)^{s/N}\, dx \right\}^{1/2} \\
&\le \varepsilon^n \left\{ \int_{\mathbb{R}^n} \frac{1}{(1+|\varepsilon x|)^{2M}} (1+|x|^2)^{s/N}\, dx \right\}^{1/2} \\
&= \varepsilon^n \left\{ \int_{|x|\le 1} \frac{(1+|x|^2)^{s/N}}{(1+|\varepsilon x|)^{2M}}\, dx + \int_{1<|x|\le 1/\varepsilon} \frac{(1+|x|^2)^{s/N}}{(1+|\varepsilon x|)^{2M}}\, dx + \int_{|x|>1/\varepsilon} \frac{(1+|x|^2)^{s/N}}{(1+|\varepsilon x|)^{2M}}\, dx \right\}^{1/2} \\
&\lesssim \varepsilon^n \left\{ \int_{|x|\le 1} 1\, dx + \int_{1<|x|\le 1/\varepsilon} |x|^{2s/N}\, dx + \int_{|x|>1/\varepsilon} |x|^{2s/N} \frac{1}{(\varepsilon|x|)^{2M}}\, dx \right\}^{1/2} .
\end{aligned}$$

For the first term of the right hand side of (5.1.10), since $s/N > 0$, we see that

$$\int_{|x|\le 1} 1\, dx = 1 \le \varepsilon^{-2s/N-n}. \tag{5.1.11}$$

For the second term of the right hand side of (5.1.10), we have

$$\begin{aligned}
\int_{1<|x|\le 1/\varepsilon} |x|^{2s/N}\, dx &\le \int_{1<|x|\le 1/\varepsilon} \left(\frac{1}{\varepsilon}\right)^{2s/N} dx \\
&= \varepsilon^{-2s/N} \int_{1<|x|\le 1/\varepsilon} 1\, dx = \varepsilon^{-2s/N} \left(\frac{1}{\varepsilon} - 1\right)^n \\
&\le \varepsilon^{-2s/N} \left(\frac{1}{\varepsilon}\right)^n = \varepsilon^{-2s/N-n}.
\end{aligned} \tag{5.1.12}$$

For the third term of the right hand side of (5.1.10), by $M > 0$ is sufficiently large, we see that

$$\begin{aligned}
\int_{|x|>1/\varepsilon} |x|^{2s/N} \frac{1}{(\varepsilon|x|)^{2M}}\, dx &= \varepsilon^{-2M} \int_{|x|>1/\varepsilon} \frac{1}{|x|^{2M-2s/N}}\, dx \\
&\lesssim \varepsilon^{-2s/N-n}.
\end{aligned} \tag{5.1.13}$$

By (5.1.9), (5.1.10), (5.1.11), (5.1.12) and (5.1.13), we have the desired estimate. □

5.2 The proof of Theorem 1.4.3

Now, we prove Theorem 1.4.3. Let $N \geq 2$, $1 < p_1, \dots, p_N < \infty$, $1/p_1 + \cdots + 1/p_N = 1/p$ and $Nn/2 < s \leq Nn$. Assume $p_i > Nn/s$ for $i = 1, \dots N$. We first prove that we can take $\alpha_1 < -n$ and $\alpha_2 > -n$ satisfying

$$\alpha_i/p_i < s/N - n/p_i\,, \quad i = 1, 2, \tag{5.2.1}$$

$$\alpha_1/p_1 + \alpha_2/p_2 > -n/p, \tag{5.2.2}$$

and

$$\alpha_1/p_1 < -n/p_1 - s/N + n/2. \tag{5.2.3}$$

Indeed, by the inequality

$$-n/p + n/p_1 + s/N - n/2 < s/N - n/p_2$$

and

$$s/N - n/p_2 > 0,$$

we can take $\alpha_2 \geq 0$ satisfying

$$-n/p + n/p_1 + s/N - n/2 < \alpha_2/p_2 < s/N - n/p_2.$$

Hence, we have (5.2.1) with $i = 2$. By the inequality,

$$-\alpha_2/p_2 - n/p < -n/p_1 - s/N + n/2,$$

we can take α_1 satisfying

$$-\alpha_2/p_2 - n/p < \alpha_1/p_1 < -n/p_1 - s/N + n/2.$$

Thus, we obtain (5.2.2) and (5.2.3). Moreover, by the inequality,

$$\alpha_1/p_1 < -n/p_1 - s/N + n/2 < -n/p_1 + s/N,$$

we have (5.2.1) with $i = 1$. Therefore, we obtain (5.2.1) (5.2.2) and (5.2.3).

For α_1 and α_2 satisfying (5.2.1), (5.2.2) and (5.2.3), we set

$$\begin{aligned} \vec{w_0} &= (w_1, w_2, w_3, \dots, w_N) = (|x|^{\alpha_1}, |x|^{\alpha_2}, 1, \dots, 1), \\ \nu_{\vec{w_0}} &= w_1^{p/p_1} \dots w_N^{p/p_N} = |x|^{p(\alpha_1/p_1 + \alpha_2/p_2)}. \end{aligned} \tag{5.2.4}$$

Let $(q_1, \dots, q_N) = (p_1 s/(Nn), \dots, p_N s/(Nn))$ and $1/q_1 + \cdots + 1/q_N = 1/q$. By $1/q = Nn/(sp)$, (5.2.1) and (5.2.2), we have $\alpha_1/q_1 + \alpha_2/q_2 > -nq$ and $\alpha_i < n(q_i - 1)$, $i = 1, 2$. By Lemma 5.1.1, we see that $\vec{w_0} \in A_{(q_1, \dots, q_N)} = A_{(p_1 s/(Nn)), \dots, p_N s/(Nn)}$.

We shall prove Theorem 1.4.3 with $\vec{w_0}$ defined by (5.2.4) by contradiction. To do this, we assume that the estimate

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w_0}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{H^{(s/N,\dots,s/N)}((\mathbb{R}^n)^N)} \tag{5.2.5}$$

holds. Let $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ be a function as in Lemma 5.1.2 with $\operatorname{supp}\widehat{\varphi} \subset \{\eta \in \mathbb{R}^n : |\eta| \le 1/(10N)\}$ and ℓ satisfying $p_1(\ell+1)+\alpha_1 > -n$. For sufficiently small $\varepsilon > 0$, we set

$$m^{(\varepsilon)}(\xi) = \widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right)\widehat{\varphi}(\xi_2)\dots\widehat{\varphi}(\xi_N), \tag{5.2.6}$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$ and $\xi = (\xi_1,\dots,\xi_N) \in (\mathbb{R}^n)^N$. By (5.2.5) and (5.2.6) and Lemma 5.1.3, it follows that

$$\begin{aligned}&\left\|\mathcal{F}^{-1}\left[\widehat{\varphi}\left(\frac{\cdot - e_1}{\varepsilon}\right)\widehat{f_1}\right]\mathcal{F}^{-1}\left[\widehat{\varphi}\widehat{f_2}\right]\dots\mathcal{F}^{-1}\left[\widehat{\varphi}\widehat{f_N}\right]\right\|_{L^p(|x|^{(\alpha_1/p_1+\alpha_2/p_2)p})}\\ &\lesssim \varepsilon^{-s/N+n/2}\prod_{i=1}^{2}\|f_i\|_{L^{p_i}(|x|^{\alpha_i})}\prod_{i=3}^{N}\|f_i\|_{L^{p_i}}\end{aligned} \tag{5.2.7}$$

for all $f_1,\dots,f_N \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathbb{R}^n$.

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\psi} = 1$ on $\operatorname{supp}\widehat{\varphi}$ and we set

$$\widehat{f_1}(\xi_1) = \varepsilon^{n/p_1 - n}\widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right), \quad \widehat{f_i}(\xi_i) = \widehat{\psi}(\xi_i), i = 2,\dots,N. \tag{5.2.8}$$

Since

$$\begin{aligned} f_1(x) &= \mathcal{F}^{-1}\left[\varepsilon^{n/p_1-n}\widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right)\right](x)\\ &= \varepsilon^{n/p_1-n}\mathcal{F}^{-1}\left[\widehat{\varphi}\left(\frac{\xi_1 - e_1}{\varepsilon}\right)\right](x)\\ &= \varepsilon^{n/p_1}e^{i\langle x,e_1\rangle}\varphi(\varepsilon x),\end{aligned}$$

by a change of variables, we see that

$$\|f_1\|_{L^{p_1}(|x|^{\alpha_1})} = \varepsilon^{-\alpha_1/p_1}\|\varphi\|_{L^{p_1}(|x|^{\alpha_1})}. \tag{5.2.9}$$

We check

$$\|\varphi\|_{L^{p_1}(|x|^{\alpha_1})} < \infty. \tag{5.2.10}$$

For $\beta \in \mathbb{Z}_+^n$, $|\beta| \le \ell$, by Lemma 5.1.2, we obtain

$$\partial^\beta\varphi(0) = \int_{\mathbb{R}^n} e^{i\langle\eta,0\rangle}\eta^\beta\widehat{\varphi}(\eta)\,d\eta = \int_{\mathbb{R}^n}\eta^\beta\widehat{\varphi}(\eta)\,d\eta = 0.$$

Combining this with Taylor's formula, we have

$$|\varphi(x)| \lesssim |x|^{\ell+1}.$$

By $p_1(\ell+1)+\alpha_1 > -n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we see that

$$\begin{aligned}\|\varphi\|_{L^{p_1}(|x|^{\alpha_1})}^{p_1} &= \int_{|x|<1} |\varphi(x)|^{p_1}|x|^{\alpha_1}\,dx + \int_{|x|\ge 1} |\varphi(x)|^{p_1}|x|^{\alpha_1}\,dx \\ &\lesssim \int_{|x|<1} |x|^{(\ell+1)p_1+\alpha_1}\,dx + \int_{|x|\ge 1} |\varphi(x)|^{p_1}|x|^{\alpha_1}\,dx \\ &< \infty.\end{aligned}$$

Hence, we obtain (5.2.10). By $\alpha_2 > -n$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$,

(5.2.11)
$$\begin{aligned}\|f_2\|_{L^{p_2}(|x|^{\alpha_2})}^{p_2} &= \int_{\mathbb{R}^n} |\psi(x)|^{p_2}|x|^{\alpha_2}\,dx \\ &= \int_{|x|<1} |\psi(x)|^{p_2}|x|^{\alpha_2}\,dx + \int_{|x|\ge 1} |\psi(x)|^{p_2}|x|^{\alpha_2}\,dx \\ &\lesssim \int_{|x|<1} |x|^{\alpha_2}\,dx + \int_{|x|\ge 1} |\psi(x)|^{p_2}|x|^{\alpha_2}\,dx \\ &< \infty.\end{aligned}$$

By $\psi \in \mathcal{S}(\mathbb{R}^n)$, we see that

(5.2.12)
$$\|f_i\|_{L^{p_i}} = \|\psi\|_{L^{p_i}} < \infty, \quad i = 3, \ldots, N.$$

We shall finish the proof. By (5.2.8) and a change of variables, we have

$$\begin{aligned}&\mathcal{F}^{-1}\left[\widehat{\varphi}\left(\frac{\cdot - e_1}{\varepsilon}\right)\widehat{f_1}\right](x)\mathcal{F}^{-1}\left[\widehat{\varphi}\widehat{f_2}\right](x)\ldots\mathcal{F}^{-1}\left[\widehat{\varphi}\widehat{f_N}\right](x) \\ &= \mathcal{F}^{-1}\left[\varepsilon^{n/p_1-n}\left\{\widehat{\varphi}\left(\frac{\cdot - e_1}{\varepsilon}\right)\right\}^2\right](x)\mathcal{F}^{-1}\left[\widehat{\varphi}\right](x)\ldots\mathcal{F}^{-1}\left[\widehat{\varphi}\right](x) \\ &= \varepsilon^{n/p_1-n}\left\{\mathcal{F}^{-1}\left[\widehat{\varphi}\left(\frac{\cdot - e_1}{\varepsilon}\right)\right] * \mathcal{F}^{-1}\left[\widehat{\varphi}\left(\frac{\cdot - e_1}{\varepsilon}\right)\right]\right\}(x)\varphi(x)^{N-1} \\ &= \varepsilon^{n/p_1}e^{i\langle x, e_1\rangle}(\varphi * \varphi)(\varepsilon x)\varphi(x)^{N-1}.\end{aligned}$$

By $\varphi \ne 0$ and $(\alpha_1/p_1 + \alpha_2/p_2)p > -n$, we can take $R_0 > 0$ satisfying

$$0 < \int_{|x|\le R_0} |\varphi(x)|^{p(N-1)}|x|^{(\alpha_1/p_1+\alpha_2/p_2)p}\,dx < \infty.$$

By Lemma 5.1.2, it follows that

$$\begin{aligned}\varphi * \varphi(0) &= \mathcal{F}^{-1}[\widehat{\varphi}] * \mathcal{F}^{-1}[\widehat{\varphi}](0) \\ &= \mathcal{F}^{-1}[\widehat{\varphi}\widehat{\varphi}](0)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle 0,\xi\rangle} \widehat{\varphi}(\xi)^2 \, d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)^2 \, d\xi \\
&\neq 0,
\end{aligned}
$$

by the continuity of $\varphi * \varphi$ at the origin, we can take $C_0 > 0$ and $\varepsilon_0 > 0$ satisfying for all ε, $0 < \varepsilon < \varepsilon_0$, x, $|x| \le R_0$, $|\varphi * \varphi(\varepsilon x)| \ge C_0$. Thus, we see that a

$$
\begin{aligned}
(5.2.13) \quad & \left\| \mathcal{F}^{-1}\left[\widehat{\varphi}\left(\frac{\cdot - e_1}{\varepsilon}\right)\widehat{f_1}\right] \mathcal{F}^{-1}\left[\widehat{\varphi}\widehat{f_2}\right] \dots \mathcal{F}^{-1}\left[\widehat{\varphi}\widehat{f_N}\right] \right\|_{L^p(|x|^{(\alpha_1/p_1+\alpha_2/p_2)p})} \\
&= \varepsilon^{n/p_1} \left\{ \int_{\mathbb{R}^n} |\varphi * \varphi(\varepsilon x)|^p |\varphi(x)|^{p(N-1)} |x|^{(\alpha_1/p_1+\alpha_2/p_2)p} \, dx \right\}^{1/p} \\
&\ge \varepsilon^{n/p_1} \left\{ \int_{|x|\le R_0} |\varphi * \varphi(\varepsilon x)|^p |\varphi(x)|^{p(N-1)} |x|^{(\alpha_1/p_1+\alpha_2/p_2)p} \, dx \right\}^{1/p} \\
&\ge \varepsilon^{n/p_1} C_0 \left\{ \int_{|x|\le R_0} |\varphi(x)|^{p(N-1)} |x|^{(\alpha_1/p_1+\alpha_2/p_2)p} \, dx \right\}^{1/p} \\
&\gtrsim \varepsilon^{n/p_1}.
\end{aligned}
$$

By (5.2.7), (5.2.9), (5.2.11), (5.2.12) and (5.2.13), we obtain

$$
\varepsilon^{n/p_1} \lesssim \varepsilon^{-s/N+n/2-\alpha_1/p_1} \|\varphi\|_{L^{p_1}(|x|^{\alpha_1})} \|\psi\|_{L^{p_2}(|x|^{\alpha_2})} \|\psi\|_{L^{p_3}} \dots \|\psi\|_{L^{p_N}}.
$$

Therefore, we have

$$
\varepsilon^{n/p_1} \lesssim \varepsilon^{-s/N+n/2-\alpha_1/p_1}.
$$

for all sufficiently small $\varepsilon > 0$. This is a contradiction (see (5.2.3)).

Chapter 6

The proof of Theorem 1.4.4

In this chapter, we study weighted norm inequalities for multilinear Fourier multipliers with Besov regularity. This result can be understood as a critical case of Theorem 1.4.1. It should be remarked that in the proof of Theorem 1.4.1, we can take q_i such that $n/s_i < q_i < \min\{2, p_i\}$, $i = 1, \dots, N$, but in this case, namely, $s_i = n/2$, we cannot take q_i like this.

We first prove the following lemma which plays an important role in the proof of Theorem 1.4.4.

6.1 Key lemma

Lemma 6.1.1 ([10, Lemma 3.1])**.** *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\phi(\xi_i) = \phi(-\xi_i)$, $\phi(\xi_i) = 1$ on $\{\xi_i \in \mathbb{R}^n : |\xi_i| \le 2\}$, $i = 1, \dots, N$. Then, the estimate*

$$|T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)|$$
$$\lesssim \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \|\Psi_{(k_1,\dots,k_N)}(D)m\|_{L^2((\mathbb{R}^n)^N)} \prod_{i=1}^{N} \left((|\phi|^2)_{(k_i-j)} * |f_i|^2(x)\right)^{1/2}$$

holds, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, where $(|\phi|^2)_{(j)}(x) = 2^{-jn}|\phi(2^{-j}x)|^2$ and $\Psi_{(k_1,\dots,k_N)}(D)$ is defined in the definition of the Besov space of product type.

Proof. Let $\{\psi_{k_1}\}_{k_1=0}^{\infty}, \dots, \{\psi_{k_N}\}_{k_N=0}^{\infty}$ be functions appearing in the definition of the Besov spaces of product type. By $\sum_{k_1=0}^{\infty} \psi_{k_1}(2^j(y_1 - x)) \times \cdots \times \sum_{k_N=0}^{\infty} \psi_{k_N}(2^j(y_N - x)) = 1$, we have

$$T_{m(\cdot/2^j)}(f_1, \dots, f_N)(x)$$
$$= \int_{(\mathbb{R}^n)^N} \mathcal{F}^{-1}[m(\cdot/2^j, \dots, \cdot/2^j)](x - y_1, \dots, x - y_N) f_1(y_1) \dots f_N(y_N)\, dy_1 \dots dy_N$$
$$= (2^{jn})^N \int_{(\mathbb{R}^n)^N} \mathcal{F}^{-1}[m] \left(2^j(x - y_1), \dots, 2^j(x - y_N)\right) f_1(y_1) \dots f_N(y_N)\, dy_1 \dots dy_N$$

$$= (2\pi)^{-Nn}(2^{jn})^N \sum_{k_1,\dots,k_N=0}^{\infty} \int_{(\mathbb{R}^n)^N} \psi_{k_1}(2^j(y_1-x))\dots\psi_{k_N}(2^j(y_N-x))$$
$$\times \widehat{m}\left(2^j(y_1-x),\dots,2^j(y_N-x)\right) f_1(y_1)\dots f_N(y_N)\,dy_1\dots dy_N.$$

Note that

$$\begin{aligned}\operatorname{supp}\psi_{k_i} &\subset \{\xi_i\in\mathbb{R}^n : 2^{k_i-1}\le|\xi_i|\le 2^{k_i+1}\}\\ &\subset \{\xi_i\in\mathbb{R}^n : |\xi_i|\le 2^{k_i+1}\},\quad i=1,\cdots,N.\end{aligned}$$

Thus, we obtain $\psi_{k_i}(\xi_i)=\psi_{k_i}(\xi_i)\phi(\xi_i/2^{k_i}), i=1,\dots,N$. Hence, we see that

$$\begin{aligned}&\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right|\\ &\lesssim (2^{jn})^N \sum_{k_1,\dots,k_N=0}^{\infty}\left|\int_{(\mathbb{R}^n)^N}\psi_{k_1}(2^j(y_1-x))\times\cdots\times\psi_{k_N}(2^j(y_N-x))\right.\\ &\quad\times\phi\left(\frac{2^j(y_1-x)}{2^{k_1}}\right)\dots\phi\left(\frac{2^j(y_N-x)}{2^{k_N}}\right)\\ &\quad\left.\times\widehat{m}(2^j(y_1-x),\dots,2^j(y_N-x))f_1(y_1)\dots f_N(y_N)\,dy_1\dots dy_N\right|.\end{aligned}$$

By Schwarz's inequality, a change of variables and $\phi(-\xi_i)=\phi(\xi_i), i=1,\cdots,N$, we have

$$\begin{aligned}&\left|T_{m(\cdot/2^j)}(f_1,\dots,f_N)(x)\right|\\ &\le (2^{jn})^N\sum_{k_1\dots k_N=0}^{\infty}\\ &\quad\times\left\|\psi_{k_1}(2^j(\cdot-x))\dots\psi_{k_N}(2^j(\cdot-x))\widehat{m}\left(2^j(\cdot-x),\dots,2^j(\cdot-x)\right)\right\|_{L^2((\mathbb{R}^n)^N)}\\ &\quad\times\left\|\phi\left(\frac{2^j(\cdot-x)}{2^{k_1}}\right)\dots\phi\left(\frac{2^j(\cdot-x)}{2^{k_N}}\right)f_1(\cdot)\dots f_N(\cdot)\right\|_{L^2((\mathbb{R}^n)^N)}\\ &=\sum_{k_1,\cdots,k_N=0}^{\infty}2^{(k_1+\cdots+k_N)n/2}\left\|\psi_{k_1}(\xi_1)\cdots\psi_{k_N}(\xi_N)\widehat{m}(\xi_1,\dots,\xi_N)\right\|_{L^2((\mathbb{R}^n)^N)}\\ &\quad\times\prod_{i=1}^N\left((|\phi|^2)_{(k_i-j)}*|f_i|^2(x)\right)^{1/2}.\end{aligned}$$

By Plancherel's theorem, we completes the proof. □

Now, we prove Theorem 1.4.4. Let $2<p_1,\dots,p_N<\infty$, $1/p_1+\cdots+1/p_N=1/p$. Assume $w_i\in A_{p_i/2}$ for $i=1,\cdots,N$ and set $w=w_1^{p/p_1}\dots w_N^{p/p_N}$. We also assume that $m\in L^\infty(\mathbb{R}^{Nn})$ satisfies $\sup_{j\in\mathbb{Z}}\|m_j\|_{B_{2,1}^{(n/2,\dots,n/2)}((\mathbb{R}^n)^N)}<\infty$, where m_j is defined by (2.2.2). By Lemma 2.4.1 (1), we decompose m as follows:

$$m(\xi)=\sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N,\\(i_1,\dots,i_N)\neq(0,\dots,0)}}\Phi_{(i_1,\dots,i_N)}(\xi)m(\xi)$$

$$= \sum_{\substack{(i_1,\dots,i_N)\in\{0,1\}^N,\\ (i_1,\dots,i_N)\neq(0,\dots,0)}} m_{(i_1,\dots,i_N)}(\xi).$$

6.2 Estimate for $m_{(1,0,\dots,0)}$ type

We first consider the case where $(i_1,\dots,i_N)$ satisfies $\sharp\{j : i_j = 1\} = 1$. Without loss of generality, we may assume that $i_1 = 1$. We simply write m instead of $m_{(1,0,\dots,0)}$. Note that by Lemma 2.4.1 (3),

$$\operatorname{supp} m \subset \{\xi = (\xi_1,\dots,\xi_N) \in (\mathbb{R}^n)^N : |\xi_i| \le |\xi_1|/N, i = 2,\dots,N\}. \tag{6.2.1}$$

It is easy to see that if $\xi = (\xi_1,\dots,\xi_N) \in \operatorname{supp} m$, then $|\xi_1 + \cdots + \xi_N| \approx |\xi_1|$.

Proof. In this case, we shall prove the estimate

$$\begin{aligned} &\|T_m(f_1,\dots,f_N)\|_{L^p(w)} \\ &\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\dots,n/2)}((\mathbb{R}^n)^N)}\right) \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)} \end{aligned} \tag{6.2.2}$$

holds, where $m^{(j)}$ will be defined later on. In Section 6.4, we shall complete the proof.

As in Section 3.2, we obtain (3.2.3) and we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$\begin{aligned} &m(\xi)\psi\left(\frac{\xi_1+\cdots+\xi_N}{2^j}\right) \\ &= m(\xi)\psi\left(\frac{\xi_1+\cdots+\xi_N}{2^j}\right)\widetilde{\psi}(\xi_1/2^j)^2\varphi(\xi_2/2^j)\cdots\varphi(\xi_N/2^j). \end{aligned}$$

Setting

$$m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\cdots\varphi(\xi_N), \tag{6.2.3}$$

we have

$$\Delta_j T_m(f_1,\dots,f_N)(x) = T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, f_2,\dots,f_N)(x), \tag{6.2.4}$$

where $\tilde{\Delta}_j f_1 = \widetilde{\psi}(D/2^j) f_1$.

By Lemma 6.1.1 and Lemma 2.4.5, we see that

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \left| T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, f_2,\dots,f_N)(x)\right|^2 \\ &\lesssim \sum_{j\in\mathbb{Z}} \left(\sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\dots,k_N)}(D) m^{(j)} \right\|_{L^2} \right. \end{aligned}$$

$$\times \left((|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2(x) \right)^{1/2} \times \prod_{i=2}^{N} \left((|\phi|^2)_{(k_i-j)} * |f_i|^2(x) \right)^{1/2} \Bigg)^2$$

$$\lesssim \prod_{i=2}^{N} M(|f_i|^2)(x)$$

$$\times \sum_{j\in\mathbb{Z}} \left(\sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} \right.$$

$$\left. \times \left((|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2(x) \right)^{1/2} \right)^2 .$$

By Schwarz's inequality, we have

$$\sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} \left((|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2(x) \right)^{1/2}$$

$$\le \left\{ \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} \right\}^{1/2}$$

$$\times \left\{ \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} (|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2(x) \right\}^{1/2} .$$

Thus, it follows that

$$\sum_{j\in\mathbb{Z}} \left| T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, f_2, \ldots, f_N)(x) \right|^2$$

$$\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\ldots,n/2)}((\mathbb{R}^n)^N)} \right) \prod_{i=2}^{N} M(|f_i|^2)(x)$$

$$\times \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} (|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2(x).$$

By Hölder's inequality, we see that

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} \left| T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, f_2, \ldots, f_N) \right|^2 \right\}^{1/2} \right\|_{L^p(w)} \tag{6.2.5}$$

$$\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\ldots,n/2)}((\mathbb{R}^n)^N)} \right)^{1/2}$$

$$\times \Bigg\| \Bigg\{ \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2}$$

$$\times\ (|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2 \Big\}^{1/2} \Bigg\|_{L^{p_1}(w_1)}$$

$$\times \prod_{i=2}^{N} \left\| M\left(|f_i|^2\right)^{1/2} \right\|_{L^{p_i}(w_i)}.$$

For the third term on the right hand side of (6.2.5), since $2 < p_i < \infty$ and $w_i \in A_{p_i/2}, i = 2, \ldots, N$, we have

$$\prod_{i=2}^{N} \left\| M\left(|f_i|^2\right)^{1/2} \right\|_{L^{p_i}(w_i)} \lesssim \prod_{i=2}^{N} \|f_i\|_{L^{p_i}(w_i)}. \tag{6.2.6}$$

For the second term on the right hand side of (6.2.5), we set as follows

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} (|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2 \right\}^{1/2} \right\|_{L^{p_1}(w_1)}$$

$$= \left\| \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} (|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2 \right\|_{L^{p_1/2}(w_1)}^{1/2}$$

$$=: \|H\|_{L^{p_1/2}(w_1)}^{1/2}.$$

We shall prove that

$$\|H\|_{L^{p_1/2}(w_1)}^{1/2} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B_{2,1}^{(n/2,\ldots,n/2)}((\mathbb{R}^n)^N)} \right)^{1/2} \|f_1\|_{L^{p_1}(w_1)}. \tag{6.2.7}$$

Once this is proved, by (3.2.3), (6.2.4), (6.2.5), (6.2.6) and (6.2.7), we obtain (6.2.2).

Let $u \in L^{(p_1/2)'}(w_1^{1-(p_1/2)'})$ be such that $\|u\|_{L^{(p_1/2)'}(w_1^{1-(p_1/2)'})} = 1$. By Lemma 2.4.5, Hölder's inequality, Lemma 2.4.3 with $w_1 \in A_{p_1/2} \subset A_{p_1}$ and $L^{(p_1/2)'}(w_1^{1-(p_1/2)'})$ -boundedness of M, it follows that

$$\left| \int_{\mathbb{R}^n_x} H(x) u(x)\, dx \right|$$

$$\leq \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2}$$

$$\times \int_{\mathbb{R}^n_x} \left\{ (|\phi|^2)_{(k_1-j)} * |\tilde{\Delta}_j f_1|^2 \right\}(x) |u(x)|\, dx$$

$$= \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}^n_x} |\tilde{\Delta}_j f_1(x)|^2 \left\{ (|\phi|^2)_{(k_1-j)} * |u| \right\}(x)\, dx \\
&\lesssim \sum_{j\in\mathbb{Z}} \sum_{k_1,\ldots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\| \Psi_{(k_1,\ldots,k_N)}(D) m^{(j)} \right\|_{L^2} \int_{\mathbb{R}^n_x} |\tilde{\Delta}_j f_1(x)|^2 Mu(x)\, dx \\
&\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\ldots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \right) \int_{\mathbb{R}^n_x} \sum_{j\in\mathbb{Z}} |\tilde{\Delta}_j f_1(x)|^2 Mu(x)\, dx \\
&\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\ldots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \right) \left\| \sum_{j\in\mathbb{Z}} |\tilde{\Delta}_j f_1|^2 \right\|_{L^{p_1/2}(w_1)} \|Mu\|_{L^{(p_1/2)'}(w_1^{1-(p_1/2)'})} \\
&\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\ldots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \right) \|f_1\|^2_{L^{p_1}(w_1)} \|u\|_{L^{(p_1/2)'}(w_1^{1-(p_1/2)'})} \\
&= \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\ldots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \right) \|f_1\|^2_{L^{p_1}(w_1)},
\end{aligned}
$$

where we have used the fact that $w_1^{1-(p_1/2)'} \in A_{(p_1/2)'}$, by Lemma 2.4.2 (2). By taking suprimum over all such as above u, we obtain (6.2.7). Therefore, we have (6.2.2)

□

6.3 Estimate for $m_{(1,1,i_3,\ldots,i_N)}$ type

Next, we consider the case where $(i_1,\ldots,i_N)$ satisfies $\sharp\{j : i_j = 1\} \geq 2$. Without loss of generality, we may assume that $i_1 = i_2 = 1$. We simply write m instead of $m_{(1,1,i_3,\ldots,i_N)}$, where $i_3,\ldots,i_N \in \{0,1\}$. Note that by Lemma 2.4.1 (3),

$$
\begin{aligned}
&\operatorname{supp} m \\
&\subset \{|\xi_1|/(4N) \leq |\xi_2| \leq 4N|\xi_1|, |\xi_i| \leq 4N|\xi_1|, i = 3,\ldots,N\}.
\end{aligned} \tag{6.3.1}
$$

Proof. In this case, we shall prove the estimate

$$
\begin{aligned}
&\|T_m(f_1,\ldots,f_N)\|_{L^p(w)} \\
&\lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\ldots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \right) \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i)}
\end{aligned} \tag{6.3.2}
$$

holds, where $m^{(j)}$ will be defined later on. In Section 6.4, we shall complete the proof.

As in Section 3.3, we can find functions $\varphi \in \mathcal{A}_0$ and $\widetilde{\psi} \in \mathcal{A}_1$ independent of j such that

$$
\begin{aligned}
&m(\xi)\psi(\xi_1/2^j) \\
&= m(\xi)\psi(\xi_1/2^j)\widetilde{\psi}(\xi_1/2^j)\widetilde{\psi}(\xi_2/2^j)^2\varphi(\xi_3/2^j)\ldots\varphi(\xi_N/2^j).
\end{aligned}
$$

Setting

$$m^{(j)}(\xi) = m(2^j\xi)\psi(\xi_1)\widetilde{\psi}(\xi_2)\varphi(\xi_3)\dots\varphi(\xi_N), \tag{6.3.3}$$

we have

$$T_m(f_1,\dots,f_N)(x) = \sum_{j\in\mathbb{Z}} T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, \tilde{\Delta}_j f_2, f_3,\dots,f_N)(x), \tag{6.3.4}$$

where $\tilde{\Delta}_j f_i = \widetilde{\psi}(D/2^j)f_i, i = 1, 2$.

By (6.3.4), Lemma 6.1.1, Lemma 2.4.5 and Schwarz's inequality, it follows that

$$\begin{aligned}
&|T_m(f_1,\dots,f_N)(x)| \\
&\le \sum_{j\in\mathbb{Z}} \left|T_{m^{(j)}(\cdot/2^j)}(\tilde{\Delta}_j f_1, \tilde{\Delta}_j f_2, f_3,\dots,f_N)(x)\right| \\
&\le \sum_{j\in\mathbb{Z}} \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \\
&\quad \times \prod_{i=1}^{2}\Big((|\phi|^2)_{(k_i-j)} * |\tilde{\Delta}_j f_i|^2(x)\Big)^{1/2} \times \prod_{i=3}^{N}\big((|\phi|^2)_{(k_i-j)} * |f_i|^2(x)\big)^{1/2} \\
&\le \prod_{i=3}^{N} M(|f_i|^2)(x)^{1/2} \\
&\quad \times \prod_{i=1}^{2}\Bigg\{\sum_{j\in\mathbb{Z}} \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \\
&\quad \times (|\phi|^2)_{(k_i-j)} * |\tilde{\Delta}_j f_i|^2(x)\Bigg\}^{1/2}.
\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}
&\|T_m(f_1,\dots,f_N)\|_{L^p(w)} \\
&\le \prod_{i=1}^{2}\Bigg\|\Bigg\{\sum_{j\in\mathbb{Z}} \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2} \\
&\quad \times\ (|\phi|^2)_{(k_i-j)} * |\tilde{\Delta}_j f_i|^2\Bigg\}^{1/2}\Bigg\|_{L^{p_i}(w_i)} \\
&\quad \times \prod_{i=3}^{N}\left\|M(|f_i|^2)^{1/2}\right\|_{L^{p_i}(w_i)} \\
&= \prod_{i=1}^{2}\Bigg\|\sum_{j\in\mathbb{Z}} \sum_{k_1,\dots,k_N=0}^{\infty} 2^{(k_1+\cdots+k_N)n/2} \left\|\Psi_{(k_1,\dots,k_N)}(D)m^{(j)}\right\|_{L^2}
\end{aligned} \tag{6.3.5}$$

$$\times\ (|\phi|^2)_{(k_i-j)} * |\tilde{\Delta}_j f_i|^2 \Big\|^{1/2}_{L^{p_i/2}(w_i)}$$

$$\times \prod_{i=3}^{N} \big\| M(|f_i|^2) \big\|^{1/2}_{L^{p_i/2}(w_i)}$$

$$=: \prod_{i=1}^{2} \|H_i\|^{1/2}_{L^{p_i/2}(w_i)} \times \prod_{i=3}^{N} \big\| M(|f_i|^2) \big\|^{1/2}_{L^{p_i/2}(w_i)} .$$

By the same way as for $m_{(1,0,\dots,0)}$, we see that

$$\|H_i\|^{1/2}_{L^{p_1/2}} \lesssim \left(\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \right)^{1/2} \|f_i\|_{L^{p_i}(w_i)}, \tag{6.3.6}$$

where $i = 1, 2$. By (6.3.5) and (6.3.6), we obtain (6.3.2). □

6.4 Completion of the proof of Theorem 1.4.4

In this section, we shall prove the estimate

$$\sup_{j\in\mathbb{Z}} \|m^{(j)}\|_{B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{(n/2,\dots,n/2)}_{2,1}((\mathbb{R}^n)^N)}$$

holds, where $m^{(j)}$ is defined by (6.2.3) or (6.3.3), and m_j is defined by (2.2.2).

Proof. We first consider the case where $m^{(j)}$ is the same as in (6.2.3). Since $\operatorname{supp} \Psi(\cdot/2^\ell) \subset \{\xi \in (\mathbb{R}^n)^N : 2^{\ell-1} \le |\xi| \le 2^{\ell+1}\}$ and

$$\operatorname{supp} \widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N) \subset \{\xi \in (\mathbb{R}^n)^N : 2^{-j_0} \le |\xi| \le 2^{j_0}\} \tag{6.4.1}$$

for some $j_0 \in \mathbb{N}$, for example, $j_0 = [\log_2 2(N+1)]$, where $[s]$ is the integer part of $s \in \mathbb{R}$, it follows from Lemma 2.4.9 (1) that

(6.4.2)

$$\begin{aligned}
&\|m^{(j)}\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&= \|m(2^j\xi)\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&\lesssim \sum_{\ell=-j_0}^{j_0} \Big\| m(2^j\xi)\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\dots+\xi_N) \\
&\quad \times \widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\Psi(\xi/2^\ell) \Big\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&\lesssim \sum_{\ell=j_0}^{j_0} \|m(2^j\xi)\Psi(\xi/2^\ell)\|_{B^{(n/2,\dots,n/2)}_{2,1}}
\end{aligned}$$

$$\times \|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B^{(n/2,\dots,n/2)}_{\infty,1}}.$$

For the first term on the right hand side of (6.4.2), by Lemma 2.4.9 (2), we have

$$\begin{aligned}
&\|m(2^j\xi)\Psi(\xi/2^\ell)\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&= \|m\left(2^{j+\ell}(\xi/2^\ell)\right)\Psi(\xi/2^\ell)\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&= \|m_{j+\ell}(\xi/2^\ell)\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&\lesssim \left(\max\{1,2^{(-\ell)n/2}\}2^{-(-\ell)n/2}\right)^N \|m_{j+\ell}\|_{B^{(n/2,\dots,n/2)}_{2,1}} \\
&\lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{B^{(n/2,\dots,n/2)}_{2,1}}.
\end{aligned}$$

For the second term on the right hand side of (6.4.2), by the Leibniz's formula, Lemma 2.4.1 (2) and (6.4.1), we see that

$$\left|\partial^\alpha_\xi\left(\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\right)\right| \le C_\alpha.$$

for all $\alpha \in \mathbb{Z}^n_+$ and $j \in \mathbb{Z}$. Therefore, we obtain

$$\sup_{j\in\mathbb{Z}} \|\Phi_{(1,0,\dots,0)}(2^j\xi)\psi(\xi_1+\cdots+\xi_N)\widetilde{\psi}(\xi_1)\varphi(\xi_2)\dots\varphi(\xi_N)\|_{B^{(n/2,\dots,n/2)}_{\infty,1}} < \infty.$$

In the case where $m^{(j)}$ is the same as in (6.3.3), the proof is similar to that of $m_{(1,0,\dots,0)}$, and we omit it. □

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