The Unramified Shintani Functions for the Reductive Symmetric Pair $(\mathbf{GSp}_4, \mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)$

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Abstract

Let F be a non-archimedean local field of *arbitrary* characteristic. We give an explicit formula of the unramified Shintani functions on $\mathbf{GSp}_4(F)$. As an application, we evaluate a local zeta integral of Murase–Sugano type, which turns out to be the spin L-factor of $\mathbf{GSp}_4$.

Contents

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1 Introduction

Proving the meromorphic continuation and the functional equation of an automorphic L-function is one of the basic problems in the theory of automorphic representations. In order to prove these, there is a distinguished technique called Rankin–Selberg method, which is a theory of an integral representation of automorphic L-functions. In 1994, Murase–Sugano [MS] introduced a new kind of global zeta integral for the symmetric pair $(\mathbf{O}_n, \mathbf{O}_{n-1})$ of the orthogonal groups over an algebraic number field. Under certain assumptions, the global zeta integral turns out to be an integral of the global Shintani function and moreover can be decomposed into an Euler product of local zeta integrals. Local Shintani functions, which are main objects in this thesis, appear in the local zeta integral as an integrand. They proved that the unramified local zeta integral represents the standard L-factor of $\mathbf{O}_n$ ([MS, Theorem 1.6]). Later, for a non-archimedean local field F' of characteristic zero or $p > 2$, Kato–Murase–Sugano [KMS] gave an explicit formula of the unramified (Whittaker–)Shintani functions for the pair $(\mathbf{SO}_n(F'), \mathbf{SO}_{n-1}(F'))$ of the split special orthogonal groups.

In this thesis we first give an explicit formula of the unramified Shintani functions for the symmetric pair $(\mathbf{GSp}_4(F), (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F))$. Here F is a non-archimedean local field of *arbitrary* characteristic. Moreover, by using our explicit formula, we prove that the unramified local zeta integral of Murase–Sugano type for $(\mathbf{GSp}_4(F), (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F))$ represents the spin L-factor of $\mathbf{GSp}_4$. This proof is more direct compared to [MS] (see Remark 5.1.3 and Remark 5.3.3). Here we note that there are two important points in this thesis. First we allow F to be of characteristic two. Our explicit formula in the case where F is of characteristic two is *not* reduced to the results in [KMS], although that in the other case is reduced to a special case of their results (see [G1]). Second, in order to give a meromorphic continuation of Shintani functionals, we employ Bernstein's rationality theorem. Note that the use of Bernstein's rationality theorem is proposed in [BFF] without detailed explanation. We supply the missing details and prove that Shintani functionals are extended to a dense subset of the space of the Satake parameters. It seems that this argument is omitted in [S] and [Z], where Bernstein's rationality theorem is used in similar situations.

We explain our result more precisely. Let $G = \mathbf{GSp}_4(F)$ be the symplectic similitude group as defined in §2.2. We denote by G_0 a certain subgroup of G which satisfies $G_0 \simeq (\mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2)(F)$. Let P (resp. P_0) be the standard minimal parabolic subgroup of G (resp. G_0) and δ (resp. δ_0) its modulus character. We denote by T (resp. T_0) the diagonal subgroup of G (resp. G_0). Then T (resp. T_0) is a Levi subgroup of P (resp. P_0). If $\mathfrak{o}$ is the ring of integers of F, then $K = \mathbf{GSp}_4(\mathfrak{o})$ (resp. $K_0 = G_0 \cap K$) is a maximal compact subgroup of G (resp. G_0). For any $(\xi, \Xi) = ((\xi_1, \xi_2, \xi_3), (\Xi_1, \Xi_2, \Xi_3)) \in X_{nr}(T_0) \times X_{nr}(T) \simeq (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3$, we define $\mathcal{S}(\xi, \Xi)$ to be the $\mathbb{C}$-vector space consisting of all the continuous functions $S : G \to \mathbb{C}$ such that

$$[L(\phi)R(\Phi)S](x) = \omega_\xi(\phi)\omega_\Xi(\Phi)S(x) \quad ({}^\forall\phi \in \mathcal{H}(G_0, K_0), {}^\forall\Phi \in \mathcal{H}(G, K)).$$

Here $\mathcal{H}(G, K)$ (resp. $\mathcal{H}(G_0, K_0)$) is the Hecke algebra of (G, K) (resp. (G_0, K_0)), and ω_Ξ (resp. ω_ξ) is the $\mathbb{C}$-algebra homomorphism $\mathcal{H}(G, K) \to \mathbb{C}$ (resp. $\mathcal{H}(G_0, K_0) \to \mathbb{C}$) corresponding to Ξ (resp. ξ) via the Satake isomorphism. We call an element of $\mathcal{S}(\xi, \Xi)$ an *unramified Shintani function of type*

(ξ, Ξ) *for the pair* (G, G_0), or simply a *Shintani function.* Our purpose is to give an explicit formula of Shintani functions. It follows from the definition that a Shintani function $S \in \mathcal{S}(\xi, \Xi)$ is determined by the values on $K_0\backslash G/K$. By the Cartan type decomposition (Theorem 3.2.1), it is enough to know the values $S(t(\lambda')\eta t(\lambda))$ for all $(\lambda', \lambda) = ((\lambda'_1, \lambda'_2, \lambda'_1), (\lambda_1, \lambda_2, \lambda_3)) \in \Lambda_0^{++} \times \Lambda^+$, where

$$\Lambda^+ = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 |\ \lambda_1 \geq \lambda_2, 2\lambda_2 \geq \lambda_3\},$$
$$\Lambda_0^{++} = \{\lambda' = (\lambda'_1, \lambda'_2, \lambda'_1) \in \mathbb{Z}^3 | \lambda'_1 \geq 0, 2\lambda'_2 \geq \lambda'_1\},$$
$$t((\lambda_1, \lambda_2, \lambda_3)) = \begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \varpi^{\lambda_2} & & \\ & & \varpi^{\lambda_3-\lambda_1} & \\ & & & \varpi^{\lambda_3-\lambda_1} \end{pmatrix}, \quad \eta = \left(\begin{array}{cc|cc} 1 & 1 & & 1 \\ & 1 & 1 & \\ \hline & & 1 & \\ & & -1 & 1 \end{array}\right).$$

For each $(\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)$, we set

$$c_S(\xi, \Xi) = \frac{\mathbf{b}(\xi, \Xi)}{\mathbf{d}'(\xi)\mathbf{d}(\Xi)},$$

where

$$\mathbf{d}(\Xi) = (1 - \Xi_1\Xi_2)(1 - \Xi_1\Xi_2^{-1})(1 - \Xi_1)(1 - \Xi_2), \quad \mathbf{d}'(\xi) = (1 - \xi_1)(1 - \xi_2),$$
$$\begin{aligned}\mathbf{b}(\xi, \Xi) = {} & (1 - q^{-1/2}\Xi_1\Xi_3\xi_1\xi_3)(1 - q^{-1/2}\Xi_1\Xi_3\xi_2\xi_3)(1 - q^{-1/2}\Xi_1\Xi_3\xi_1\xi_2\xi_3)(1 - q^{-1/2}\Xi_2\Xi_3\xi_1\xi_2\xi_3) \\ & \times (1 - q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_1\xi_3)(1 - q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_2\xi_3)(1 - q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_3)(1 - q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_1\xi_2\xi_3).\end{aligned}$$

Then our main result is as follows.

Theorem A (Theorem 4.5.4) *Let* (ξ, Ξ) *be any element of* $X_{nr}(T_0) \times X_{nr}(T)$*. Then we have*

$$\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) = \begin{cases} 1 & (\textit{if } (\xi\Xi)|_Z \equiv 1), \\ 0 & (\textit{otherwise}). \end{cases}$$

If $(\xi\Xi)|_Z \equiv 1$*, for any nonzero element* $S \in \mathcal{S}(\xi, \Xi)$ *we have* $S(1_4) \neq 0$*, and the Shintani function* $W_{\xi,\Xi} \in \mathcal{S}(\xi, \Xi)$ *with* $W_{\xi,\Xi}(1_4) = 1$ *is given by*

$$W_{\xi,\Xi}(t(\lambda')\eta t(\lambda)) = \frac{(\Xi_1\Xi_2\Xi_3^2)^{\lambda_3}}{(1 - q^{-2})^2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi) \left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda)) \left((w'\xi)^{-1}\delta_0^{1/2}\right)(t(\lambda'))$$

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$*. Here* 1_4 *is the identity element of* G*,* Z *is the center of* G *and* W *(resp.* W_0*) is the Weyl group of* (G, T) *(resp.* (G_0, T_0)*).*

We shall explain the outline of our proof of Theorem A by describing the contents of each section. In Section 2, we introduce basic notation and objects which will be used throughout this thesis. In Section 3, we recall the definition of the Shintani functions on $\mathbf{GSp}_4(F)$ and prove the Cartan type decomposition of $\mathbf{GSp}_4(F)$. Also, we prove the uniqueness of the Shintani functions by using a system of difference equations satisfied by Shintani functions. In Section 4, we prove the explicit formula of the Shintani function. In §4.1, we construct a nonzero Shintani functional $\Omega_{\xi,\Xi}$ for any element (ξ, Ξ) of a certain domain $U_c^\sharp \subset (\mathbb{C}^\times)^5$ by using *relative invariants*, which are introduced in

[KMS]. In §4.4, we give a meromorphic continuation of $\Omega_{\xi,\Xi}$. This is the most technical part of the proof of Theorem A. In order to do that, instead of following the method of [KMS] closely, we apply Bernstein's rationality theorem (Theorem 4.4.4) combining with a simple measure theoretic argument (Proposition 4.3.17). The necessary verifications for applying this theorem is done in §4.2 (Rank one calculation) and §4.3 (Uniqueness of Shintani functionals). In §4.5, we prove Theorem A by using the method employed to prove that of the unramified Whittaker functions in [CS] (see also [KMS]). The meromorphic continuation of $\Omega_{\xi,\Xi}$ enables us to get a $(W_0 \times W)$-invariance of Shintani functions. In Section 5, we evaluate a local zeta integral of Murase–Sugano type, which turns out to be the spin L-factor of $\mathbf{GSp}_4$, as an application of our explicit formula.

We note that besides the paper [KMS] mentioned above there are several papers studying (Whittaker–)Shintani functions on $\mathbf{GSp}_4(F)$ or related groups. For example, an explicit formula of Whittaker–Shintani functions for $(\mathbf{Sp}_{2n}(F), \text{Jacobi group})$ was given by Murase [M] for $n = 2$. Later Murase's result was generalized to any n by Shen [S]. Also, Bump–Friedberg–Furusawa [BFF] proved an explicit formula of Bessel functions on $\mathbf{GSp}_4(F)$ and Hironaka [H2] proved that of Shintani functions for $(\mathbf{Sp}_4(F), \mathbf{SL}_2(F) \times \mathbf{SL}_2(F))$.

The main results in this thesis have been announced in [G1] and the contents from Section 2 to Section 4 are going to be published as a paper [G2].

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2 Preliminaries

We denote by $\mathbb{Z}$ and $\mathbb{C}$ the ring of rational integers and the complex number field, respectively. For $z \in \mathbb{C}$, we denote by $\|z\|$ the absolute value of z. Throughout this thesis F is a non-archimedean local field of arbitrary characteristic. We denote by $\mathfrak{o} = \mathfrak{o}_F$ the ring of integers of F, and we let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$. Once and for all, we fix a generator ϖ of $\mathfrak{p}$. We normalize the $\mathfrak{p}$-adic absolute value $|x|$ of $x \in F$ so that $|\varpi| = q^{-1}$. For an affine algebraic group $\mathbf{H}$ over F, we denote by H the locally compact group $\mathbf{H}(F)$. We denote by 1_n the identity matrix of size n.

2.1 Unramified principal series representations of reductive groups.

In this subsection, we recall properties of an unramified principal series representation of a connected reductive algebraic group. Let $\mathbf{H}$ be a connected reductive algebraic group over F. For simplicity, we assume that $\mathbf{H}$ is split over F. Let $\mathbf{T_H}$ be a maximal split torus of $\mathbf{H}$ and $\mathbf{P_H}$ a minimal parabolic subgroup of $\mathbf{H}$ containing $\mathbf{T_H}$. Then we have the Levi decomposition $\mathbf{P_H} = \mathbf{T_H}\mathbf{N_H}$. Here $\mathbf{N_H}$ is the unipotent radical of $\mathbf{P_H}$. We denote by $\Sigma = \Sigma(\mathbf{H}, \mathbf{T_H})$ the root system of $(\mathbf{H}, \mathbf{T_H})$, by $\Sigma^+ = \Sigma^+(\mathbf{H}, \mathbf{T_H})$ the set of positive roots corresponding to $\mathbf{P_H}$ and by $\Delta = \Delta(\mathbf{H}, \mathbf{T_H}, \mathbf{P_H})$ the set of simple roots determined by Σ^+. We denote by $\alpha^\vee$ the coroot corresponding to $\alpha \in \Sigma$. We set $a_\alpha := \alpha^\vee(\varpi)$ for $\alpha \in \Sigma$. Since $\mathbf{H}$ is split, $\mathbf{H}$ is defined over $\mathfrak{o}$ (see [T, 1.10], for example). Also, since $\mathbf{P_H}, \mathbf{T_H}$ and $\mathbf{N_H}$ are split, $\mathbf{P_H}, \mathbf{T_H}$ and $\mathbf{N_H}$ are defined over $\mathfrak{o}$. We set $K_H = \mathbf{H}(\mathfrak{o})$. Then K_H is a maximal compact subgroup of $H = \mathbf{H}(F)$. We have the Iwasawa decomposition $H = P_H K_H$. Let $B_H \subset K_H$ be the Iwahori subgroup corresponding to Σ^+, that is,

$$B_H = (\text{the inverse image of } \mathbf{P_H}(\mathfrak{o}/\mathfrak{p}) \subset \mathbf{H}(\mathfrak{o}/\mathfrak{p}) \text{ via the natural map } K_H \twoheadrightarrow \mathbf{H}(\mathfrak{o}/\mathfrak{p})).$$

Let W_H be the Weyl group of (H, T_H). Then we have the following three Bruhat type decompositions:

$$H = P_H W_H P_H, \quad H = P_H W_H B_H, \quad K_H = B_H W_H B_H.$$

We denote by $\ell(w)$ the length of $w \in W_H$ corresponding to Σ^+ and by w_ℓ the longest element of W_H. We denote by $w_\alpha \in W_H$ the reflection corresponding to $\alpha \in \Sigma$.

We fix an isomorphism $t : (\mathbf{GL}_1)^{\dim(\mathbf{T_H})} \overset{\sim}{\to} \mathbf{T_H}, (t_1, \cdots, t_{\dim(\mathbf{T_H})}) \mapsto t(t_1, \cdots, t_{\dim(\mathbf{T_H})})$. Let $X^*(\mathbf{T_H})$ be the $\mathbb{Z}$-module consisting of algebraic homomorphisms $\mathbf{T_H} \to \mathbf{GL}_1$ and $X_*(\mathbf{T_H})$ the $\mathbb{Z}$-module consisting of algebraic homomorphisms $\mathbf{GL}_1 \to \mathbf{T_H}$. Then we have the canonical pairing $\langle \cdot, \cdot \rangle : X^*(\mathbf{T_H}) \times X_*(\mathbf{T_H}) \to \mathbb{Z}$ given by composition. Let $\{e_i\}_{i=1}^{\dim(\mathbf{T_H})}$ be the standard basis of $X^*(\mathbf{T_H})$ given by $e_i(t(t_1, \cdots, t_{\dim(\mathbf{T_H})})) = t_i$ $(i = 1, \cdots, \dim(\mathbf{T_H}))$ and $\{d_i\}_{i=1}^{\dim(\mathbf{T_H})}$ the basis of $X_*(\mathbf{T_H})$ dual to $\{e_i\}_{i=1}^{\dim(\mathbf{T_H})}$ with respect to the canonical pairing. For $\sigma, \tau \in X^*(\mathbf{T_H})$, we write $\sigma \geq \tau$ if $\sigma - \tau$ is a linear combination of positive roots with nonnegative coefficients.

A character of the group T_H is called *unramified* if it is trivial on $T_H \cap K_H$. Let $X_{nr}(T_H)$ be the group of unramified characters of T_H. Then the modulus character δ_{P_H} of P_H is an element of $X_{nr}(T_H)$. If we denote by $X_{nr}(F^\times)$ the group of unramified characters of $F^\times$, we often identify $X_{nr}(T_H)$ with $X_{nr}(F^\times)^{\dim(T_H)}$ via

$$X_{nr}(T_H) \to X_{nr}(F^\times)^{\dim(T_H)}, \chi \mapsto (\chi_1, \cdots, \chi_{\dim(T_H)}) := (\chi \circ d_1, \cdots, \chi \circ d_{\dim(T_H)})$$

or $(\mathbb{C}^\times)^{\dim(T_H)}$ via

$$X_{nr}(T_H) \to (\mathbb{C}^\times)^{\dim(T_H)}, \chi \mapsto (\chi_1(\varpi), \cdots, \chi_{\dim(T_H)}(\varpi)).$$

Remark 2.1.1 *Via the above identify $X_{nr}(T_H) \simeq (\mathbb{C}^\times)^{\dim(T_H)}$, we often regard $X_{nr}(T_H)$ as a complex manifold or an affine algebraic variety.*

For $\chi \in X_{nr}(T_H)$, we denote by $i_{P_H}^H(\chi)$ the normalized unramified principal series representation of H. The standard model of this representation is the vector space consisting of locally constant functions $f \in C^\infty(H)$ which satisfy $f(px) = (\chi\delta_{P_H}^{1/2})(p)f(x)$ $((p, x) \in P_H \times H)$. The group H acts on $i_{P_H}^H(\chi)$ by the right translation $R = R_\chi$, where $[R(h)f](x) = f(xh)$. We note that $i_{P_H}^H(\chi)$ is an admissible representation of H. Let $\mathcal{P}_\chi$ be the intertwining operator $C_c^\infty(H) \to i_{P_H}^H(\chi)$ defined by

$$\mathcal{P}_\chi(f)(x) := \int_{P_H} (\chi^{-1}\delta_{P_H}^{1/2})(p)f(px)d_l p \quad ({}^\forall f \in C_c^\infty(H)).$$

Here $d_l p$ is the left invariant measure of P_H with $\mathrm{vol}(P_H \cap K_H; d_l p) = 1$. It is well-known that $\mathcal{P}_\chi$ is surjective. We set $\phi_{K_H,\chi} := \mathcal{P}_\chi(\mathrm{ch}_{K_H})$. Here ch_A is the characteristic function of a subset $A \subset H$. Then we have $i_{P_H}^H(\chi)^{K_H} = \mathbb{C}\phi_{K_H,\chi}$. The restriction of $f \in i_{P_H}^H(\chi)$ to K_H induces an isomorphism $i_{P_H}^H(\chi) \overset{\sim}{\to} C^\infty(P_H \cap K_H \backslash K_H)$ as a $\mathbb{C}$-vector space. Indeed, its inverse map $C^\infty(P_H \cap K_H \backslash K_H) \ni f \mapsto f_\chi \in i_{P_H}^H(\chi)$ is given by

$$f_\chi(pk) := (\chi\delta_{P_H}^{1/2})(p)f(k) \quad ({}^\forall(p, k) \in P_H \times K_H)$$

via the Iwasawa decomposition. We denote by R_χ a group homomorphism $H \to GL(C^\infty(P_H \cap K_H \backslash K_H))$ so that the diagram

$$\begin{array}{ccc} i_{P_H}^H(\chi) & \xrightarrow{\text{res.}} & C^\infty(P_H \cap K_H\backslash K_H) \\ {\scriptstyle R(h)}\downarrow & & \downarrow{\scriptstyle R_\chi(h)} \\ i_{P_H}^H(\chi) & \xrightarrow{\text{res.}} & C^\infty(P_H \cap K_H\backslash K_H) \end{array}$$

commutes for all $h \in H$.

Let $\mathcal{H}(H, K_H)$ be the Hecke algebra of (H, K_H) over $\mathbb{C}$, that is,

$$\mathcal{H}(H, K_H) = \{\varphi \in C_c(H) |\ \varphi(k_1 x k_2) = \varphi(x)\ (^\forall x \in H, {}^\forall k_1, k_2 \in K_H)\}$$

with the multiplication given by convolution

$$(\varphi_1 * \varphi_2)(x) = \int_H \varphi_1(xh^{-1})\varphi_2(h)dh \quad (^\forall \varphi_1, \varphi_2 \in \mathcal{H}(H, K_H)).$$

Here dh is the Haar measure of H with $\mathrm{vol}(K_H; dh) = 1$. We note that the identity element of $\mathcal{H}(H, K_H)$ is ch_{K_H} for the multiplication. For an unramified representation (π, V) of H, the Hecke algebra $\mathcal{H}(H, K_H)$ acts on V^{K_H} by

$$\pi(\varphi)v := \int_H \varphi(h)\pi(h)v dh \quad (^\forall v \in V^{K_H}, {}^\forall \varphi \in \mathcal{H}(H, K_H)).$$

In particular, $\mathcal{H}(H, K_H)$ acts on $i_{P_H}^H(\chi)^{K_H}$. Since $i_{P_H}^H(\chi)^{K_H} = \mathbb{C}\phi_{K_H,\chi}$, there exists a $\mathbb{C}$-algebra homomorphism $\omega_\chi : \mathcal{H}(H, K_H) \to \mathbb{C}$ such that

$$R(\varphi)\phi_{K_H,\chi} = \omega_\chi(\varphi)\phi_{K_H,\chi} \quad (^\forall \varphi \in \mathcal{H}(H, K_H)).$$

We recall the Satake isomorphism using the above notation. Let $\mathbb{C}[T_H/T_H \cap K_H]$ be the group algebra of $T_H/T_H \cap K_H$. Since a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[T_H/T_H \cap K_H] \to \mathbb{C}$ is determined by the image of the generator $d_1(\varpi), \cdots, d_{\dim(\mathbf{T_H})}(\varpi)$ of $\mathbb{C}[T_H/T_H \cap K_H]$, we have $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[T_H/T_H \cap K_H], \mathbb{C}) \simeq (\mathbb{C}^\times)^{\dim \mathbf{T_H}} \simeq X_{nr}(T_H)$. The Weyl group W_H acts on T_H by $w \cdot t := wtw^{-1} (w \in W_H, t \in T_H)$. The action is extended linearly to an action of W_H on $\mathbb{C}[T_H/T_H \cap K_H]$.

Theorem 2.1.2 *There exists a unique $\mathbb{C}$-algebra homomorphism $\omega : \mathcal{H}(H, K_H) \to \mathbb{C}[T_H/T_H \cap K_H]$ with the following properties*:

i) $\omega : \mathcal{H}(H, K_H) \xrightarrow{\sim} \mathbb{C}[T_H/T_H \cap K_H]^{W_H} := \{f \in \mathbb{C}[T_H/T_H \cap K_H] |\ w \cdot f = f(^\forall w \in W_H)\}$;

ii) *For all $\chi \in X_{nr}(T_H)$, $\chi \circ \omega = \omega_\chi$.*

For two unramified characters $\chi, \chi' \in X_{nr}(T_H)$, there exists $w \in W_H$ such that $\chi = w\chi'$ if and only if $\omega_\chi = \omega_{\chi'}$. Here W_H acts on $X_{nr}(T_H)$ by $(w\chi)(t) := \chi(w^{-1} \cdot t)(t \in T_H, w \in W_H, \chi \in X_{nr}(T_H))$. Therefore we have an isomorphism

$$X_{nr}(T_H)/W_H \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(H, K_H), \mathbb{C}), \quad \chi \mapsto \omega_\chi.$$

An unramified character χ of T_H is called *regular* if $w\chi \neq \chi$ for all $w \in W_H - \{1\}$. We denote by $X_{nr}^{reg}(T_H)$ the set of regular characters in $X_{nr}(T_H)$. We assume χ to be a regular character until the end of this subsection. For any $w \in W_H$, we set $\phi_w := \mathcal{P}_\chi(\mathrm{ch}_{B_H w B_H}) \in i_{P_H}^H(\chi)$. Then $\{\phi_w\}_{w \in W_H}$ is a basis of $i_{P_H}^H(\chi)^{B_H}$. It is well-known that there exists a set $\{T_{w,\chi} : i_{P_H}^H(\chi) \to i_{P_H}^H(w\chi)\}_{w \in W_H, \chi \in X_{nr}^{reg}(T_H)}$ of intertwining operators such that

i) $T_{1,\chi} = \mathrm{id}$;

ii) $T_{w,\chi}(\phi_{K_H,\chi}) = c_w(\chi)\phi_{K_H,w\chi}$;

iii) If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then $T_{w_1w_2,\chi} = T_{w_1,w_2\chi} \circ T_{w_2,\chi} \quad ({}^\forall w_1, w_2 \in W_H)$;

iv) $T_{w_\alpha}(\phi_{w_\alpha w} + \phi_w) = c_\alpha(\chi)(\phi_{w_\alpha w} + \phi_w) \quad ({}^\forall \alpha \in \Delta)$.

Here

$$c_\alpha(\chi) := \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}, \quad c_w(\chi) := \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} c_\alpha(\chi)$$

for $\alpha \in \Sigma^+$. We call c_α a *c-function* for $\alpha \in \Sigma^+$. See [C2, §3] and [C1, §6.4] for more detail.

Proposition 2.1.3 ([C2, §3]. See also [KMS, 1.10]) *There is another basis $\{g_w\}_{w \in W_H}$ of $i_{P_H}^H(\chi)^{B_H}$ such that*

i) $R(\mathrm{ch}_{B_H t^{-1} B_H}) g_w = \mathrm{vol}(B_H t B_H)\left((w\chi)^{-1}\delta_{P_H}^{1/2}\right)(t) g_w \quad ({}^\forall t \in T_H^+)$;

ii) $g_1 = \phi_1$;

iii) $\phi_{K_H,\chi} = q^{\ell(w_\ell)} \sum_{w \in W_H} \overline{c}_w(\chi) g_w$,

where $T_H^+ = \{t \in T_H |\ |\alpha(t)| \leq 1({}^\forall \alpha \in \Sigma^+)\}$ *and* $\overline{c}_w(\chi) := \prod_{\alpha > 0, w\alpha > 0} c_\alpha(\chi)$.

The two bases $\{\phi_w\}_{w \in W_H}$ and $\{g_w\}_{w \in W_H}$ play an important role in our proof of an explicit formula of Shintani functions (see §4.2 and §4.5).

2.2 Basic objects

Let $\mathbf{G}$ be an affine algebraic group over F defined by

$$\mathbf{G} = \mathbf{GSp}_4 = \{g \in \mathbf{GL}_4 |\ {}^t gJg = \nu(g)J, \nu(g) \in \mathbf{GL}_1\}, \quad J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}$$

and $\mathbf{P}$ its standard minimal parabolic subgroup defined by

$$\mathbf{P} = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ & * & * & * \\ \hline & & * & \\ & & * & * \end{array} \right) \in \mathbf{G} \right\}.$$

We note that $\nu : \mathbf{G} \to \mathbf{GL}_1$ is a homomorphism of algebraic groups, which is called the similitude character of $\mathbf{G}$. Let $\mathbf{P} = \mathbf{TN}$ be the Levi decomposition of $\mathbf{P}$, where $\mathbf{T}$ is the maximal (split) torus of $\mathbf{G}$ defined by

$$\mathbf{T} = \{t(t_1, t_2, t_3) := \mathrm{diag}(t_1, t_2, t_3 t_1^{-1}, t_3 t_2^{-1}) |\ t_1, t_2, t_3 \in \mathbf{GL}_1\}$$

and $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$. Then $X^*(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbf{GL}_1)$ and $X_*(\mathbf{T}) = \mathrm{Hom}(\mathbf{GL}_1, \mathbf{T})$ are $\mathbb{Z}$-modules of rank three. Let $\{e_i\}_{i=1}^3$ be the standard basis of $X^*(\mathbf{T})$ given by $e_i(t(t_1, t_2, t_3)) = t_i$ and

$\{d_i\}_{i=1}^3$ the basis of $X_*(\mathbf{T})$ dual to $\{e_i\}_{i=1}^3$ with respect to the canonical pairing $X^*(\mathbf{T}) \times X_*(\mathbf{T}) \to \mathbb{Z}, (\chi, \mu) \mapsto \langle \chi, \mu \rangle$. We often identify $X_*(\mathbf{T})$ with $\mathbb{Z}^3$ via

$$X_*(\mathbf{T}) \ni \mu \mapsto (\langle e_1, \mu \rangle, \langle e_2, \mu \rangle, \langle e_3, \mu \rangle) \in \mathbb{Z}^3.$$

Let $\Sigma \subset X^*(\mathbf{T})$ be the root system of $(\mathbf{G}, \mathbf{T})$ and Σ^+ the set of positive roots with respect to $\mathbf{P}$. Then we have $\Sigma^+ = \{e_1 - e_2, e_1 + e_2 - e_3, 2e_1 - e_3, 2e_2 - e_3\}$ and the set of simple roots is given by $\Delta = \{\alpha_1 := e_1 - e_2, \alpha_2 := 2e_2 - e_3\}$. The coroot of $\alpha \in \Sigma^+$ is given by

$$\alpha^\vee = \begin{cases} d_1 - d_2 & (\text{if } \alpha = \alpha_1 = e_1 - e_2), \\ d_1 + d_2 & (\text{if } \alpha = e_1 + e_2 - e_3), \\ d_1 & (\text{if } \alpha = 2e_1 - e_3), \\ d_2 & (\text{if } \alpha = 2e_2 - e_3). \end{cases}$$

We define homomorphisms $x_\alpha : \mathbf{G}_a \to \mathbf{G} (\alpha \in \Sigma)$ by

$$x_{e_1-e_2}(t) = {}^t x_{-e_1+e_2}(t) = \left(\begin{array}{cc|cc} 1 & t & & \\ & 1 & & \\ \hline & & 1 & \\ & & -t & 1 \end{array}\right), \quad x_{e_1+e_2-e_3}(t) = {}^t x_{-e_1-e_2+e_3}(t) = \left(\begin{array}{cc|cc} 1 & & & t \\ & 1 & t & \\ \hline & & 1 & \\ & & & 1 \end{array}\right),$$

$$x_{2e_1-e_3}(t) = {}^t x_{-2e_1+e_3}(t) = \left(\begin{array}{cc|cc} 1 & & t & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right), \quad x_{2e_2-e_3}(t) = {}^t x_{-2e_2+e_3}(t) = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & t \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

and set $\mathbf{N}_\alpha := \{x_\alpha(t) | t \in \mathbf{G}_a\}$. We denote by $\mathbf{N}^-$ the group generated by $\{x_{-\alpha}(t) |\ \alpha \in \Sigma^+, t \in \mathbf{G}_a\}$. We note that $\mathbf{N}$ is generated by $\{x_\alpha(t) |\ \alpha \in \Sigma^+, t \in \mathbf{G}_a\}$.

Let $\mathbf{G}_0$ be an affine algebraic group over F defined by

$$\mathbf{G}_0 = \mathbf{GL}_2 \times_{\mathbf{GL}_1} \mathbf{GL}_2 = \{(g_1, g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2 | \det(g_1) = \det(g_2)\}.$$

We often identify $\mathbf{G}_0$ with a subgroup of $\mathbf{G}$ via the embedding

$$\mathbf{G}_0 \ni (\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) \mapsto \left(\begin{array}{cc|cc} a_1 & & b_1 & \\ & a_2 & & b_2 \\ \hline c_1 & & d_1 & \\ & c_2 & & d_2 \end{array}\right) \in \mathbf{G}.$$

Then $\mathbf{P}_0 = \mathbf{P} \cap \mathbf{G}_0$ is a minimal parabolic subgroup of $\mathbf{G}_0$. Let $\mathbf{P}_0 = \mathbf{T}_0\mathbf{N}_0$ be the Levi decomposition of $\mathbf{P}_0$, where $\mathbf{T}_0$ is the maximal (split) torus of $\mathbf{G}_0$ defined by

$$\mathbf{T}_0 = \left\{ (\begin{pmatrix} t'_1 & \\ & t'_3 t'^{-1}_1 \end{pmatrix}, \begin{pmatrix} t'_2 & \\ & t'_3 t'^{-1}_2 \end{pmatrix}) \,\middle|\, t'_1, t'_2, t'_3 \in \mathbf{GL}_1 \right\}$$

and $\mathbf{N}_0$ is the unipotent radical of $\mathbf{P}_0$. Then $\mathbf{T}_0$ is identified with $\mathbf{T}$ via the embedding. Let $\{e'_i\}_{i=1}^3$ be the standard basis of $X^*(\mathbf{T}_0)$ given by $e'_i(t(t'_1, t'_2, t'_3)) = t'_i$ and $\{d'_i\}_{i=1}^3$ the basis of $X_*(\mathbf{T})$ dual to $\{e'_i\}_{i=1}^3$ with respect to the canonical pairing. We often identify $X_*(\mathbf{T}_0)$ with $\mathbb{Z}^3$ in the same way as $X_*(\mathbf{T})$. Let $\Sigma_0 \subset X^*(\mathbf{T}_0)$ be the root system of $(\mathbf{G}_0, \mathbf{T}_0)$ and Σ_0^+ the set of positive roots with respect to $\mathbf{P}_0$. Then we have $\Sigma_0^+ = \{\beta_1 := 2e_1' - e_3', \beta_2 := 2e_2' - e_3'\}$. We note that the set of simple roots coincides with Σ_0^+. The coroot of $\beta \in \Sigma_0^+$ is given by

$$\beta^\vee = \begin{cases} d_1' & (\text{if } \beta = \beta_1 = 2e_1' - e_3'), \\ d_2' & (\text{if } \beta = \beta_2 = 2e_2' - e_3'). \end{cases}$$

We define homomorphisms $x_\beta' : \mathbf{G}_a \to \mathbf{G}_0 \subset \mathbf{G} (\beta \in \Sigma_0)$ by

$$x'_{2e_1'-e_3'}(t) = {}^t x'_{-2e_1'+e_3'}(t) = \left(\begin{array}{cc|cc} 1 & & t & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right), \quad x'_{2e_2'-e_3'}(t) = {}^t x'_{-2e_2'+e_3'}(t) = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & t \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

and set $\mathbf{N}_{0,\beta} := \{x_\beta'(t) | t \in \mathbf{G}_a\}$. We denote by $\mathbf{N}_0^-$ the group generated by $\{x_{-\beta}(t) |\ \alpha \in \Sigma_0^+, t \in \mathbf{G}_a\}$. We note that $\mathbf{N}_0$ is generated by $\{x_\beta'(t) |\ \alpha \in \Sigma^+, t \in \mathbf{G}_a\}$ We define $\varpi_i \in X^*(\mathbf{T}), \varpi_i' \in X^*(\mathbf{T}_0)$ $(i = 1, 2)$ by

$$\varpi_1 := e_1, \quad \varpi_2 := e_1 + e_2, \quad \varpi_1' := e_1' - e_3', \quad \varpi_2' := e_2' - e_3'.$$

Each ϖ_i (resp. ϖ_i') is called the fundamental weight attached to the simple root α_i (resp β_i).

We set $K := \mathbf{G}(\mathfrak{o})$. Then K is a maximal compact subgroup of $G = \mathbf{G}(F)$. Similarly, $K_0 := \mathbf{G}_0(\mathfrak{o})$ is a maximal compact subgroup of $G_0 = \mathbf{G}_0(F)$. For a subgroup $\mathbf{H}$ of $\mathbf{G}$ defined over $\mathfrak{o}$, we set

$$H_{(0)} := \mathbf{H}(\mathfrak{o}), \quad H_{(1)} := \ker(H_{(0)} \twoheadrightarrow \mathbf{H}(\mathfrak{o}/\mathfrak{p})).$$

Here $H_{(0)} \twoheadrightarrow \mathbf{H}(\mathfrak{o}/\mathfrak{p})$ is the canonical surjection.

2.3 Decompositions for $\mathbf{GSp}_4(F)$ and its subgroups.

In this subsection, we recall several important decompositions of G and its subgroups. For any closed subgroup H of G, we normalize the measure dh of H so that $\mathrm{vol}(H \cap K; dh) = 1$. Let dg, dk, dt and dn be the Haar measures of G, K, T and N, respectively. We denote by $d_l p$ the left invariant measure $dtdn$ of $P = TN$ and by $\delta : P \to \mathbb{C}^\times$ the modulus character of P: $\delta(p) = |t_1^4 t_2^2 t_3^{-3}| ({}^\forall p = t(t_1, t_2, t_3)n \in P)$. Similarly, let dg', dk', dt' and dn' be the Haar measures of G_0, K_0, T_0 and N_0, respectively. We denote by $d_l p'$ the left invariant measure $dt'dn'$ of $P_0 = T_0 N_0$ and by $\delta_0 : P_0 \to \mathbb{C}^\times$ the modulus character of P_0: $\delta_0(p') = |t_1^2 t_2^2 t_3^{-2}| ({}^\forall p' = t(t_1, t_2, t_3)n' \in P_0)$.

Proposition 2.3.1 (Iwasawa decomposition)

$$\begin{aligned} G &= PK = KP, \quad dg = d_l p dk = \delta(p) dk d_l p, \\ G_0 &= P_0 K_0 = K_0 P_0, \quad dg' = d_l p' dk' = \delta_0(p') dk' d_l p'. \end{aligned}$$

Let W be the Weyl group of (G, T), that is, $W = \{g \in G | gTg^{-1} = T\}/T$. Then W is generated by $w_1 T$ and $w_2 T$, where

$$w_1 = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right), \quad w_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & & & 1 \\ \hline & & 1 & \\ & -1 & & \end{array}\right).$$

A complete set of representatives of W is given by

$$\{1_4,\ w_1,\ w_2,\ w_1w_2,\ w_2w_1,\ w_1w_2w_1,\ w_2w_1w_2,\ w_2w_1w_2w_1\}.$$

We set

$$w_\ell := \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right) \in w_2w_1w_2w_1T.$$

Similarly, let W_0 be the Weyl group of (G_0, T_0), which is defined in the same way as W. Then W_0 is generated by $w_1'T_0$ and $w_2'T_0$, where

$$w_1' = \left(\begin{array}{cc|cc} & & 1 & \\ & 1 & & \\ \hline -1 & & & \\ & & & 1 \end{array}\right), \quad w_2' = w_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & & & 1 \\ \hline & & 1 & \\ & -1 & & \end{array}\right).$$

A complete set of representatives of W_0 is given by $\{1_4,\ w_1',\ w_2',\ w_1'w_2'\}$.

Proposition 2.3.2 (Bruhat decomposition)

$$G = \bigsqcup_{w\in W} PwP = \bigsqcup_{w\in W} PwN = \bigsqcup_{w\in W} NwP,$$
$$G_0 = \bigsqcup_{w'\in W_0} P_0w'P_0 = \bigsqcup_{w'\in W_0} P_0w'N_0 = \bigsqcup_{w'\in W_0} N_0w'P_0.$$

Let B be the Iwahori subgroup of G corresponding to Σ^+ and B_0 the Iwahori subgroup of G_0 corresponding to Σ_0^+.

Proposition 2.3.3 (Bruhat type decomposition)

$$K = \bigsqcup_{w\in W} BwB = \bigsqcup_{w\in W} N_{(0)}T_{(0)}wB,$$
$$K_0 = \bigsqcup_{w'\in W_0} B_0w'B_0 = \bigsqcup_{w'\in W_0} N_{0,(0)}T_{0,(0)}w'B_0.$$

Proposition 2.3.4 (Iwahori factorization)

$$B = N^-_{(1)}T_{(0)}N_{(0)}, \quad B_0 = N^-_{0,(1)}T_{0,(0)}N_{0,(0)}.$$

3 Shintani functions

In this section, we prove the uniqueness of Shintani functions on G. In §3.1, we introduce the space of Shintani functions. In §3.2, we prove a Cartan type decomposition of G. In §3.3, we show that the dimension of the space of Shintani functions is at most one by using a system of difference equations which Shintani functions satisfy.

3.1 The space of Shintani functions.

In this subsection, we introduce the space of Shintani functions on G and investigate fundamental properties of Shintani functions. Let $X_{nr}(T)$(resp. $X_{nr}(T_0)$) be the group consisting of unramified characters of T(resp. T_0). We note that $\delta \in X_{nr}(T), \delta_0 \in X_{nr}(T_0)$. For $\xi \in X_{nr}(T_0), \Xi \in X_{nr}(T)$, we define $\mathcal{S}(\xi, \Xi)$ to be the $\mathbb{C}$-vector space consisting of all continuous functions $S : G \to \mathbb{C}$ such that

$$\begin{aligned}[L(\phi)R(\Phi)S](x) &:= \int_{G_0} dg' \int_G dg\ \phi(g')S(g'^{-1}xg)\Phi(g) \\ &= \omega_\xi(\phi)\omega_\Xi(\Phi)S(x)\end{aligned}$$

for all $(\phi, \Phi) \in \mathcal{H}(G_0, K_0) \times \mathcal{H}(G, K)$. We call an element of $\mathcal{S}(\xi, \Xi)$ an *unramified Shintani function of type* (ξ, Ξ), or simply a *Shintani function.* Let Z (resp. Z_0) be the center of G (resp. G_0). We note that $Z \subset Z_0 \simeq Z \times \{\pm 1\}$.

Lemma 3.1.1 *Every Shintani function $S \in \mathcal{S}(\xi, \Xi)$ has the following properties.*

i) $S(k'xk) = S(x) \quad (^\forall(k', x, k) \in K_0 \times G \times K)$;

ii) $S(z_0xz) = \xi(z_0)^{-1}\Xi(z)S(x) \quad (^\forall(z_0, x, z) \in Z_0 \times G \times Z)$.

In particular, we have $\mathcal{S}(\xi, \Xi) = \{0\}$ if $(\xi\Xi)|_Z \not\equiv 1$.

Proof.

i) Since $\mathrm{ch}_K \in \mathcal{H}(G, K), \mathrm{ch}_{K_0} \in \mathcal{H}(G_0, K_0)$, we have

$$[L(\mathrm{ch}_{K_0})R(\mathrm{ch}_K)S](x) = \int_{K_0} dk' \int_K dk S(k'xk).$$

On the other hand, by definition of Shintani functions we have

$$\begin{aligned}[L(\mathrm{ch}_{K_0})R(\mathrm{ch}_K)S](x) &= \omega_\xi(\mathrm{ch}_{K_0})\omega_\Xi(\mathrm{ch}_K)S(x) \\ &= S(x).\end{aligned}$$

ii) Since $\mathrm{ch}_{zK} \in \mathcal{H}(G, K), \mathrm{ch}_{z_0K_0} \in \mathcal{H}(G_0, K_0)$, we have

$$\begin{aligned}[L(\mathrm{ch}_{z_0K_0})R(\mathrm{ch}_{zK})S](x) &= \int_{G_0} dg' \int_G dg\ \mathrm{ch}_{z_0K_0}(g')S(g'^{-1}xg)\mathrm{ch}_{zK}(g) \\ &= \int_{K_0} dk' \int_K dk\ S(k'^{-1}z_0^{-1}xzk) \\ &= S(z_0^{-1}xz).\end{aligned}$$

On the other hand, by definition of the Shintani functions we have

$$\begin{aligned}[L(\mathrm{ch}_{z_0K_0})R(\mathrm{ch}_{zK})S](x) &= \omega_\xi(\mathrm{ch}_{z_0K_0})\omega_\Xi(\mathrm{ch}_{zK})S(x) \\ &= \xi(z_0)\Xi(z)S(x).\end{aligned}$$

□

From Lemma 3.1.1 (i), it follows that a Shintani function $S \in \mathcal{S}(\xi, \Xi)$ is determined by its values on $K_0\backslash G/K$.

3.2 A Cartan type decomposition of $\mathbf{GSp}_4(F)$

In this subsection, we shall prove the following theorem, called a *Cartan type decomposition* of G. The proof is almost the same as that of [KMS] for the split special orthogonal group $\mathbf{SO}_n$.

Theorem 3.2.1 *We have the double coset decomposition*

$$G = \bigsqcup_{\substack{\mu \in \Lambda^+ \\ \mu' \in \Lambda_0^{++}}} K_0 g(\mu', \mu) K$$

with

$$g(\mu', \mu) = t(\mu')\eta t(\mu), \quad t(\mu) = t(\varpi^{\mu_1}, \varpi^{\mu_2}, \varpi^{\mu_3}).$$

From Theorem 3.2.1, it follows that a Shintani function $S \in \mathcal{S}(\xi, \Xi)$ is determined by its values on $\{g(\mu', \mu) \mid (\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+\}$.

We set $K_0^* := K_0 \cdot \langle w_1 \rangle$, which is a subset of K. In order to prove Theorem 3.2.1, we shall first prove the following decomposition of G.

Proposition 3.2.2

$$G = \bigcup_{\substack{\lambda \in \Lambda^+ \\ \lambda' \in \Lambda_0^{++}}} K_0^* g(\lambda', \lambda) K.$$

By [IM, Corollary 2.35], we have

$$\begin{aligned} G &= \bigsqcup_{\lambda \in \Lambda^+} BWt(\lambda)K \\ &= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}BWt(\lambda)K \\ &= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{(1)}^- Wt(\lambda)K. \end{aligned}$$

Here we set

$$\mathcal{V} := K_0^* \left\{\eta(y_1, y_2) \mid (y_1, y_2) \in \mathfrak{o}^2\right\} \supset N_{(0)}T_{(0)}.$$

We can easily check the following decomposition:

$$\begin{aligned} G &= \bigcup_{\lambda \in \Lambda^+} \bigcup_{w \in W} \mathcal{V}N_{(1)}^- wt(\lambda)K \\ &= \bigcup_{\lambda \in \Lambda^+} \bigcup_{w \in W} \mathcal{V}N_{-e_1+e_2,(1)}N_{-e_1-e_2+e_3,(1)}wt(\lambda)K. \end{aligned}$$

For $w \in W$, we set

$$\mathcal{U}_w := \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}wt(\lambda)K.$$

We note that

$$x_{-e_1+e_2}(a)t(t_1, t_2, t_3) = t(t_1, t_2, t_3)x_{-e_1+e_2}(at_1t_2^{-1}),$$

$$x_{-e_1-e_2+e_3}(a)t(t_1,t_2,t_3) = t(t_1,t_2,t_3)x_{-e_1-e_2+e_3}(at_1t_2t_3^{-1}).$$

In particular, we have

$$\mathcal{U} := \mathcal{U}_{1_4} = \bigcup_{\lambda\in\Lambda^+} \mathcal{V}t(\lambda)K.$$

First we shall see the following lemma to prove Proposition 3.2.2.

Lemma 3.2.3 *For all $w \in W$, we have $\mathcal{U} \supset \mathcal{U}_w$.*

Proof. We may assume that $w \neq 1_4$, that is, $\ell(w) \neq 0$. Since for each $1_4 \neq w \in W$ there exist a simple root α and $w' \in W$ such that $w = w_\alpha w'$ and $\ell(w') < \ell(w)$. Then we shall see that $\mathcal{U}_w \subset \mathcal{U}_{w'}$. We note that $w_{\alpha_1} = w_1$ and $w_{\alpha_2} = w_2$.

i) We assume that $\alpha = \alpha_1$. We can easily check the following equalities:

$$\begin{aligned}
&x_{-e_1+e_2}(a)w_1 = w_1x_{e_1-e_2}(a),\\
&x_{-e_1-e_2+e_3}(a)w_1 = w_1x_{-e_1-e_2+e_3}(a),\\
&x_{-e_1-e_2+e_3}(a)x_{e_1-e_2}(b) = x_{-2e_2+e_3}(2ab)x_{e_1-e_2}(b)x_{-e_1-e_2+e_3}(a).
\end{aligned}$$

Noting that $w_1 \in \mathcal{V}$, we have

$$\begin{aligned}
\mathcal{U}_w &= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}wt(\lambda)K\\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w_1w't(\lambda)K\\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}w_1N_{-e_1-e_2+e_3,(1)}N_{e_1-e_2,(1)}w't(\lambda)K\\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{e_1-e_2,(1)}w't(\lambda)K\\
&\subset \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-2e_2+e_3,(1)}N_{e_1-e_2,(1)}N_{-e_1-e_2+e_3,(1)}w't(\lambda)K\\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}w't(\lambda)K\\
&\subset \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w't(\lambda)K = \mathcal{U}_{w'}.
\end{aligned}$$

ii) We assume that $\alpha = \alpha_2$. We can easily check the following equalities:

$$\begin{aligned}
&x_{-e_1+e_2}(a)w_2 = w_2x_{-e_1-e_2+e_3}(a),\\
&x_{-e_1+e_2}(a)x_{-e_1-e_2+e_3}(b) = x_{-2e_1+e_3}(-2ab)x_{-e_1-e_2+e_3}(b)x_{-e_1+e_2}(a).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{U}_w &= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}wt(\lambda)K\\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w_2w't(\lambda)K
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V} w_2 N_{-e_1+e_2,(1)} N_{-e_1-e_2+e_3,(1)} w' t(\lambda) K \\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V} N_{-e_1+e_2,(1)} N_{-e_1-e_2+e_3,(1)} w' t(\lambda) K \\
&\subset \bigcup_{\lambda\in\Lambda^+} \mathcal{V} N_{-2e_1+e_3,(1)} N_{-e_1-e_2+e_3,(1)} N_{-e_1+e_2,(1)} w' t(\lambda) K \\
&= \bigcup_{\lambda\in\Lambda^+} \mathcal{V} N_{-e_1-e_2+e_3,(1)} N_{-e_1+e_2,(1)} w' t(\lambda) K = \mathcal{U}_{w'}.
\end{aligned}
$$

Hence we have the assertion. □

For $g_1, g_2 \in G$, if $g_1 \in K_0^* g_2 K$, then we write $g_1 \sim g_2$.

Lemma 3.2.4 *For $(y_1, y_2) \in \mathfrak{o}^2, \mu = (\mu_1, \mu_2, \mu_3) \in \Lambda^+$ and $u_1, u_2 \in \mathfrak{o}^\times$, we have*

$$
\begin{aligned}
&\eta(y_1, 0) t(\mu) \sim \eta(y_1, \varpi^{\mu_1+\mu_2-\mu_3}) t(\mu), \\
&\eta(0, y_2) t(\mu) \sim \eta(\varpi^{\mu_1-\mu_2}, y_2) t(\mu), \\
&\eta(u_1 y_1, u_2 y_2) t(\mu) \sim \eta(y_1, y_2) t(\mu).
\end{aligned}
$$

Proof. The following equalities are easily checked by direct computation:

$$
\begin{aligned}
&\eta(y_1, 0) t(\mu) = x_{2e_1'-e_3'}(\varpi^{\mu_1+\mu_2-\mu_3} y_1) \eta(y_1, \varpi^{\mu_1+\mu_2-\mu_3}) t(\mu) x_{e_1+e_2-e_3}(-1), \\
&\eta(0, y_2) t(\mu) = x_{2e_1'-e_3'}(-\varpi^{\mu_1-\mu_2} y_2) \eta(\varpi^{\mu_1-\mu_2}, y_2) t(\mu) x_{e_1-e_2}(-1), \\
&\eta(u_1 y_1, u_2 y_2) t(\mu) = t(1, u_1, u_1 u_2)^{-1} \eta(y_1, y_2) t(\mu) t(1, u_1, u_1 u_2).
\end{aligned}
$$

□

Proposition 3.2.2 follows immediately from Lemma 3.2.3 and the following lemma.

Lemma 3.2.5 *For any $(y_1, y_2) \in \mathfrak{o}^2$ and $\mu \in \Lambda^+$, there exist $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$ such that $\eta(y_1, y_2) t(\mu) \sim g(\lambda', \lambda) = t(\lambda') \eta t(\lambda)$.*

Proof. By Lemma 3.2.4, we may assume that $\mathbf{y} = (y_1, y_2) = (\varpi^{\nu_1}, \varpi^{\nu_2}), \nu_1, \nu_2 \geq 0$. Let $\mathbf{y} = (y_1, y_2) = (\varpi^{\nu_1}, \varpi^{\nu_2}), \nu_1, \nu_2 \geq 0$.

i) If $\nu_1 > \nu_2$, then

$$
x_{-2e_1'+e_3'}(\varpi^{\nu_1-\nu_2} - 1) \eta(\mathbf{y}) t(\mu) = \eta(\mathbf{y}_1) t(\mu) x_{-2e_1'+e_3'}(\varpi^{2\mu_2-\mu_3}(\varpi^{\nu_1-\nu_2} - 1)) \quad (\mathbf{y}_1 = (\varpi^{\nu_2}, \varpi^{\nu_2})),
$$

that is, $\eta(\mathbf{y}) t(\mu) \sim \eta(\mathbf{y}_1) t(\mu)$. Hence we may assume that $\nu_2 \geq \nu_1$.

ii) If $\mu_1 - \mu_2 < \nu_1$, then

$$
\begin{aligned}
x_{2e_1'-e_3'}(\varpi^{\mu_1-\mu_2+\nu_2}(1 - \varpi^{\nu_1-\mu_1+\mu_2})) \eta(\mathbf{y}) t(\mu) &= \eta(\mathbf{y}_2) t(\mu) x_{e_1-e_2}(\varpi^{\nu_1-\mu_1+\mu_2} - 1) \\
&\qquad (\mathbf{y}_2 = (\varpi^{\mu_1-\mu_2}, \varpi^{\nu_2})),
\end{aligned}
$$

that is, $\eta(\mathbf{y}) t(\mu) \sim \eta(\mathbf{y}_2) t(\mu)$. Hence we may assume that $\mu_1 - \mu_2 \geq \nu_1$.

iii) If $\nu_2-\nu_1>2\mu_2-\mu_3$, then

$$x_{2e_2'-e_3'}(\varpi^{2\mu_2-\mu_3}(\varpi^{\nu_2-\nu_1-2\mu_2+\mu_3}-1))\eta(\mathbf{y})t(\mu)=\eta(\mathbf{y}_3)t(\mu)x_{2e_2-e_3}(\varpi^{\nu_2-\nu_1-2\mu_2+\mu_3}-1)$$
$$(\mathbf{y}_3=(\varpi^{\nu_1},\varpi^{2\mu_2-\mu_3+\nu_1})),$$

that is, $\eta(\mathbf{y})t(\mu)\sim\eta(\mathbf{y}_3)t(\mu)$. Hence we may assume that $\nu_2-\nu_1\leq 2\mu_2-\mu_3$.

If $\nu_2\geq\nu_1, \mu_1-\mu_2\geq\nu_1$ and $2\mu_2-\mu_3\geq\nu_2-\nu_1$, we have a factorization

$$\eta(\mathbf{y})t(\mu)=t(\begin{pmatrix}\nu_1+\nu_2\\ \nu_2\\ \nu_1+\nu_2\end{pmatrix})\eta t(\begin{pmatrix}\mu_1-\nu_1-\nu_2\\ \mu_2-\nu_2\\ \mu_3-\nu_1-\nu_2\end{pmatrix}).$$

We have completed the proof of the assertion. □

Proof of Theorem 3.2.1. First we shall prove the decomposition

$$G=\bigcup_{\substack{\lambda\in\Lambda^+\\ \lambda'\in\Lambda_0^{++}}}K_0 g(\lambda',\lambda)K. \tag{3.1}$$

The disjointness of the above decomposition is proved in the next subsection. By Proposition 3.2.2, it is enough to show that for any $(\mu',\mu)\in\Lambda_0^{++}\times\Lambda^+$ there exist $(\lambda',\lambda)\in\Lambda_0^{++}\times\Lambda^+$ and $(k',k)\in K_0\times K$ such that $w_1g(\mu',\mu)=k'g(\lambda',\lambda)k$. We can easily check that

$$\begin{aligned}w_1g(\mu',\mu)&=w_1t(\mu')\eta t(\mu)\\ &=w_1t(\mu')w_1^{-1}w_1\eta t(\mu)\\ &=t(w_1\cdot\mu')w_1\eta t(\mu),\end{aligned}$$

where $w_1\cdot\mu'=(\mu_2',\mu_1',\mu_1')$. Putting

$$\widehat{\mu'}=\begin{pmatrix}2\mu_2'-\mu_1'\\ \mu_2'\\ 2\mu_2'-\mu_1'\end{pmatrix},\quad \mu_c'=\begin{pmatrix}\mu_1'-\mu_2'\\ \mu_1'-\mu_2'\\ 2(\mu_1'-\mu_2')\end{pmatrix},$$

we have $w_1\cdot\mu'=\widehat{\mu'}+\mu_c'$. We note that $\widehat{\mu'}\in\Lambda_0^{++},\mu+\mu_c'\in\Lambda^+$ and $t(\mu_c')\in Z$. Hence we have

$$w_1g(\mu',\mu)=t(\widehat{\mu'})t(\mu_c')w_1\eta t(\mu)=t(\widehat{\mu'})w_1\eta t(\mu+\mu_c').$$

Since

$$\begin{aligned}w_1\eta&=w_1x_{2e_1-e_3}(1)x_{e_1+e_2-e_3}(1)x_{e_1-e_2}(1)\\ &=x_{2e_2-e_3}(1)x_{e_1+e_2-e_3}(1)x_{-e_1+e_2}(1)w_1,\end{aligned}$$

we have

$$\begin{aligned}&w_1g(\mu',\mu)\\ &=t(\widehat{\mu'})w_1\eta t(\mu+\mu_c')\\ &=\Big(t(\widehat{\mu'})x_{2e_2-e_3}(1)x_{2e_1-e_3}(-1)t(\widehat{\mu'})^{-1}\Big)\,g(\widehat{\mu'},\mu+\mu_c')\Big(t(\mu+\mu_c')^{-1}x_{e_1-e_2}(-1)x_{-e_1+e_2}(1)w_1t(\mu+\mu_c')\Big).\end{aligned}$$

Noting that

$$t(\widehat{\mu'})x_{2e_2-e_3}(1)x_{2e_1-e_3}(-1)t(\widehat{\mu'})^{-1}\in K_0,\quad t(\mu+\mu_c')^{-1}x_{e_1-e_2}(-1)x_{-e_1+e_2}(1)w_1t(\mu+\mu_c')\in K,$$

we have $w_1g(\mu',\mu)\sim g(\widehat{\mu'},\mu+\mu_c')$. Hence we have the decomposition (3.1). □

3.3 Uniqueness of Shintani functions.

Let $(\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)$. In this subsection, we prove the uniqueness of Shintani functions (Theorem 3.3.1). We also prove the disjointness of the decomposition in Theorem 3.2.1.

Theorem 3.3.1 *For any Shintani function $S \in \mathcal{S}(\xi, \Xi)$, we have $S = 0$ if $S(1_4) = 0$. In particular, we have*

$$\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) \leq 1.$$

Remark 3.3.2 *In §4.5, we shall prove that* $\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) = 1$ *if and only if* $(\xi\Xi)|_Z \equiv 1$ *(see Theorem 4.5.4).*

We denote by $F[\mathbf{G}]$ the ring of polynomial functions on G, which represents the group scheme $\mathbf{G}$. First we define elements $\alpha_1, \alpha_2, \beta_1, \beta_2$ of $F[\mathbf{G}]$, which are called *relative invariants* on G.

Remark 3.3.3 *Although we use the same symbols $\alpha_1, \alpha_2, \beta_1$ and β_2 for relative invariants as the simple roots introduced in §2.1, this will not cause any confusion.*

For $I = \{i_1, \cdots, i_s\}, J = \{j_1, \cdots, j_s\} \subset \{1, 2, 3, 4\}$, we set

$$\Delta_{I,J}(g) := \det(g_{I,J}), \quad g_{I,J} := (g_{i_k.j_l})_{1 \leq k,l \leq s}.$$

We define relative invariants $\alpha_i (i = 1, 2)$ on G by

$$\alpha_1(g) = \Delta_{\{1\},\{1\}}(w_\ell g), \quad \alpha_2(g) = \Delta_{\{1,2\},\{1,2\}}(w_\ell g).$$

Here w_ℓ is the longest element of the Weyl group W defined in §2.3. Then, for $(y_1, y_2) \in F^2, t = t(t_1, t_2, t_3), t' = t(t'_1, t'_2, t'_3) \in T, n \in N$, we can easily check the following properties of $\alpha_i (i = 1, 2)$:

i) $\alpha_i(\eta(y_1, y_2)w_\ell) = 1 \quad (i = 1, 2)$;

ii) $\alpha_1(tng) = \varpi'_1(t)^{-1}\alpha_1(g) = t_1'^{-1}t'_3\alpha_1(g)$;

iii) $\alpha_2(tng) = (\varpi'_1 + \varpi'_2)(t)^{-1}\alpha_2(g) = t_1'^{-1}t_2'^{-1}t_3'^2\alpha_2(g)$;

iv) $\alpha_1(gtn) = \varpi_1(t)\alpha_1(g) = t_1\alpha_1(g)$;

v) $\alpha_2(gtn) = \varpi_2(t)\alpha_2(g) = t_1t_2\alpha_2(g)$.

We recall the Bruhat decomposition $G = \bigsqcup_{w \in W} PwP$. The double coset $Pw_\ell P$ has the following properties.

Lemma 3.3.4 i) $Pw_\ell P = \{g \in G | \alpha_1(g)\alpha_2(g) \neq 0\}$;

ii) *The double coset $Pw_\ell P$ is an open dense subset of G with respect to the $\mathfrak{p}$-adic topology;*

iii) *The double coset $Pw_\ell P = Pw_\ell N$ is homeomorphic to $P \times N$ via $pw_\ell n \mapsto (p, n)$.*

Proof. The assertions (ii) and (iii) are well-known. The assertion (i) follows from the Bruhat decomposition of G. □

We define relative invariants $\beta_j (j = 1, 2)$ by

$$\beta_1(g) := \Delta_{\{2\},\{1\}}(w_\ell g), \quad \beta_2(g) := \Delta_{\{2,4\},\{1,2\}}(w_\ell g).$$

Then, for $(y_1, y_2) \in F^2, t = t(t_1, t_2, t_3), t' = t(t'_1, t'_2, t'_3) \in T, n \in N, n' \in N_0$, we can easily check the following properties of β_i:

i) $|\beta_i(\eta(y_1,y_2)w_\ell| = |y_i| \quad (i=1,2)$;

ii) $\beta_1(t'n'g) = \varpi_2'(t')^{-1}\beta_1(g) = t_2'^{-1}t_3'\beta_1(g)$;

iii) $\beta_2(t'n'g) = e_3'(t')\beta_2(g) = t_3'\beta_2(g)$;

iv) $\beta_1(gtn) = \varpi_1(t)\beta_1(g) = t_1\beta_1(g)$;

v) $\beta_2(gtn) = \varpi_2(t)\beta_2(g) = t_1t_2\beta_2(g)$.

Lemma 3.3.5 *We have a decomposition*

$$G = \bigcup_{\substack{(\varepsilon_1,\varepsilon_2)\in\{0,1\}^2 \\ w\in W}} P_0\eta(\varepsilon_1,\varepsilon_2)wP.$$

Proof. We note that

$$\eta(y_1,y_2) = \begin{cases} 1_4 & (\text{if } (y_1,y_2)=(0,0)), \\ t(1,y_1,1)^{-1}\eta(1,0)t(1,y_1,1) & (\text{if } y_1\neq 0, y_2=0), \\ t(1,1,y_2)^{-1}\eta(0,1)t(1,1,y_2) & (\text{if } y_1=0, y_2\neq 0), \\ t(1,y_1,y_1y_2)^{-1}\eta t(1,y_1,y_1y_2) & (\text{if } (y_1,y_2)\neq(0,0)). \end{cases}$$

Hence, by the Bruhat decomposition of G, we have

$$G = \bigcup_{\substack{(y_1,y_2)\in F^2 \\ w\in W}} P_0\eta(y_1,y_2)wP = \bigcup_{\substack{(\varepsilon_1,\varepsilon_2)\in\{0,1\}^2 \\ w\in W}} P_0\eta(\varepsilon_1,\varepsilon_2)wP.$$

□

Remark 3.3.6 *We note that the union in Lemma 3.3.5 is not disjoint (see Lemma 4.3.8).*

The double coset $P_0\eta w_\ell P$ has several properties which are similar to those of $Pw_\ell P$.

Lemma 3.3.7 i) $P_0\eta w_\ell P = \{g\in G | \alpha_1(g)\alpha_2(g)\beta_1(g)\beta_2(g)\neq 0\}$;

ii) *The double coset $P_0\eta w_\ell P$ is an open dense subset of G with respect to the $\mathfrak{p}$-adic topology;*

iii) *The double coset $P_0\eta w_\ell P = P_0^\sharp\eta w_\ell P$ is homeomorphic to $P_0^\sharp\times P$ via $p_0\eta w_\ell p\mapsto(p_0,p)$. Here*

$$P_0^\sharp := T^\sharp N_0 \subset P_0, \quad T^\sharp := \{t(t_1,t_2,t_1) |\ t_1,t_2\in F^\times\}.$$

Proof. The assertion (i) follows from Lemma 3.3.4 (i). Since $\alpha_1\alpha_2\beta_1\beta_2$ is a surjective continuous function from G to F, it follows from (i) that the double coset $P_0\eta w_\ell P$ is an open subset of G. To prove (ii), we shall see that $P_0\eta w_\ell P$ is dense in G. Since $Pw_\ell P$ is dense in G, it suffices to show that $P_0\eta w_\ell P$ is a dense subset of $Pw_\ell P$. It follows from the remark mentioned in the proof of Lemma 3.3.5. It is easy to see the assertion (iii). □

The following lemma is used to obtain an integral expression of Shintani functions in §4.1.

Lemma 3.3.8 *Let* $g = n't(t_1', t_2', t_1')\eta w_\ell t(t_1, t_2, t_3)n \in P_0^\sharp \eta w_\ell P$. *Then we have*

$$|t_1| = |\alpha_1(g)|, \quad |t_2| = \left|\frac{\alpha_2(g)}{\beta_1(g)}\right|, \quad |t_3| = \left|\frac{\alpha_1(g)\alpha_2(g)\nu(g)}{\beta_1(g)\beta_2(g)}\right|,$$
$$|t_1'| = \left|\frac{\beta_1(g)\beta_2(g)}{\alpha_1(g)\alpha_2(g)}\right|, \quad |t_2'| = \left|\frac{\beta_2(g)}{\alpha_2(g)}\right|.$$

Here $\nu : G \to F^\times$ *is the similitude character of* G.

In order to prove Theorem 3.3.1, we introduce a partial order $\geq_S$ on the set $\Lambda_0^{++} \times \Lambda^+$.

Definition 3.3.9 *For* $(\mu', \mu), (\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$, *we write* $(\mu', \mu) \geq_{\mathrm{S}} (\lambda', \lambda)$ *if all of the following five conditions hold:*

i) $\mu_1 - \mu_3 \geq \lambda_1 - \lambda_3$;

ii) $\mu_1 + \mu_2 - 2\mu_3 - \mu_1' + \mu_2' \geq \lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_1' + \lambda_2'$;

iii) $\mu_1 - \mu_3 - \mu_1' + \mu_2' \geq \lambda_1 - \lambda_3 - \lambda_1' + \lambda_2'$;

iv) $\mu_1 + \mu_2 - 2\mu_3 - \mu_1' \geq \lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_1'$;

v) $\mu_3 + \mu_1' = \lambda_3 + \lambda_1'$.

Remark 3.3.10 *We note that* $(\mu', \mu) \geq_{\mathrm{S}} (\lambda', \lambda)$ *if and only if all of the following five conditions hold:*

i)′ $\mu_1 + \mu_1' \geq \lambda_1 + \lambda_1'$;

ii)′ $\mu_1 + \mu_2 + \mu_1' + \mu_2' \geq \lambda_1 + \lambda_2 + \lambda_1' + \lambda_2'$;

iii)′ $\mu_1 + \mu_2' \geq \lambda_1 + \lambda_2'$;

iv)′ $\mu_1 + \mu_2 + \mu_1' \geq \lambda_1 + \lambda_2 + \lambda_1'$;

v)′ $\mu_3 + \mu_1' = \lambda_3 + \lambda_1'$.

Then we have the following lemma.

Lemma 3.3.11 *Let* $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$.

i) *If* $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$ *satisfies*

$$K_0 t(\mu')Kt(\mu)^{-1}z(\mu_3)K \cap K_0 t(\lambda')\eta w_\ell t(\lambda)^{-1}z(\lambda_3)K \neq \emptyset,$$

then $(\mu', \mu) \geq_{\mathrm{S}} (\lambda', \lambda)$ *holds.*

ii) *The number of* $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$ *which satisfies* $(\mu', \mu) \geq_{\mathrm{S}} (\lambda', \lambda)$ *is finite.*

Remark 3.3.12 *We can easily check the following two equalities* :

$$K_0 t(\mu')Kt(\mu)^{-1}z(\mu_3)K = K_0 t(\mu')Kt(\mu)K,$$
$$K_0 t(\lambda')\eta w_\ell t(\lambda)^{-1}z(\lambda_3)K = K_0 g(\lambda', \lambda)K.$$

Before proving Lemma 3.3.11, we shall see the disjointness of the decomposition in Theorem 3.2.1.

Proof of the disjointness of Theorem 3.2.1. Let $g(\mu',\mu) \in K_0 g(\lambda',\lambda)K$. Since $g(\mu',\mu) \in K_0 t(\mu')\eta t(\mu)K \subset K_0 t(\mu')Kt(\mu)K$, we have

$$g(\mu',\mu) \in K_0 t(\mu')Kt(\mu)K \cap K_0 g(\lambda',\lambda)K.$$

From Lemma 3.3.11 (and Remark 3.3.12), we have $(\mu',\mu) \geq_S (\lambda',\lambda)$. Similarly we have $(\lambda',\lambda) \geq_S (\mu',\mu)$, that is, $(\mu',\mu) = (\lambda',\lambda)$. We have completed the proof of Theorem 3.2.1. $\square$

We denote by $\mathfrak{o}[\mathbf{G}]$ the $\mathfrak{o}$-structure of $F[\mathbf{G}]$. We note that $\mathfrak{o}[\mathbf{G}]$ is a Hopf algebra over $\mathfrak{o}$. If $m^* : \mathfrak{o}[\mathbf{G}] \to \mathfrak{o}[\mathbf{G}] \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbf{G}]$ is the coproduct of $\mathfrak{o}[\mathbf{G}]$, we denote by Δ an $\mathfrak{o}$-algebra homomorphism defined by the composite

$$\Delta : \mathfrak{o}[\mathbf{G}] \xrightarrow{m^*} \mathfrak{o}[\mathbf{G}] \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbf{G}] \xrightarrow{m^*\otimes \mathrm{Id}} \mathfrak{o}[\mathbf{G}] \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbf{G}] \otimes_{\mathfrak{o}} \mathfrak{o}[\mathbf{G}].$$

Proof of Lemma 3.3.11. We fix $(\mu',\mu) \in \Lambda_0^{++} \times \Lambda^+$.

i) Let $g = t(\lambda')\eta w_\ell t(\lambda)^{-1} z(\lambda_3)$ for some $(\lambda',\lambda) \in \Lambda_0^{++} \times \Lambda^+$. By the assumption, there exist $k, k_1 \in K$ and $k' \in K_0$ such that $g = k' t(\mu')kt(\mu)^{-1} z(\mu_3)k_1$. By comparing the similitudes of both sides, we have $\mu_3 + \mu_1' = \lambda_3 + \lambda_1'$. This is (v) of the definition of the partial order $\geq_S$. We put $f = \alpha_1 \in \mathfrak{o}[\mathbf{G}] \subset F[\mathbf{G}]$. Since we have

$$\begin{aligned} f(g) &= f(t(\lambda')\eta w_\ell t(\lambda)^{-1} z(\lambda_3)) \\ &= \varpi_1'(t(\lambda'))^{-1} \varpi_1(t(\lambda)z(\lambda_3)^{-1})^{-1} f(\eta w_\ell) \\ &= \varpi_1(t(\lambda)z(\lambda_3)^{-1})^{-1}, \end{aligned}$$

we obtain

$$v(f(g)) = -\langle \varpi_1, \lambda - (\lambda_3,\lambda_3,2\lambda_3)\rangle. \tag{3.2}$$

Here, for any $x \in F^\times$, the value $v(x)$ is defined by $x \in \varpi^{v(x)}\mathfrak{o}^\times$. We note that if $\Delta(f) = \sum_i f_{(1),i} \otimes f_{(2),i} \otimes f_{(3),i}$ ($f_{(j),i} \in \mathfrak{o}[\mathbf{G}]$), then we have

$$f(g) = \sum_i f_{(1),i}(k') f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu_3)) f_{(3),i}(k_1).$$

We may assume that each $f_{(j),i}$ is a nonzero element of $\mathfrak{o}[\mathbf{G}]$. Since $\mathfrak{o}[\mathbf{G}]$ has a basis consisting of weight vectors, we may assume that each $f_{(2),i}$ is a weight vector. Namely, there exist $\sigma_i' \in X^*(\mathbf{T}_0), \sigma_i \in X^*(\mathbf{T})$ such that

$$f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu_3)) = \sigma_i'(t(\mu'))^{-1}\sigma_i(t(\mu)z(\mu_3)^{-1})^{-1} f_{(2),i}(k).$$

We note that $G \times G$ acts on $F[\mathbf{G}]$ by the left translation L and the right translation R. Then f is a highest weight vector belonging to a finite dimensional $(G\times G)$-submodule of $F[\mathbf{G}]$. Let V_f be the $(G\times G)$-submodule of $F[\mathbf{G}]$ generated by f. Since, for $n, n' \in N^-$, there exist $A_i(n',n) \in F$ such that

$$[L(n')^{-1}R(n)f](g) = f(n'gn) = \sum_i A_i(n',n) f_{(2),i}(g),$$

$\{f_{(2),i}\}$ is a basis of V_f. Since f is the highest weight vector of V_f, we have $\varpi_1' \geq \sigma_i', \varpi_1 \geq \sigma_i$. Therefore we have

$$\begin{aligned}
v(f(g)) &= v(\sum_i f_{(1),i}(k')f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu_3))f_{(3),i}(k_1)) \\
&\geq \inf\{v(f_{(1),i}(k')f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu_3))f_{(3),i}(k_1))|i\} \\
&\geq \inf\{v(f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu_3)))|i\} \\
&= \inf\{-\langle\sigma_i', \mu'\rangle - \langle\sigma_i, \mu - (\mu_3, \mu_3, 2\mu_3)\rangle|i\} \\
&\geq -\langle\varpi_1', \mu'\rangle - \langle\varpi_1, \mu - (\mu_3, \mu_3, 2\mu_3)\rangle \\
&= -\langle\varpi_1, \mu - (\mu_3, \mu_3, 2\mu_3)\rangle.
\end{aligned}$$

Comparing the equation (3.2) and the above, we have

$$\langle\varpi_1, \mu - (\mu_3, \mu_3, 2\mu_3)\rangle \geq \langle\varpi_1, \lambda - (\lambda_3, \lambda_3, 2\lambda_3)\rangle,$$

that is, $\mu_1 - \mu_3 \geq \lambda_1 - \lambda_3$. This is (i) of the definition of the partial order $\geq_S$. By applying the same way as $f = \alpha_1$ to $f = \alpha_2, \beta_1, \beta_2$, the conditions (ii),(iii) and (iv) of the definition of $\geq_S$ are also obtained. However note that we must consider V_f to be a $(G_0 \times G)$-submodule of $F[\mathbf{G}]$ for $f = \beta_1$ or β_2.

ii) This assertion follows from the definitions of $\Lambda^+, \Lambda_0^{++}$ and the partial order $\geq_S$. □

For $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$, we denote by $m(\mu', \mu)$ the number of $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$ which satisfies $(\mu', \mu) \geq_{\mathrm{S}} (\lambda', \lambda)$. Then we have $1 \leq m(\mu', \mu) < \infty$ for all $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$ from Lemma 3.3.11 (ii). The following lemma is easily checked.

Lemma 3.3.13 *If $(\mu', \mu) \geq_{\mathrm{S}} (\lambda', \lambda)$ and $(\mu', \mu) \neq (\lambda', \lambda)$, then $m(\mu', \mu) > m(\lambda', \lambda)$.*

Proof of Theorem 3.3.1. For any Shintani function $S \in \mathcal{S}(\xi, \Xi)$ and $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$, we put $S(\mu', \mu) = S(g(\mu', \mu))$. We assume that

$$K_0 t'(\mu')Kt(\mu)K = \bigsqcup_{i=0}^{M} K_0 g(\lambda'_{(i)}, \lambda_{(i)})K, \quad g(\lambda'_{(i)}, \lambda_{(i)}) = t'(\lambda'_{(i)})\eta t(\lambda_{(i)}).$$

Since $K_0 t'(\mu')Kt(\mu)K$ is a compact subset of G, the above union is finite. We may assume that $(\lambda'_{(0)}, \lambda_{(0)}) = (\mu', \mu)$ without loss of generality. It follows from Lemma 3.3.11 (i) that each $(\lambda'_{(i)}, \lambda_{(i)})$ satisfies $(\mu', \mu) \geq_S (\lambda'_{(i)}, \lambda_{(i)})$. By definition of the Shintani functions, we have

$$\begin{aligned}
&\omega_\xi(\mathrm{ch}_{K_0 t'(\mu')^{-1}K_0})\omega_\Xi(\mathrm{ch}_{Kt(\mu)K})S(\mathbf{0}, \mathbf{0}) \\
&= C^{(0)}_{\mu',\mu}\mathrm{vol}\big(K_0 g(\mu', \mu)K; dg\big)S(\mu', \mu) + \sum_{i=1}^{M} C^{(i)}_{\mu',\mu}\mathrm{vol}\big(K_0 g(\lambda'_{(i)}, \lambda_{(i)})K; dg\big)S(\lambda'_{(i)}, \lambda_{(i)}).
\end{aligned} \tag{3.3}$$

Here $\mathbf{0} := (0, 0, 0) \in \mathbb{Z}^3$ and, for all $i = 0, \cdots, m$, $C^{(i)}_{\mu',\mu}$ is a positive integer given as the number of elements of the inverse image of $K_0 g(\lambda'_{(i)}, \lambda_{(i)})K$ under the multiplication map

$$K_0\backslash K_0 t'(\mu')K_0 \times Kt(\mu)K/K \to K_0\backslash K_0 t'(\mu')Kt(\mu)K/K.$$

We note that $\mathrm{vol}\big(K_0 g(\mu',\mu)K; dg\big)$ is nonzero. In particular, if $S(\mathbf{0},\mathbf{0}) = 0$, then $S(\mu',\mu) = 0$ for all $(\mu',\mu) \in \Lambda_0^{++} \times \Lambda^+$. We shall prove it by induction on $m(\mu',\mu) \geq 1$. We take $(\mu',\mu) \in \Lambda_0^{++} \times \Lambda^+$ and assume that $S(\mathbf{0},\mathbf{0}) = 0$.

First we assume that $m(\mu',\mu) = 1$. Then, since $K_0 t'(\mu')Kt(\mu)K = K_0 g(\mu',\mu)K$, we have

$$0 = \omega_\xi(\mathrm{ch}_{K_0 t'(\mu')^{-1}K_0})\omega_\Xi(\mathrm{ch}_{Kt(\mu)K})S(\mathbf{0},\mathbf{0}) = C_{\mu',\mu}\mathrm{vol}\big(K_0 g(\mu',\mu)K; dg\big)S(\mu',\mu).$$

Here $C_{\mu',\mu}$ is a positive integer. Hence we have $S(\mu',\mu) = 0$.

Next we assume that $S(\lambda',\lambda) = 0$ for all (λ',λ) such that $m(\mu',\mu) > m(\lambda',\lambda)$. If

$$K_0 t'(\mu')Kt(\mu)K = \bigsqcup_{i=0}^{M} K_0 g(\lambda'_{(i)},\lambda_{(i)})K, \quad (\lambda'_{(0)},\lambda_{(0)}) = (\mu',\mu)$$

then $m(\mu',\mu) > m(\lambda'_{(i)},\lambda_{(i)})$ for all $i = 1,\cdots,M$ from Lemma 3.3.13. Hence the equality (3.3) implies $S(\mu',\mu) = 0$ by the induction hypothesis. □

4 An explicit formula of Shintani functions

In this section, we shall prove one of the main results in this thesis. In §4.1, we construct a nonzero intertwining operator $\Omega_{\xi,\Xi} \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes_{\mathbb{C}} i_P^G(\Xi), \mathbb{C})$ for any element (ξ,Ξ) of a certain domain $U_c^\sharp \subset (\mathbb{C}^\times)^5$ and give an integral expression of Shintani functions by using $\Omega_{\xi,\Xi}$. In §4.2, we calculate the image of the intertwining operator $\Omega_{\xi,\Xi}$ for several elements in $i_{P_0}^{G_0}(\xi) \otimes_{\mathbb{C}} i_P^G(\Xi)$. In §4.3, we prove that there exists a nonempty subset $\widetilde{U}_c^\sharp$ of $X_{nr}(T_0) \times X_{nr}(T)$ such that $\mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$ is one dimensional for any element $(\xi,\Xi) \in \widetilde{U}_c^\sharp$ (Corollary 4.3.16). The uniqueness is indispensable for a meromorphic continuation of $\Omega_{\xi,\Xi}$ in §4.4. In §4.5, we prove an explicit formula of Shintani functions in the same way as [KMS] for the unramified (Whittaker-)Shintani function.

4.1 An integral expression of Shintani functions.

In this subsection, we give an integral expression of Shintani functions. For an intertwining operator $\Omega \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$, we set

$$S_\Omega(x) := \Omega(\phi_{K_0,\xi} \otimes R(x)\phi_{K,\Xi}) \quad ({}^\forall x \in G).$$

Proposition 4.1.1 *Let $\Omega \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$. Then $S_\Omega \in \mathcal{S}(\xi,\Xi)$, that is, S_Ω is a Shintani function of type (ξ,Ξ).*

Proof. Let $x \in G$. The we have

$$\begin{aligned} L(\phi)S_\Omega(x) &= \int_{G_0} \phi(g')S_\Omega(g'^{-1}x)dg' \\ &= \int_{G_0} \phi(g')\Omega(\phi_{K_0,\xi} \otimes R(g'^{-1}x)\phi_{K,\Xi})dg' \\ &= \int_{G_0} \phi(g')\Omega(R(g')\phi_{K_0,\xi} \otimes R(x)\phi_{K,\Xi})dg' \\ &= \Omega(\int_{G_0} \phi(g')R(g')\phi_{K_0,\xi}dg' \otimes R(x)\phi_{K,\Xi}) \end{aligned}$$

$$\begin{aligned}&= \Omega(R(\phi)\phi_{K_0,\xi} \otimes R(x)\phi_{K,\Xi})\\&= \omega_\xi(\phi)S_\Omega(x)\end{aligned}$$

for all $\phi \in \mathcal{H}(G_0, K_0)$. Also, we have $R(\Phi)S_\Omega(x) = \omega_\Xi(\Phi)S_\Omega(x)$ for all $\Phi \in \mathcal{H}(G, K)$ in a similar way. $\square$

We shall construct an intertwining operator $\Omega \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$ concretely by using the relative invariants on G. Let $\xi \in X_{nr}(T_0), \Xi \in X_{nr}(T)$ be unramified characters such that $(\xi\Xi)|_Z \equiv 1$. Then we define a function $\Upsilon_{\xi,\Xi} : P_0\eta w_\ell P \to \mathbb{C}$ by the following properties:

i) $\Upsilon_{\xi,\Xi}(p_0xp) = (\xi^{-1}\delta_0^{1/2})(p_0)(\Xi\delta^{-1/2})(p)\Upsilon_{\xi,\Xi}(x) \quad ({}^\forall(p_0, x, p) \in P_0 \times P_0\eta w_\ell P \times P)$;

ii) $\Upsilon_{\xi,\Xi}(\eta w_\ell) = 1$.

Remark 4.1.2 *We note that $P_0 \cap \eta w_\ell P(\eta w_\ell)^{-1} = Z$. Hence the condition $(\xi\Xi)|_Z \not\equiv 1$ implies every function $F : P_0\eta w_\ell P \to \mathbb{C}$ which has the property*

$$F(p_0xp) = (\xi^{-1}\delta_0^{1/2})(p_0)(\Xi\delta^{-1/2})(p)F(x) \quad ({}^\forall(p_0, x, p) \in P_0 \times P_0\eta w_\ell P \times P)$$

is identically zero.

Lemma 4.1.3 *For $g \in P_0\eta w_\ell P = P_0^\sharp\eta w_\ell P$, we have*

$$\begin{aligned}\Upsilon_{\xi,\Xi}(g) = {}&(\Xi_1\Xi_3\xi_1\xi_3|\cdot|^{-1/2})(\alpha_1(g))(\Xi_2\Xi_3\xi_1\xi_2\xi_3|\cdot|^{-1/2})(\alpha_2(g))\\&\times(\Xi_2^{-1}\Xi_3^{-1}\xi_1^{-1}\xi_3^{-1}|\cdot|^{-1/2})(\beta_1(g))(\Xi_3^{-1}\xi_1^{-1}\xi_2^{-1}\xi_3^{-1}|\cdot|^{-1/2})(\beta_2(g))(\Xi_3|\cdot|^{3/2})(\nu(g)).\end{aligned}$$

Proof. From Lemma 3.3.8, we obtain the assertion. $\square$

We extend $\Upsilon_{\xi,\Xi} : P_0\eta w_\ell P \to \mathbb{C}$ to the whole G by setting $\Upsilon_{\xi,\Xi} \equiv 0$ on $G - P_0\eta w_\ell P$. We note that $\Upsilon_{\xi,\Xi}$ is not necessarily continuous on the whole G.

Proposition 4.1.4 *Let U_c be a nonempty open subset of the complex manifold $X_{nr}(T_0) \times X_{nr}(T) = (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3$ given by*

$$U_c = \left\{ (\xi, \Xi) \in (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3 \;\middle|\; \begin{array}{l} \|\xi_1\xi_3\Xi_1\Xi_3\| < q^{-1/2} \\ \|\xi_1\xi_2\xi_3\Xi_2\Xi_3\| < q^{-1/2} \\ \|\xi_1^{-1}\xi_3^{-1}\Xi_2^{-1}\Xi_3^{-1}\| < q^{-1/2} \\ \|\xi_1^{-1}\xi_2^{-1}\xi_3^{-1}\Xi_3^{-1}\| < q^{-1/2} \end{array} \right\}.$$

Then the function $\Upsilon_{\xi,\Xi}$ is continuous on G for any $(\xi, \Xi) \in U_c$.

Proof. The set U_c is nonempty, because $((q^{-2}, q^{-2}, q^5), (q^{-4}, q^{-2}, 1)) \in U_c$, for example. Let $(\xi, \Xi) \in U_c$. We see that $\Upsilon_{\xi,\Xi}$ is continuous at each $x \in G$. Since it is obvious that $\Upsilon_{\xi,\Xi}$ is continuous on $P_0\eta w_\ell P$, we may assume that $x \in G - P_0\eta w_\ell P$. Then we have $\Upsilon_{\xi,\Xi}(x) = 0$. We consider a sequence $\{x_n\}_{n=1}^\infty$ of elements in G which converges to x. Then we may see that $\lim_{n\to\infty}\Upsilon_{\xi,\Xi}(x_n) = 0$. Now, since $x \notin P_0\eta w_\ell P$, at least one of $\alpha_1(x) = 0, \alpha_2(x) = 0, \beta_1(x) = 0$ or $\beta_2(x) = 0$ holds. We note that $\alpha_1, \alpha_2, \beta_1$ and β_2 are continuous on G. We consider the two subsequences of $\{x_n\}_{n=1}^\infty$ given by $\{x_{n_i}^{(1)}\}_i := \{x_n\}_{n=1}^\infty \cap P_0\eta w_\ell P, \{x_{n_i}^{(2)}\}_i := \{x_n\}_{n=1}^\infty \cap (G - P_0\eta w_\ell P)$. Obviously, we have $\{x_n\}_{n=1}^\infty = \{x_{n_i}^{(1)}\}_i \sqcup \{x_{n_i}^{(2)}\}_i$.

i) We assume that the sequence $\{x_{n_i}^{(1)}\}$ is a finite set, that is, $x_n \notin P_0\eta w_\ell P$ for $n \gg 0$. Then we have $\lim_{n\to\infty} \Upsilon_{\xi,\Xi}(x_n) = 0$.

ii) We assume that the sequence $\{x_{n_i}^{(2)}\}$ is a finite set, that is, $x_n \in P_0\eta w_\ell P$ for $n \gg 0$. Then Lemma 4.1.3 yields that

$$\begin{aligned}\|\lim_{n\to\infty} \Upsilon_{\xi,\Xi}(x_n)\| &= \lim_{n\to\infty} \|(\Xi_1\Xi_3\xi_1\xi_3|\cdot|^{-1/2})(\alpha_1(x_n))(\Xi_2\Xi_3\xi_1\xi_2\xi_3|\cdot|^{-1/2})(\alpha_2(x_n)) \\ &\quad \times (\Xi_2^{-1}\Xi_3^{-1}\xi_1^{-1}\xi_3^{-1}|\cdot|^{-1/2})(\beta_1(x_n))(\Xi_3^{-1}\xi_1^{-1}\xi_2^{-1}\xi_3^{-1}|\cdot|^{-1/2})(\beta_2(x_n))(\Xi_3|\cdot|^{3/2})(\nu(x_n))\| \\ &= 0.\end{aligned}$$

This means that $\lim_{n\to\infty} \Upsilon_{\xi,\Xi}(x_n) = 0$.

iii) We assume that both the sequences $\{x_{n_i}^{(1)}\}$ and $\{x_{n_i}^{(2)}\}$ are the infinite sets. Since $\lim_{i\to\infty} x_{n_i}^{(1)} = \lim_{i\to\infty} x_{n_i}^{(2)} = x$ holds, we have $\lim_{i\to\infty} \Upsilon_{\xi,\Xi}(x_{n_i}^{(1)}) = \lim_{i\to\infty} \Upsilon_{\xi,\Xi}(x_{n_i}^{(2)}) = 0$ from the cases (i) and (ii). Hence we have $\lim_{n\to\infty} \Upsilon_{\xi,\Xi}(x_n) = 0$.

Therefore we have completed the proof of the proposition. $\square$

For $(\xi,\Xi) \in U_c$, we define a continuous function $Y_{\xi,\Xi} : G \to \mathbb{C}$ by

$$Y_{\xi,\Xi}(g) = \Upsilon_{\xi,\Xi}(g^{-1}) \quad ({}^\forall g \in G).$$

From Remark 4.1.2, if $(\xi\Xi)|_Z \not\equiv 1$, we have $Y_{\xi,\Xi} \equiv 0$ on G. Since $\eta^{-1} = t(1,-1,1)\eta t(1,-1,1) \in T_{(0)}\eta T_{(0)}$, we have the following lemma from the definition.

Lemma 4.1.5 *For $(\xi,\Xi) \in U_c$ such that $(\xi\Xi)|_Z \equiv 1$, we have*

i) $Y_{\xi,\Xi}(pgp_0) = (\Xi^{-1}\delta^{1/2})(p)(\xi\delta_0^{-1/2})(p_0)Y_{\xi,\Xi}(g) \quad ({}^\forall(p,p_0) \in P \times P_0)$;

ii) $Y_{\xi,\Xi}(w_\ell\eta) = 1$.

Let $(\xi,\Xi) \in U_c$. We define an intertwining operator $\Omega_{\xi,\Xi} \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$ by

$$\begin{aligned}\Omega_{\xi,\Xi}(\mathcal{P}_\xi(\phi') \otimes \mathcal{P}_\Xi(\phi)) &:= \int_{G\times G_0} \phi(g)\phi'(g')Y_{\xi,\Xi}(gg'^{-1})dgdg' \\ &= \int_{K\times K_0} \mathcal{P}(\phi)(k)\mathcal{P}(\phi')(k')Y_{\xi,\Xi}(kk'^{-1})dkdk'\end{aligned}$$

for all $(\phi',\phi) \in C_c^\infty(G_0) \times C_c^\infty(G)$. Then $S_{\xi,\Xi} := S_{\Omega_{\xi,\Xi}}$ has the following expression.

Proposition 4.1.6 *For $(\xi,\Xi) \in U_c$, we have*

$$S_{\xi,\Xi}(x) = \int_{K\times K_0} Y_{\xi,\Xi}(kx^{-1}k')dkdk' \quad ({}^\forall x \in G).$$

Proof. Let $x \in G$. Then we have

$$\begin{aligned}S_{\xi,\Xi}(x) &= \Omega_{\xi,\Xi}(\mathcal{P}_\xi(\mathrm{ch}_{K_0}) \otimes R(x)\mathcal{P}_\Xi(\mathrm{ch}_K)) \\ &= \Omega_{\xi,\Xi}(\mathcal{P}_\xi(\mathrm{ch}_{K_0}) \otimes \mathcal{P}_\Xi(R(x)\mathrm{ch}_K))\end{aligned}$$

$$= \int_{G\times G_0} \mathrm{ch}_K(gx)\mathrm{ch}_{K_0}(g')Y_{\xi,\Xi}(gg'^{-1})dgdg'$$
$$= \int_{G\times G_0} \mathrm{ch}_K(g)\mathrm{ch}_{K_0}(g')Y_{\xi,\Xi}(gx^{-1}g'^{-1})dgdg'$$
$$= \int_{K\times K_0} Y_{\xi,\Xi}(kx^{-1}k'^{-1})dkdk'$$
$$= \int_{K\times K_0} Y_{\xi,\Xi}(kx^{-1}k')dkdk'.$$

□

Remark 4.1.7 *In the next subsection, we shall see that for $(\xi,\Xi)\in U_c$ such that $(\xi\Xi)|_Z\equiv 1$ the intertwining operator $\Omega_{\xi,\Xi}$ is not identically zero.*

4.2 Rank one calculation.

Let $(\xi,\Xi)\in X_{nr}(T_0)\times X_{nr}(T)$. For $w\in W$(resp. $w'\in W_0$), we set $\Phi_w := \mathcal{P}_\Xi(\mathrm{ch}_{BwB})\in i_P^G(\Xi)$(resp. $\phi_{w'} := \mathcal{P}_\xi(\mathrm{ch}_{B_0w'B_0})\in i_{P_0}^{G_0}(\xi)$). From the Bruhat type decomposition (Proposition 2.3.3), the set $\{\Phi_w\}_{w\in W}$(resp. $\{\phi_{w'}\}_{w'\in W_0}$) is a basis of $i_P^G(\Xi)^B$(resp. $i_{P_0}^{G_0}(\xi)^{B_0}$). We fix $(\xi,\Xi)=(\xi_1,\xi_2,\xi_3,\Xi_1,\Xi_2,\Xi_3)\in U_c$ such that $(\xi\Xi)|_Z\equiv 1$ untill the end of this subsection. We note that $\xi_1\xi_2\xi_3^2\Xi_1\Xi_2\Xi_3^2=1$. The purpose of this subsection is to calculate the following integrals:

$$I_i := \mathrm{vol}(B;dg)^{-1}\mathrm{vol}(B_0;dg')^{-1}\Omega_{\xi,\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_i}))\quad (i=1,2),$$
$$J_i := \mathrm{vol}(B;dg)^{-1}\mathrm{vol}(B_0;dg')^{-1}\Omega_{\xi,\Xi}((\phi_{1_4}+\phi_{w_i'})\otimes R(\eta w_\ell)\Phi_{1_4})\quad (i=1,2).$$

Refer to section 2 for notation $T_{(0)}, N_{(1)}, x_\alpha$ and so on.

Proposition 4.2.1

i) $I_1 = (q-1)\dfrac{1-q^{-1}\Xi_1\Xi_2^{-1}}{(1-q^{-1/2}\Xi_1\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_1\Xi_3\xi_2\xi_3)}$;

ii) $I_2 = (q-1)\dfrac{1-q^{-1}\Xi_2}{(1-q^{-1/2}\Xi_2\Xi_3\xi_1\xi_2\xi_3)(1-q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_3)}$;

iii) $J_1 = (q-1)\dfrac{1-q^{-1}\xi_1}{(1-q^{-1/2}\Xi_1\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_2\Xi_3\xi_1\xi_2\xi_3)}$;

iv) $J_2 = (q-1)\dfrac{1-q^{-1}\xi_2}{(1-q^{-1/2}\Xi_2\Xi_3\xi_1\xi_2\xi_3)(1-q^{-1/2}\Xi_1\Xi_3\xi_2\xi_3)}$.

First we calculate the value $\Omega_{\xi,\Xi}(\phi_{1_4}\otimes R(w_\ell\eta)\Phi_{1_4})$.

Proposition 4.2.2 $\Omega_{\xi,\Xi}(\phi_{1_4}\otimes R(w_\ell\eta)\Phi_{1_4}) = \mathrm{vol}(B)\mathrm{vol}(B')$.

To prove Proposition 4.2.2, we use the following lemma.

Lemma 4.2.3 i) *$w_\ell\eta N_{0,(1)}^- \subset Bw_\ell\eta$;*

ii) $N_{(1)}\eta \subset T_{(0)}\eta T_{(0)} N_{0,(1)}$;

iii) $N^-_{(1)} w_\ell \eta \subset T_{(0)} w_\ell \eta T_{(0)} N_{0,(1)}$.

Proof.

i) Let $x_{-2e_1+e_3}(a)x_{-2e_2+e_3}(b) \in N^-_{0,(1)}$, that is, $a, b \in \mathfrak{p}$. Then we have

$$\begin{aligned}
&w_\ell \eta x_{-2e_1+e_3}(a)x_{-2e_2+e_3}(b)\\
&= x_{-e_1+e_2}(-b)x_{-2e_1+e_3}(-b)x_{-2e_2+e_3}(a/(ab-a-1))\\
&\quad \times t(1,(-ab+a+1)^{-1},1)x_{e_1-e_2}(-a)x_{2e_1-e_3}(-a/(ab-a-1))\\
&\qquad \times x_{e_1+e_2-e_3}(a(1-b)/(ab-a-1))x_{2e_2-e_3}((ab-a-b)/(ab-a-1))\\
&\in Bw_\ell\eta.
\end{aligned}$$

ii) Let $x_{e_1-e_2}(a_1)x_{2e_1-e_3}(a_2)x_{e_1+e_2-e_3}(a_3)x_{2e_2-e_3}(a_4) \in N_{(1)}$, that is, $a_1, a_2, a_3, a_4 \in \mathfrak{p}$. Then we have

$$\begin{aligned}
&x_{e_1-e_2}(a_1)x_{2e_1-e_3}(a_2)x_{e_1+e_2-e_3}(a_3)x_{2e_2-e_3}(a_4)\eta\\
&= t(1,(a_1+1)^{-1},(a_1+1)^{-1}(1+a_3-a_4)^{-1})\eta\\
&\quad \times t(1,a_1+1,(a_1+1)(1+a_3-a_4)))x_{2e_1-e_3}(a_1(1+a_3-a_4)+a_2-a_3)x_{2e_2-e_3}(a_4)\\
&\in T_{(0)}\eta T_{(0)}N_{0,(1)}.
\end{aligned}$$

iii) This assertion follows immediately from (ii).

□

Proof of Proposition 4.2.2. By definition, we have

$$\Omega_{\xi,\Xi}(\phi_{1_4} \otimes R(w_\ell\eta)\Phi_{1_4}) = \int_{B\times B_0} Y_{\xi,\Xi}(xw_\ell\eta x'^{-1})dxdx'.$$

From Lemma 4.2.3, we have

$$\begin{aligned}
Bw_\ell\eta B_0 &\subset Bw_\ell\eta N^-_{0,(1)}T_{(0)}N_{0,(0)}\\
&\subset Bw_\ell\eta T_{(0)}N_{0,(0)}\\
&\subset T_{(0)}N_{(0)}w_\ell\eta T_{(0)}N_{0,(0)}\\
&\subset P_{(0)}w_\ell\eta P_{0,(0)}.
\end{aligned}$$

Since $Y_{\xi,\Xi}|_{P_{(0)}w_\ell\eta P_{0,(0)}} \equiv 1$, we obtain Proposition 4.2.2. □

Let $d\tau$ be the normalized Haar measure of F so that $\mathrm{vol}(\mathfrak{o}; d\tau) = 1$. Next we prove the following proposition.

Proposition 4.2.4 i) *For each $i = 1, 2$, we have*

$$I_i = 1 + q\cdot\int_{\mathfrak{o}} Y_{\xi,\Xi}(w_i x_{\alpha_i}(\tau)w_\ell\eta)d\tau;$$

ii) *For each* $i = 1, 2$, *we have*

$$J_i = 1 + q \cdot \int_{\mathfrak{o}} Y_{\xi,\Xi}(w_\ell \eta x_{\beta_i}(\tau) w_i'^{-1}) d\tau.$$

To prove Proposition 4.2.4, we use Proposition 4.2.2 and the following two lemmas.

Lemma 4.2.5 i) $Bw_iBw_\ell\eta \subset P_{(0)}w_iN_{\alpha_i,(0)}w_\ell\eta P_{0,(0)}$ $(i = 1, 2)$;

ii) $w_\ell\eta B_0 w_i'^{-1} B_0 \subset P_{(0)}w_\ell\eta N_{\beta_i,(0)} w_i'^{-1} P_{0,(0)}$ $(i = 1, 2)$.

Proof.

i) By the Iwahori factorization, we have

$$\begin{aligned} Bw_iB &\subset T_{(0)}N_{(0)}w_iB \\ &\subset T_{(0)}N_{(0)}w_iN_{(0)}N^-_{(1)} \\ &\subset T_{(0)}N_{(0)}w_iN_{\alpha_i,(0)}N^-_{(1)}. \end{aligned}$$

Hence, from Lemma 4.2.3 (iii), we have

$$\begin{aligned} Bw_iBw_\ell\eta &\subset T_{(0)}N_{(0)}w_iN_{\alpha_i,(0)}T_{(0)}w_\ell\eta T_{(0)}N_{0,(1)} \\ &\subset T_{(0)}N_{(0)}w_iN_{\alpha_i,(0)}w_\ell\eta T_{(0)}N_{0,(1)} \\ &\subset P_{(0)}w_iN_{\alpha_i,(0)}w_\ell\eta P_{0,(0)}. \end{aligned}$$

ii) By the Iwahori factorization, we have

$$\begin{aligned} B_0w_i'^{-1}B_0 &\subset B_0w_i'^{-1}T_{(0)}N_{0,(0)} \\ &\subset N^-_{0,(1)}N_{0,(0)}w_i'^{-1}T_{(0)}N_{0,(0)} \\ &\subset N^-_{0,(1)}N_{\beta_i,(0)}w_i'^{-1}T_{(0)}N_{0,(0)}. \end{aligned}$$

Hence, from Lemma 4.2.3 (i),(iii), we have

$$\begin{aligned} w_\ell\eta B_0w_i'^{-1}B_0 &\subset Bw_\ell\eta N_{\beta_i,(0)}w_i'^{-1}T_{(0)}N_{0,(0)} \\ &\subset T_{(0)}w_\ell\eta T_{(0)}N_{0,(1)}N_{\beta_i,(0)}w_i'^{-1}T_{(0)}N_{0,(0)} \\ &\subset T_{(0)}w_\ell\eta N_{0,(1)}N_{\beta_i,(0)}w_i'^{-1}T_{(0)}N_{0,(0)} \\ &\subset T_{(0)}w_\ell\eta N_{\beta_i,(0)}N_{\beta_j,(1)}w_i'^{-1}T_{(0)}N_{0,(0)} \quad (1 \leq i \neq j \leq 2) \\ &\subset T_{(0)}w_\ell\eta N_{\beta_i,(0)}w_i'^{-1}T_{(0)}N_{0,(0)} \\ &\subset P_{(0)}w_\ell\eta N_{\beta_i,(0)}w_i'^{-1}P_{0,(0)}. \end{aligned}$$

□

The next lemma is easily checked.

Lemma 4.2.6 i) *We have the following two double coset decompositions*

$$Bw_1B = \bigsqcup_{a \in \mathcal{O}/\mathfrak{p}} \left(\begin{array}{cc|cc} a & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & -a \end{array}\right) B, \quad Bw_2B = \bigsqcup_{a \in \mathcal{O}/\mathfrak{p}} \left(\begin{array}{c|ccc} 1 & & & \\ \hline & a & & 1 \\ & & 1 & \\ & -1 & & \end{array}\right) B.$$

In particular, we have $\mathrm{vol}(Bw_iB; dg) = q \cdot \mathrm{vol}(B; dg)$ *for each* $i = 1, 2$.

ii) *We have the following two double coset decompositions*

$$B_0 w_1' B_0 = \bigsqcup_{a \in \mathcal{O}/\mathfrak{p}} \left(\begin{array}{ccc|c} a & & 1 & \\ & 1 & & \\ -1 & & & \\ \hline & & & 1 \end{array}\right) B_0, \quad B_0 w_2' B_0 = \bigsqcup_{a \in \mathcal{O}/\mathfrak{p}} \left(\begin{array}{c|ccc} 1 & & & \\ \hline & a & & 1 \\ & & 1 & \\ & -1 & & \end{array}\right) B_0.$$

In particular, we have $\mathrm{vol}(B_0 w_i' B_0; dg') = q \cdot \mathrm{vol}(B_0; dg')$ *for each* $i = 1, 2$.

Proof of Proposition 4.2.4.

i) From the definition of $\Omega_{\xi,\Xi}$ and Lemma 4.2.3 (iii), we have

$$\begin{aligned} \Omega_{\xi,\Xi}(\phi_{14} \otimes R(\eta w_\ell)\Phi_{w_i}) &= \int_{G \times G_0} \phi_{14}(x')\Phi_{w_i}(x\eta w_\ell) Y_{\xi,\Xi}(x x'^{-1}) dx dx' \\ &= \int_{G \times G_0} \phi_{14}(x')\Phi_{w_i}(x) Y_{\xi,\Xi}(x(\eta w_\ell)^{-1} x'^{-1}) dx dx' \\ &= \int_{G \times G_0} \phi_{14}(x'^{-1})\Phi_{w_i}(x) Y_{\xi,\Xi}(x w_\ell \eta x') dx dx' \\ &= \int_{B w_i B \times B_0} Y_{\xi,\Xi}(x w_\ell \eta x') dx dx' \\ &= \int_{B w_i B \times B_0} Y_{\xi,\Xi}(x w_\ell \eta) dx dx' \\ &= \mathrm{vol}(B_0; dg') \int_{B w_i B} Y_{\xi,\Xi}(x w_\ell \eta) dx. \end{aligned}$$

Since Lemma 4.2.5 (i) and Lemma 4.2.6 (i) yield that

$$\begin{aligned} \Omega_{\xi,\Xi}(\phi_{14} \otimes R(\eta w_\ell)\Phi_{w_i}) &= \mathrm{vol}(B_0; dg') \int_{B w_i B} Y_{\xi,\Xi}(x w_\ell \eta) dx \\ &= q \cdot \mathrm{vol}(B; dg) \mathrm{vol}(B_0; dg') \int_{\mathfrak{o}} Y_{\xi,\Xi}(w_i x_{\alpha_i}(\tau) w_\ell \eta) d\tau, \end{aligned}$$

we obtain the assertion (i) by combining with Proposition 4.2.2.

ii) From the definition of $\Omega_{\xi,\Xi}$ and Lemma 4.2.3 (iii), we have

$$\begin{aligned} \Omega_{\xi,\Xi}(\phi_{w_i'} \otimes R(\eta w_\ell)\Phi_{14}) &= \int_{G \times G_0} \phi_{w_i'}(x')\Phi_{14}(x\eta w_\ell) Y_{\xi,\Xi}(x x'^{-1}) dx dx' \\ &= \int_{G \times G_0} \phi_{w_i'}(x')\Phi_{14}(x) Y_{\xi,\Xi}(x w_\ell \eta x'^{-1}) dx dx' \\ &= \int_{B \times B_0 w_i' B_0} Y_{\xi,\Xi}(x w_\ell \eta x'^{-1}) dx dx' \\ &= \int_{B \times B_0 w_i' B_0} Y_{\xi,\Xi}(w_\ell \eta x'^{-1}) dx dx' \\ &= \mathrm{vol}(B; dg) \int_{B_0 w_i'^{-1} B_0} Y_{\xi,\Xi}(w_\ell \eta x') dx'. \end{aligned}$$

Since Lemma 4.2.5 (ii) and Lemma 4.2.6 (ii) yield that

$$\Omega_{\xi,\Xi}(\phi_{14} \otimes R(\eta w_\ell)\Phi_{w_i}) = \mathrm{vol}(B;dg)\int_{Bw_i^{-1}B} Y_{\xi,\Xi}(w_\ell \eta x')dx'$$
$$= q\cdot \mathrm{vol}(B;dg)\mathrm{vol}(B_0;dg')\int_{\mathfrak{o}} Y_{\xi,\Xi}(w_\ell\eta x_{\beta_i}(\tau)w_i'^{-1})d\tau,$$

we obtain the assertion (ii) by combining with Proposition 4.2.2. □

Now we shall see Proposition 4.2.1. We set

$$\kappa(g) := (\alpha_1(g^{-1}), \alpha_2(g^{-1}), \beta_1(g^{-1}), \beta_2(g^{-1})) \in F^4 \quad ({}^\forall g \in G).$$

We note that $\kappa(g) \in (F^\times)^4$ if and only if $g \in Pw_\ell\eta P_0$. The next lemma is easily checked by direct calculation.

Lemma 4.2.7 i)

$$\kappa(w_i x_{\alpha_i}(\tau)w_\ell\eta) = \begin{cases}(-\tau,-1,1-\tau,-1) & (\textit{if } i=1),\\ (1,-\tau,1,1-\tau) & (\textit{if } i=2).\end{cases}$$

ii)

$$\kappa(w_\ell\eta x_{\beta_i}(\tau)w_i'^{-1}) = \begin{cases}(\tau,\tau-1,1,1) & (\textit{if } i=1),\\ (1,\tau,\tau+1,1) & (\textit{if } i=2).\end{cases}$$

We extend an unramified character $\chi \in X_{nr}(F^\times)$ to a function on F by setting $\chi(0)=0$. We can easily check the following lemma, which is useful for our purpose.

Lemma 4.2.8 *For unramified characters $\chi,\chi' \in X_{nr}(F^\times)$ such that $\|\chi(\varpi)\|, \|\chi'(\varpi)\| < q$, we have*

$$1 + q\cdot\int_{\mathfrak{o}} \chi(\tau)\chi'(1+\tau)d\tau = (q-1)\frac{1-q^{-2}(\chi\chi')(\varpi)}{(1-q^{-1}\chi(\varpi))(1-q^{-1}\chi'(\varpi))}.$$

Proof of Proposition 4.2.1. We have the formula

$$I_1 = 1 + q\cdot\int_{\mathfrak{o}} Y_{\xi,\Xi}(w_i x_{\alpha_1}(\tau)w_\ell\eta)d\tau$$

from Proposition 4.2.4. From Lemma 4.1.3 and Lemma 4.2.7, we have

$$I_1 = 1 + q\cdot\int_{\mathfrak{o}}(\Xi_1\Xi_3\xi_1\xi_3|\cdot|^{-1/2})(-\tau)(\Xi_2^{-1}\Xi_3^{-1}\xi_1^{-1}\xi_3^{-1}|\cdot|^{-1/2})(1-\tau)d\tau$$
$$= 1 + q\cdot\int_{\mathfrak{o}}(\Xi_1\Xi_3\xi_1\xi_3|\cdot|^{-1/2})(\tau)(\Xi_2^{-1}\Xi_3^{-1}\xi_1^{-1}\xi_3^{-1}|\cdot|^{-1/2})(1+\tau)d\tau.$$

Hence it follows from Lemma 4.2.8 that

$$I_1 = (q-1)\frac{1-q^{-1}\Xi_1\Xi_2^{-1}}{(1-q^{-1/2}\Xi_1\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_2^{-1}\Xi_3^{-1}\xi_1^{-1}\xi_3^{-1})}.$$

The assertions related to I_2, J_1 and J_2 follow also in the same way as I_1. □

4.3 Uniqueness of Shintani functionals.

In this subsection, we shall prove that the dimension of the intertwining space $\mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi)\otimes i_P^G(\Xi),\mathbb{C})$ is exactly one under the suitable assumption on (ξ,Ξ) (Corollary 4.3.16). This result is indispensable for a meromorphic continuation of the intertwining operator $\Omega_{\xi,\Xi}\in\mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi)\otimes i_P^G(\Xi),\mathbb{C})$ in the next subsection.

First we recall the definition of an l-space and some properties from [BZ].

Definition 4.3.1 i) *We call a locally compact Hausdorff topological space X an l-space if each point of X has a fundamental system consisting of compact open neighborhoods.*

ii) *We call a topological group which is an l-space a locally profinite group.*

The following two results are well-known (see [BZ, Lemma 1.2, Proposition 1.5]).

Lemma 4.3.2 *Let X be an l-space and Y a locally closed subset of X. Then Y is an l-space in the induced topology.*

Proposition 4.3.3 *Let X be an l-space and H a locally profinite group acting on X by a continuous left action. We assume that*

i) *H is countable at infinity, that is, H is covered by its countable compact subsets;*

ii) *The number of H-orbits in X is finite.*

Then there exists an open H-orbit $X_0\subset X$ such that for any point $x_0\in X_0$ the map $H\to X_0$ given by $h\mapsto hx_0$ is open.

Then we have the following proposition.

Proposition 4.3.4 *Along with the assumptions of Proposition 4.3.3, let the number of H-orbits in X be $m+1$. Then there exist distinct H-orbits $X_0,\cdots,X_m$ such that $X_0\cup\cdots\cup X_i$ is an open subset of X for any $i=0,\cdots,m$.*

Proof. Let $X_0,\cdots,X_{j-1}\subset X$ be distinct H-orbits such that $X_0\cup\cdots\cup X_i$ is an open subset of X for any $i=0,\cdots,j-1$. We put $Y_{j-1}=X_0\cup\cdots\cup X_{j-1}$. Since $X-Y_{j-1}$ is a closed subset of X, it is an l-space by Lemma 4.3.2. We note that H acts on $X-Y_{j-1}$. Hence Proposition 4.3.3 implies that there exists an open H-orbit X_j in $X-Y_{j-1}$. Then $X_0\cup\cdots\cup X_j$ is an open subset of X. Indeed, since there exists an open subset U of X such that $X_j=U\cap(X-Y_{j-1})$, we have

$$X_0\cup\cdots\cup X_j=Y_{j-1}\cup(U\cap(X-Y_{j-1}))=Y_{j-1}\cup U.$$

By induction, we have the assertion. □

Let us return to our situation. We set

$$K^{(n)}:=\{k\in K|\ k-1_4\in\varpi^nM_4(\mathfrak{o})\}\quad({}^\forall n\geq 0).$$

Then $\{K^{(n)}\}_{n\geq 0}$ is a fundamental system consisting of open compact neighborhoods of the identity element of G. Hence G is a locally profinite group. Let $\mathcal{U}$ be a locally closed subset of G invariant under the left and right translations by P and P_0, respectively. For $\sigma\in X_{nr}(T)$, we denote by

$I_c^\infty(\sigma,\mathcal{U})$ the vector space consisting of $f \in C^\infty(\mathcal{U})$ with compact support modulo P, such that $f(px) = (\sigma\delta^{1/2})(p)f(x)$ for $(p,x) \in P \times G$. Then P_0 acts on $I_c^\infty(\sigma,\mathcal{U})$ by the right translation. We set $\mathcal{O}_0 := (P_0\eta w_\ell P)^{-1} = Pw_\ell\eta P_0$. Then $\mathcal{O}_0$ is an open dense subset of G by Lemma 3.3.7. We define an action of $P \times P_0$ on G by $(P \times P_0) \times G \ni ((p,p_0),x) \mapsto pxp_0^{-1}$. From Lemma 3.3.5, the number of $(P \times P_0)$-orbits in G is finite. If the number of $(P \times P_0)$-orbits in G is $m+1$, then Proposition 4.3.4 yields that there exist distinct $(P \times P_0)$-orbits $\mathcal{O}_1, \cdots, \mathcal{O}_m$ such that $\widetilde{\mathcal{O}_i} := \mathcal{O}_0 \cup \cdots \cup \mathcal{O}_i$ is an open subset of G for any $i = 0, \cdots, m$. We note that $\widetilde{\mathcal{O}_0} = \mathcal{O}_0, \widetilde{\mathcal{O}_m} = G$ and $\mathcal{O}_i = \widetilde{\mathcal{O}_i} \setminus \widetilde{\mathcal{O}_{i-1}}$. In particular, the third equality implies that $\mathcal{O}_i$ is a closed subset of $\widetilde{\mathcal{O}_i}$. Hence each orbit $\mathcal{O}_i$ is a locally closed subset of G. By [C1, Lemma 6.1.1.], we have the following proposition.

Proposition 4.3.5 *Let $\sigma \in X_{nr}(T)$. For each $i = 1, \cdots, m$, the sequence of the P_0-modules*

$$0 \to I_c^\infty(\sigma, \widetilde{\mathcal{O}_{i-1}}) \to I_c^\infty(\sigma, \widetilde{\mathcal{O}_i}) \to I_c^\infty(\sigma, \mathcal{O}_i) \to 0$$

is exact.

We note that $I_c^\infty(\sigma, \widetilde{\mathcal{O}_m}) = i_P^G(\sigma)$. For each $g \in G$, we set $\mathcal{O}_g := PgP_0$ and denote by δ_g the modulus character of $P_0 \cap g^{-1}Pg$.

Proposition 4.3.6 *Let $\sigma \in X_{nr}(T)$ be an unramified character and $\rho : P_0 \to \mathbb{C}^\times$ a one-dimensional representation of P_0. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\sigma, \mathcal{O}_g), \rho) \le 1 \quad ({}^\forall g \in G).$$

Proof. For $\sigma \in X_{nr}(T)$, it follows from definition that the P_0-module $I_c^\infty(\sigma, \mathcal{O}_g)$ is isomorphic to the compact induction $c\text{-Ind}_{P_0 \cap g^{-1}Pg}^{P_0}(\sigma \otimes \delta^{1/2})$ via $I_c^\infty(\sigma, \mathcal{O}_g) \ni f \mapsto L(g^{-1})f \in c\text{-Ind}_{P_0 \cap g^{-1}Pg}^{P_0}(\sigma \otimes \delta^{1/2})$, where $L(g^{-1})f(x) := f(gx)$ for any $g, x \in G$. Hence, for each $g \in G$, we have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\sigma, \mathcal{O}_g), \rho) &= \dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\sigma, \mathcal{O}_g) \otimes \rho^{-1}, \mathbb{C}) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(c\text{-Ind}_{P_0 \cap g^{-1}Pg}^{P_0}(\sigma\delta^{1/2} \circ \mathrm{Ad}(g)) \otimes \rho^{-1}, \mathbb{C}) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{P_0 \cap g^{-1}Pg}((\sigma\delta^{1/2} \circ \mathrm{Ad}(g)) \otimes \rho^{-1} \otimes \delta_0\delta_g^{-1}, \mathbb{C}) \\ &\le 1. \end{aligned}$$

□

Corollary 4.3.7 *Let $\sigma \in X_{nr}(T)$ be an unramified character and $\rho : P_0 \to \mathbb{C}^\times$ a one-dimensional representation of P_0. Then the condition $\dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\sigma, \mathcal{O}_g), \rho) = 1$ is equivalent to*

$$\left((\sigma\delta^{1/2} \circ \mathrm{Ad}(g)) \otimes \rho^{-1} \otimes \delta_0\delta_g^{-1}\right)|_{P_0 \cap g^{-1}Pg}(x) = 1 \quad ({}^\forall x \in P_0 \cap g^{-1}Pg).$$

In particular, $(\sigma\rho^{-1})|_Z \equiv 1$ if and only if $\dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\sigma, \mathcal{O}_0), \rho) = 1$.

Proof. The first part of the assertion follows immediately from the proof of Proposition 4.3.6. We note that $P_0 \cap (w_\ell\eta)^{-1}Pw_\ell\eta = Z$ and $\delta_{w_\ell\eta} \equiv 1$. Then, for $z \in Z = P_0 \cap (w_\ell\eta)^{-1}Pw_\ell\eta$, we have

$$\begin{aligned} \left((\sigma\delta^{1/2} \circ \mathrm{Ad}(w_\ell\eta)) \otimes \rho^{-1} \otimes \delta_0\delta_{w_\ell\eta}^{-1}\right)(z) &= \left((\sigma\delta^{1/2}) \otimes \rho^{-1}\right)(z) \\ &= (\sigma\rho^{-1})(z). \end{aligned}$$

Hence we have the assertion. □

We have the following two lemmas by direct calculation.

Lemma 4.3.8 *Let $w_\ell \neq w \in W$. Then we have*

$$Pw\eta P_0 = \begin{cases} PwP_0 & (w = 1_4, w_2), \\ Pw\eta(1,0)P_0 & (w = w_1, w_2w_1, w_1w_2w_1), \\ Pw\eta(0,1)P_0 & (w = w_1w_2, w_2w_1w_2). \end{cases}$$

Lemma 4.3.9 *For $w \in W, \varepsilon_i = 0, 1 (i = 1, 2)$, we put $g = w\eta(\varepsilon_1, \varepsilon_2)$. Then we have*

$$T \cap g^{-1}Tg = \begin{cases} T & (\textit{if } (\varepsilon_1, \varepsilon_2) = (0,0)), \\ \{t(t_1, t_1, t_2) \in T\} & (\textit{if } (\varepsilon_1, \varepsilon_2) = (1,0)), \\ \{t(t_1, t_2, t_1t_2) \in T\} & (\textit{if } (\varepsilon_1, \varepsilon_2) = (0,1)), \\ Z & (\textit{if } (\varepsilon_1, \varepsilon_2) = (1,1)). \end{cases}$$

Lemma 3.3.5 implies that

$$G = \bigcup_{\substack{w \in W \\ \varepsilon_1, \varepsilon_2 = 0,1}} Pw\eta(\varepsilon_1, \varepsilon_2)P_0.$$

Let $g = w\eta(\varepsilon_1, \varepsilon_2)$ and $\mathcal{O}_g = PgP_0$. We assume that $\mathcal{O}_g \neq \mathcal{O}_0$. By Lemma 4.3.8, we may assume that $\eta(\varepsilon_1, \varepsilon_2) \neq \eta$. Hence the subgroup $T \cap g^{-1}Tg$ contains a torus which is properly larger than Z from Lemma 4.3.9. We set $R := \{w\eta(\varepsilon_1, \varepsilon_2) |\ w \in W, \varepsilon_1, \varepsilon_2 = 0, 1, \varepsilon_1\varepsilon_2 = 0\}$. Then the set R is the union of R_1 and R_2, where

$$\begin{aligned} R_1 &:= \{g \in R |\ (T \cap g^{-1}Tg) - Z \text{ has an element of the form } t(t_1, t_1, t_2)\}, \\ R_2 &:= \{g \in R |\ (T \cap g^{-1}Tg) - Z \text{ has an element of the form } t(t_1, t_2, t_1t_2)\}. \end{aligned}$$

We set $X := X_{nr}(T_0) \times X_{nr}(T) \simeq (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3$ and, for each $g \in R$,

$$X_g := X - \left\{(\xi, \Xi) \in X \;\middle|\; \left((\Xi\delta^{1/2} \circ \mathrm{Ad}(g)) \otimes (\xi\delta_0^{-1/2}) \otimes \delta_0\delta_g^{-1}\right)(x) = 1 \ (^\forall x \in P_0 \cap g^{-1}Pg)\right\}.$$

Then X_g is a Zariski open set of X.

Theorem 4.3.10 *Let $(\xi, \Xi) \in \bigcap_{g \in R} X_g$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(i_P^G(\Xi), \xi^{-1}\delta_0^{1/2}) = \dim_{\mathbb{C}} \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C}) \leq 1.$$

In particular, if $(\xi\Xi)|_Z = (\xi\Xi\delta_0^{-1/2})|_Z \not\equiv 1$, we have $\mathrm{Hom}_{P_0}(i_P^G(\Xi), \xi^{-1}\delta_0^{1/2}) = \{0\}$.

Remark 4.3.11 *We shall see that the Zariski open set $\bigcap_{g \in R} X_g$ is nonempty (see Proposition 4.3.14).*

Before proving Theorem 4.3.10, we shall see the following lemma.

Lemma 4.3.12 *Let $g \in R$ and $(\xi, \Xi) \in X_g$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\Xi, \mathcal{O}_g), \xi^{-1}\delta_0^{1/2}) = 0.$$

Proof. From Corollary 4.3.7, the condition $\dim_{\mathbb{C}} \mathrm{Hom}_{P_0}(I_c^\infty(\Xi, \mathcal{O}_g), \xi^{-1}\delta_0^{1/2}) \neq 0$ is equivalent to

$$\left((\sigma\delta^{1/2} \circ \mathrm{Ad}(g)) \otimes \xi\delta_0^{-1/2} \otimes \delta_0\delta_g^{-1}\right)(x) = 1 \quad ({}^\forall x \in P_0 \cap g^{-1}Pg).$$

$\square$

Proof of Theorem 4.3.10. Let $(\xi, \Xi) \in \bigcap_{g\in R} X_g$. We note that the Hom-functor $\mathrm{Hom}_{P_0}(\,\cdot\,, \xi^{-1}\delta_0^{1/2})$ is a left exact contravariant functor. We put $m = \sharp\,(P\backslash G/P_0) - 1$. Then, from Proposition 4.3.5, we have an exact sequence

$$0 \to \mathrm{Hom}_{P_0}(I_c^\infty(\Xi, \mathcal{O}_i), \xi^{-1}\delta_0^{1/2}) \to \mathrm{Hom}_{P_0}(I_c^\infty(\Xi, \widetilde{\mathcal{O}_i}), \xi^{-1}\delta_0^{1/2}) \to \mathrm{Hom}_{P_0}(I_c^\infty(\Xi, \widetilde{\mathcal{O}_{i-1}}), \xi^{-1}\delta_0^{1/2})$$

for each $i = 1, \cdots, m$. Since $I_c^\infty(\Xi, \widetilde{\mathcal{O}_m}) = i_P^G(\Xi)$, we have an exact sequence

$$0 \to \mathrm{Hom}_{P_0}(i_P^G(\Xi), \xi^{-1}\delta_0^{1/2}) \to \mathrm{Hom}_{P_0}(I_c^\infty(\Xi, \mathcal{O}_0), \xi^{-1}\delta_0^{1/2})$$

by Lemma 4.3.12. We obtain the assertion from Lemma 4.3.6 and Corollary 4.3.7. $\square$

Corollary 4.3.13 *Let $(\xi, \Xi) \in \bigcap_{g\in R} X_g$. Then we have*

i) $\dim_{\mathbb{C}} \mathrm{Hom}_{G_0}(i_P^G(\Xi), i_{P_0}^{G_0}(\xi^{-1})) = \dim_{\mathbb{C}} \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C}) \leq 1.$

ii) *If $(\xi\Xi)|_Z \not\equiv 1$, we have* $\mathrm{Hom}_{G_0}(i_P^G(\Xi), i_{P_0}^{G_0}(\xi^{-1})) = \{0\}$.

Proof. It follows immediately from Frobenius reciprocity and Theorem 4.3.10. $\square$

For $(\xi, \Xi) \in X = (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3$, the condition $(\xi\Xi)|_Z \equiv 1$ means that (ξ, Ξ) is a zero of the polynomial $Y_1Y_2Y_3^2Z_1Z_2Z_3^2 - 1 \in \mathbb{C}[X] := \mathbb{C}[Y_1^\pm, Y_2^\pm, Y_3^\pm, Z_1^\pm, Z_2^\pm, Z_3^\pm]$. We consider the algebraic set $X^\sharp$ of X defined by

$$X^\sharp = \{(\xi, \Xi) \in X \mid \xi_1\xi_2\xi_3^2\Xi_1\Xi_2\Xi_3^2 = 1\} \simeq (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3.$$

We note that modulus characters $\delta \in X_{nr}(T)$ and $\delta_0 \in X_{nr}(T_0)$ are identified with (q^{-4}, q^{-2}, q^3) and (q^{-2}, q^{-2}, q^2), respectively. We set $U_c^\sharp := U_c \cap X^\sharp$ and $X_g^\sharp := X_g \cap X^\sharp$. We note that $U_c^\sharp$ is a nonempty open subset of $X^\sharp$ in the Euclidean topology. Indeed, we have $((q^{-2}, q^{-2}, q^5), (q^{-4}, q^{-2}, 1)) \in U_c^\sharp$. We set $\widetilde{U}_c^\sharp := U_c^\sharp \cap \bigcap_{g\in R} X_g^\sharp$. Then $\widetilde{U}_c^\sharp$ is an open subset of $X^\sharp$ in the Euclidean topology.

Proposition 4.3.14 *The set $\bigcap_{g\in R} X_g^\sharp$ is an open dense subset of $X^\sharp$ in the Euclidean topology. In particular, we have $\widetilde{U}_c^\sharp \neq \emptyset$.*

We use the following lemma, which is easily checked by direct computation, to prove Proposition 4.3.14.

Lemma 4.3.15 *For $g = w\eta(\varepsilon_1, \varepsilon_2)g \in R$, we put $t' = g^{-1}tg \in T \cap g^{-1}Tg$.*

i) *If $g \in R_1, t' = t(\varpi^{\lambda_1}, \varpi^{\lambda_1}, \varpi^{\lambda_2})$, we have*

$$t = \begin{cases} t' & (\text{if } w = 1_4, w_1, w_2w_1w_2, w_\ell), \\ t(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \varpi^{\lambda_1+\lambda_2}) & (\text{if } w = w_2, w_1w_2, w_2w_1, w_1w_2w_1). \end{cases}$$

ii) *If* $g \in R_2, t' = t(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \varpi^{\lambda_1+\lambda_2})$, *we have*

$$t = \begin{cases} t' & (\text{if } w = 1_4, w_1, w_2w_1w_2, w_\ell), \\ t(\varpi^{\lambda_1}, \varpi^{\lambda_1}, \varpi^{\lambda_2}) & (\text{if } w = w_2, w_1w_2, w_2w_1, w_1w_2w_1). \end{cases}$$

Proof of Proposition 4.3.14. We note that for each $g \in R$ there exist $\gamma_i(g) \in \mathbb{Z}(i = 1, 2, 3)$ such that δ_g^{-1} is identified with $(q^{\gamma_1(g)}, q^{\gamma_2(g)}, q^{\gamma_3(g)}) \in (\mathbb{C}^\times)^3$. For any $\lambda, \mu \in \mathbb{Z}^3$, we consider the element in $\mathbb{C}[X]$ given by

$$F_{\mu,\lambda}(Y_1^\pm, \cdots, Z_3^\pm) = 1 - (q^{-2}Z_1)^{\mu_1}(q^{-1}Z_2)^{\mu_2}(q^{\gamma_1(g)-1}Y_1)^{\lambda_1}(q^{\gamma_2(g)-1}Y_2)^{\lambda_2}(q^{\gamma_3(g)+5/2}Y_3Z_3)^{\lambda_3}$$

and its image $F^\sharp_{\mu,\lambda}$ of the canonical surjection $\mathbb{C}[X] \to \mathbb{C}[X^\sharp] \simeq \mathbb{C}[X]/(Y_1Y_2Y_3^2Z_1Z_2Z_3^2 - 1)$, that is,

$$F^\sharp_{\mu,\lambda}(Y_2^\pm, \cdots, Z_3^\pm) = 1 - q^A Z_1^{\mu_1-\lambda_1} Z_2^{\mu_2-\lambda_1} Y_2^{\lambda_2-\lambda_1}(Y_3Z_3)^{\lambda_3-2\lambda_1} \bmod (Y_1Y_2Y_3^2Z_1Z_2Z_3^2 - 1),$$

where $A = A(g)$ is a certain rational number. Let $g \in R$. We note that there exists $t_0 \in T \cap g^{-1}Tg$ such that

$$\left((\Xi\delta^{1/2} \circ \mathrm{Ad}(g)) \otimes (\xi\delta_0^{-1/2}) \otimes \delta_0\delta_g^{-1}\right)(t_0) \neq 1 \tag{4.1}$$

if and only if there exist $\lambda, \mu = \mu(\lambda) \in \mathbb{Z}^3$ and $u \in T_{(0)}$ such that

$$t(\lambda) = g^{-1}t(\mu)ug, \quad F_{\lambda,\mu}(\xi, \Xi) \neq 0.$$

For each $g \in R$, we take $\lambda(g), \mu(g) \in \mathbb{Z}^3$ such that $t(\lambda(g)) = g^{-1}t(\mu(g))ug \in T \cap g^{-1}Tg$ and consider the algebraic set $\mathcal{V}_g$ of $X^\sharp$ given by

$$\mathcal{V}_g = \{(\xi, \Xi) \in X^\sharp = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3 |\ F^\sharp_{\mu(g),\lambda(g)}(\xi, \Xi) = 0\}.$$

Here $u \in T_{(0)}$. We note that $X^\sharp \setminus \mathcal{V}_g \subset X_g^\sharp$.

We shall prove that $X^\sharp \setminus \bigcup_{g\in R} \mathcal{V}_g = \bigcap_{g\in R} X^\sharp \setminus \mathcal{V}_g \subset \bigcap_{g\in R} X_g^\sharp$ is a dense subset of $X^\sharp$. By Proposition 4.3.17 described below in this subsection, it is enough to show that $\mathcal{V}_g \neq X^\sharp$ for any $g \in R$. For each $g \in R$, we take $\lambda(g) \in \mathbb{Z}^3$ so that

$$\lambda(g) = \begin{cases} (0, 0, 1) & (\text{if } g \in R_1), \\ (0, 1, 1) & (\text{if } g \in R_2). \end{cases}$$

Then Lemma 4.3.15 implies that for each $g \in R$ we have $F^\sharp_{\lambda(g),\mu(g)} \neq 0$. This means that $\mathcal{V}_g \neq X^\sharp$. □

The following corollary is crucial for our proof of a meromorphic continuation of the functional $\Omega_{\xi,\Xi}$ (see Theorem 4.4.1).

Corollary 4.3.16 *For* $(\xi, \Xi) \in \widetilde{U}_c^\sharp$, *we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_0}(i_P^G(\Xi), i_{P_0}^{G_0}(\xi^{-1})) = \dim_{\mathbb{C}} \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C}) = 1.$$

Proof. Since $\Omega_{\xi,\Xi} \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$, Proposition 4.2.2 yields the assertion. $\square$

Finally we prove the above-mentioned proposition, which are also repeatedly used in §4.4 and §4.5.

Proposition 4.3.17 *Let $\mathcal{X}$ be an open subvariety of the affine variety $\mathbb{C}^n$ and $\{\mathcal{V}_i\}_i$ a countable family consisting of proper Zariski closed subsets of $\mathcal{X}$. Then $\mathcal{X} - \bigcup_i \mathcal{V}_i$ is a dense subset of $\mathcal{X}$ in the Euclidean topology.*

Although Proposition 4.3.17 is a special case of Baire category theorem ([K, p.200, Theorem 34]), we will give a proof without using the theorem.

Lemma 4.3.18 *Let $\mathcal{X}$ be a Hausdorff topological space and $\mathfrak{B}$ the Borel algebra of $\mathcal{X}$. Also, let μ be a strictly positive measure on $(\mathcal{X}, \mathfrak{B})$. If C is an element of $\mathfrak{B}$ such that $\mu(C) = 0$, then $\mathcal{X} - C$ is a dense subset of $\mathcal{X}$.*

Proof. We put $\mathcal{Y} = \overline{\mathcal{X} - C}$. Then we have $\mathcal{X} - \mathcal{Y} \subset C$. Since $\mu(C) = 0$ and $\mathcal{X} - \mathcal{Y}$ is an open subset of $\mathcal{X}$, we have $\mathcal{X} - \mathcal{Y} = \emptyset$. Namely, $\mathcal{X} = \mathcal{Y}$. $\square$

Lemma 4.3.19 *Let f be a nonzero holomorphic function on a domain $\mathcal{D} \subset \mathbb{C}^n$. Then the set $\{z \in \mathcal{D} \mid f(z) = 0\}$ has $2n$-dimensional Lebesgue measure zero.*

Proof. See [GR, p.9, Corollary 10], for example. $\square$

Proof of Proposition 4.3.17. We note that Lebesgue measure is strictly positive. Thus Proposition 4.3.17 immediately follows from Lemma 4.3.18, Lemma 4.3.19 and the countable additivity of Lebesgue measure. $\square$

4.4 Continuation of Shintani functionals.

We put $V = C^\infty(P_0 \cap K_0 \backslash K_0) \otimes_{\mathbb{C}} C^\infty(P \cap K \backslash K)$. Then we have $V \simeq i_{P_0}^{G_0}(\xi) \otimes_{\mathbb{C}} i_P^G(\Xi)$. In this subsection, we shall give a "meromorphic continuation" of the functional $\Omega_{\xi,\Xi}$.

Theorem 4.4.1 i) *For each $v \in V$, the function $\widetilde{U}_c^\sharp \ni (\xi, \Xi) \mapsto \Omega_{\xi,\Xi}(v) \in \mathbb{C}$ extends to a rational function on $X^\sharp$;*

ii) *There exists a subset $X^\dagger \subset X^\sharp$, which is the complement of a countable union of hypersurfaces, such that the rational function $X^\sharp \ni (\xi, \Xi) \mapsto \Omega_{\xi,\Xi}(v) \in \mathbf{P}^1(\mathbb{C})$ has no poles on $X^\dagger$ for every $v \in V$;*

iii) *For $(\xi, \Xi) \in X^\dagger$, the above extended map $\Omega_{\xi,\Xi} : V \to \mathbb{C}$ induces an intertwining operator $\Omega_{\xi,\Xi} \in \mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C})$ which satisfies $\Omega_{\xi,\Xi}(\phi_{1_4} \otimes R(w_\ell \eta)\Phi_{1_4}) = \mathrm{vol}(B)\mathrm{vol}(B_0)$.*

Corollary 4.4.2 *There exists a dense subset $D^\sharp \subset X^\sharp$ which satisfies the following two properties:*

i) *The set $D^\sharp$ is the complement of a countable union of proper Zariski closed subsets of $X^\sharp$ and stable under the action of the Weyl group $W_0 \times W$;*

ii) *For any element $(\xi, \Xi) \in D^\sharp$, we have*

$$\mathrm{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_P^G(\Xi), \mathbb{C}) = \mathbb{C} \cdot \Omega_{\xi,\Xi}.$$

There are several ways to give a meromorphic continuation of generalized spherical functions, like Whittaker functions and Bessel functions, or the intertwining operators associated with such functions. For example, Bump–Friedberg–Furusawa applied Hartogs' theorem to give the analytic continuation of the unramified Bessel functions on the split special orthogonal group of odd degree (see [BFF, p.153]). Also, Kato–Murase–Sugano introduced a new rationality argument to give a meromorphic continuation of an intertwining operator attached to generalized spherical functions on connected split reductive groups (see [KMS, section 2]). We also note that another method was used in [H1] to prove a rationality of p-adic integrals (see [H1, Remark 1.1]. See also [D]). In this thesis we apply Bernstein's rationality theorem to give a meromorphic continuation of the Shintnai functional. First we recall a remarkable theorem for establishing a meromorphic continuation of a functional with good properties, which was proved by Bernstein in a letter to Piatetski-Shapiro ([Be]), and obtain a meromorphic continuation of the intertwining operator $\Omega_{\xi,\Xi}$ by using the theorem.

We shall recall Bernstein's rationality theorem. Let k be a field, V a k-vector space and V^* its dual space. We take a subset Θ of $V \times k$. The set Θ is called a *system of equations in* $V \times k$, or simply a *system*. Then we call $\lambda \in V^*$ a *solution to the system* Θ if $\lambda(v) = s$ for any $(v, s) \in \Theta$.

Let V be a $\mathbb{C}$-vector space and $\mathcal{X}$ an irreducible affine variety over $\mathbb{C}$. We denote by $\mathbb{C}[\mathcal{X}]$ the ring of regular functions on $\mathcal{X}$. We consider $V_{\mathbb{C}[\mathcal{X}]} := \mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}} V$ as $\mathbb{C}[\mathcal{X}]$-module. Each $\chi \in \mathcal{X}$ induces a linear map $V_{\mathbb{C}[\mathcal{X}]} \to V, \phi \otimes v \mapsto \phi(\chi)v$. We denote also by χ the map induced from $\chi \in \mathcal{X}$. For a map $p : \mathcal{X} \to V$, we say that p is *regular* if there exists $\hat{v} \in V_{\mathbb{C}[\mathcal{X}]}$ such that $p(\chi) = \chi(\hat{v})$ for all $\chi \in \mathcal{X}$. Hence such a map $p : \mathcal{X} \to V$ is identified with an element of $V_{\mathbb{C}[\mathcal{X}]}$. For each $\chi \in \mathcal{X}$, we take a system $\Theta_\chi = \{(v_{r,\chi}, s_{r,\chi}) \in V \times \mathbb{C} |\ r \in \mathcal{R}\}$, where $\mathcal{R}$ is a common indexing set for any χ. We consider the family of the systems $\Theta_{\mathcal{X}} = \{\Theta_\chi | \chi \in \mathcal{X}\}$. We say that $\Theta_{\mathcal{X}}$ is *regular* if both $\chi \mapsto v_{r,\chi}$ and $\chi \mapsto s_{r,\chi}$ are regular for all $r \in \mathcal{R}$. We denote by $\mathbb{C}(\mathcal{X})$ the field of fractions of $\mathbb{C}[\mathcal{X}]$. Let $V_{\mathbb{C}(\mathcal{X})} := \mathbb{C}(\mathcal{X}) \otimes_{\mathbb{C}} V$ and $V^*_{\mathbb{C}(\mathcal{X})} := \mathrm{Hom}_{\mathbb{C}(\mathcal{X})}(V_{\mathbb{C}(\mathcal{X})}, \mathbb{C}(\mathcal{X}))$. If a family of systems $\Theta_{\mathcal{X}}$ in $V \times \mathbb{C}$ is regular, for each $r \in \mathcal{R}$ there exist $\hat{v_r} \in V_{\mathbb{C}[\mathcal{X}]} \subset V_{\mathbb{C}(\mathcal{X})}$ and $s_r \in \mathbb{C}[\mathcal{X}] \subset \mathbb{C}(\mathcal{X})$ such that

$$\chi \mapsto \chi(\hat{v_r}) = v_{r,\chi}, \quad \chi \mapsto s_r(\chi) = s_{r,\chi}.$$

Hence we can regard such a family of systems $\Theta_{\mathcal{X}}$ as a single system Θ in $V_{\mathbb{C}(\mathcal{X})} \times \mathbb{C}(\mathcal{X})$:

$$\Theta := \{(\hat{v_r}, s_r) \in V_{\mathbb{C}(\mathcal{X})} \times \mathbb{C}(\mathcal{X}) |\ r \in \mathcal{R}\}.$$

We note that $\lambda \in V^*_{\mathbb{C}(\mathcal{X})}$ is determined by its values on elements of the form $1 \otimes v (v \in V)$. For $\lambda \in V^*_{\mathbb{C}(\mathcal{X})}$ and $\chi \in \mathcal{X}$, we define $\lambda_\chi \in V^*$ by $\lambda_\chi(v) = \lambda(1 \otimes v)(\chi)$ ($^\forall v \in V$). We note that $\lambda(\hat{v}) \in \mathbb{C}(\mathcal{X})$ has no poles at χ for all $\hat{v} \in V_{\mathbb{C}[\mathcal{X}]}$ if $\lambda(1 \otimes v) \in \mathbb{C}(\mathcal{X})$ has no poles at χ for all $v \in V$.

Lemma 4.4.3 *For $\hat{v} \in V_{\mathbb{C}[\mathcal{X}]}$, we have $\lambda_\chi(\chi(\hat{v})) = \lambda(\hat{v})(\chi)$. In particular, if $\lambda \in V^*_{\mathbb{C}(\mathcal{X})}$ is a solution to the system Θ, then λ_χ is a solution to the system Θ_χ for all $\chi \in \mathcal{X}$.*

With the notation above, we state Bernstein's rationality theorem. For the proof, refer to [Ba] or [G2, Appendix].

Theorem 4.4.4 (Bernstein[Be]) *Along with the assumptions above, suppose that V has a countable basis over $\mathbb{C}$, and there exists a nonempty subset $\mathcal{X}_0 \subset \mathcal{X}$, which is open in the Euclidean topology, such that Θ_χ has a unique solution for all $\chi \in \mathcal{X}_0$. Then*

i) *Θ has a unique solution $\lambda \in V^*_{\mathbb{C}(\mathcal{X})}$;*

ii) *There exists a subset $\mathcal{X}^\dagger \subset \mathcal{X}$, which is the complement of a countable union of hypersurfaces, such that, for any $\chi \in \mathcal{X}^\dagger$, $\lambda_\chi \in V^*$ is defined and is the unique solution to Θ_χ.*

Remark 4.4.5 *It seems that Proposition 4.3.17 is implicitly used to prove the uniqueness of the solution $\lambda \in V^*_{\mathbb{C}(\mathcal{X})}$ in Bernstein's rationality theorem in [Ba]. We also note that it is also possible to prove it without using Proposition 4.3.17 (see [G2, Appendix]).*

Let us return to our situation. We put $V = C^\infty(P_0 \cap K_0 \backslash K_0) \otimes_{\mathbb{C}} C^\infty(P \cap K \backslash K)$. First we see the following lemma to prove Theorem 4.4.1.

Lemma 4.4.6 *For $f' \otimes f \in V$ and $(g', g) \in G_0 \times G$, the map $X \to V, (\xi, \Xi) \mapsto R_\xi(g)f' \otimes R_\Xi(g)f$ is regular in the sense of the above.*

Proof. For any $(f, g) \in C^\infty(P \cap K \backslash K) \times G$ and $(f', g') \in C^\infty(P_0 \cap K_0 \backslash K_0) \times G_0$, it is enough to show that the maps $X_{nr}(T) \ni \Xi \mapsto R_\Xi(g)f \in C^\infty(P \cap K \backslash K)$ and $X_{nr}(T_0) \ni \xi \mapsto R_\xi(g')f' \in C^\infty(P_0 \cap K_0 \backslash K_0)$ are regular, respectively. We see that $X_{nr}(T) \ni \Xi \mapsto R_\Xi(g)f \in C^\infty(P \cap K \backslash K)$ is regular. Let $g \in G$. We note that

$$C^\infty(P \cap K \backslash K) = \bigcup_{l \geq 0} C^\infty(P \cap K \backslash K)^{K^{(l)}}.$$

Since $\{K^{(l)}\}_{l \geq 0}$ is the fundamental system of the identity element of G, for any $l \geq 0$ there exists $l_0 > l$ such that $K^{(l_0)} \subset gK^{(l)}g^{-1} \cap K^{(l)}$. Hence, for $f \in C^\infty(P \cap K \backslash K)^{K^{(l)}}$ and $k \in K^{(l_0)}$, we have

$$[R_\Xi(k)R_\Xi(g)f](x) = f_\Xi(xkg) = f_\Xi(xgg^{-1}kg) = f_\xi(xg) = [R_\Xi(g)f](x) \quad ({}^\forall x \in K).$$

Namely, we have $R_\Xi(g)f \in C^\infty(P \cap K \backslash K)^{K^{(l_0)}}$. We note that $m(l_0) := \sharp\left(P \cap K \backslash K / K^{(l_0)}\right) < \infty$. This means that $C^\infty(P \cap K \backslash K)^{K^{(l_0)}}$ is an $m(l_0)$-dimensional vector space over $\mathbb{C}$. We consider the basis of $C^\infty(P \cap K \backslash K)^{K^{(l_0)}}$ given by $f_i^{(l_0)}(u_j) = \delta_{ij}(i, j = 1, \cdots, m(l_0))$. Then we can wright $R_\Xi(g)f = \sum_{n=1}^m c_n(\Xi)f_n^{(l_0)}$. We shall see that $c_i \in \mathbb{C}[X_{nr}(T)]$ for all $i = 1, \cdots, m(l_0)$. Let $u_1, \cdots, u_{m(l_0)} \in K$ be a complete system of representatives of $P \backslash G / K^{(l_0)}$. Then, for each u_i, there exist $p_i \in P, k_i \in K^{(l_0)} \subset K^{(l)}$ and $j(i) = 1, \cdots, m(l_0)$ such that $u_i g = p_i u_{j(i)} k_i$. Hence we have

$$\begin{aligned}
[R_\Xi(g)f](u_i) &= f_\Xi(p_i u_{j(i)} k_i) = (\Xi\delta^{1/2})(p_i) f(u_{j(i)}) \\
&= \sum_{n=1}^{m(l_0)} c_n(\Xi) f_n^{(l_0)}(u_i) \\
&= c_i(\Xi).
\end{aligned}$$

If $p_i = t(\mu^{(i)})$ mod $(T \cap K)N$ for some $\mu^{(i)} \in \mathbb{Z}^3$, the above equality yields that

$$c_i(\Xi) = (\Xi\delta^{1/2})(p_i) = q^{-2\mu_1^{(i)} - \mu_2^{(i)} + (3/2)\mu_3^{(i)}} \Xi^{\mu_1^{(i)}} \Xi^{\mu_2^{(i)}} \Xi^{\mu_3^{(i)}}$$

for all $i = 1, \cdots, m(l_0)$, that is, $c_i \in \mathbb{C}[X_{nr}(T)]$. Therefore the map $\Xi \mapsto R_\Xi(g)f$ is regular. It follows that the map $X_{nr}(T_0) \ni \xi \mapsto R_\xi(g')f' \in C^\infty(P_0 \cap K_0 \backslash K_0)$ is regular in the same way as above. $\square$

Now we prove Theorem 4.4.1 as an application of Theorem 4.4.4.

Proof of Theorem 4.4.1. Since $i_P^G(\Xi)$ (resp. $i_{P_0}^{G_0}(\xi)$) is an admissible representation of G (resp. G_0), $i_P^G(\Xi)$ (resp. $i_{P_0}^{G_0}(\xi)$) has a countable basis $\{F_i\}_{i\in\mathbb{Z}}$ (resp. $\{f_j\}_{j\in\mathbb{Z}}$) over $\mathbb{C}$. Then $\{f_i\otimes F_j\}_{i,j\in\mathbb{Z}}$ is a countable basis of V. For any $(\xi,\Xi)\in X^\sharp$, we define a system $\Theta_{\xi,\Xi}=\Theta_{(\xi,\Xi)}\subset V\times\mathbb{C}$ by

$$\Theta_{\xi,\Xi}:=\{\big(R_\xi(g')f_i\otimes R_\Xi(g')F_j-f_i\otimes F_j,0\big)\mid g'\in G_0,i,j\in\mathbb{Z}\}\cup\{(\phi_{1_4}\otimes R(w_\ell\eta)\Phi_{1_4},\mathrm{vol}(B)\mathrm{vol}(B_0))\}.$$

Let $\Theta=\Theta_{X^\sharp}=\{\Theta_{\xi,\Xi}|\ (\xi,\Xi)\in X^\sharp\}$ be the family of the systems. Then Θ is regular from Lemma 4.4.6. For all $(\xi,\Xi)\in\widetilde{U}_c^\sharp$, Proposition 4.2.2 and Corollary 4.3.16 imply that the functional $\Omega_{\xi,\Xi}\in V^*$ is a unique solution to $\Theta_{\xi,\Xi}$. Since $\widetilde{U}_c^\sharp$ is the nonempty open subset of $X^\sharp$ in the Euclidean topology, Theorem 4.4.4 yields that

a) Θ has a unique solution $\Omega\in V^*_{\mathbb{C}(X^\sharp)}$,

b) There exists a subset $X^\dagger\subset X^\sharp$, which is the complement of a countable union of hypersurfaces, such that, for any $(\xi,\Xi)\in X^\dagger$, $\Omega_{(\xi,\Xi)}\in V^*$ is defined and is the unique solution to $\Theta_{\xi,\Xi}$. In particular, we have $\Omega_{(\xi,\Xi)}=\Omega_{\xi,\Xi}$ for $(\xi,\Xi)\in X^\dagger\cap\widetilde{U}_c^\sharp$.

For $(\xi,\Xi)\in X^\dagger$, we set $\Omega_{\xi,\Xi}(v)=\Omega_{(\xi,\Xi)}(v)$ for all $v\in V$. Then we have a family $\{\Omega_{\xi,\Xi}\}_{(\xi,\Xi)\in X^\dagger}$ of intertwining operators. Hence we obtain the assertions. □

Finally we construct a subset $D^\sharp\subset X^\sharp$ which satisfies the properties in Corollary 4.4.2. We take and fix the set $X^\dagger$ considered in Theorem 4.4.1. For each subset $Y\subset X^\sharp$, we set

$$\begin{aligned}Y_w&:=\{\chi\in Y|\ w\cdot\chi\in Y\}=Y\cap w^{-1}\cdot Y\quad(w\in W_0\times W),\\ Y_{W_0\times W}&:=\bigcap_{w\in W_0\times W}Y_w=\cap_{w\in W}w\cdot Y.\end{aligned}$$

We note that the Weyl group $W_0\times W$ acts on $X^\sharp$. Indeed, if $(\xi,\Xi)\in X^\sharp$, then

$$(w'\xi)(z)(w\Xi)(z)=\xi(w'^{-1}zw')\Xi(w^{-1}zw)=\xi(z)\Xi(z)=1$$

for $(w',w)\in W_0\times W$ and $z\in Z$. The following lemma, which is easily checked, is useful in §5.3.

Lemma 4.4.7 *Let $\Xi=(\Xi_1,\Xi_2,\Xi_3)\in X_{nr}(T),\xi=(\xi_1,\xi_2,\xi_3)\in X_{nr}(T_0)$. Then we have*

$$w\Xi=\begin{cases}(\Xi_2,\Xi_1,\Xi_3) & (\textit{if } w=w_1),\\ (\Xi_1,\Xi_2^{-1},\Xi_2\Xi_3) & (\textit{if } w=w_2),\\ (\Xi_2^{-1},\Xi_1,\Xi_2\Xi_3) & (\textit{if } w=w_1w_2),\\ (\Xi_2,\Xi_1^{-1},\Xi_1\Xi_3) & (\textit{if } w=w_2w_1),\\ (\Xi_1^{-1},\Xi_2,\Xi_1\Xi_3) & (\textit{if } w=w_1w_2w_1),\\ (\Xi_2^{-1},\Xi_1^{-1},\Xi_1\Xi_2\Xi_3) & (\textit{if } w=w_2w_1w_2),\\ (\Xi_1^{-1},\Xi_2^{-1},\Xi_1\Xi_2\Xi_3) & (\textit{if } w=w_\ell),\end{cases}\qquad w'\xi=\begin{cases}(\xi_1^{-1},\xi_2,\xi_1\xi_3) & (\textit{if } w'=w_1'),\\ (\xi_1,\xi_2^{-1},\xi_2\xi_3) & (\textit{if } w'=w_2'),\\ (\xi_1^{-1},\xi_2^{-1}\xi_1\xi_2\xi_3) & (\textit{if } w'=w_1'w_2').\end{cases}$$

We set

$$D^\sharp:=X^\dagger_{W_0\times W}\cap_{g\in R}X_g^\sharp\cap X_{nr}^{reg}(T_0)\times X_{nr}^{reg}(T)\subset X^\sharp.$$

We note that $D^\sharp$ is stable under the action of the Weyl group $W_0\times W$. In order to prove Corollary 4.4.2, it is enough to show the following lemma. Indeed, Corollary 4.4.2 is immediately obtained from Theorem 4.4.1 .

Lemma 4.4.8 *The set $D^\sharp$ is the complement of a countable union of proper Zariski closed subsets of $X^\sharp$. In particular, $D^\sharp$ is a dense subset of $X^\sharp$ in the Euclidean topology.*

Proof. We note that $X^\dagger_{W_0\times W} = \bigcap_{w\in W_0\times W} w\cdot X^\dagger$ is the complement of a countable union of hypersurfaces. Since the set $X^\sharp \cap X^{reg}_{nr}(T_0)\times X^{reg}_{nr}(T)$ is the complement of a finite union of proper Zariski closed subsets of $X^\sharp$, the set $D^\sharp = X^\dagger_{W_0\times W} \cap (X^\sharp \cap X^{reg}_{nr}(T_0)\times X^{reg}_{nr}(T))$ is the complement of a countable union of proper Zariski closed subsets of $X^\sharp$. Hence we have the assertion by Proposition 4.3.17. □

4.5 An explicit formula of Shintani functions.

In this subsection, we shall prove an explicit formula of the Shintani function. For any $(\xi,\Xi) = ((\xi_1,\xi_2,\xi_3),(\Xi_1,\Xi_2,\Xi_3)) \in X^\sharp$, we set

$$\zeta(\xi,\Xi) := \frac{\mathbf{e}'(\xi)\mathbf{e}(\Xi)}{\mathbf{b}(\xi,\Xi)},$$

where

$$\begin{aligned}
&\mathbf{e}'(\xi) := (1-q^{-1}\xi_1)(1-q^{-1}\xi_2), \quad \mathbf{e}(\Xi) := (1-q^{-1}\Xi_1\Xi_2)(1-q^{-1}\Xi_1\Xi_2^{-1})(1-q^{-1}\Xi_1)(1-q^{-1}\Xi_2),\\
&\mathbf{b}(\xi,\Xi) := (1-q^{-1/2}\Xi_1\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_1\Xi_3\xi_2\xi_3)(1-q^{-1/2}\Xi_1\Xi_3\xi_1\xi_2\xi_3)(1-q^{-1/2}\Xi_2\Xi_3\xi_1\xi_2\xi_3)\\
&\qquad \times (1-q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_2\xi_3)(1-q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_3)(1-q^{-1/2}\Xi_1\Xi_2\Xi_3\xi_1\xi_2\xi_3).
\end{aligned}$$

First we shall prove the next proposition, which is an analogue of Jacquet's functional equation for the unramified Whittaker function in [J] (see also [CS]).

Proposition 4.5.1 *Let $g\in G$. For $(\xi,\Xi)\in D^\sharp$ such that $\zeta(\xi,\Xi)\neq 0$, the value $S_{\xi,\Xi}(g)/\zeta(\xi,\Xi)$ is $W_0\times W$-invariant as a function of (ξ,Ξ).*

We take an element $(w',w)\in W_0\times W$. Then we have

$$(T_{w',\xi}\otimes T_{w,\Xi})^*\Omega_{w'\xi,w\Xi} := \Omega_{w'\xi,w\Xi}\circ(T_{w',\xi}\otimes T_{w,\Xi}) \in \mathrm{Hom}_{G_0}(i^{G_0}_{P_0}(\xi)\otimes i^G_P(\Xi),\mathbb{C})$$

for any $(\xi,\Xi)\in D^\sharp$. Hence it follows from Corollary 4.4.2 that there exists a scalar factor $a_{w',w}(\xi,\Xi)\in\mathbb{C}$ such that

$$(T_{w',\xi}\otimes T_{w,\Xi})^*\Omega_{w'\xi,w\Xi} = c'_{w'}(\xi)c_w(\Xi)a_{w',w}(\xi,\Xi)\Omega_{\xi,\Xi}$$

for any $(\xi,\Xi)\in D^\sharp$ which satisfies $c'_{w'}(\xi)c_w(\Xi)\neq 0$. Here

$$c'_{w'}(\xi) = \prod_{\substack{\beta\in\Sigma_0^+\\ w'\beta<0}} c'_\beta(\xi), \quad c_w(\Xi) = \prod_{\substack{\alpha\in\Sigma^+\\ w\alpha<0}} c_\alpha(\Xi).$$

First we consider the case where $w = w_{\alpha_i} = w_i$ and $w' = 1_4$. Then, since $T_{w_i,\Xi}(\Phi_{1_4}+\Phi_{w_i}) = c_{\alpha_i}(\Xi)(\Phi_{1_4}+\Phi_{w_i})$, we have

$$\begin{aligned}
&c_{\alpha_i}(\Xi)a_{1_4,w_i}(\xi,\Xi)\Omega_{\xi,\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_i}))\\
&= (\mathrm{id}\otimes T_{w_i,\Xi})^*\Omega_{\xi,w_i\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_i}))\\
&= c_{\alpha_i}(\Xi)\Omega_{\xi,w_i\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_i})),
\end{aligned}$$

that is,

$$a_{1_4,w_i}(\xi,\Xi)=\frac{\Omega_{\xi,w_i\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_i}))}{\Omega_{\xi,\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_i}))}.$$

Next we consider the case where $w=1_4$ and $w'=w_{\beta_i}=w'_i$. Then, since $T_{w'_i,\xi}(\phi_{1_4}+\phi_{w'_i})=c_{\beta_i}(\xi)(\phi_{1_4}+\phi_{w'_i})$, we have

$$a_{w'_i,1_4}(\xi,\Xi)=\frac{\Omega_{w'_i\xi,\Xi}((\phi_{1_4}+\phi_{w'_i})\otimes R(\eta w_\ell)\Phi_{1_4})}{\Omega_{\xi,\Xi}((\phi_{1_4}+\phi_{w'_i})\otimes R(\eta w_\ell)\Phi_{1_4})}$$

in the same way as the first case. In order to prove Proposition 4.5.1, we prove the following lemma.

Lemma 4.5.2 *For $i=1,2$, we have*

$$\zeta(\xi,w_i\Xi)=a_{1_4,w_i}(\xi,\Xi)\zeta(\xi,\Xi)\quad \zeta(w'_i\xi,\Xi)=a_{w'_i,1_4}(\xi,\Xi)\zeta(\xi,\Xi).$$

Proof. From Proposition 4.2.1, we have

$$\begin{aligned}a_{1_4,w_1}(\xi,\Xi)&=\frac{\Omega_{\xi,w_1\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_1}))}{\Omega_{\xi,\Xi}(\phi_{1_4}\otimes R(\eta w_\ell)(\Phi_{1_4}+\Phi_{w_1}))}\\&=\frac{(1-q^{-1}\Xi_1^{-1}\Xi_2)(1-q^{-1/2}\Xi_1\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_1\Xi_3\xi_2\xi_3)}{(1-q^{-1}\Xi_1\Xi_2^{-1})(1-q^{-1/2}\Xi_2\Xi_3\xi_1\xi_3)(1-q^{-1/2}\Xi_2\Xi_3\xi_2\xi_3)}\\&=\frac{\zeta(\xi,w_1\Xi)}{\zeta(\xi,\Xi)}.\end{aligned}$$

The other cases are obtained in the same way as above. □

Proof of Proposition 4.5.1. Since $T_{w_i,\Xi}(\phi_{K,\Xi})=c_{w_i}(\Xi)\phi_{K,\Xi}$ holds for $i=1,2$, we have

$$\begin{aligned}\frac{S_{\xi,w_i\Xi}(g)}{\zeta(\xi,w_i\Xi)}&=\frac{\Omega_{\xi,w_i\Xi}(\phi_{K_0,\xi}\otimes R(g)\phi_{K,\Xi})}{\zeta(\xi,w_i\Xi)}\\&=c_{w_i}(\Xi)^{-1}\frac{\Omega_{\xi,w_i\Xi}(\phi_{K_0,\xi}\otimes T_{w_i,\Xi}(R(g)\phi_{K,\Xi}))}{\zeta(\xi,w_i\Xi)}\\&=c_{w_i}(\Xi)^{-1}\frac{(\mathrm{id}\otimes T_{w_i,\Xi})^*\Omega_{\xi,w_i\Xi}(\phi_{K_0,\xi}\otimes R(g)\phi_{K,\Xi})}{\zeta(\xi,w_i\Xi)}\\&=a_{1_4,w_i}(\xi,\Xi)\frac{\Omega_{\xi,\Xi}(\phi_{K_0,\xi}\otimes R(g)\phi_{K,\Xi})}{\zeta(\xi,w_i\Xi)}\\&=\frac{\Omega_{\xi,\Xi}(\phi_{K_0,\xi}\otimes R(g)\phi_{K,\Xi})}{\zeta(\xi,\Xi)}\\&=\frac{S_{\xi,\Xi}(g)}{\zeta(\xi,\Xi)}.\end{aligned}$$

This means that the value $S_{\xi,\Xi}(g)/\zeta(\xi,\Xi)$ is W-invariant as a function of Ξ. The W_0-invariance follows in the same manner. □

Next we prove an explicit formula of the Shintani function $S_{\xi,\Xi}$ for $(\xi,\Xi)\in X^\sharp$ such that $\zeta(\xi,\Xi)\neq 0$. By Theorem 3.2.1, it suffices to know the value

$$S_{\xi,\Xi}(t(\lambda')\eta t(\lambda))=S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1}z(\lambda_3))$$

$$= \omega_\Xi(z(\lambda_3))S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})$$
$$= (\Xi_1\Xi_2\Xi_3^2)^{\lambda_3} S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})$$

for $(\lambda, \lambda') \in \Lambda^+ \times \Lambda_0^{++}$. For $(\xi, \Xi) \in X^\sharp$, we set

$$c_S(\xi, \Xi) := \frac{\prod_{\beta\in\Sigma_0^+} c_\beta(\xi) \prod_{\alpha\in\Sigma^+} c_\alpha(\Xi)}{\zeta(\xi,\Xi)} = \frac{\mathbf{b}(\xi,\Xi)}{\mathbf{d}'(\xi)\mathbf{d}(\Xi)},$$

where

$$\mathbf{d}'(\xi) := (1-\xi_1)(1-\xi_2), \quad \mathbf{d}(\Xi) := (1-\Xi_1\Xi_2)(1-\Xi_1\Xi_2^{-1})(1-\Xi_1)(1-\Xi_2).$$

Theorem 4.5.3 *For any element $(\xi, \Xi) \in X^\sharp$ such that $\zeta(\xi, \Xi) \neq 0$, we have*

$$\frac{S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})}{\zeta(\xi,\Xi)}$$
$$= q^6 \cdot \mathrm{vol}(B)\mathrm{vol}(B_0) \sum_{\substack{w\in W \\ w'\in W_0}} c_S(w'\xi, w\Xi) \left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda)) \left((w'\xi^{-1})\delta_0^{1/2}\right)(t(\lambda')).$$

Proof. We can see that

$$B_0 t(\lambda')B_0\eta w_\ell B t(\lambda)^{-1} B \subset K_0 t(\lambda')\eta w_\ell t(\lambda)^{-1} K$$

in the same way as Lemma 4.2.3. Hence we have

$$\begin{aligned}
L(\mathrm{ch}_{B_0t(\lambda')^{-1}B_0})R(\mathrm{ch}_{Bt(\lambda)^{-1}B})S_{\xi,\Xi}(\eta w_\ell) &= \int_G dg \int_{G_0} dg\ \mathrm{ch}_{B_0t(\lambda')^{-1}B_0}(g')S_{\xi,\Xi}(g'^{-1}\eta w_\ell g)\mathrm{ch}_{Bt(\lambda)^{-1}B}(g) \\
&= \int_G dg \int_{G_0} dg\ \mathrm{ch}_{B_0t(\lambda')^{-1}B_0}(g'^{-1})S_{\xi,\Xi}(g'\eta w_\ell g)\mathrm{ch}_{Bt(\lambda)^{-1}B}(g) \\
&= \int_{Bt(\lambda)^{-1}B} dg \int_{B_0t(\lambda')B_0} dg'\ S_{\xi,\Xi}(g'\eta w_\ell g) \\
&= \mathrm{vol}(Bt(\lambda)^{-1}B; dg)\mathrm{vol}(B_0t(\lambda')B_0; dg')S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1}) \\
&= \mathrm{vol}(Bt(\lambda)B; dg)\mathrm{vol}(B_0t(\lambda')B_0; dg')S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1}).
\end{aligned}$$

Let $\{g_w\}_{w\in W}$ and $\{g'_{w'}\}_{w'\in W_0}$ be the bases of $i_P^G(\Xi)^B$ and $i_{P_0}^{G_0}(\xi)^{B_0}$ obtained by Proposition 2.1.3, respectively. We note that these bases satisfy

$$\phi_{K,\Xi} = q^4 \sum_{w\in W} \overline{c}_w(\Xi) g_w, \quad \phi_{K_0,\xi} = q^2 \sum_{w'\in W_0} \overline{c}'_{w'}(\xi) g'_{w'},$$

where

$$\overline{c}_w(\Xi) := \prod_{\substack{\alpha\in\Sigma^+ \\ w\alpha>0}} c_\alpha(\Xi), \quad \overline{c}'_{w'}(\xi) := \prod_{\substack{\beta\in\Sigma_0^+ \\ w'\beta>0}} c'_\beta(\xi).$$

Hence $S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})$ can be expressed as

$$S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})$$

$$
\begin{aligned}
&= \mathrm{vol}(Bt(\lambda)B;dg)^{-1}\mathrm{vol}(B_0t(\lambda')B_0;dg')^{-1}L(\mathrm{ch}_{B_0t(\lambda')^{-1}B_0})R(\mathrm{ch}_{Bt(\lambda)^{-1}B})S_{\xi,\Xi}(\eta w_\ell)\\
&= \mathrm{vol}(Bt(\lambda)B;dg)^{-1}\mathrm{vol}(B_0t(\lambda')B_0;dg')^{-1}\\
&\qquad \times L(\mathrm{ch}_{B_0t(\lambda')^{-1}B_0})R(\mathrm{ch}_{Bt(\lambda)^{-1}B})\Omega_{\xi,\Xi}(\phi_{K_0,\xi}\otimes R(\eta w_\ell)\phi_{K,\Xi})\\
&= q^6\cdot \mathrm{vol}(Bt(\lambda)B;dg)^{-1}\mathrm{vol}(B_0t(\lambda')B_0;dg')^{-1}\\
&\qquad \times \sum_{\substack{w\in W\\ w'\in W_0}} \overline{c}_w(\Xi)\overline{c}'_{w'}(\xi)L(\mathrm{ch}_{B_0t(\lambda')^{-1}B_0})R(\mathrm{ch}_{Bt(\lambda)^{-1}B})\Omega_{\xi,\Xi}(g'_{w'}\otimes R(\eta w_\ell)g_w)\\
&= q^6\cdot \mathrm{vol}(Bt(\lambda)B;dg)^{-1}\mathrm{vol}(B_0t(\lambda')B_0;dg')^{-1}\\
&\qquad \times \sum_{\substack{w\in W\\ w'\in W_0}} \overline{c}_w(\Xi)\overline{c}'_{w'}(\xi)\Omega_{\xi,\Xi}(R(\mathrm{ch}_{B_0t(\lambda')^{-1}B_0})g'_{w'}\otimes R(\eta w_\ell)R(\mathrm{ch}_{Bt(\lambda)^{-1}B})g_w)\\
&= q^6 \sum_{\substack{w\in W\\ w'\in W_0}} \overline{c}_w(\Xi)\overline{c}'_{w'}(\xi)\left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda))\left((w'\xi)^{-1}\delta_0^{1/2}\right)(t(\lambda'))\Omega_{\xi,\Xi}(g'_{w'}\otimes R(\eta w_\ell)g_w).
\end{aligned}
$$

Since $\Phi_{1_4}=g_{1_4}$ and $\phi_{1_4}=g'_{1_4}$, we have

$$\Omega_{\xi,\Xi}(g'_{1_4}\otimes R(\eta w_\ell)g_{1_4})=\mathrm{vol}(B)\mathrm{vol}(B_0)$$

by Proposition 4.2.2. Therefore we have

$$
\begin{aligned}
&\frac{S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})}{\zeta(\xi,\Xi)}\\
&= q^6\cdot \mathrm{vol}(B;dg)\mathrm{vol}(B_0;dg)\frac{\overline{c}_{1_4}(\Xi)\overline{c}'_{1_4}(\xi)}{\zeta(\xi,\Xi)}\left(\Xi^{-1}\delta^{1/2}\right)(t(\lambda))\left(\xi^{-1}\delta_0^{1/2}\right)(t(\lambda'))\\
&\quad + q^6 \sum_{(w,w')\in W\times W_0-\{(1_4,1_4)\}} \frac{\overline{c}_w(\Xi)\overline{c}'_{w'}(\xi)}{\zeta(\xi,\Xi)}\left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda))\left((w'\xi)^{-1}\delta_0^{1/2}\right)(t(\lambda'))\\
&\qquad\qquad \times \Omega_{\xi,\Xi}(g'_{w'}\otimes R(\eta w_\ell)g_w)\\
&= q^6\cdot \mathrm{vol}(B;dg)\mathrm{vol}(B_0;dg')c_S(\xi,\Xi)\left(\Xi^{-1}\delta^{1/2}\right)(t(\lambda))\left(\xi^{-1}\delta_0^{1/2}\right)(t(\lambda'))\\
&\quad + q^6 \sum_{(w,w')\in W\times W_0-\{(1_4,1_4)\}} \frac{\overline{c}_w(\Xi)\overline{c}'_{w'}(\xi)}{\zeta(\xi,\Xi)}\left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda))\left((w'\xi)^{-1}\delta_0^{1/2}\right)(t(\lambda'))\\
&\qquad\qquad \times \Omega_{\xi,\Xi}(g'_{w'}\otimes R(\eta w_\ell)g_w).
\end{aligned}
$$

Since Ξ and ξ are regular, the set $\{(w\Xi)^{-1}\delta^{1/2}\boxtimes(w'\xi)^{-1}\delta_0^{1/2}\mid (w',w)\in W_0\times W\}$ of characters of $T_0\times T$ is linearly independent. Hence Proposition 4.5.1 implies that

$$
\begin{aligned}
&\frac{S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})}{\zeta(\xi,\Xi)}\\
&= q^6\cdot \mathrm{vol}(B;dg)\mathrm{vol}(B_0;dg')\sum_{\substack{w\in W\\ w'\in W_0}} c_S(w'\xi,w\Xi)\left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda))\left((w'\xi^{-1})\delta_0^{1/2}\right)(t(\lambda')).
\end{aligned}
$$

$\square$

Finally we prove our main theorem, which is an explicit formula of the Shintani function of type (ξ, Ξ) for *any* $(\xi, \Xi) \in X^\sharp$.

Theorem 4.5.4 *Let (ξ, Ξ) be any element of $X_{nr}(T_0) \times X_{nr}(T)$. Then we have*

$$\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) = \begin{cases} 1 & (\textit{if } (\xi, \Xi) \in X^\sharp), \\ 0 & (\textit{otherwise}). \end{cases}$$

If $(\xi, \Xi) \in X^\sharp$, for any nonzero element $S \in \mathcal{S}(\xi, \Xi)$ we have $S(1_4) \neq 0$, and the Shintani function $W_{\xi,\Xi} \in \mathcal{S}(\xi, \Xi)$ with $W_{\xi,\Xi}(1_4) = 1$ is given by

$$\begin{aligned} W_{\xi,\Xi}(g(\lambda', \lambda)) &= (\Xi_1\Xi_2\Xi_3^2)^{\lambda_3} W_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1}) \\ &= \frac{(\Xi_1\Xi_2\Xi_3^2)^{\lambda_3}}{(1-q^{-2})^2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi) \left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda)) \left((w'\xi)^{-1}\delta_0^{1/2}\right)(t(\lambda')) \end{aligned}$$

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$.

In order to prove Theorem 4.5.4, we first compute the value

$$S_{\xi,\Xi}(1_4) = S_{\xi,\Xi}(\eta w_\ell) = q^6 \cdot \zeta(\xi, \Xi)\mathrm{vol}(B; dg)\mathrm{vol}(B_0; dg') \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi)$$

at the identity and normalize the Shintani function. The following lemma can be easily checked by direct computation.

Lemma 4.5.5

$$\begin{aligned} S_{\xi,\Xi}(1_4) &= S_{\xi,\Xi}(\eta w_\ell) \\ &= q^6(1-q^{-2})^2\zeta(\xi, \Xi)\mathrm{vol}(B; dg)\mathrm{vol}(B_0; dg'). \end{aligned}$$

Hence we have immediately the following proposition.

Proposition 4.5.6 *For any $(\xi, \Xi) \in D^\sharp$ such that $\zeta(\xi, \Xi) \neq 0$, the basis of $\mathcal{S}(\xi, \Xi)$, $W_{\xi,\Xi} \in \mathcal{S}(\xi, \Xi)$ with $W_{\xi,\Xi}(1_4) = 1$, is given by*

$$\begin{aligned} W_{\xi,\Xi}(g(\lambda', \lambda)) &= (\Xi_1\Xi_2\Xi_3^2)^{\lambda_3} S_{\xi,\Xi}(t(\lambda')\eta w_\ell t(\lambda)^{-1})/S_{\xi,\Xi}(1_4) \\ &= \frac{(\Xi_1\Xi_2\Xi_3^2)^{\lambda_3}}{(1-q^{-2})^2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi) \left((w\Xi)^{-1}\delta^{1/2}\right)(t(\lambda)) \left((w'\xi)^{-1}\delta_0^{1/2}\right)(t(\lambda')) \end{aligned}$$

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$.

For any $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$, we denote by $F_{\xi,\Xi}(\lambda', \lambda)$ a rational function on $X^\sharp$ given by the right hand side of the formula in Proposition 4.5.6. We set $F_{\xi,\Xi}(k_0 g(\lambda', \lambda)k) = F_{\xi,\Xi}(\lambda', \lambda)$ for any $(k_0, \lambda', \lambda, k) \in K_0 \times \Lambda_0^{++} \times \Lambda^+ \times K$. Then, for $(\xi, \Xi) \in D^\sharp$ such that $\zeta(\xi, \Xi) \neq 0$, the function $F_{\xi,\Xi}$ is a Shintani function with $F_{\xi,\Xi}(1_4) = 1$. Theorem 4.5.4 follow from Theorem 3.3.1, Proposition 4.5.6 and the following proposition.

Proposition 4.5.7 i) *For any $x \in G$, the rational function $F_{\xi,\Xi}(x)$ is regular on the whole $X^\sharp$;*

ii) *For any $(\xi,\Xi) \in X^\sharp$, the function $F_{\xi,\Xi}$ on G is the Shintani function of type (ξ,Ξ) with $F_{\xi,\Xi}(1_4) = 1$.*

Proof. From Theorem 3.2.1, it is enough to show that $F_{\xi,\Xi}(\mu',\mu)$ is regular on $X^\sharp$ for any $(\mu',\mu) \in \Lambda_0^{++} \times \Lambda^+$. By induction on $m(\mu',\mu) \geq 1$, we shall see the assertion (i). We put

$$U_0 = \left\{(\xi,\Xi) \in D^\sharp \mid \zeta(\xi,\Xi) \neq 0\right\}.$$

Then $X^\sharp - U_0$ is a countable union of proper Zariski closed subsets in $X^\sharp$ (see Corollary 4.4.2). In particular, U_0 is a dense subset of $X^\sharp$ by Proposition 4.3.17.

First we assume that $m(\mu',\mu) = 1$. Then, since

$$K_0 t'(\mu') K t(\mu) K = K_0 g(\mu',\mu) K,$$

for all $(\xi,\Xi) \in U_0$ we have

$$\omega_\xi(\mathrm{ch}_{K_0 t'(\mu')^{-1} K_0})\omega_\Xi(\mathrm{ch}_{K t(\mu) K}) = C_{\mu',\mu}\mathrm{vol}\big(K_0 g(\mu',\mu)K; dg\big) F_{\xi,\Xi}(\mu',\mu),$$

where $C_{\mu',\mu}$ is a certain positive integer, which does not depended on (ξ,Ξ). Hence $F_{\xi,\Xi}(\mu',\mu)$ is regular on $X^\sharp$.

Next we assume that $F_{\xi,\Xi}(\lambda',\lambda)$ is regular on $X^\sharp$ for any $(\lambda',\lambda) \in \Lambda_0^{++} \times \Lambda^+$ which satisfies $m(\mu',\mu) > m(\lambda',\lambda)$. If

$$K_0 t'(\mu') K t(\mu) K = \bigsqcup_{i=0}^{M} K_0 g(\lambda'_{(i)},\lambda_{(i)}) K, \quad (\lambda'_{(0)},\lambda_{(0)}) = (\mu',\mu),$$

then we have an equality

$$\begin{aligned}
&\omega_\xi(\mathrm{ch}_{K_0 t'(\mu')^{-1} K_0})\omega_\Xi(\mathrm{ch}_{K t(\mu) K}) \\
&= C^{(0)}_{\mu',\mu}\mathrm{vol}\big(K_0 g(\mu',\mu)K; dg\big) F_{\xi,\Xi}(\mu',\mu) + \sum_{i=1}^{M} C^{(i)}_{\mu',\mu}\mathrm{vol}\big(K_0 g(\lambda'_{(i)},\lambda_{(i)})K; dg\big) F_{\xi,\Xi}(\lambda'_{(i)},\lambda_{(i)})
\end{aligned}$$

for all $(\xi,\Xi) \in U_0$ (see (3.3)). Here $C^{(i)}_{\mu',\mu}$ is a certain positive integer which does not depend on (ξ,Ξ) for all $i = 0,\cdots,M$. Since $m(\mu',\mu) > m(\lambda'_{(i)},\lambda_{(i)})$ for $i = 1,\cdots,M$ by Lemma 3.3.13, each $F_{\xi,\Xi}(\lambda'_{(i)},\lambda_{(i)})$ is regular on $X^\sharp$ by the induction hypothesis. Hence $F_{\xi,\Xi}(\mu',\mu)$ is regular on $X^\sharp$.

We show the assertion (ii). The assertion (i) implies that, for all $(\mu',\mu) \in \Lambda_0^{++} \times \Lambda^+$ and all $(\phi,\Phi) \in \mathcal{H}(G_0,K_0) \times \mathcal{H}(G,K)$, the function $X^\sharp \ni (\xi,\Xi) \mapsto [L(\phi)R(\Phi)F_{\xi,\Xi}](g(\mu',\mu)) \in \mathbb{C}$ is regular on $X^\sharp$. If for any $(\mu',\mu) \in \Lambda_0^{++} \times \Lambda^+$ we put $f_{\mu',\mu}(\xi,\Xi) = [L(\phi)R(\Phi)F_{\xi,\Xi}](g(\mu',\mu)) - \omega_\xi(\phi)\omega_\Xi(\Phi)F_{\xi,\Xi}(g(\mu',\mu))$, then $f_{\mu',\mu}$ is regular on $X^\sharp$. Since $f_{\mu',\mu}$ is zero on the dense subset U_0 of $X^\sharp$, we have $f_{\mu',\mu} = 0$. Namely, for any $(\xi,\Xi) \in X^\sharp$, the function $F_{\xi,\Xi}$ is the Shintani function of type (ξ,Ξ) with $F_{\xi,\Xi}(1_4) = 1$. □

5 Evaluation of a local zeta integral of Murase–Sugano type

In this section, we introduce a local zeta integral of Murase–Sugano type (*cf.* [MS]) and evaluate it as an application of our explicit formula of the Shintani function on G.

5.1 A local zeta integral of Murase–Sugano type.

In this subsection, we define a local zeta integral of Murase–Sugano type and state that the zeta integral represents the local spin L-factor of $\mathbf{GSp_4}$. Let G_1 be the subgroup of $GL_4(F)$ defined by

$$G_1 := \{g \in GL_4(F) |\ \det(g) \in (F^\times)^2\}$$

and P_{22} the maximal parabolic subgroup of G_1 given by

$$P_{22} = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ & * & & * \\ \hline * & * & * & * \\ & * & & * \end{array} \right) \in G_1 \right\}.$$

We note that $G \subset G_1$. We put

$$\kappa = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \in G_1.$$

Let $P_{22} = M_{22}N_{22}$ be the Levi decomposition of P_{22}, where

$$M_{22} = \left\{ \mathbf{m}_1(a,b) := \kappa \begin{pmatrix} a & \\ & b \end{pmatrix} \kappa^{-1} \ \middle|\ a, b \in GL_2(F), \det(ab) \in (F^\times)^2 \right\},$$

$$N_{22} = \left\{ \left(\begin{array}{cc|cc} 1 & * & & * \\ & 1 & & \\ \hline & * & 1 & * \\ & & & 1 \end{array} \right) \in G_1 \right\}.$$

We note that every $\mathbf{m}_1(a,b) \in M_{22}$ has a factorization

$$\mathbf{m}_1(a,b) = \kappa \begin{pmatrix} \alpha^{-1} \cdot a & \\ & b \end{pmatrix} \begin{pmatrix} \alpha \cdot 1_2 & \\ & 1_2 \end{pmatrix} \kappa^{-1}, \quad \alpha^2 = \frac{\det(ab)}{\det(b)^2} = \frac{\det(a)}{\det(b)}.$$

Namely, for any $m_1 \in M_{22}$ we have a factorization $m_1 = \beta(m_1)\mathrm{diag}(\alpha(m_1), 1, \alpha(m_1), 1)$ for some $(\beta(m_1), \alpha(m_1)) \in G_0 \times F^\times$. We note that such a factorization of m_1 is not unique. We set $K_1 := G_1 \cap GL_4(\mathfrak{o})$. Then every $g \in G_1$ has an Iwasawa decomposition

$$\begin{aligned} g &= m_1(g)n_1(g)k_1(g) \\ &= \beta(m_1(g))\mathrm{diag}(\alpha(m_1(g)), 1, \alpha(m_1(g)), 1)n_1(g)k_1(g) \end{aligned}$$

for some $(m_1(g), n_1(g), k_1(g)) \in M_{22} \times N_{22} \times K_1$. For every $g \in G_1$, we fix such a factorization and set $\beta(g) := \beta(m_1(g))$ and $\alpha(g) := \alpha(m_1(g))$. The following lemma is easily checked by direct calculation.

Lemma 5.1.1

$$P_{22} \cap K_1 = \kappa \begin{pmatrix} GL_2(\mathfrak{o}) & M_2(\mathfrak{o}) \\ & GL_2(\mathfrak{o}) \end{pmatrix} \kappa^{-1} \cap K_1.$$

Let $(\xi, \Xi) \in X^\sharp$. For any Shintani function $S \in \mathcal{S}(\xi, \Xi)$, we define a local zeta integral of Murase–Sugano type by

$$Z_{MS}(s; S) := \int_{G_0\backslash G} W(\beta(g)^{-1}g)|\alpha(g)|^s d\dot{g} \quad (s \in \mathbb{C}), \tag{5.1}$$

where $d\dot{g}$ is the right invariant measure of $G_0\backslash G$. Since a Shintani function $S \in \mathcal{S}(\xi, \Xi)$ can be regarded as a function on $K_0\backslash G/K$, it follows from Lemma 5.2.3, described in §5.2, that the value $S(\beta(g)^{-1}g)|\alpha(g)|^s$ is independent of a choice of an Iwasawa decomposition of $g \in G \subset G_1$. For $\chi = (\chi_1, \chi_2, \chi_3) \in (\mathbb{C}^\times)^3, s \in \mathbb{C}$, we set

$$L(\chi; s) := (1 - \chi_3 q^{-s})^{-1}(1 - \chi_1\chi_3 q^{-s})^{-1}(1 - \chi_2\chi_3 q^{-s})^{-1}(1 - \chi_1\chi_2\chi_3 q^{-s})^{-1}.$$

As an application of the explicit formula of Shintani functions, we shall prove the following theorem in §5.3.

Theorem 5.1.2 *Let $(\xi, \Xi) \in X^\sharp$. For the Shintani function $S \in \mathcal{S}(\xi, \Xi)$ such that $S(1_4) = 1$, the zeta integral (5.1) is absolutely convergent if*

$$\mathrm{Re}(s) > s_\Xi := \max\left\{ \log_q \|\Xi_3\|, \log_q \|\Xi_1\Xi_3\|, \log_q \|\Xi_2\Xi_3\|, \log_q \|\Xi_1\Xi_2\Xi_3\| \right\}.$$

Then the zeta integral (5.1) can be evaluated as

$$Z_{MS}(s; S) = \frac{L(\Xi; s)}{L(\xi^{-1}; s + 1/2)} \quad (\mathrm{Re}(s) > s_\Xi).$$

Remark 5.1.3 *Theorem 5.1.2 is generalization of [MS, Theorem 1.6] for the pair $(\mathbf{SO}_5(F_0), \mathbf{SO}_4(F_0))$ of split special orthogonal groups. Here F_0 is a non-archimedean local field of characteristic 0. While they proved their result without using the explicit formula of Shintani functions for $(\mathbf{SO}_5(F_0), \mathbf{SO}_4(F_0))$, we compute the local zeta integral (5.1) using that for (G, G_0).*

5.2 A double coset decomposition.

In this subsection, we prove the following theorem, which is necessary for our computation of the zeta integral $Z_{MS}(s; S)$.

Theorem 5.2.1 *We have*

$$G = \bigsqcup_{l \geq 0} G_0 a(l) K, \quad a(l) := g(\mathbf{0}, (l, l, l)).$$

Remark 5.2.2 *An Iwasawa decomposition of $a(l)$ is given by*

$$a(l) = \left(\begin{array}{cc|cc} -\varpi^l & & -\varpi^l - 1 & \\ & & & 1 \\ \hline -1 & & -1 & \\ & 1 & & \end{array}\right) \begin{pmatrix} \varpi^l & & & \\ & 1 & & \\ & & \varpi^l & \\ & & & 1 \end{pmatrix} \left(\begin{array}{cc|cc} 1 & \varpi^{-l} & & \\ & 1 & & \\ \hline & -\varpi^{-l} & 1 & -\varpi^{-l} \\ & & & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 1 & -1 & \\ & & -1 & 1 \\ \hline -1 & & 1 & \\ & \varpi^l & 1 & \varpi^l \end{array}\right).$$

We note that

$$\left(\begin{array}{cc|cc} -\varpi^l & & -\varpi^l - 1 & \\ & & & 1 \\ \hline -1 & & -1 & \\ & 1 & & \end{array}\right) \in K_0.$$

First we prove the following lemma.

Lemma 5.2.3 *Let $g \in G$. We assume that $g \in G_0 a(l) K$ for some $l \geq 0$. Then the following assertions hold :*

i) $|\alpha(g)| = q^{-l}$;

ii) $\beta(g)^{-1} g \in K_0 a(l) K$.

Proof. Let $g = ha(l)k$ for some $(h,k) \in G_0 \times K$. We note that $a(l) \in K_0 \mathrm{diag}(\varpi^l, 1, \varpi^l, 1) N_{22} K_1$ (see Remark 5.2.2). Hence we have

$$\beta(g)^{-1} g = \mathrm{diag}(\alpha(g), 1, \alpha(g), 1) n_1(g) k_1(g) \in \beta(g)^{-1} h K_0 \mathrm{diag}(\varpi^l, 1, \varpi^l, 1) N_{22} K_1.$$

In particular, $\beta(g)^{-1} h \in K_0$ and $\alpha(g) \in \varpi^l \mathfrak{o}^\times$ from Lemma 5.1.1. □

Next we show the following lemma to prove the disjointness of Theorem 5.2.1.

Lemma 5.2.4 *Let $g \in G$. If $|\alpha(g)| = q^{-l}$ and $g = ha(l')k$ for some $(h,k) \in G_0 \times K$, then $l = l'$.*

Proof. We have

$$\mathrm{diag}(\alpha(g), 1, \alpha(g), 1) n_1(g) k_1(g) \in K_0 a(l') K$$

by Lemma 5.2.3 (ii). By comparing the determinants of both sides, we have $q^{-2l} = |\alpha(g)|^2 = q^{-2l'}$. □

Proof of Theorem 5.2.1. By the Iwasawa decomposition of G, G can be written as

$$\begin{aligned}
G &= G_0 N K \\
&= G_0 K \cup G_0 \left\{ \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) \;\middle|\; x_0 \neq 0 \right\} K \\
&\quad \cup G_0 \left\{ \left(\begin{array}{cc|cc} 1 & & & x_2 \\ & 1 & x_2 & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \;\middle|\; x_2 \neq 0 \right\} K \cup G_0 \left\{ \left(\begin{array}{cc|cc} 1 & x_0 & & x_2 \\ & 1 & x_2 & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) \;\middle|\; x_0, x_2 \neq 0 \right\} K.
\end{aligned}$$

Since $w_2 = w_2' \in G_0 \cap K = K_0$, we have

$$G_0 \left(\begin{array}{cc|cc} 1 & x_0 & & x_2 \\ & 1 & x_2 & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) K = G_0 \left(\begin{array}{cc|cc} 1 & x_2 & & x_0 \\ & 1 & x_0 & \\ \hline & & 1 & \\ & & -x_2 & 1 \end{array} \right) K.$$

Hence we have

$$G = G_0 K \cup G_0 \left\{ \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) \;\middle|\; x_0 \neq 0 \right\} K \cup G_0 \left\{ \left(\begin{array}{cc|cc} 1 & x_0 & & x_2 \\ & 1 & x_2 & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array} \right) \;\middle|\; x_0, x_2 \neq 0 \right\} K$$

$$= G_0K \cup \bigcup_{k\in\mathbb{Z}} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^k & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\varpi^k & 1 \end{array}\right) K \cup \bigcup_{k,l\in\mathbb{Z}} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^k & & \varpi^l \\ & 1 & \varpi^l & \\ \hline & & 1 & \\ & & -\varpi^k & 1 \end{array}\right) K$$

$$= G_0K \cup \bigcup_{k>0} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K \cup \bigcup_{k,l>0} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \varpi^{-l} \\ & 1 & \varpi^{-l} & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K$$

$$= G_0K \cup \bigcup_{k>0} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K \cup \bigcup_{k\geq l>0} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \varpi^{-l} \\ & 1 & \varpi^{-l} & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K.$$

Since

$$G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & 1 \\ & 1 & 1 & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K = G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \\ & 1 & & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K,$$

we have

$$G = \bigcup_{k\geq l\geq 0} G_0 \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \varpi^{-l} \\ & 1 & \varpi^{-l} & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) K.$$

If $k \geq l \geq 0$, we have

$$a(k) = \left(\begin{array}{cc|cc} \varpi^k & & -1-\varpi^k & \\ & & & \varpi^k \\ \hline & & -1 & \\ & 1 & & \varpi^{k-l} \end{array}\right) \left(\begin{array}{cc|cc} 1 & \varpi^{-k} & & \varpi^{-l} \\ & 1 & \varpi^{-l} & \\ \hline & & 1 & \\ & & -\varpi^{-k} & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & 1 & -1 & \\ & -\varpi^{k-l} & -1 & 1 \\ \hline & & -1 & \\ & 1 & & \end{array}\right).$$

Hence we obtain

$$G = \bigcup_{l\geq 0} G_0 a(l) K.$$

From Remark 5.2.2 and Lemma 5.2.4, if $a(l') \in G_0 a(l) K$, then $l = l'$. This means that the above union is disjoint. We have completed the proof of Theorem 5.2.1.

□

5.3 Evaluation of the local zeta integral.

In this subsection, we evaluate the zeta integral $Z_{MS}(s;S)$ by using our explicit formula of the Shintani function. Theorem 5.2.1 yields that

$$\int_{G_0\backslash G} F(g)d\dot{g} = \sum_{l=0}^{\infty} F(a(l))v_l, \quad v_l := \mathrm{vol}(G_0 \cap a(l)Ka(l)^{-1}; dg')^{-1}$$

for any function $F : G_0\backslash G/K \to \mathbb{C}$. We note that the integrand $S(\beta(g)^{-1}g)|\alpha(g)|^s$ of the zeta integral is a function on $G_0\backslash G/K$. Hence we have

$$\begin{aligned} Z_{MS}(s;S) &= \sum_{l=0}^{\infty} S(\beta(a(l))^{-1}a(l))|\alpha(a(l))|^s v_l \\ &= \sum_{l=0}^{\infty} S(\beta(a(l))^{-1}a(l)) v_l q^{-ls}. \end{aligned}$$

Since $\beta(a(l))^{-1}a(l) \in K_0 a(l) K$, it is enough to compute the volume v_l and the value

$$S(\beta(a(l))^{-1}a(l)) = S(a(l)).$$

First we calculate the volume v_l. From the definition of the measure dg' of G_0, we have $v_0^{-1} = \mathrm{vol}(G_0 \cap \eta K \eta^{-1}; dg') = \mathrm{vol}(K_0; dg') = 1$. Let $l \geq 0$. We put

$$\widetilde{a}(l) = \left(\begin{array}{cc|cc} 1 & & -1 & \\ & & & 1 \\ \hline & & -1 & \\ & 1 & & \end{array}\right) a(l) \left(\begin{array}{cc|cc} 1 & & & -1 \\ & & & 1 \\ \hline & & -1 & \\ & 1 & -1 & \end{array}\right) = \left(\begin{array}{cc|cc} \varpi^l & 1 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -1 & \varpi^l \end{array}\right).$$

We note that $\widetilde{a}(l) \in K_0 a(l) K$. By the invariance of the Haar measure dg', we have

$$\begin{aligned} v_l^{-1} &= \mathrm{vol}(G_0 \cap \widetilde{a}(l) K \widetilde{a}(l)^{-1}; dg') \\ &= \mathrm{vol}(K_0^{(l)}; dg'), \end{aligned}$$

where $K_0^{(l)} := G_0 \cap \widetilde{a}(l) w_2 K w_2^{-1} \widetilde{a}(l)^{-1}$. Then $K_0^{(l)}$ is the group consisting of elements

$$h = (\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}) \tag{5.2}$$

of K_0 which satisfy the following four congruences:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \equiv \begin{pmatrix} a_2 & -b_2 \\ -c_2 & d_2 \end{pmatrix} \pmod{\mathfrak{p}^l} \tag{5.3}$$

We note that $K_0^{(0)} = K_0$ and $K_0^{(l)} \supset K_0^{(l+1)} (l \geq 0)$. In order to compute the volume v_l, it is enough to compute the index $[K_0^{(l)} : K_0^{(l+1)}]$ for all $l \geq 0$.

Proposition 5.3.1 *We have disjoint unions*

$$\begin{aligned} K_0 = & \bigsqcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} u & x \\ & u^{-1} \end{pmatrix}, 1_2) \sqcup \bigsqcup_{u,v \not\equiv 0 \bmod \mathfrak{p}} K_0^{(1)}(\begin{pmatrix} u & \\ v & u^{-1} \end{pmatrix}, 1_2) \\ & \sqcup \bigsqcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} x & u \\ -u^{-1} & \end{pmatrix}, 1_2) \sqcup \bigsqcup_{u,v \not\equiv 0 \bmod \mathfrak{p}} K_0^{(1)}(\begin{pmatrix} & u \\ -u^{-1} & v \end{pmatrix}, 1_2) \end{aligned}$$

$$\sqcup \bigsqcup_{\substack{u_1, u_2 \not\equiv 0 \bmod \mathfrak{p} \\ u_1 u_2 \not\equiv 1 \bmod \mathfrak{p} \\ v_1 \not\equiv 0 \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} u_1 & (u_1u_2-1)v_1^{-1} \\ v_1 & u_2 \end{pmatrix}, 1_2)$$

and

$$\begin{aligned} K_0^{(l)} = & \bigsqcup_{x, y \bmod \mathfrak{p}} K_0^{(l+1)}(\begin{pmatrix} 1+\varpi^l x & \varpi^l y \\ & (1+\varpi^l x)^{-1} \end{pmatrix}, 1_2) \\ & \sqcup \bigsqcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(l+1)}(\begin{pmatrix} 1+\varpi^l x & \\ \varpi^l u & (1+\varpi^l x)^{-1} \end{pmatrix}, 1_2) \\ & \sqcup \bigsqcup_{\substack{x \bmod \mathfrak{p} \\ u, v \not\equiv 0 \bmod \mathfrak{p}}} K_0^{(l+1)}(\begin{pmatrix} 1+\varpi^l x & \varpi^l u \\ \varpi^l v & \frac{1+\varpi^{2l}uv}{1+\varpi^l x} \end{pmatrix}, 1_2) \end{aligned}$$

for any $l > 0$.

Before proving Proposition 5.3.1, we calculate the volume v_l for all $l \geq 0$ and prove Theorem 5.1.2.

Corollary 5.3.2 *For $l \geq 0$, we have*

$$v_l = \begin{cases} 1 & (\text{if } l = 0), \\ q^{3l}(1-q^{-2}) & (\text{if } l > 0). \end{cases}$$

In particular, the generating function for the sequence $\{v_l\}_{l \geq 0}$ is given by

$$\sum_{l=0}^{\infty} v_l t^l = \frac{1-qt}{1-q^3 t}.$$

The domain of convergence of the above power series is $\|t\| < q^{-3}$.

Proof. Since

$$[K_0 : K_0^{(1)}] = q(q^2-1) = q^3(1-q^{-2}), \quad [K_0^{(l)} : K_0^{(l+1)}] = q^3 \quad (l > 0)$$

from Proposition 5.3.1, we obtain $v_l^{-1} = \mathrm{vol}(K_0^{(l)}; dg') = q^{-3l}(1-q^{-2})^{-1}$ for all $l > 0$. □

Remark 5.3.3 *Murase–Sugano proved Corollary 5.3.2 as a corollary of computation of their local zeta integral (see [MS, Lemma 1.12]).*

Proof of Theorem 5.1.2. Let $S \in \mathcal{S}(\xi, \Xi)$ be the Shintani function such that $S(1_4) = 1$. From Theorem 4.5.4, we have

$$S(a(l)) = \frac{(\Xi_1 \Xi_2 \Xi_3^2)^l}{(1-q^{-2})^2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi) \left((w\Xi)^{-1} \delta^{1/2} \right) (t((l,l,l)))$$

$$= \frac{q^{-3l}(\Xi_1\Xi_2\Xi_3^2)^l}{(1-q^{-2})^2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi)(w\Xi)^{-1}(t((l,l,l))).$$

Hence we have

$$\begin{aligned} Z_{MS}(s;S) &= \sum_{l=0}^{\infty} S(a(l))q^{-ls}v_l \\ &= (1-q^{-2})^{-2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi)\zeta_w(s), \end{aligned}$$

where

$$\zeta_w(s) = \sum_{l=0}^{\infty}(\Xi_1\Xi_2\Xi_3^2)^l(w\Xi)^{-1}(t((l,l,l)))q^{-l(s+3)}v_l$$

for any $w \in W$. From Corollary 5.3.2, we have

$$\zeta_w(s) = \begin{cases} \dfrac{1-q^{-s-2}\Xi_3}{1-q^{-s}\Xi_3} & (\text{if } w = 1_4, w_1), \\ \dfrac{1-q^{-s-2}\Xi_2\Xi_3}{1-q^{-s}\Xi_2\Xi_3} & (\text{if } w = w_2, w_1w_2), \\ \dfrac{1-q^{-s-2}\Xi_1\Xi_3}{1-q^{-s}\Xi_1\Xi_3} & (\text{if } w = w_2w_1, w_1w_2w_1), \\ \dfrac{1-q^{-s-2}\Xi_1\Xi_2\Xi_3}{1-q^{-s}\Xi_1\Xi_2\Xi_3} & (\text{if } w = w_2w_1w_2, w_\ell) \end{cases}$$

for $\mathrm{Re}(s) > s_\Xi$. Hence we obtain

$$\begin{aligned} Z_{MS}(s;S) &= (1-q^{-2})^{-2} \sum_{\substack{w \in W \\ w' \in W_0}} c_S(w'\xi, w\Xi)\zeta_w(s) \\ &= \frac{(1-q^{-s-1/2}\xi_1^{-1}\xi_3^{-1})(1-q^{-s-1/2}\xi_2^{-1}\xi_3^{-1})(1-q^{-s-1/2}\xi_1^{-1}\xi_2^{-1}\xi_3^{-1})(1-q^{-s-1/2}\xi_3^{-1})}{(1-q^{-s}\Xi_1\Xi_3)(1-q^{-s}\Xi_2\Xi_3)(1-q^{-s}\Xi_1\Xi_2\Xi_3)(1-q^{-s}\Xi_3)} \\ &= \frac{L(\Xi;s)}{L(\xi^{-1};s+1/2)} \end{aligned}$$

for $\mathrm{Re}(s) > s_\Xi$. □

Finally we prove Proposition 5.3.1.

Proof of Proposition 5.3.1. For $g \in GL_2(\mathfrak{o})$, we put

$$g^* = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} g \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1} \in GL_2(\mathfrak{o}).$$

We note that $(g^*)^{-1} = (g^{-1})^*$ and $(g^*)^* = g$. For any $l > 0$, every element of $K_0^{(l)}$ can be written in the form $(g + \varpi^l X, g^*)$ for some $(g, X) \in GL_2(\mathfrak{o}) \times M_2(\mathfrak{o})$. In particular, $(g, g^*) \in K_0^{(l)}$ for any $g \in GL_2(\mathfrak{o})$. Since for any $(k_1, k_2) \in K_0$ the equality

$$K_0^{(l)}(k_1,k_2) = K_0^{(l)}((k_2^{-1})^*, k_2^{-1})(k_1,k_2)$$

$$= K_0^{(l)}((k_2^{-1})^* k_1, 1_2)$$

holds, we have

$$K_0^{(l)} = \bigcup_{\substack{k \in SL_2(\mathfrak{o}) \\ k \equiv 1_2 \bmod \mathfrak{p}^l}} K_0^{(l+1)}(k, 1_2)$$

for any $l \geq 0$. Let $l \geq 0$. We take $k = (k_{ij}) \in SL_2(\mathfrak{o})$ such that $k \equiv 1_2 \bmod \mathfrak{p}^l$.

i) We handle the case where $k_{21} = 0$. We note that $k_{11}, k_{22} \in \mathfrak{o}^\times$. First we assume that $l > 0$. Since $k_{11} = k_{22}^{-1}$, we have

$$\begin{aligned}
K_0^{(l+1)}(k, 1_2) &= K_0^{(l+1)}(\begin{pmatrix} 1 & k_{11}k_{12} \\ & 1 \end{pmatrix} \begin{pmatrix} k_{11} & \\ & k_{11}^{-1} \end{pmatrix}, 1_2) \\
&= K_0^{(l+1)}(\begin{pmatrix} 1 & \varpi^l b \\ & 1 \end{pmatrix} \begin{pmatrix} k_{11} & \\ & k_{11}^{-1} \end{pmatrix}, 1_2) \quad (^\exists b \bmod \mathfrak{p}) \\
&= K_0^{(l+1)}(\begin{pmatrix} k_{11} & \\ & k_{11}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varpi^l b k_{11}^{-2} \\ & 1 \end{pmatrix}, 1_2) \\
&= K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l a & \\ & (1 + \varpi^l a)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varpi^l b k_{11}^{-2} \\ & 1 \end{pmatrix}, 1_2) \quad (^\exists a \bmod \mathfrak{p}).
\end{aligned}$$

We note that the last equality follows from $(1+\mathfrak{p}^l)/(1+\mathfrak{p}^{l+1}) \simeq \mathfrak{o}/\mathfrak{p}$. Hence there exist $a, b \bmod \mathfrak{p}$ such that

$$K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l a & \varpi^l b \\ & (1 + \varpi^l a)^{-1} \end{pmatrix}, 1_2).$$

Next we assume that $l = 0$. Then we have

$$\begin{aligned}
K_0^{(1)}(k, 1_2) &= K_0^{(1)}(\begin{pmatrix} 1 & k_{11}k_{12} \\ & 1 \end{pmatrix} \begin{pmatrix} k_{11} & \\ & k_{11}^{-1} \end{pmatrix}, 1_2) \\
&= K_0^{(1)}(\begin{pmatrix} 1 & \varpi^l b \\ & 1 \end{pmatrix} \begin{pmatrix} k_{11} & \\ & k_{11}^{-1} \end{pmatrix}, 1_2) \quad (^\exists b \bmod \mathfrak{p}) \\
&= K_0^{(1)}(\begin{pmatrix} k_{11} & \\ & k_{11}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varpi^l b k_{11}^{-2} \\ & 1 \end{pmatrix}, 1_2) \\
&= K_0^{(1)}(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varpi^l b k_{11}^{-2} \\ & 1 \end{pmatrix}, 1_2) \quad (^\exists a \not\equiv 0 \bmod \mathfrak{p}).
\end{aligned}$$

We note that the last equality follows from $\mathfrak{o}^\times/(1+\mathfrak{p}) \simeq (\mathfrak{o}/\mathfrak{p})^\times$. Hence there exist $a \not\equiv 0, b \bmod \mathfrak{p}$ such that

$$K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}(\begin{pmatrix} a & \varpi^l b \\ & a^{-1} \end{pmatrix}, 1_2).$$

ii) We handle the case where $k_{21} \neq 0$. Since $k_{21} \equiv 0 \bmod \mathfrak{p}^l$, there exist $u \in \mathfrak{o}^\times$ and $n \geq 0$ such that $k_{21} = \varpi^{l+n} u$. If $n \geq 1$, then

$$K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}(\begin{pmatrix} k_{11} & k_{12} \\ \varpi^{l+n} u & k_{22} \end{pmatrix}, 1_2)$$

$$= K_0^{(l+1)}(\begin{pmatrix} k_{11} & k_{12} \\ & k_{22} \end{pmatrix}, 1_2).$$

This case is reduced to the case (i). We may assume that $n = 0$. Then we have

$$\begin{aligned} K_0^{(l+1)}(k, 1_2) &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & k_{12} \\ \varpi^l u & k_{22} \end{pmatrix}, 1_2) \\ &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & k_{12} \\ \varpi^l v_1 & k_{22} \end{pmatrix}, 1_2) \quad (^\exists v_1 \not\equiv 0 \bmod \mathfrak{p}). \end{aligned}$$

We now consider the case (ii-i) $k_{12} = 0$ and the case (ii-ii) $k_{12} \neq 0$, separately.

ii-i) We assume that $k_{12} = 0$. If $l > 0$, we have

$$K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}(\begin{pmatrix} 1+\varpi^l a & \\ \varpi^l v_1' & (1+\varpi^l a)^{-1} \end{pmatrix}, 1_2) \quad (^\exists v_1' \not\equiv 0 \bmod \mathfrak{p}, a \bmod \mathfrak{p}).$$

If $l = 0$, we have

$$K_0^{(1)}(k, 1_2) = K_0^{(1)}(\begin{pmatrix} a & \\ \varpi^l v_1' & a^{-1} \end{pmatrix}, 1_2) \quad (^\exists v_1', a \not\equiv 0 \bmod \mathfrak{p}).$$

We note that $\mathfrak{o}^\times/(1+\mathfrak{p}) \simeq (\mathfrak{o}/\mathfrak{p})^\times$.

ii-ii) We assume that $k_{12} \neq 0$. Since $k_{12} \equiv 0 \bmod \mathfrak{p}^l$, there exist $u \in \mathfrak{o}^\times$ and $n \geq 0$ such that $k_{12} = \varpi^{l+n} u$, that is,

$$K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}(\begin{pmatrix} k_{11} & \varpi^{l+n} u \\ \varpi^l v_1 & k_{22} \end{pmatrix}, 1_2).$$

If $n \geq 1$, we have

$$K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}(\begin{pmatrix} k_{22}^{-1} & \\ \varpi^l v_1 & k_{22} \end{pmatrix}, 1_2).$$

This case is reduced to the case (ii-i). We may assume that $n = 0$. Then we have

$$\begin{aligned} K_0^{(l+1)}(k, 1_2) &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & \varpi^l u \\ \varpi^l v_1 & k_{22} \end{pmatrix}, 1_2) \\ &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & \varpi^l v_2 \\ \varpi^l v_1 & k_{22} \end{pmatrix}, 1_2) \quad (^\exists v_2 \not\equiv 0 \bmod \mathfrak{p}). \end{aligned}$$

Let $l > 0$. Then, since $k_{11}, k_{22} \equiv 1 \bmod \mathfrak{p}^l$, we have

$$\begin{aligned} K_0^{(l+1)}(k, 1_2) &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & \varpi^l v_2 \\ \varpi^l v_1 & k_{22} \end{pmatrix}, 1_2) \\ &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & \\ & k_{22} \end{pmatrix} \begin{pmatrix} 1 & \varpi^l v_2 k_{11}^{-1} \\ \varpi^l v_1 k_{22}^{-1} & 1 \end{pmatrix}, 1_2) \\ &= K_0^{(l+1)}(\begin{pmatrix} k_{11} & \\ & k_{22} \end{pmatrix} \begin{pmatrix} 1 & \varpi^l v_2' \\ \varpi^l v_1' & 1 \end{pmatrix}, 1_2) \quad (^\exists v_1', v_2' \not\equiv 0 \bmod \mathfrak{p}) \end{aligned}$$

$$= K_0^{(l+1)}(\begin{pmatrix} 1+\varpi^l a & \\ & 1+\varpi^l d \end{pmatrix}\begin{pmatrix} 1 & \varpi^l v_2' \\ \varpi^l v_1' & 1 \end{pmatrix}, 1_2) \quad ({}^\exists a, d \bmod \mathfrak{p}).$$

Let $l=0$. We note that

$$k = \begin{pmatrix} k_{11} & v_2 \\ v_1 & k_{22} \end{pmatrix} = \begin{pmatrix} v_2 & -k_{11} \\ k_{22} & -v_1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

If $k_{11}k_{22}=0$, we have

$$\begin{aligned} &K_0^{(1)}(k, 1_2) \\ &= \begin{cases} K_0^{(l+1)}(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1_2) \quad ({}^\exists a \not\equiv 0 \bmod \mathfrak{p}, b \bmod \mathfrak{p}) \\ \hfill (\text{if } k_{22}=0), \\ K_0^{(1)}(\begin{pmatrix} a & \\ v_1' & a^{-1} \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1_2) \quad ({}^\exists v_1', a \not\equiv 0 \bmod \mathfrak{p}) \\ \hfill (\text{if } k_{11}=0 \text{ and } k_{22}\neq 0), \end{cases} \\ &= \begin{cases} K_0^{(l+1)}(\begin{pmatrix} b' & a \\ -a^{-1} & \end{pmatrix}, 1_2) \quad ({}^\exists b' \bmod \mathfrak{p}) \qquad (\text{if } k_{22}=0), \\ K_0^{(1)}(\begin{pmatrix} & a \\ -a^{-1} & v_1' \end{pmatrix}, 1_2) \qquad (\text{if } k_{11}=0 \text{ and } k_{22}\neq 0). \end{cases} \end{aligned}$$

There remains only the case where $k_{11}k_{22}\neq 0$. Then there exist $m,n\geq 0, u, u' \in \mathfrak{o}^\times$ such that

$$K_0^{(1)}(k, 1_2) = K_0^{(1)}(\begin{pmatrix} v_2 & -\varpi^m u \\ \varpi^n u' & -v_1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1_2).$$

If $m>0$ (resp. $n>0$), the case is reduced to $k_{11}=0$ (resp. $k_{22}=0$). We may assume that $m=n=0$. Then we have

$$\begin{aligned} K_0^{(1)}(k, 1_2) &= K_0^{(1)}(\begin{pmatrix} v_2 & -u \\ u' & -v_1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, 1_2) \\ &= K_0^{(1)}(\begin{pmatrix} u & \\ & u' \end{pmatrix}\begin{pmatrix} 1 & v_2 u^{-1} \\ v_1 u'^{-1} & 1 \end{pmatrix}, 1_2) \\ &= K_0^{(1)}\begin{pmatrix} u_1 & v_2' \\ v_1' & u_2 \end{pmatrix}, 1_2) \quad ({}^\exists u_1, u_2, v_1', v_2' \not\equiv 0 \bmod \mathfrak{p}). \end{aligned}$$

We note that $u_1u_2 - v_1'v_2' = 1$.

Therefore the results for the cases (i) and (ii) yield that

$$K_0 = \bigcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} u & x \\ & u^{-1} \end{pmatrix}, 1_2) \cup \bigcup_{u, v \not\equiv 0 \bmod \mathfrak{p}} K_0^{(1)}(\begin{pmatrix} u & \\ v & u^{-1} \end{pmatrix}, 1_2)$$

$$
\begin{aligned}
&\cup \bigcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} x & u \\ -u^{-1} & \end{pmatrix}, 1_2) \cup \bigcup_{u, v \not\equiv 0 \bmod \mathfrak{p}} K_0^{(1)}(\begin{pmatrix} & u \\ -u^{-1} & v \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{u_1, u_2 \not\equiv 0 \bmod \mathfrak{p} \\ v_1, v_2 \not\equiv 0 \bmod \mathfrak{p} \\ u_1 u_2 - v_1 v_2 = 1}} K_0^{(1)}(\begin{pmatrix} u_1 & v_2 \\ v_1 & u_2 \end{pmatrix}, 1_2) \\
&= \bigcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} u & x \\ & u^{-1} \end{pmatrix}, 1_2) \cup \bigcup_{u, v \not\equiv 0 \bmod \mathfrak{p}} K_0^{(1)}(\begin{pmatrix} u & \\ v & u^{-1} \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} x & u \\ -u^{-1} & \end{pmatrix}, 1_2) \cup \bigcup_{u, v \not\equiv 0 \bmod \mathfrak{p}} K_0^{(1)}(\begin{pmatrix} & u \\ -u^{-1} & v \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{u_1, u_2 \not\equiv 0 \bmod \mathfrak{p} \\ u_1 u_2 \not\equiv 1 \bmod \mathfrak{p} \\ v_1 \not\equiv 0 \bmod \mathfrak{p}}} K_0^{(1)}(\begin{pmatrix} u_1 & (u_1 u_2 - 1) v_1^{-1} \\ v_1 & u_2 \end{pmatrix}, 1_2)
\end{aligned}
$$

and, for $l > 0$,

$$
\begin{aligned}
K_0^{(l)} &= \bigcup_{x, y \bmod \mathfrak{p}} K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l x & \varpi^l y \\ & (1 + \varpi^l x)^{-1} \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l x & \\ \varpi^l u & (1 + \varpi^l x)^{-1} \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{x, y \bmod \mathfrak{p} \\ u, v \not\equiv 0 \bmod \mathfrak{p} \\ x + y + \varpi^l (xy - uv) = 0}} K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l x & \varpi^l u \\ \varpi^l v & 1 + \varpi^l y \end{pmatrix}, 1_2) \\
&= \bigcup_{x, y \bmod \mathfrak{p}} K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l x & \varpi^l y \\ & (1 + \varpi^l x)^{-1} \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{u \not\equiv 0 \bmod \mathfrak{p} \\ x \bmod \mathfrak{p}}} K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l x & \\ \varpi^l u & (1 + \varpi^l x)^{-1} \end{pmatrix}, 1_2) \\
&\cup \bigcup_{\substack{x \bmod \mathfrak{p} \\ u, v \not\equiv 0 \bmod \mathfrak{p}}} K_0^{(l+1)}(\begin{pmatrix} 1 + \varpi^l x & \varpi^l u \\ \varpi^l v & \frac{1 + \varpi^{2l} uv}{1 + \varpi^l x} \end{pmatrix}, 1_2).
\end{aligned}
$$

We can easily check that these union is disjoint. $\square$

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