The Unramified Shintani Functions for the Reductive Symmetric Pair (GSp4, GL2×GL1 GL2)

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The Unramified Shintani Functions for the Reductive Symmetric Pair 
\( (\text{GSp}_4, \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2) \)

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Abstract

Let \( F \) be a non-archimedean local field of arbitrary characteristic. We give an explicit formula of the unramified Shintani functions on \( \text{GSp}_4(F) \). As an application, we evaluate a local zeta integral of Murase–Sugano type, which turns out to be the spin \( L \)-factor of \( \text{GSp}_4 \).

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of Shintani functions. It follows from the definition that a Shintani function $S \in \mathcal{S}(\xi, \Xi)$ is determined by the values on $K_0 \backslash G/K$. By the Cartan type decomposition (Theorem 3.2.1), it is enough to know the values $S(t(\lambda')\eta(\lambda))$ for all $(\lambda', \lambda) = ((\lambda_1', \lambda_2', \lambda_1), (\lambda_1, \lambda_2, \lambda_3)) \in \Lambda_0^{++} \times \Lambda^+$, where

\[
\Lambda^+ = \{ \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}_{>0}^3 | \lambda_1 \geq 2\lambda_2 \geq 3, \lambda_3 \geq 3 \},
\]

\[
\Lambda_0^{++} = \{ \lambda' = (\lambda_1', \lambda_2', \lambda_1) \in \mathbb{Z}_+^3 | \lambda_1' \geq 0, 2\lambda_2' \geq 3 \},
\]

\[
t((\lambda_1, \lambda_2, \lambda_3)) = \begin{pmatrix} \varpi^{\lambda_1} \\ \varpi^{\lambda_2} \\ \varpi^{\lambda_3-\lambda_1} \\ \varpi^{\lambda_1-\lambda_1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.
\]

For each $(\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)$, we set

\[
c_S(\xi, \Xi) = \frac{b(\xi, \Xi)}{d(\xi)d(\Xi)},
\]

where

\[
d(\Xi) = (1 - \Xi_1 \Xi_2)(1 - \Xi_1 \Xi_2^{-1})(1 - \Xi_1)(1 - \Xi_2), \quad d'(\xi) = (1 - \xi_1)(1 - \xi_2),
\]

\[
b(\xi, \Xi) = (1 - q^{-1/2} \Xi_1 \Xi_3 \xi_1 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_3 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_3 \xi_1 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_3 \xi_1 \xi_2 \xi_3) \\
\times (1 - q^{-1/2} \Xi_1 \Xi_2 \xi_3 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \xi_3 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \xi_3 \xi_3).
\]

Then our main result is as follows.

**Theorem A (Theorem 4.5.4)** Let $(\xi, \Xi)$ be any element of $X_{nr}(T_0) \times X_{nr}(T)$. Then we have

\[
\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) = \begin{cases} 1 & \text{if } (\xi \Xi)|_Z \equiv 1, \\ 0 & \text{otherwise}. \end{cases}
\]

If $(\xi \Xi)|_Z \equiv 1$, for any nonzero element $S \in \mathcal{S}(\xi, \Xi)$ we have $S(1_4) \neq 0$, and the Shintani function $W_{\xi, \Xi} \in \mathcal{S}(\xi, \Xi)$ with $W_{\xi, \Xi}(1_4) = 1$ is given by

\[
W_{\xi, \Xi}(t(\lambda')\eta(\lambda)) = \frac{(\Xi_1 \Xi_2 \Xi_3 \Xi_3)^{\lambda_1}}{(1 - q^{-1/2})^2} \sum_{w \in W} \sum_{w' \in W_0} c_S(w' \xi, w \Xi) \left( (w \Xi)^{-1/2} t(\lambda) \right) \left( (w' \xi)^{-1} \delta_0^{1/2} \right) (t(\lambda'))
\]

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$. Here $1_4$ is the identity element of $G$, $Z$ is the center of $G$ and $W$ (resp. $W_0$) is the Weyl group of $(G, T)$ (resp. $(G_0, T_0)$).

We shall explain the outline of our proof of Theorem A by describing the contents of each section. In Section 2, we introduce basic notation and objects which will be used throughout this thesis. In Section 3, we recall the definition of the Shintani functions on $\text{GSp}_4(F)$ and prove the Cartan type decomposition of $\text{GSp}_4(F)$. Also, we prove the uniqueness of the Shintani functions by using a system of difference equations satisfied by Shintani functions. In Section 4, we prove the explicit formula of the Shintani function. In §4.1, we construct a nonzero Shintani functional $\Omega_{\xi, \Xi}$ for any element $(\xi, \Xi)$ of a certain domain $U_\mathcal{E}^+ \subset (\mathbb{C}^*)^5$ by using relative invariants, which are introduced in
In §4.4, we give a meromorphic continuation of $\Omega_{\xi, \Xi}$. This is the most technical part of the proof of Theorem A. In order to do that, instead of following the method of [KMS] closely, we apply Bernstein’s rationality theorem (Theorem 4.4.4) combining with a simple measure theoretic argument (Proposition 4.3.17). The necessary verifications for applying this theorem is done in §4.2 (Rank one calculation) and §4.3 (Uniqueness of Shintani functionals). In §4.5, we prove Theorem A by using the method employed to prove that of the unramified Whittaker functions in [CS] (see also [KMS]). The meromorphic continuation of $\Omega_{\xi, \Xi}$ enables us to get a $(W_0 \times W)$-invariance of Shintani functions. In Section 5, we evaluate a local zeta integral of Murase–Sugano type, which turns out to be the spin $L$-factor of $\text{GSp}_4$, as an application of our explicit formula.

We note that besides the paper [KMS] mentioned above there are several papers studying (Whittaker–)Shintani functions on $\text{GSp}_4(F)$ or related groups. For example, an explicit formula of Whittaker–Shintani functions for $(\text{Sp}_{2n}(F), \text{Jacobi group})$ was given by Murase [M] for $n = 2$. Later Murase’s result was generalized to any $n$ by Shen [S]. Also, Bump–Friedberg–Furusawa [BFF] proved an explicit formula of Bessel functions on $\text{GSp}_4(F)$ and Hironaka [H2] proved that of Shintani functions for $(\text{Sp}_4(F), \text{SL}_2(F) \times \text{SL}_2(F))$.

The main results in this thesis have been announced in [G1] and the contents from Section 2 to Section 4 are going to be published as a paper [G2].

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2 Preliminaries

We denote by $\mathbb{Z}$ and $\mathbb{C}$ the ring of rational integers and the complex number field, respectively. For $z \in \mathbb{C}$, we denote by $|z|$ the absolute value of $z$. Throughout this thesis $F$ is a non-archimedean local field of arbitrary characteristic. We denote by $\mathfrak{O} = \mathfrak{o}_F$ the ring of integers of $F$, and we let $p$ be the maximal ideal of $\mathfrak{O}$. Let $q$ be the number of elements of $\mathfrak{o}/p$. Once and for all, we fix a generator $\varpi$ of $p$. We normalize the $p$-adic absolute value $|x|$ of $x \in F$ so that $|\varpi| = q^{-1}$. For an affine algebraic group $H$ over $F$, we denote by $H$ the locally compact group $H(F)$. We denote by $1_n$ the identity matrix of size $n$.

2.1 Unramified principal series representations of reductive groups.

In this subsection, we recall properties of an unramified principal series representation of a connected reductive algebraic group. Let $H$ be a connected reductive algebraic group over $F$. For simplicity, we assume that $H$ is split over $F$. Let $T_H$ be a maximal split torus of $H$ and $P_H$ a minimal parabolic subgroup of $H$ containing $T_H$. Then we have the Levi decomposition $P_H = T_H N_H$. Here $N_H$ is the unipotent radical of $P_H$. We denote by $\Sigma = \Sigma(H, T_H)$ the root system of $(H, T_H)$, by $\Sigma^+ = \Sigma^+(H, T_H)$ the set of positive roots corresponding to $P_H$ and by $\Delta = \Delta(H, T_H, P_H)$ the set of simple roots determined by $\Sigma^+$. We denote by $\alpha^\vee$ the coroot corresponding to $\alpha \in \Sigma$. We set $a_\alpha := \alpha^\vee(\varpi)$ for $\alpha \in \Sigma$. Since $H$ is split, $H$ is defined over $\mathfrak{O}$ (see [T, 1.10], for example). Also, since $P_H, T_H$ and $N_H$ are split, $P_H, T_H$ and $N_H$ are defined over $\mathfrak{O}$. We set $K_H = H(\mathfrak{O})$. Then $K_H$ is a maximal compact
subgroup of $H = H(F)$. We have the Iwasawa decomposition $H = P_H K_H$. Let $B_H \subset K_H$ be the Iwahori subgroup corresponding to $\Sigma^+$, that is,

$$B_H = (\text{the inverse image of } P_H W_P H o/p) \subset H o/p) \text{ via the natural map } K_H \twoheadrightarrow H o/p).$$

Let $W_H$ be the Weyl group of $(H, T_H)$. Then we have the following three Bruhat type decompositions:

$$H = P_H W_H P_H, \quad H = P_H W_H B_H, \quad K_H = B_H W_H B_H.$$

We denote by $\ell(w)$ the length of $w \in W_H$ corresponding to $\Sigma^+$ and by $w_l$ the longest element of $W_H$. We denote by $w_0 \in W_H$ the reflection corresponding to $\alpha \in \Sigma$.

We fix an isomorphism $t : (GL_1)^{\dim(T_H)} \cong T_H, (t_1, \ldots, t_{\dim(T_H)}) \mapsto t(t_1, \ldots, t_{\dim(T_H)})$. Let $X^*(T_H)$ be the $\mathbb{Z}$-module consisting of algebraic homomorphisms $T_H \rightarrow GL_1$ and $X_*(T_H)$ the $\mathbb{Z}$-module consisting of algebraic homomorphisms $GL_1 \rightarrow T_H$. Then we have the canonical pairing $\langle \cdot, \cdot \rangle : X^*(T_H) \times X_*(T_H) \rightarrow \mathbb{Z}$ given by composition. Let $\{e_i\}_{i=1}^{\dim(T_H)}$ be the standard basis of $X^*(T_H)$ given by $e_i(t(t_1, \ldots, t_{\dim(T_H)})) = t_i$ ($i = 1, \ldots, \dim(T_H)$) and $\{d_i\}_{i=1}^{\dim(T_H)}$ the basis of $X_*(T_H)$ dual to $\{e_i\}_{i=1}^{\dim(T_H)}$ with respect to the canonical pairing. For $\sigma, \tau \in X^*(T_H)$, we write $\sigma \geq \tau$ if $\sigma - \tau$ is a linear combination of positive roots with nonnegative coefficients.

A character of the group $T_H$ is called unramified if it is trivial on $T_H \cap K_H$. Let $X_{nr}(T_H)$ be the group of unramified characters of $T_H$. Then the modulus character $\delta_{P_H}$ of $P_H$ is an element of $X_{nr}(T_H)$. If we denote by $X_{nr}(F^\times)$ the group of unramified characters of $F^\times$, we often identify $X_{nr}(T_H)$ with $X_{nr}(F^\times)^{\dim(T_H)}$ via

$$X_{nr}(T_H) \rightarrow X_{nr}(F^\times)^{\dim(T_H)}, \chi \mapsto (\chi_1, \ldots, \chi_\dim(T_H)) := (\chi \circ d_1, \ldots, \chi \circ d_\dim(T_H))$$

or $(\mathbb{C}^\times)^{\dim(T_H)}$ via

$$X_{nr}(T_H) \rightarrow (\mathbb{C}^\times)^{\dim(T_H)}, \chi \mapsto (\chi_1(\varpi), \ldots, \chi_\dim(T_H)(\varpi)).$$

**Remark 2.1.1** Via the above identify $X_{nr}(T_H) \cong (\mathbb{C}^\times)^{\dim(T_H)}$, we often regard $X_{nr}(T_H)$ as a complex manifold or an affine algebraic variety.

For $\chi \in X_{nr}(T_H)$, we denote by $i_H^{\tilde{P}_H}(\chi)$ the normalized unramified principal series representation of $H$. The standard model of this representation is the vector space consisting of locally constant functions $f \in C^\infty(H)$ which satisfy $f(px) = (\chi^f \delta_{P_H}^f)(p) f(x)$ $(p, x \in P_H \times H)$. The group $H$ acts on $i_H^{\tilde{P}_H}(\chi)$ by the right translation $R = R_X$, where $[R(h)f](x) = f(xh)$. We note that $i_H^{\tilde{P}_H}(\chi)$ is an admissible representation of $H$. Let $P_\chi$ be the intertwining operator $C^\infty_c(H) \rightarrow i_H^{\tilde{P}_H}(\chi)$ defined by

$$P_\chi(f)(x) := \int_{P_H} (\chi^{-1} \delta_{P_H}^f)(p) f(px) d\mu_p \quad (\chi \in C^\infty_c(H)).$$

Here $d\mu_p$ is the left invariant measure of $P_H$ with $\text{vol}(P_H \cap K_H; d\mu_p) = 1$. It is well-known that $P_\chi$ is surjective. We set $\phi_{K_H, X} := P_\chi(\text{ch}_{K_H})$. Here $\text{ch}_A$ is the characteristic function of a subset $A \subset H$. Then we have $i_H^{\tilde{P}_H}(\chi)^{K_H} \cong \mathbb{C} \phi_{K_H, X}$. The restriction of $f \in i_H^{\tilde{P}_H}(\chi)$ to $K_H$ induces an isomorphism $i_H^{\tilde{P}_H}(\chi) \cong C^\infty(P_H \cap K_H \setminus K_H)$ as a $\mathbb{C}$-vector space. Indeed, its inverse map $C^\infty(P_H \cap K_H \setminus K_H) \ni f \mapsto f_\chi \in i_H^{\tilde{P}_H}(\chi)$ is given by

$$f_\chi(pk) := (\chi \delta_{P_H}^{f(k)})(p) f(k) \quad (\chi \in C^\infty_c(H), p, k \in P_H \times K_H).$$
via the Iwasawa decomposition. We denote by $R_\chi$ a group homomorphism $H \to GL(C^\infty(P_H \cap K_H \backslash K_H))$ so that the diagram
\[
i^H_P(\chi) \xrightarrow{\text{res}} C^\infty(P_H \cap K_H \backslash K_H)
\]
\[
\begin{array}{ccc}
R(h) \\
\downarrow \\
i^H_P(\chi) \xrightarrow{\text{res}} C^\infty(P_H \cap K_H \backslash K_H)
\end{array}
\]
commutes for all $h \in H$.

Let $\mathcal{H}(H, K_H)$ be the Hecke algebra of $(H, K_H)$ over $\mathbb{C}$, that is,
\[
\mathcal{H}(H, K_H) = \{ \varphi \in C_c(H) \mid \varphi(k_1 x k_2) = \varphi(x) \; (\forall x \in H, \forall k_1, k_2 \in K_H) \}
\]
with the multiplication given by convolution
\[
(\varphi_1 \ast \varphi_2)(x) = \int_H \varphi_1(x h^{-1} \varphi_2(h) dh \quad (\forall \varphi_1, \varphi_2 \in \mathcal{H}(H, K_H)).
\]
Here $dh$ is the Haar measure of $H$ with $\text{vol}(K_H; dh) = 1$. We note that the identity element of $\mathcal{H}(H, K_H)$ is $\text{ch}_{K_H}$ for the multiplication. For an unramified representation $(\pi, V)$ of $H$, the Hecke algebra $\mathcal{H}(H, K_H)$ acts on $V^{K_H}$ by
\[
\pi(\varphi)v := \int_H \varphi(h)\pi(h)v dh \quad (\forall v \in V^{K_H}, \forall \varphi \in \mathcal{H}(H, K_H)).
\]
In particular, $\mathcal{H}(H, K_H)$ acts on $i^H_P(\chi)^{K_H}$. Since $i^H_P(\chi)^{K_H} = \mathbb{C}\phi_{K_H, \chi}$, there exists a $\mathbb{C}$-algebra homomorphism $\omega_\chi : \mathcal{H}(H, K_H) \to \mathbb{C}$ such that
\[
R(\varphi)\phi_{K_H, \chi} = \omega_\chi(\varphi)\phi_{K_H, \chi} \quad (\forall \varphi \in \mathcal{H}(H, K_H)).
\]

We recall the Satake isomorphism using the above notation. Let $\mathbb{C}[T_H/T_H \cap K_H]$ be the group algebra of $T_H/T_H \cap K_H$. Since a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[T_H/T_H \cap K_H] \to \mathbb{C}$ is determined by the image of the generator $d_1(\pi), \ldots, d_{\dim(T_H)}(\pi)$ of $\mathbb{C}[T_H/T_H \cap K_H]$, we have $\text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[T_H/T_H \cap K_H], \mathbb{C}) \simeq (\mathbb{C}^\times)^{\dim T_H} \simeq X_{nr}(T_H)$. The Weyl group $W_H$ acts on $T_H$ by $w \cdot t := wtw^{-1}(w \in W_H, t \in T_H)$. The action is extended linearly to an action of $W_H$ on $\mathbb{C}[T_H/T_H \cap K_H]$.

**Theorem 2.1.2** There exists a unique $\mathbb{C}$-algebra homomorphism $\omega : \mathcal{H}(H, K_H) \to \mathbb{C}[T_H/T_H \cap K_H]$ with the following properties:

i) $\omega : \mathcal{H}(H, K_H) \isom \mathbb{C}[T_H/T_H \cap K_H]^{W_H} := \{ f \in \mathbb{C}[T_H/T_H \cap K_H] \mid w \cdot f = f(\omega w \in W_H) \}$;

ii) For all $\chi \in X_{nr}(T_H)$, $\chi \circ \omega = \omega_\chi$.

For two unramified characters $\chi, \chi' \in X_{nr}(T_H)$, there exists $w \in W_H$ such that $\chi = \omega \chi'$ if and only if $\omega_\chi = \omega_{\chi'}$. Here $W_H$ acts on $X_{nr}(T_H)$ by $(w \chi)(t) := \chi(w^{-1}t)(t \in T_H, w \in W_H, \chi \in X_{nr}(T_H))$. Therefore we have an isomorphism
\[
X_{nr}(T_H)/W_H \isom \text{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(H, K_H), \mathbb{C}), \quad \chi \mapsto \omega_\chi.
\]

An unramified character $\chi$ of $T_H$ is called regular if $w \chi \neq \chi$ for all $w \in W_H - \{1\}$. We denote by $X_{nr}^{\text{reg}}(T_H)$ the set of regular characters in $X_{nr}(T_H)$. We assume $\chi$ to be a regular character until the end of this subsection. For any $w \in W_H$, we set $\phi_w := P_\chi(ch_{B_HwB_H}) \in i^H_P(\chi)$. Then $\{\phi_w\}_{w \in W_H}$ is a basis of $i^H_P(\chi)^{B_H}$. It is well-known that there exists a set $\{ T_{w, \chi} : i^H_P(\chi) \to i^H_P(\chi) \}_{w \in W_H, \chi \in X_{nr}^{\text{reg}}(T_H)}$ of intertwining operators such that
The unramified Shintani functions

i) $T_{1,X} = \text{id}$;

ii) $T_{w,X}(\phi_{K_{H},X}) = c_w(\chi)\phi_{K_{H},w_{\chi}}$;

iii) If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then $T_{w_1w_2,X} = T_{w_1,X} \circ T_{w_2,X}$ $(\forall w_1, w_2 \in W_H)$;

iv) $T_{w_{\alpha}}(\phi_{w_{\alpha}w} + \phi_{w}) = c_{\alpha}(\chi)(\phi_{w_{\alpha}w} + \phi_{w})$ $(\forall \alpha \in \Delta)$.

Here

$$c_{\alpha}(\chi) := \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}, \quad c_w(\chi) := \prod_{\alpha > 0, w_{\alpha} < 0} c_{\alpha}(\chi)$$

for $\alpha \in \Sigma^+$. We call $c_\alpha$ a c-function for $\alpha \in \Sigma^+$. See [C2, §3] and [C1, §6.4] for more detail.

**Proposition 2.1.3** ([C2, §3]. See also [KMS, 1.10]) There is another basis $\{g_w\}_{w \in W_H}$ of $T^*_H(\chi)^{B_H}$ such that

i) $R(\text{ch}_{B_{H}^{-1}B_{H}})g_{w} = \text{vol}(B_{H}tB_{H})\left((w_{\chi})^{-1}\delta_{t}^{1/2}\right)(t)g_{w}$ $(\forall t \in T^+_H)$;

ii) $g_1 = \phi_1$;

iii) $\phi_{K_{H}X} = q^{\ell(w_{\alpha})}\sum_{w \in W_H} \bar{c}_{w}(\chi)g_w$,

where $T^+_H = \{t \in T_H \mid |\alpha(t)| \leq 1(\forall \alpha \in \Sigma^+)\}$ and $\bar{c}_{w}(\chi) := \prod_{\alpha > 0, w_{\alpha} < 0} c_{\alpha}(\chi)$.

The two bases $\{\phi_w\}_{w \in W_H}$ and $\{g_w\}_{w \in W_H}$ play an important role in our proof of an explicit formula of Shintani functions (see §4.2 and §4.5).

### 2.2 Basic objects

Let $G$ be an affine algebraic group over $F$ defined by

$$G = \text{GSp}_4 = \{g \in \text{GL}_4 \mid ^tJg = \nu(g)J, \nu(g) \in \text{GL}_1\}, \quad J = \begin{pmatrix} & & & 12 \\ & & 1 \end{pmatrix}$$

and $P$ its standard minimal parabolic subgroup defined by

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \in G \right\}.$$

We note that $\nu : G \rightarrow \text{GL}_1$ is a homomorphism of algebraic groups, which is called the similitude character of $G$. Let $P = TN$ be the Levi decomposition of $P$, where $T$ is the maximal (split) torus of $G$ defined by

$$T = \{t(t_1, t_2, t_3) := \text{diag}(t_1, t_2, t_3^{-1}, t_3^{-1}) \mid t_1, t_2, t_3 \in \text{GL}_1\}$$

and $N$ is the unipotent radical of $P$. Then $X^*(T) = \text{Hom}(T, \text{GL}_1)$ and $X_*(T) = \text{Hom}(\text{GL}_1, T)$ are $\mathbb{Z}$-modules of rank three. Let $\{e_i\}_{i=1}^3$ be the standard basis of $X^*(T)$ given by $e_i(t(t_1, t_2, t_3)) = t_i$ and
Let $\Sigma \subset X^*(T)$ be the root system of $(G, T)$ and $\Sigma^+$ the set of positive roots with respect to $P$. Then we have $\Sigma^+ = \{e_1 - e_2, e_1 + e_2 - e_3, 2e_1 - e_3, 2e_2 - e_3\}$ and the set of simple roots is given by $\Delta = \{\alpha_1 := e_1 - e_2, \alpha_2 := 2e_2 - e_3\}$. The coroot of $\alpha \in \Sigma^+$ is given by

$$\alpha^\vee = \begin{cases} d_1 - d_2 & (\text{if } \alpha = \alpha_1 = e_1 - e_2), \\ d_1 + d_2 & (\text{if } \alpha = e_1 + e_2 - e_3), \\ d_1 & (\text{if } \alpha = 2e_1 - e_3), \\ d_2 & (\text{if } \alpha = 2e_2 - e_3). \end{cases}$$

We define homomorphisms $x_\alpha : G_\alpha \to G(\alpha \in \Sigma)$ by

$$x_{e_1 - e_2}(t) = t x_{-e_1 + e_2}(t) = \begin{pmatrix} 1 & t \\ 1 & 1 \\ -t & 1 \end{pmatrix}, \quad x_{e_1 + e_2 - e_3}(t) = t x_{-e_1 - e_2 + e_3}(t) = \begin{pmatrix} 1 & t \\ 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$x_{e_2 - e_1 - e_3}(t) = t x_{-e_2 + e_1 + e_3}(t) = \begin{pmatrix} 1 & t \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad x_{2e_2 - e_3}(t) = t x_{-2e_2 + e_3}(t) = \begin{pmatrix} 1 & t \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and set $N_\alpha := \{x_\alpha(t) | t \in G_\alpha\}$. We denote by $N^-$ the group generated by $\{x_{-\alpha}(t) | \alpha \in \Sigma^+, t \in G_\alpha\}$. We note that $N$ is generated by $\{x_{\alpha}(t) | \alpha \in \Sigma^+, t \in G_\alpha\}$.

Let $G_0$ be an affine algebraic group over $F$ defined by

$$G_0 = GL_2 \times GL_1, \quad GL_2 = \{(g_1, g_2) \in GL_2 \times GL_2 | \det(g_1) = \det(g_2)\}.$$ 

We often identify $G_0$ with a subgroup of $G$ via the embedding

$$G_0 \ni \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right) \mapsto \left( \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ c_1 & d_1 \\ c_2 & d_2 \end{array} \right) \in G.$$ 

Then $P_0 = P \cap G_0$ is a minimal parabolic subgroup of $G_0$. Let $P_0 = T_0 N_0$ be the Levi decomposition of $P_0$, where $T_0$ is the maximal (split) torus of $G_0$ defined by

$$T_0 = \left\{ \left( \begin{array}{cc} t_1 & t_2 \\ t_3 & t_4 \end{array} \right), \left( \begin{array}{cc} t_5 & t_6 \\ t_7 & t_8 \end{array} \right) \right| t_1, t_2, t_3 \in GL_1 \right\}$$

and $N_0$ is the unipotent radical of $P_0$. Then $T_0$ is identified with $T$ via the embedding. Let $\{e_i\}_{i=1}^3$ be the standard basis of $X^*(T_0)$ given by $e_i(t(t_1', t_2', t_3')) = t_i'$ and $\{d_i\}_{i=1}^3$ the basis of $X_*(T)$ dual to $\{e_i\}_{i=1}^3$ with respect to the canonical pairing. We often identify $X_*(T_0)$ with $\mathbb{Z}^3$ in the same way as $X_*(T)$. Let $\Sigma_0 \subset X^*(T_0)$ be the root system of $(G_0, T_0)$ and $\Sigma_0^+$ the set of positive roots with respect
We define homomorphisms to \(P\). The unramified Shintani functions coincide with \(\Sigma^+\). Similarly, let \(P\) be a subgroup \(G\) in this subsection, we recall several important decompositions of by \(d\). Here

\[
\text{Proposition 2.3.1 (Iwasawa decomposition)}
\]

\[
G = PK = KP, \quad dg = d_0 pdk = \delta(p)dkd_p, \\
G_0 = P_0 K_0 = K_0 P_0, \quad dg' = d_0 p'dk' = \delta_0(p')dk'd_0p'.
\]

Let \(W\) be the Weyl group of \((G, T)\), that is, \(W = \{g \in G | gTg^{-1} = T\} / T\). Then \(W\) is generated by \(w_1 T\) and \(w_2 T\), where

\[
w_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.
\]
A complete set of representatives of $W$ is given by
\[
\{1, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1, w_2w_1w_2, w_2w_1w_2w_1\}.
\]

We set
\[
w_\ell := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \ell \\ 1 \\ 1 \\ 1 \end{pmatrix} \in w_2w_1w_2T.
\]

Similarly, let $W_0$ be the Weyl group of $(G_0, T_0)$, which is defined in the same way as $W$. Then $W_0$ is generated by $w_1'T_0$ and $w_2'T_0$, where
\[
w_1' = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad w_2' = w_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.
\]

A complete set of representatives of $W_0$ is given by $\{1, w_1', w_2', w_1'w_2'\}$.

**Proposition 2.3.2 (Bruhat decomposition)**

\[
G = \bigsqcup_{w \in W} PwP = \bigsqcup_{w \in W} PwN = \bigsqcup_{w \in W} NwP,
\]

\[
G_0 = \bigsqcup_{w' \in W_0} P_0w'P_0 = \bigsqcup_{w' \in W_0} P_0w'N_0 = \bigsqcup_{w' \in W_0} N_0w'P_0.
\]

Let $B$ be the Iwahori subgroup of $G$ corresponding to $\Sigma^+$ and $B_0$ the Iwahori subgroup of $G_0$ corresponding to $\Sigma^+_0$.

**Proposition 2.3.3 (Bruhat type decomposition)**

\[
K = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} N(0)T(0)wB,
\]

\[
K_0 = \bigsqcup_{w' \in W_0} B_0w'B_0 = \bigsqcup_{w' \in W_0} N_0(0)T_0(0)w'B_0.
\]

**Proposition 2.3.4 (Iwahori factorization)**

\[
B = N_{(1)}^-T(0)N(0), \quad B_0 = N_{0,(1)}^-T_0(0)N_{0,(0)}.
\]

### 3 Shintani functions

In this section, we prove the uniqueness of Shintani functions on $G$. In §3.1, we introduce the space of Shintani functions. In §3.2, we prove a Cartan type decomposition of $G$. In §3.3, we show that the dimension of the space of Shintani functions is at most one by using a system of difference equations which Shintani functions satisfy.
3.1 The space of Shintani functions.

In this subsection, we introduce the space of Shintani functions on $G$ and investigate fundamental properties of Shintani functions. Let $X_{nr}(T)$ (resp. $X_{nr}(T_0)$) be the group consisting of unramified characters of $T$ (resp. $T_0$). We note that $\delta \in X_{nr}(T), \delta_0 \in X_{nr}(T_0)$. For $\xi \in X_{nr}(T_0), \Xi \in X_{nr}(T)$, we define $S(\xi, \Xi)$ to be the $\mathbb{C}$-vector space consisting of all continuous functions $S : G \rightarrow \mathbb{C}$ such that

$$[L(\phi)R(\Phi)S](x) := \int_{G_0} dg' \int_G dg \phi(g')S(g^{-1}xg)\Phi(g)$$

$$= \omega_{\xi}(\phi)\omega_{\Xi}(\Phi)S(x)$$

for all $(\phi, \Phi) \in \mathcal{H}(G_0, K_0) \times \mathcal{H}(G, K)$. We call an element of $S(\xi, \Xi)$ an unramified Shintani function of type $(\xi, \Xi)$, or simply a Shintani function. Let $Z$ (resp. $Z_0$) be the center of $G$ (resp. $G_0$). We note that $Z \subset Z_0 \simeq Z \times \{\pm 1\}$.

**Lemma 3.1.1** Every Shintani function $S \in S(\xi, \Xi)$ has the following properties.

i) $S(k'xk) = S(x)$ \quad ($'k', x, k) \in K_0 \times G \times K$;

ii) $S(z_0xz) = \xi(z_0)^{-1}\Xi(z)S(x)$ \quad ($'z_0, x, z) \in Z_0 \times G \times Z$.

In particular, we have $S(\xi, \Xi) = \{0\}$ if $(\xi \Xi) \mid Z \neq 1$.

**Proof.**

i) Since $\text{ch}_K \in \mathcal{H}(G, K), \text{ch}_{K_0} \in \mathcal{H}(G_0, K_0)$, we have

$$[L(\text{ch}_{K_0})R(\text{ch}_K)S](x) = \int_{K_0} dk' \int_K dkS(k'xk).$$

On the other hand, by definition of Shintani functions we have

$$[L(\text{ch}_{K_0})R(\text{ch}_K)S](x) = \omega_{\xi}(\text{ch}_{K_0})\omega_{\Xi}(\text{ch}_K)S(x) = S(x).$$

ii) Since $\text{ch}_{zK} \in \mathcal{H}(G, K), \text{ch}_{z_0K_0} \in \mathcal{H}(G_0, K_0)$, we have

$$[L(\text{ch}_{z_0K_0})R(\text{ch}_{zK})S](x) = \int_{G_0} dg' \int_G dg \text{ch}_{z_0K_0}(g')S(g'^{-1}xg)\text{ch}_{zK}(g)$$

$$= \int_{K_0} dk' \int_K dk S(k'^{-1}z_0^{-1}xz)$$

$$= S(z_0^{-1}xz).$$

On the other hand, by definition of the Shintani functions we have

$$[L(\text{ch}_{z_0K_0})R(\text{ch}_{zK})S](x) = \omega_{\xi}(\text{ch}_{z_0K_0})\omega_{\Xi}(\text{ch}_{zK})S(x) = \xi(z_0)\Xi(z)S(x).$$

From Lemma 3.1.1 (i), it follows that a Shintani function $S \in S(\xi, \Xi)$ is determined by its values on $K_0 \backslash G/K$. 


3.2 A Cartan type decomposition of $\text{GSp}_4(F)$

In this subsection, we shall prove the following theorem, called a Cartan type decomposition of $G$. The proof is almost the same as that of [KMS] for the split special orthogonal group $\text{SO}_n$.

**Theorem 3.2.1** We have the double coset decomposition

$$G = \bigsqcup_{\mu \in \Lambda^+, \mu' \in \Lambda_0^+} K_0 g(\mu', \mu) K$$

with

$$g(\mu', \mu) = t(\mu') \eta t(\mu), \quad t(\mu) = t(\omega^{\mu_1}, \omega^{\mu_2}, \omega^{\mu_3}).$$

From Theorem 3.2.1, it follows that a Shintani function $S \in S(\xi, \Xi)$ is determined by its values on

$$\{ g(\mu', \mu) | (\mu', \mu) \in \Lambda_0^+ \times \Lambda^+ \}.$$  

We set $K_0^* := K_0 \cdot \langle w_1 \rangle$, which is a subset of $K$. In order to prove Theorem 3.2.1, we shall first prove the following decomposition of $G$.

**Proposition 3.2.2**

$$G = \bigsqcup_{\lambda \in \Lambda^+, \lambda' \in \Lambda_0^+} K_0^* g(\lambda', \lambda) K.$$  

By [IM, Corollary 2.35], we have

$$G = \bigsqcup_{\lambda \in \Lambda^+}BW t(\lambda) K = \bigsqcup_{\lambda \in \Lambda^+} VBW t(\lambda) K = \bigsqcup_{\lambda \in \Lambda^+} VN^{-1} W t(\lambda) K.$$  

Here we set

$$V := K_0^* \{ \eta(y_1, y_2) | (y_1, y_2) \in \mathfrak{so}^2 \} \supset N(0) T(0).$$  

We can easily check the following decomposition:

$$G = \bigsqcup_{\lambda \in \Lambda^+} \bigcup_{w \in W} VN^{-1} w t(\lambda) K = \bigsqcup_{\lambda \in \Lambda^+} \bigcup_{w \in W} VN_{-e_1+e_2,(1)} N_{-e_1-e_2+e_3,(1)} w t(\lambda) K.$$  

For $w \in W$, we set

$$U_w := \bigsqcup_{\lambda \in \Lambda^+} VN_{-e_1-e_2+e_3,(1)} N_{-e_1+e_2,(1)} w t(\lambda) K.$$  

We note that

$$x_{-e_1+e_2}(a) t(t_1, t_2, t_3) = t(t_1, t_2, t_3) x_{-e_1+e_2}(at_1 t_2^{-1}).$$
In particular, we have

$$x_{-e_1-e_2+e_3}(a)t(t_1, t_2, t_3) = t(t_1, t_2, t_3)x_{-e_1-e_2+e_3}(at_1t_2t_3^{-1}).$$

In particular, we have

$$\mathcal{U} := \mathcal{U}_{14} = \bigcup_{\lambda \in \Lambda^+} \mathcal{V}t(\lambda)K.$$ 

First we shall see the following lemma to prove Proposition 3.2.2.

**Lemma 3.2.3** For all $w \in W$, we have $\mathcal{U} \supset \mathcal{U}_w$.

**Proof.** We may assume that $w \neq 1_4$, that is, $\ell(w) \neq 0$. Since for each $1_4 \neq w \in W$ there exist a simple root $\alpha$ and $w' \in W$ such that $w = w_\alpha w'$ and $\ell(w') < \ell(w)$. Then we shall see that $\mathcal{U}_w \subset \mathcal{U}_{w'}$.

We note that $w_{\alpha_1} = w_1$ and $w_{\alpha_2} = w_2$.

i) We assume that $\alpha = \alpha_1$. We can easily check the following equalities:

$$x_{-e_1+e_2}(a)w_1 = w_1x_{e_1-e_2}(a),$$

$$x_{-e_1-e_2+e_3}(a)w_1 = w_1x_{-e_1-e_2+e_3}(a),$$

$$x_{-e_1-e_2+e_3}(a)x_{e_1-e_2}(b) = x_{-2e_2+3e_3}(2ab)x_{e_1-e_2}(b)x_{-e_1-e_2+e_3}(a).$$

Noting that $w_1 \in V$, we have

$$\mathcal{U}_w = \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w_1t(\lambda)K$$

$$= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w_1w't(\lambda)K$$

$$= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}w_1N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w't(\lambda)K$$

$$= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w't(\lambda)K$$

$$\subset \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-2e_2+e_3,(1)}N_{-e_1+e_2+(1)}N_{-e_1-e_2+e_3,(1)}w't(\lambda)K$$

$$= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}w't(\lambda)K$$

$$\subset \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w't(\lambda)K = \mathcal{U}_w.$$

ii) We assume that $\alpha = \alpha_2$. We can easily check the following equalities:

$$x_{-e_1+e_2}(a)w_2 = w_2x_{-e_1-e_2+e_3}(a),$$

$$x_{-e_1+e_2}(a)x_{-e_1-e_2+e_3}(b) = x_{-2e_1+3e_3}(-2ab)x_{-e_1-e_2+e_3}(b)x_{-e_1+e_2}(a).$$

Hence we have

$$\mathcal{U}_w = \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-e_1+e_2,(1)}w(\lambda)K$$

$$= \bigcup_{\lambda \in \Lambda^+} \mathcal{V}N_{-e_1-e_2+e_3,(1)}N_{-1+e_2,(1)}w_2w'(\lambda)K.$$
\[
\begin{align*}
\mathcal{U}_{t_1} &= \bigcup_{\lambda \in \Lambda^+} \mathcal{U} \nu_2 N_{-e_1 + e_2, (1)} N_{-e_1 - e_2 + e_3, (1)} w'(\lambda) K \\
\mathcal{U} &= \bigcup_{\lambda \in \Lambda^+} \mathcal{U} \nu_2 N_{-e_1 + e_2, (1)} N_{-e_1 - e_2 + e_3, (1)} w'(\lambda) K \\
\mathcal{U} &= \bigcup_{\lambda \in \Lambda^+} \mathcal{U} \nu_2 N_{-e_1 - e_2 + e_3, (1)} N_{-e_1 + e_2, (1)} w'(\lambda) K = \mathcal{U}_{w^*}.
\end{align*}
\]

Hence we have the assertion. \(\blacksquare\)

For \(g_1, g_2 \in G\), if \(g_1 \in K_0^* g_2 K\), then we write \(g_1 \sim g_2\).

**Lemma 3.2.4** For \((y_1, y_2) \in \sigma^2, \mu = (\mu_1, \mu_2, \mu_3) \in \Lambda^+\) and \(u_1, u_2 \in \sigma^\times\), we have

\[
\begin{align*}
\eta(y_1, 0)t(\mu) &\sim \eta(y_1, \nu_1 + \mu_2 - \mu_3)t(\mu), \\
\eta(0, y_2)t(\mu) &\sim \eta(\nu_2 - \mu_1, y_2)t(\mu), \\
\eta(u_1 y_1, u_2 y_2)t(\mu) &\sim \eta(y_1, y_2)t(\mu).
\end{align*}
\]

**Proof.** The following equalities are easily checked by direct computation:

\[
\begin{align*}
\eta(y_1, 0)t(\mu) &= x_{2e_1 - e_3} (\varepsilon_1 + \mu_2 - \mu_3) \eta(y_1, \nu_1 + \mu_2 - \mu_3)t(\mu) x_{e_1 + e_2 - e_3}(-1), \\
\eta(0, y_2)t(\mu) &= x_{2e_1 - e_3} (-\varepsilon_2 + \mu_1 - \mu_2) \eta(\nu_1 - \mu_1, y_2)t(\mu) x_{e_1 - e_3}(-1), \\
\eta(u_1 y_1, u_2 y_2)t(\mu) &= t(1, u_1, u_1 u_2)^{-1} \eta(y_1, y_2)t(\mu)t(1, u_1, u_1 u_2).
\end{align*}
\]

\(\blacksquare\)

**Lemma 3.2.5** For any \((y_1, y_2) \in \sigma^2\) and \(\mu \in \Lambda^+\), there exist \((\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+\) such that \(\eta(y_1, y_2)t(\mu) \sim g(\lambda', \lambda) = t(\lambda')\eta t(\lambda)\).

**Proof.** By Lemma 3.2.4, we may assume that \(y = (y_1, y_2) = (\nu_1, \nu_2), \nu_1, \nu_2 \geq 0\). Let \(y = (y_1, y_2) = (\varepsilon_1 + \varepsilon_2, \nu_1, \nu_2), \nu_1, \nu_2 \geq 0\).

i) If \(\nu_1 > \nu_2\), then

\[
x_{2e_1 + e_3} (\varepsilon_1 + \nu_2 - 1) \eta(y)t(\mu) = \eta(y_1)t(\mu)x_{2e_1 + e_3} (\varepsilon_1 + \nu_2 - 1) \quad \text{(} y_1 = (\varepsilon_2, \varepsilon_1 + \varepsilon_2) \text{)},
\]

that is, \(\eta(y)t(\mu) \sim \eta(y_1)t(\mu)\). Hence we may assume that \(\nu_2 \geq \nu_1\).

ii) If \(\mu_1 - \mu_2 < \nu_1\), then

\[
x_{2e_1 - e_3} (\varepsilon_1 - \nu_1 + \mu_2) (1 - \varepsilon_1 + \mu_1 + \mu_2) \eta(y)t(\mu) = \eta(y_2)t(\mu)x_{e_1 - e_3} (\varepsilon_1 - \nu_1 + \mu_2 - 1) \quad \text{(} y_2 = (\varepsilon_1 - \mu_2, \varepsilon_1 - \mu_2) \text{)},
\]

that is, \(\eta(y)t(\mu) \sim \eta(y_2)t(\mu)\). Hence we may assume that \(\mu_1 - \mu_2 \geq \nu_1\).
iii) If $\nu_2 - \nu_1 > 2\mu_2 - \mu_3$, then
\[
x_{2e_2 - e_3}(\omega^{2\mu_2 - \mu_3}(\omega^{\nu_2 - \nu_1 - 2\mu_2 + \mu_3} - 1))\eta(y)t(\mu) = \eta(y_3)t(\mu)x_{2e_2 - e_3}(\omega^{\nu_2 - \nu_1 - 2\mu_2 + \mu_3} - 1)
\]
that is, $\eta(y)t(\mu) \sim \eta(y_3)t(\mu)$. Hence we may assume that $\nu_2 - \nu_1 \leq 2\mu_2 - \mu_3$.

If $\nu_2 \geq \nu_1$, $\mu_1 - \mu_2 \geq \nu_2$ and $2\mu_2 - \mu_3 \geq \nu_2 - \nu_1$, we have a factorization
\[
\eta(y)t(\mu) = t\left(\begin{array}{c}
\nu_1 + \nu_2 \\
2
\end{array}\right) \eta t\left(\begin{array}{c}
\mu_1 - \nu_2 \\
2
\end{array}\right).
\]
We have completed the proof of the assertion. \(\square\)

**Proof of Theorem 3.2.1.** First we shall prove the decomposition
\[(3.1)\quad G = \bigcup_{\lambda \in \Lambda^+} K_0 g(\lambda', \lambda) K.
\]
The disjointness of the above decomposition is proved in the next subsection. By Proposition 3.2.2, it is enough to show that for any $(\mu', \mu) \in \Lambda_0^+ + \Lambda^+$ there exist $(\lambda', \lambda) \in \Lambda_0^+ + \Lambda^+$ and $(k', k) \in K \times K$ such that $w_1 g(\mu', \mu) = k' g(\lambda', \lambda) k$. We can easily check that
\[
w_1 g(\mu', \mu) = w_1 t(\mu')\eta t(\mu)
= w_1 t(\mu')w_1^{-1} w_1 \eta t(\mu)
= t(\mu \cdot \mu')w_1 \eta t(\mu),
\]
where $\mu \cdot \mu' = (\mu_2', \mu_1', \mu_1')$. Putting
\[
\hat{\mu}' = \left(\begin{array}{c}
2\mu_2' - \mu_1' \\
\mu_2' \\
2\mu_2' - \mu_1'
\end{array}\right), \quad \mu_1' = \left(\begin{array}{c}
\mu_1' - \mu_2' \\
\mu_1' - \mu_2' \\
2(\mu_1' - \mu_2')
\end{array}\right),
\]
we have $w_1 \cdot \mu' = \hat{\mu}' + \mu_1'$. We note that $\hat{\mu}' \in \Lambda_0^+ + \Lambda^+$, $\mu + \mu_1' \in \Lambda^+$ and $t(\mu_1') \in Z$. Hence we have
\[
w_1 g(\mu', \mu) = t(\mu')t(\mu_1')w_1 \eta t(\mu) = t(\mu')w_1 \eta t(\mu + \mu_1').
\]
Since
\[
w_1 \eta = w_1 x_{2e_2 - e_3}(1)x_{e_1 + e_2 - e_3}(1)x_{e_1 - e_2}(1)
= x_{2e_2 - e_3}(1)x_{e_1 + e_2 - e_3}(1)x_{e_1 + e_2}(1)w_1,
\]
we have
\[
w_1 g(\mu', \mu)
= t(\hat{\mu}')w_1 \eta t(\mu + \mu_1')
= \left(t(\hat{\mu}')x_{2e_2 - e_3}(1)x_{e_1 - e_2}(-1)t(\hat{\mu}')^{-1}\right) g(\hat{\mu}', \mu + \mu_1')\left(t(\mu + \mu_1')^{-1} x_{e_1 - e_2}(-1)x_{e_1 + e_2}(1)w_1 t(\mu + \mu_1')\right).
\]
Noting that
\[
t(\hat{\mu}')x_{2e_2 - e_3}(1)x_{e_1 - e_2}(-1) t(\hat{\mu}')^{-1} \in K_0, \quad t(\mu + \mu_1')^{-1} x_{e_1 - e_2}(-1)x_{e_1 + e_2}(1)w_1 t(\mu + \mu_1') \in K,
\]
we have $w_1 g(\mu', \mu) \sim g(\hat{\mu}', \mu + \mu_1')$. Hence we have the decomposition (3.1). \(\square\)
3.3 Uniqueness of Shintani functions.

Let \((\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)\). In this subsection, we prove the uniqueness of Shintani functions (Theorem 3.3.1). We also prove the disjointness of the decomposition in Theorem 3.2.1.

**Theorem 3.3.1** For any Shintani function \(S \in \mathcal{S}(\xi, \Xi)\), we have \(S = 0\) if \(S(1) = 0\). In particular, we have
\[
\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) \leq 1.
\]

**Remark 3.3.2** In §4.5, we shall prove that \(\dim_{\mathbb{C}} \mathcal{S}(\xi, \Xi) = 1\) if and only if \((\xi \Xi)|_Z \equiv 1\) (see Theorem 4.5.4).

We denote by \(F[G]\) the ring of polynomial functions on \(G\), which represents the group scheme \(G\). First we define elements \(\alpha_1, \alpha_2, \beta_1, \beta_2\) of \(F[G]\), which are called relative invariants on \(G\).

**Remark 3.3.3** Although we use the same symbols \(\alpha_1, \alpha_2, \beta_1, \beta_2\) for relative invariants as the simple roots introduced in §2.1, this will not cause any confusion.

For \(I = \{i_1, \cdots, i_s\}, J = \{j_1, \cdots, j_s\} \subset \{1, 2, 3, 4\}\), we set
\[
\Delta_{I,J}(g) := \det(g_{i,j}), \quad g_{i,j} := (g_{i_k, j_l})_{1 \leq k, l \leq s}.
\]

We define relative invariants \(\alpha_i(i = 1, 2)\) on \(G\) by
\[
\alpha_1(g) = \Delta_{\{1\},\{1\}}(g t_1 g), \quad \alpha_2(g) = \Delta_{\{1,2\},\{1,2\}}(g t_2 g).
\]

Here \(w_t\) is the longest element of the Weyl group \(W\) defined in §2.3. Then, for \((y_1, y_2) \in F^2, t = t(t_1, t_2, t_3), t' = t(t'_1, t'_2, t'_3) \in T, n \in \mathbb{N}\), we can easily check the following properties of \(\alpha_i(i = 1, 2)\):

i) \(\alpha_i(\eta(y_1, y_2)w_t) = 1\) \((i = 1, 2)\);

ii) \(\alpha_1(t g_{\mathfrak{n}} g) = \omega'_1(t)^{-1} \alpha_1(g) = t_1^{-1} t_3 \alpha_1(g)\);

iii) \(\alpha_2(t g_{\mathfrak{n}} g) = (\omega'_1 + \omega'_2(t)^{-1} \alpha_2(g) = t_1^{-1} t_2^{-1} t_3 \alpha_2(g)\);

iv) \(\alpha_1(g t_{\mathfrak{n}}) = \omega_1(t) \alpha_1(g) = t_1 \alpha_1(g)\);

v) \(\alpha_2(g t_{\mathfrak{n}}) = \omega_2(t) \alpha_2(g) = t_1 t_2 \alpha_2(g)\).

We recall the Bruhat decomposition \(G = \bigsqcup_{w \in W} P w P\). The double coset \(P w t P\) has the following properties.

**Lemma 3.3.4** i) \(P w_t P = \{g \in G| \alpha_1(g) \alpha_2(g) \neq 0\}\);

ii) The double coset \(P w_t P\) is an open dense subset of \(G\) with respect to the \(p\)-adic topology;

iii) The double coset \(P w_t P = P w_t N\) is homeomorphic to \(P \times N\) via \(p w_t n \mapsto (p, n)\).

**Proof.** The assertions (ii) and (iii) are well-known. The assertion (i) follows from the Bruhat decomposition of \(G\).

We define relative invariants \(\beta_j(j = 1, 2)\) by
\[
\beta_1(g) := \Delta_{\{2\},\{1\}}(g t_1 g), \quad \beta_2(g) := \Delta_{\{2,4\},\{1,2\}}(g t_2 g).
\]

Then, for \((y_1, y_2) \in F^2, t = t(t_1, t_2, t_3), t' = t(t'_1, t'_2, t'_3) \in T, n \in \mathbb{N}, n' \in \mathbb{N}_0\), we can easily check the following properties of \(\beta_j\):
Hence, by the Bruhat decomposition of $G$,

\[ |\beta_i(\eta_1, \eta_2)\omega_i| = |\eta_i| \quad (i = 1, 2); \]

ii) $\beta_1(t' n' g) = \omega'_2(t')^{-1} \beta_1(g) = t_2' t_3' \beta_1(g);$ 

iii) $\beta_2(t' n' g) = e'_3(t') \beta_2(g) = t_3' \beta_2(g);$ 

iv) $\beta_1(\eta_1, \eta_2) = \omega_1(t) \beta_1(g) = t_1 \beta_1(g);$ 

v) $\beta_2(\eta_1, \eta_2) = \omega_2(t) \beta_2(g) = t_1 t_2 \beta_2(g).$

**Lemma 3.3.5** We have a decomposition

\[ G = \bigcup_{(\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2} P_0 \eta_1 \varepsilon_1 \varepsilon_2 \omega w P. \]

**Proof.** We note that

\[ \eta_1(\eta_1, \eta_2) = \begin{cases} 1 & (\text{if } (\eta_1, \eta_2) = (0, 0)), \\ t(1, y_1, 1)^{-1} \eta_1(1, 0) t(1, y_1, 1) & (\text{if } y_1 \neq 0, y_2 = 0), \\ t(1, 1, y_2)^{-1} \eta_2(0, 1) t(1, 1, y_2) & (\text{if } y_1 = 0, y_2 \neq 0), \\ t(1, y_1, y_2)^{-1} \eta_1(1, y_1, y_2) & (\text{if } (y_1, y_2) \neq (0, 0)). \end{cases} \]

Hence, by the Bruhat decomposition of $G$, we have

\[ G = \bigcup_{(\eta_1, \eta_2) \in F^2} P_0 \eta_1 \varepsilon_1 \varepsilon_2 \omega w P = \bigcup_{(\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2} P_0 \eta_1 \varepsilon_1 \varepsilon_2 \omega w P. \]

\[ \square \]

**Remark 3.3.6** We note that the union in Lemma 3.3.5 is not disjoint (see Lemma 4.3.8).

The double coset $P_0 \eta \omega P$ has several properties which are similar to those of $\omega P$.

**Lemma 3.3.7**

i) $P_0 \eta \omega P = \{g \in G|\alpha_1(g) \alpha_2(g) \beta_1(g) \beta_2(g) \neq 0\};$

ii) The double coset $P_0 \eta \omega P$ is an open dense subset of $G$ with respect to the $\mathfrak{p}$-adic topology;

iii) The double coset $P_0 \eta \omega P = P_0^\varepsilon \eta \omega P$ is homeomorphic to $P_0^\varepsilon \times \mathcal{P}$ via $p_0 \eta \omega P \mapsto (p_0, p).$ Here

\[ P_0^\varepsilon := T^2 N_0 \subset P_0, \quad T^2 := \{t(t_1, t_2, t_3) | t_1, t_2 \in F^x\}. \]

**Proof.** The assertion (i) follows from Lemma 3.3.4 (i). Since $\alpha_1 \alpha_2 \beta_1 \beta_2$ is a surjective continuous function from $G$ to $F$, it follows from (i) that the double coset $P_0 \eta \omega P$ is an open subset of $G$. To prove (ii), we shall see that $P_0 \eta \omega P$ is dense in $G$. Since $P \omega P$ is dense in $G$, it suffices to show that $P_0 \eta \omega P$ is a dense subset of $P \omega P$. It follows from the remark mentioned in the proof of Lemma 3.3.5. It is easy to see the assertion (iii).

\[ \square \]

The following lemma is used to obtain an integral expression of Shintani functions in §4.1.
Lemma 3.3.8 Let \( g = n't(t_1', t_2', t_3')\eta \omega t(t_1, t_2, t_3)n \in P_0^t \eta \omega t P \). Then we have

\[
|t_1| = |\alpha_1(g)|, \quad |t_2| = \left| \frac{\alpha_2(g)}{\beta_1(g)} \right|, \quad |t_3| = \left| \frac{\alpha_1(g)\alpha_2(g)\nu(g)}{\beta_1(g)\beta_2(g)} \right|,
\]

\[
|t_1'| = \left| \frac{\beta_1(g)\beta_2(g)}{\alpha_1(g)\alpha_2(g)} \right|, \quad |t_2'| = \left| \frac{\beta_2(g)}{\alpha_2(g)} \right|.
\]

Here \( \nu : G \to F^* \) is the similitude character of \( G \).

In order to prove Theorem 3.3.1, we introduce a partial order \( \geq_S \) on the set \( \Lambda_0^{++} \times \Lambda^+ \).

Definition 3.3.9 For \((\mu', \mu), (\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+ \), we write \((\mu', \mu) \geq_S (\lambda', \lambda)\) if all of the following five conditions hold:

i) \( \mu_1 - \mu_3 \geq \lambda_1 - \lambda_3 \);

ii) \( \mu_1 + \mu_2 - 2\mu_3 - \mu_1' + \mu_2' \geq \lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_1' + \lambda_2' \);

iii) \( \mu_1 - \mu_3 - \mu_1' + \mu_2' \geq \lambda_1 - \lambda_3 - \lambda_1' + \lambda_2' \);

iv) \( \mu_1 + \mu_2 - 2\mu_3 - \mu_1' \geq \lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_1' \);

v) \( \mu_3 + \mu_1' = \lambda_3 + \lambda_1' \).

Remark 3.3.10 We note that \((\mu', \mu) \geq_S (\lambda', \lambda)\) if and only if all of the following five conditions hold:

i') \( \mu_1 + \mu_1' \geq \lambda_1 + \lambda_1' \);

ii') \( \mu_1 + \mu_2 + \mu_1' + \mu_2' \geq \lambda_1 + \lambda_2 + \lambda_1' + \lambda_2' \);

iii') \( \mu_1 + \mu_2' \geq \lambda_1 + \lambda_2' \);

iv') \( \mu_1 + \mu_2 + \mu_1' \geq \lambda_1 + \lambda_2 + \lambda_1' \);

v') \( \mu_3 + \mu_1' = \lambda_3 + \lambda_1' \).

Then we have the following lemma.

Lemma 3.3.11 Let \((\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+ \).

i) If \((\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+ \) satisfies

\[
K_0t(\mu')Kt(\mu)^{-1}z(\mu_3)K \cap K_0t(\lambda')\eta \omega t(\lambda)^{-1}z(\lambda_3)K \neq \emptyset,
\]

then \((\mu', \mu) \geq_S (\lambda', \lambda)\) holds.

ii) The number of \((\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+ \) which satisfies \((\mu', \mu) \geq_S (\lambda', \lambda)\) is finite.

Remark 3.3.12 We can easily check the following two equalities :

\[
K_0t(\mu')Kt(\mu)^{-1}z(\mu_3)K = K_0t(\mu')Kt(\mu)K,
K_0t(\lambda')\eta \omega t(\lambda)^{-1}z(\lambda_3)K = K_0g(\lambda', \lambda)K.
\]
Before proving Lemma 3.3.11, we shall see the disjointness of the decomposition in Theorem 3.2.1.

**Proof of the disjointness of Theorem 3.2.1.** Let \( g(\mu', \mu) \in K_0(\lambda', \lambda)K \). Since \( g(\mu', \mu) \in K_0 t(\mu')\eta(t(\mu))K \subset K_0 t(\mu')Kt(\mu)K \), we have

\[
g(\mu', \mu) \in K_0 t(\mu')Kt(\mu)K \cap K_0 g(\lambda', \lambda)K.
\]

From Lemma 3.3.11 (and Remark 3.3.12), we have \( (\mu', \mu) \geq_\Sigma (\lambda', \lambda) \). Similarly we have \( (\lambda', \lambda) \geq_\Sigma (\mu', \mu) \), that is, \( (\mu', \mu) = (\lambda', \lambda) \). We have completed the proof of Theorem 3.2.1. \( \square \)

We denote by \( \mathfrak{o}[G] \) the \( \mathfrak{o} \)-structure of \( F[G] \). We note that \( \mathfrak{o}[G] \) is a Hopf algebra over \( \mathfrak{o} \). If \( m^* : \mathfrak{o}[G] \to \mathfrak{o}[G] \otimes_{\mathfrak{o}} \mathfrak{o}[G] \) is the coproduct of \( \mathfrak{o}[G] \), we denote by \( \Delta \) an \( \mathfrak{o} \)-algebra homomorphism defined by the composite

\[
\Delta : \mathfrak{o}[G] \xrightarrow{m^*} \mathfrak{o}[G] \otimes_{\mathfrak{o}} \mathfrak{o}[G] \xrightarrow{\text{Id} \otimes \text{Id}} \mathfrak{o}[G] \otimes_{\mathfrak{o}} \mathfrak{o}[G] \otimes_{\mathfrak{o}} \mathfrak{o}[G].
\]

**Proof of Lemma 3.3.11.** We fix \( (\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+ \).

1. Let \( g = t(\lambda')\eta w t(\lambda)^{-1}z(\lambda_3) \) for some \( (\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+ \). By the assumption, there exist \( k, k_1 \in K \) and \( k' \in K_0 \) such that \( g = k' t(\mu')k t(\mu)^{-1}z(\mu_3)k_1 \). By comparing the similitudes of both sides, we have \( \mu_3 + \mu_1' = \lambda_3 + \lambda_1' \). This is (v) of the definition of the partial order \( \geq_\Sigma \). We put \( f = \alpha_1 \in \mathfrak{o}[G] \subset F[G] \). Since we have

\[
f(g) = f(t(\lambda')\eta w t(\lambda)^{-1}z(\lambda_3))
= \varpi'_1(t(\lambda'))^{-1}\varpi_1(t(\lambda)z(\lambda_3)^{-1})^{-1}f(\eta w)
= \varpi_1(t(\lambda)z(\lambda_3)^{-1})^{-1},
\]

we obtain

\[
(3.2) \quad v(f(g)) = -\langle \varpi_1, \lambda - (\lambda_3, 2\lambda_3) \rangle.
\]

Here, for any \( x \in F^x \), the value \( v(x) \) is defined by \( x \in \varpi^x \mathfrak{o}^x \). We note that if \( \Delta(f) = \sum_i f_{(1), i} \otimes f_{(2), i} \otimes f_{(3), i} \) \( (f_{(j), i} \in \mathfrak{o}[G]) \), then we have

\[
f(g) = \sum_i f_{(1), i}(k') f_{(2), i}(t(\mu')k t(\mu)^{-1}z(\mu_3)) f_{(3), i}(k_1).
\]

We may assume that each \( f_{(j), i} \) is a nonzero element of \( \mathfrak{o}[G] \). Since \( \mathfrak{o}[G] \) has a basis consisting of weight vectors, we may assume that each \( f_{(2), i} \) is a weight vector. Namely, there exist \( \sigma_i' \in X^*(T_0), \sigma_i \in X^*(T) \) such that

\[
f_{(2), i}(t(\mu')k t(\mu)^{-1}z(\mu_3)) = \sigma_i'(t(\mu'))^{-1}\sigma_i(t(\mu)z(\mu_3)^{-1})^{-1}f_{(2), i}(k).
\]

We note that \( G \times G \) acts on \( F[G] \) by the left translation \( L \) and the right translation \( R \). Then \( f \) is a highest weight vector belonging to a finite dimensional \( (G \times G) \)-submodule of \( F[G] \). Let \( V_f \) be the \( (G \times G) \)-submodule of \( F[G] \) generated by \( f \). Since, for \( n, n' \in N^- \), there exist \( A_i(n', n) \in F \) such that

\[
[L(n')^{-1} R(n)f](g) = f(n'gn) = \sum_i A_i(n', n) f_{(2), i}(g),
\]
Lemma 3.3.13

The following lemma is easily checked.

Since \( K_0 \) is a compact subset of \( V_f \), we have \( \omega' \geq \sigma' \), \( \omega_1 \geq \sigma_1 \).
Therefore we have

\[
v(f(g)) = v\left( \sum_i f_{(1),i}(k')f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu'))f_{(3),i}(k_1) \right)
\]

\[
\geq \inf \left\{ v\left( f_{(1),i}(k')f_{(2),i}(t(\mu')kt(\mu)^{-1}z(\mu'))f_{(3),i}(k_1) \right) \mid i \right\}
\]

\[
\geq \inf \{ \langle \sigma', \mu' \rangle - \langle \sigma, \mu - (\mu_3, \mu_3, 2\mu_3) \rangle \mid i \}
\]

\[
= -\langle \omega', \mu' \rangle - \langle \omega, \mu - (\mu_3, \mu_3, 2\mu_3) \rangle
\]

Comparing the equation (3.2) and the above, we have

\[
\langle \omega, \mu - (\mu_3, \mu_3, 2\mu_3) \rangle \geq \langle \omega, \lambda - (\lambda_3, \lambda_3, 2\lambda_3) \rangle,
\]

that is, \( \mu_1 - \mu_3 \geq \lambda_1 - \lambda_3 \). This is (i) of the definition of the partial order \( \geq \). By applying the same way as \( f = a_1 \) to \( f = a_2, \beta_1, \beta_2 \), the conditions (ii),(iii) and (iv) of the definition of \( \geq \) are also obtained. However note that we must consider \( V_f \) to be a \((G_0 \times G)\)-submodule of \( F[G] \) for \( f = \beta_1 \) or \( \beta_2 \).

ii) This assertion follows from the definitions of \( \Lambda^+, \Lambda^{++}_0 \) and the partial order \( \geq \).

For \((\mu', \mu) \in \Lambda^{++}_0 \times \Lambda^+ \), we denote by \( m(\mu', \mu) \) the number of \((\lambda', \lambda) \in \Lambda^{++}_0 \times \Lambda^+ \) which satisfies \((\mu', \mu) \geq \) \((\lambda', \lambda) \). Then we have \( 1 \leq m(\mu', \mu) < \infty \) for all \((\mu', \mu) \in \Lambda^{++}_0 \times \Lambda^+ \) from Lemma 3.3.11 (ii).

The following lemma is easily checked.

Lemma 3.3.13 If \((\mu', \mu) \geq \) \((\lambda', \lambda) \) and \((\mu', \mu) \neq \) \((\lambda', \lambda) \), then \( m(\mu', \mu) > m(\lambda', \lambda) \).

Proof of Theorem 3.3.1. For any Shintani function \( S \in S(\xi, \Xi) \) and \((\mu', \mu) \in \Lambda^{++}_0 \times \Lambda^+ \), we put \( S(\mu', \mu) = S(g(\mu', \mu)) \). We assume that

\[
K_0 t'(\mu') K t(\mu) K = \bigcup_{i=0}^M K_0 g(\lambda'_{(i)}, \lambda_{(i)}) K, \quad g(\lambda'_{(i)}, \lambda_{(i)}) = t'(\lambda'_{(i)}) \eta t(\lambda_{(i)}).
\]

Since \( K_0 t'(\mu') K t(\mu) K \) is a compact subset of \( G \), the above union is finite. We may assume that \((\lambda'_{(0)}, \lambda_{(0)}) = (\mu', \mu) \) without loss of generality. It follows from Lemma 3.3.11 (i) that each \((\lambda'_{(i)}, \lambda_{(i)}) \) satisfies \((\mu', \mu) \geq \) \((\lambda'_{(i)}, \lambda_{(i)}) \). By definition of the Shintani functions, we have

\[
\omega_k (ch_{K_0 t'(\mu')}^{-1} K_0) \omega_k (ch_{K t(\mu) K}) S(0, 0)
\]

\[
= C_{\mu', \mu}^{(0)} \text{vol}\left( K_0 g(\mu', \mu) K; dg \right) S(\mu', \mu) + \sum_{i=1}^M C_{\mu', \mu}^{(i)} \text{vol}\left( K_0 g(\lambda'_{(i)}, \lambda_{(i)}) K; dg \right) S(\lambda'_{(i)}, \lambda_{(i)}).
\]

Here \( 0 := (0, 0, 0) \in \mathbb{Z}^3 \) and, for all \( i = 0, \ldots, m \), \( C_{\mu', \mu}^{(i)} \) is a positive integer given as the number of elements of the inverse image of \( K_0 g(\lambda'_{(i)}, \lambda_{(i)}) K \) under the multiplication map

\[
K_0 \setminus K_0 t'(\mu') K_0 \times K t(\mu) K/K \to K_0 \setminus K_0 t'(\mu') K t(\mu) K/K.
\]
We note that $\text{vol}(K_0g(\mu', \mu)K; dg)$ is nonzero. In particular, if $S(0, 0) = 0$, then $S(\mu', \mu) = 0$ for all $(\mu', \mu) \in \Lambda_0^+ \times \Delta^+$. We shall prove it by induction on $m(\mu', \mu) \geq 1$. We take $(\mu', \mu) \in \Lambda_0^+ \times \Lambda^+$ and assume that $S(0, 0) = 0$.

First we assume that $m(\mu', \mu) = 1$. Then, since $K_0t'(\mu')Kt(\mu)K = K_0g(\mu', \mu)K$, we have

$$0 = \omega_\xi(ch_{K_0t'(\mu')^{-1}K_0})\omega_\Xi(ch_{Kt(\mu)K})S(0, 0) = C_{\mu', \mu}\text{vol}(K_0g(\mu', \mu)K; dg)S(\mu', \mu).$$

Here $C_{\mu', \mu}$ is a positive integer. Hence we have $S(\mu', \mu) = 0$.

Next we assume that $S(\lambda', \lambda) = 0$ for all $(\lambda', \lambda)$ such that $m(\lambda', \lambda) > m(\lambda', \lambda)$. If

$$K_0t'(\mu')Kt(\mu)K = \bigcup_{i=0}^M K_0g(\lambda', \lambda)K, \quad (\lambda_{(0)}', \lambda_{(0)}) = (\mu', \mu)$$

then $m(\mu', \mu) > m(\lambda_{(i)}', \lambda_{(i)})$ for all $i = 1, \cdots, M$ from Lemma 3.3.13. Hence the equality (3.3) implies $S(\mu', \mu) = 0$ by the induction hypothesis.

\[ \square \]

4 An explicit formula of Shintani functions

In this section, we shall prove one of the main results in this thesis. In §4.1, we construct a nonzero intertwining operator $\Omega_{\xi, \Xi} \in \text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_\mu^G(\Xi), \mathbb{C})$ for any element $(\xi, \Xi)$ of a certain domain $U_\xi \subset (\mathbb{C}^\times)^5$ and give an integral expression of Shintani functions by using $\Omega_{\xi, \Xi}$. In §4.2, we calculate the image of the intertwining operator $\Omega_{\xi, \Xi}$ for several elements in $i_{P_0}^{G_0}(\xi) \otimes i_\mu^G(\Xi)$. In §4.3, we prove that there exists a nonempty subset $\tilde{U}_\xi$ of $X_{nr}(T_0) \times X_{nr}(T)$ such that $\text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_\mu^G(\Xi), \mathbb{C})$ is one dimensional for any element $(\xi, \Xi) \in \tilde{U}_\xi$ (Corollary 4.3.16). The uniqueness is indispensable for a meromorphic continuation of $\Omega_{\xi, \Xi}$ in §4.4. In §4.5, we prove an explicit formula of Shintani functions in the same way as [KMS] for the unramified (Whittaker-)Shintani function.

4.1 An integral expression of Shintani functions.

In this subsection, we give an integral expression of Shintani functions. For an intertwining operator $\Omega \in \text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_\mu^G(\Xi), \mathbb{C})$, we set

$$S_\Omega(x) := \Omega(\phi_{K_0, \xi} \otimes R(x)\phi_{K, \Xi}) \quad (\forall x \in G).$$

Proposition 4.1.1 Let $\Omega \in \text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_\mu^G(\Xi), \mathbb{C})$. Then $S_\Omega \in \mathcal{S}(\xi, \Xi)$, that is, $S_\Omega$ is a Shintani function of type $(\xi, \Xi)$.

Proof. Let $x \in G$. The we have

$$L(\phi)S_\Omega(x) = \int_{G_0} \phi(g')S_\Omega(g^{-1}x)dg'$$

$$= \int_{G_0} \phi(g')\Omega(\phi_{K_0, \xi} \otimes R(g'^{-1}x)\phi_{K, \Xi})dg'$$

$$= \int_{G_0} \phi(g')\Omega(R(g')\phi_{K_0, \xi} \otimes R(x)\phi_{K, \Xi})dg'$$

$$= \Omega(\int_{G_0} \phi(g')R(g')\phi_{K_0, \xi}dg' \otimes R(x)\phi_{K, \Xi})$$
Then the function \( \Phi(x) = \omega(x)S_\Omega(x) \) for all \( \phi \in \mathcal{H}(G_0, K_0) \). Also, we have \( R(\Phi)S_\Omega(x) = \omega_\Xi(\Phi)S_\Omega(x) \) for all \( \Phi \in \mathcal{H}(G, K) \) in a similar way.

We shall construct an intertwining operator \( \Omega \in \text{Hom}_{G_0} (G_0 \xi, \mathcal{C}) \) concretely by using the relative invariants on \( G \). Let \( \xi \in \mathcal{X}_{nr}(T_0), \Xi \in \mathcal{X}_{nr}(T) \) be unramified characters such that \( (\xi, \Xi) \in 1 \). Then we define a function \( \Upsilon_{\xi, \Xi} : P_0 \eta \nu \nu P \to \mathbb{C} \) by the following properties:

1. \( \Upsilon_{\xi, \Xi}(\nu \phi) = (\xi^{-1} \phi_0^{1/2})(\nu \phi)(\Xi \phi^{-1/2})(\nu \phi) \Upsilon_{\xi, \Xi}(\phi) \quad (\nu \phi, \phi \in P_0 \times P_0 \eta \nu \nu P \times P) \);
2. \( \Upsilon_{\xi, \Xi}(\eta \nu) = 1 \).

**Remark 4.1.2** We note that \( P_0 \cap \eta \nu \nu P(\eta \nu)^{-1} = Z \). Hence the condition \( (\xi, \Xi) \in 1 \) implies every function \( F : P_0 \eta \nu \nu P \to \mathbb{C} \) which has the property

\[
F(\nu \phi) = (\xi^{-1} \phi_0^{1/2})(\nu \phi)(\Xi \phi^{-1/2})(\nu \phi) F(\phi) \quad (\nu \phi, \phi \in P_0 \times P_0 \eta \nu \nu P \times P)
\]

is identically zero.

**Lemma 4.1.3** For \( g \in P_0 \eta \nu \nu P = P_0^2 \eta \nu \nu P \), we have

\[
\Upsilon_{\xi, \Xi}(g) = (\Xi_1 \xi_3 \xi_1 \xi_3 \cdot |^{-1/2})(\alpha_1(g))(\Xi_2 \xi_2 \xi_2 \xi_3 \cdot |^{-1/2})(\alpha_2(g)) \times (\Xi_2^{-1} \xi_1 \xi_2 \xi_3 \cdot |^{-1/2})(\beta_1(g))(\Xi_3^{-1} \xi_1 \xi_2 \xi_3 \cdot |^{-1/2})(\beta_2(g))(\Xi_3 \cdot |^{3/2})(\nu(g)).
\]

**Proof.** From Lemma 3.3.8, we obtain the assertion.

We extend \( \Upsilon_{\xi, \Xi} : P_0 \eta \nu \nu P \to \mathbb{C} \) to the whole \( G \) by setting \( \Upsilon_{\xi, \Xi} \equiv 0 \) on \( G - P_0 \eta \nu \nu P \). We note that \( \Upsilon_{\xi, \Xi} \) is not necessarily continuous on the whole \( G \).

**Proposition 4.1.4** Let \( U_c \) be a nonempty open subset of the complex manifold \( X_{nr}(T_0) \times X_{nr}(T) = (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3 \) given by

\[
U_c = \left\{ (\xi, \Xi) \in (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3 \mid \begin{array}{l}
|\xi_1 \xi_3 \xi_1 \Xi_3| < q^{-1/2} \\
|\xi_2 \xi_2 \xi_3 \Xi_3| < q^{-1/2} \\
|\xi_1 \xi_3 \xi_2 \Xi_3| < q^{-1/2} \\
|\xi_1 \xi_2 \xi_3 | < q^{-1/2}
\end{array} \right\}.
\]

Then the function \( \Upsilon_{\xi, \Xi} \) is continuous on \( G \) for any \( (\xi, \Xi) \in U_c \).

**Proof.** The set \( U_c \) is nonempty, because \( (q^{-2}, q^{-2}, q^0), (q^{-1}, q^{-2}, 1) \) \( \in U_c \), for example. Let \( (\xi, \Xi) \in U_c \). We see that \( \Upsilon_{\xi, \Xi} \) is continuous at each \( x \in G \). Since it is obvious that \( \Upsilon_{\xi, \Xi} \) is continuous on \( P_0 \eta \nu \nu P \), we may assume that \( x \in G - P_0 \eta \nu \nu P \). Then we have \( \Upsilon_{\xi, \Xi}(x) = 0 \). We consider a sequence \( \{x_n\}_{n=1}^\infty \) of elements in \( G \) which converges to \( x \). Then we may see that \( \lim_{n \to \infty} \Upsilon_{\xi, \Xi}(x_n) = 0 \). Now, since \( x \notin P_0 \eta \nu \nu P \), at least one of \( \alpha_1(x) = 0, \alpha_1(x) \neq 0, \beta_1(x) = 0, \beta_2(x) \neq 0 \) holds. We note that \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are continuous on \( G \). We consider the two subsequences of \( \{x_n\}_{n=1}^\infty \) given by

\[
\{x_n\}_{i=1}^{(1)} : = \{x_n\}_{i=1}^\infty \cap P_0 \eta \nu \nu P, \{x_n\}_{i=2}^{(2)} : = \{x_n\}_{i=1}^\infty \cap (G - P_0 \eta \nu \nu P).
\]

Obviously, we have \( \{x_n\}_{i=1}^{(1)} \cap \{x_n\}_{i=2}^{(2)} = \emptyset \).
The unramified Shintani functions

i) We assume that the sequence \( \{x_{n_i}^{(1)}\} \) is a finite set, that is, \( x_n \notin P_0 \eta w t P \) for \( n \gg 0 \). Then we have \( \lim_{n \to \infty} Y_{\xi, \Xi}(x_n) = 0 \).

ii) We assume that the sequence \( \{x_{n_i}^{(2)}\} \) is a finite set, that is, \( x_n \in P_0 \eta w t P \) for \( n \gg 0 \). Then Lemma 4.1.3 yields that

\[
\| \lim_{n \to \infty} Y_{\xi, \Xi}(x_n) \| = \lim_{n \to \infty} \| (\Xi_1 \Xi_2 \xi_1 \xi_2 | -^{1/2}(\alpha_1(x_n)) (\Xi_2 \Xi_2 \xi_2 \xi_3 | -^{1/2}(\alpha_2(x_n)) \times (\Xi_2^{-1} \Xi_3^{-1} \xi_2^{-1} | -^{1/2}(\beta_1(x_n)) (\Xi_3^{-1} \xi_1^{-1} \xi_3^{-1} | -^{1/2}(\beta_2(x_n)) (\Xi_3 | -^{3/2}(\nu(x_n))) \|
\]

\( = 0 \).

This means that \( \lim_{n \to \infty} Y_{\xi, \Xi}(x_n) = 0 \).

iii) We assume that both the sequences \( \{x_{n_i}^{(1)}\} \) and \( \{x_{n_i}^{(2)}\} \) are the infinite sets. Since \( \lim_{i \to \infty} x_{n_i}^{(1)} = \lim_{i \to \infty} x_{n_i}^{(2)} = x \) holds, we have \( \lim_{i \to \infty} Y_{\xi, \Xi}(x_{n_i}^{(1)}) = \lim_{i \to \infty} Y_{\xi, \Xi}(x_{n_i}^{(2)}) = 0 \) from the cases (i) and (ii). Hence we have \( \lim_{n \to \infty} Y_{\xi, \Xi}(x_n) = 0 \).

Therefore we have completed the proof of the proposition. \( \square \)

For \( (\xi, \Xi) \in U_c \), we define a continuous function \( Y_{\xi, \Xi} : G \to \mathbb{C} \) by

\[
Y_{\xi, \Xi}(g) = T_{\xi, \Xi}(g^{-1}) \quad (\forall g \in G).
\]

From Remark 4.1.2, if \( (\xi, \Xi)|_Z \neq 1 \), we have \( Y_{\xi, \Xi} \equiv 0 \) on \( G \). Since \( \eta^{-1} = t(1, -1, 1) t(1, -1, 1) \in T(0) \eta T(0) \), we have the following lemma from the definition.

**Lemma 4.1.5** For \( (\xi, \Xi) \in U_c \) such that \( (\xi, \Xi)|_Z \equiv 1 \), we have

i) \( Y_{\xi, \Xi}(pg p_0) = (\Xi^{-1} \delta^{1/2})(p)(\xi \delta_0^{-1/2})(p_0) Y_{\xi, \Xi}(g) \) \( (\forall (p, p_0) \in P \times P_0) \);

ii) \( Y_{\xi, \Xi}(\omega t \eta) = 1 \).

Let \( (\xi, \Xi) \in U_c \). We define an intertwining operator \( \Omega_{\xi, \Xi} \in \text{Hom}_{G_0}(i^{G_0}_{P_0}(\xi) \otimes i^{P_0}_{P}(\Xi), \mathbb{C}) \) by

\[
\Omega_{\xi, \Xi}(P\xi(\phi') \otimes P\Xi(\phi)) := \int_{G \times G_0} \phi(g) \phi'(g') Y_{\xi, \Xi}(gg^{-1}) dg dg' = \int_{K \times K_0} P(\phi)(k) P(\phi')(k') Y_{\xi, \Xi}(kk^{-1}) dk dk'
\]

for all \( (\phi', \phi) \in C^\infty_c(G_0) \times C^\infty_c(G) \). Then \( S_{\xi, \Xi} := S_{\Omega_{\xi, \Xi}} \) has the following expression.

**Proposition 4.1.6** For \( (\xi, \Xi) \in U_c \), we have

\[
S_{\xi, \Xi}(x) = \int_{K \times K_0} Y_{\xi, \Xi}(k x^{-1} k') dk dk' \quad (\forall x \in G).
\]

**Proof.** Let \( x \in G \). Then we have

\[
S_{\xi, \Xi}(x) = \Omega_{\xi, \Xi}(P\xi(ch K_0) \otimes R(x) P\Xi(ch K)) = \Omega_{\xi, \Xi}(P\xi(ch K_0) \otimes P\Xi(R(x)ch K))
\]
Remark 4.1.7 In the next subsection, we shall see that for $(\xi, \Xi) \in U_c$ such that $(\xi \Xi) |_Z \equiv 1$ the intertwining operator $\Omega_{\xi \Xi}$ is not identically zero.

4.2 Rank one calculation.

Let $(\xi, \Xi) \in X_{nr}(T_0) \times X_{nr}(T)$. For $w \in W$ (resp. $w' \in W_0$), we set $\Phi_w := \mathcal{P}_w(\chi_{B_0B}) \in i_P^G(\Xi)$ (resp. $\phi_{w'} := \mathcal{P}_{w'}(\chi_{B_0B_0}) \in i_{P_0}^{G_0}(\xi)$). From the Bruhat type decomposition (Proposition 2.3.3), the set $(\Phi_w)_{w \in W}$ (resp. $(\phi_{w'})_{w' \in W_0}$) is a basis of $i_P^G(\Xi)^B$ (resp. $i_{P_0}^{G_0}(\xi)^{B_0}$). We fix $(\xi, \Xi) = (\xi_1, \xi_2, \xi_3, \Xi_1, \Xi_2, \Xi_3) \in U_c$ such that $(\xi \Xi) |_Z \equiv 1$ until the end of this subsection. We note that $\xi_1 \xi_2 \xi_3^2 \Xi_1 \Xi_2 \Xi_3^2 = 1$. The purpose of this subsection is to calculate the following integrals:

$$I_i := \int \Omega_{\xi \Xi}(w_1 \Phi_{14} \otimes R(\eta w_l)(\Phi_{14} \otimes \Phi_{w_i})) \quad (i = 1, 2),$$

$$J_i := \int \Omega_{\xi \Xi}((\Phi_{14} + \phi_{w'}) \otimes R(\eta w_l)\Phi_{14}) \quad (i = 1, 2).$$

Refer to section 2 for notation $T(0), N(1), x_\alpha$ and so on.

Proposition 4.2.1

i) $I_1 = (q - 1) \frac{1 - q^{-1}\Xi_1 \Xi_2^{-1}}{(1 - q^{-1/2}\Xi_1 \Xi_3 \xi_1 \xi_3)(1 - q^{-1/2}\Xi_1 \Xi_3 \xi_2 \xi_3)}$;

ii) $I_2 = (q - 1) \frac{1 - q^{-1}\Xi_2}{(1 - q^{-1/2}\Xi_2 \Xi_3 \xi_1 \xi_3)(1 - q^{-1/2}\Xi_2 \Xi_3 \xi_2 \xi_3)}$;

iii) $J_1 = (q - 1) \frac{1 - q^{-1}\xi_1}{(1 - q^{-1/2}\Xi_1 \Xi_3 \xi_1 \xi_3)(1 - q^{-1/2}\Xi_1 \Xi_3 \xi_2 \xi_3)}$;

iv) $J_2 = (q - 1) \frac{1 - q^{-1}\xi_2}{(1 - q^{-1/2}\Xi_2 \Xi_3 \xi_1 \xi_3)(1 - q^{-1/2}\Xi_2 \Xi_3 \xi_2 \xi_3)}$.

First we calculate the value $\Omega_{\xi \Xi}(\Phi_{14} \otimes R(w_l\eta)(\Phi_{14})$.

Proposition 4.2.2 $\Omega_{\xi \Xi}(\Phi_{14} \otimes R(w_l\eta)(\Phi_{14}) = \int B \int B'$.

To prove Proposition 4.2.2, we use the following lemma.

Lemma 4.2.3 i) $w_l \eta N_{0,(1)} \subset B w_l \eta$;
Proof.

i) Let $x_{-2e_1+e_3}(a)x_{-2e_2+e_3}(b) \in N_{0,(1)}^-$, that is, $a, b \in \mathfrak{p}$. Then we have
\[
wx_τx_{-2e_1+e_3}(a)x_{-2e_2+e_3}(b)
= x_{-e_1+e_2}(-b)x_{-2e_1+e_3}(-b)x_{-2e_2+e_3}(a/(ab - a - 1))
\times x_{e_1-e_2}(-a)x_{2e_1-e_3}(-a/(ab - a - 1))
\times x_{e_1+e_2-e_3}(a(1-b)/(ab-a-1))x_{2e_2-e_3}((ab-b-a)/(ab-a-1))
\in Bwτη.
\]

ii) Let
\[
x_{e_1-e_2}(a_1)x_{e_1-e_3}(a_2)x_{e_1+e_2-e_3}(a_3)x_{2e_2-e_3}(a_4) \in N_{(1)}
\]
that is, $a_1, a_2, a_3, a_4 \in \mathfrak{p}$. Then we have
\[
x_{e_1-e_2}(a_1)x_{e_1-e_3}(a_2)x_{e_1+e_2-e_3}(a_3)x_{2e_2-e_3}(a_4)η
= t(1, (a_1 + 1)^{-1}, (a_1 + 1)^{-1}(1 + a_3 - a_4)^{-1})η
\times t(1, a_1 + 1, (a_1 + 1)(1 + a_3 - a_4))x_{2e_2-e_3}(a_1(1 + a_3 - a_4) + a_2 - a_3)x_{2e_2-e_3}(a_4)
\in T_0N_{0,(1)}.
\]

iii) This assertion follows immediately from (ii).

\[\square\]

Proof of Proposition 4.2.2. By definition, we have
\[
Ω_{\xi, ξ}(Φ_{14} ⊗ R(wτη)Φ_{14}) = \int_{B × B_0} Y_{ξ, ξ}(xwτηx'^{-1})dx dx'.
\]
From Lemma 4.2.3, we have
\[
BwτηB_0 \subset BwτηN_{0,(1)}^-T_0N_{0,(0)}
\subset BwτηT_0N_{0,(0)}
\subset T_0N_{0}wτηT_0N_{0,(0)}
\subset P_{0}wτηP_{0,(0)}.
\]
Since $Y_{ξ, ξ}|_{P_{0}wτηP_{0,(0)}} ≡ 1$, we obtain Proposition 4.2.2.

Let $dτ$ be the normalized Haar measure of $F$ so that $\text{vol}(σ; dτ) = 1$. Next we prove the following proposition.

**Proposition 4.2.4** i) For each $i = 1, 2$, we have
\[
I_i = 1 + q \cdot \int_0 Y_{ξ, ξ}(w_ix_{α_i}(τ)wτη)dτ;
\]
ii) For each $i = 1, 2$, we have
\[ J_i = 1 + q \cdot \int_0^\infty Y_{\xi, \xi}(\tau) w_i^{t-1} d\tau. \]

To prove Proposition 4.2.4, we use Proposition 4.2.2 and the following two lemmas.

**Lemma 4.2.5**

i) $Bw_i Bw_i \eta \subset P(0) w_i N_{\alpha, (0)} w_i \eta P_0(0)$ ($i = 1, 2$);

ii) $w_i \eta B_0 w_i^{t-1} B_0 \subset P(0) w_i \eta N_{\beta, (0)} w_i^{t-1} P_0(0)$ ($i = 1, 2$).

**Proof.**

i) By the Iwahori factorization, we have
\[ Bw_i B \subset T(0) N(0) w_i B \]
\[ \subset T(0) N(0) w_i N(0)^{-1} N_{(1)} \]
\[ \subset T(0) N(0) w_i N_{\alpha, (0)}^{-1} N_{(1)}^{-1}. \]

Hence, from Lemma 4.2.3 (iii), we have
\[ Bw_i Bw_i \eta \subset T(0) N(0) w_i N_{\alpha, (0)} T(0) w_i \eta T(0) N_0(0) \]
\[ \subset T(0) N(0) w_i N_{\alpha, (0)} w_i \eta T(0) N_0(0) \]
\[ \subset P(0) w_i N_{\alpha, (0)} w_i \eta P_0(0). \]

ii) By the Iwahori factorization, we have
\[ B_0 w_i^{t-1} B_0 \subset B_0 w_i^{t-1} T(0) N_0(0) \]
\[ \subset N_{(1)}^{-1} N_0(0) w_i^{t-1} T(0) N_0(0) \]
\[ \subset N_{(1)}^{-1} N_{\beta, (0)} w_i^{t-1} T(0) N_0(0). \]

Hence, from Lemma 4.2.3 (i),(iii), we have
\[ w_i \eta B_0 w_i^{t-1} B_0 \subset Bw_i \eta N_{\beta, (0)} w_i^{t-1} T(0) N_0(0) \]
\[ \subset T(0) w_i \eta T(0) N_{\beta, (0)} w_i^{t-1} T(0) N_0(0) \]
\[ \subset T(0) w_i \eta N_{\beta, (0)} w_i^{t-1} T(0) N_0(0) \]
\[ \subset T(0) w_i \eta N_{\beta, (0)} w_i^{t-1} T(0) N_0(0) \]
\[ \subset P(0) w_i \eta N_{\beta, (0)} w_i^{t-1} P_0(0). \]

The next lemma is easily checked.

**Lemma 4.2.6**

i) We have the following two double coset decompositions
\[ Bw_1 B = \bigcup_{a \in \mathcal{O}/p} \begin{pmatrix} a & 1 \\ 1 & -a \end{pmatrix} B, \quad Bw_2 B = \bigcup_{a \in \mathcal{O}/p} \begin{pmatrix} 1 & a \\ 1 & -1 \end{pmatrix} B. \]

In particular, we have $\text{vol}(Bw_i B; dg) = q \cdot \text{vol}(B; dg)$ for each $i = 1, 2$. 

ii) We have the following two double coset decompositions

\[ B_0 w_i B_0 = \bigsqcup_{a \in \mathcal{O}/p} \begin{pmatrix} a & 1 \\ -1 & 1 \end{pmatrix} B_0, \quad B_0 w'_i B_0 = \bigsqcup_{a \in \mathcal{O}/p} \begin{pmatrix} 1 & 1 \\ -1 & a \end{pmatrix} B_0. \]

In particular, we have \( \text{vol}(B_0 w_i B_0; d\gamma') = q \cdot \text{vol}(B_0; d\gamma') \) for each \( i = 1, 2 \).

**Proof of Proposition 4.2.4.**

i) From the definition of \( \Omega_{\xi, \Theta} \) and Lemma 4.2.3 (iii), we have

\[
\Omega_{\xi, \Theta}(\phi_{14} \otimes R(\eta \omega \ell)\Phi_{w_i}) = \int_{G \times G_0} \phi_{14}(x')\Phi_{w_i}(x\eta \omega \ell)Y_{\xi, \Theta}(xx'^{-1})dxdx'
\]

\[
= \int_{G \times G_0} \phi_{14}(x')\Phi_{w_i}(x)Y_{\xi, \Theta}(x(\eta \omega \ell)^{-1}x'^{-1})dxdx'
\]

\[
= \int_{G \times G_0} \phi_{14}(x'^{-1})\Phi_{w_i}(x)Y_{\xi, \Theta}(x\omega \eta x')dxdx'
\]

\[
= \int_{B \times B_0} Y_{\xi, \Theta}(x\omega \eta x')dxdx'
\]

\[
= \int_{B \times B_0} Y_{\xi, \Theta}(x\omega \eta)dx
\]

Since Lemma 4.2.5 (i) and Lemma 4.2.6 (i) yield that

\[
\Omega_{\xi, \Theta}(\phi_{14} \otimes R(\eta \omega \ell)\Phi_{w_i}) = \text{vol}(B_0; d\gamma') \int_{B, \omega_1} Y_{\xi, \Theta}(x \omega \eta)dx
\]

\[
= q \cdot \text{vol}(B; d\gamma)\text{vol}(B_0; d\gamma') \int_\rho Y_{\xi, \Theta}(w_1 x_{\alpha_1}(\tau)w \eta) d\tau,
\]

we obtain the assertion (i) by combining with Proposition 4.2.2.

ii) From the definition of \( \Omega_{\xi, \Theta} \) and Lemma 4.2.3 (iii), we have

\[
\Omega_{\xi, \Theta}(\phi_{w'} \otimes R(\eta \omega \ell)\Phi_{14}) = \int_{G \times G_0} \phi_{w'}(x')\Phi_{14}(x\eta \omega \ell)Y_{\xi, \Theta}(xx'^{-1})dxdx'
\]

\[
= \int_{G \times G_0} \phi_{w'}(x')\Phi_{14}(x)Y_{\xi, \Theta}(x\omega \eta x'^{-1})dxdx'
\]

\[
= \int_{B \times B_0w'_i B_0} Y_{\xi, \Theta}(x\omega \eta x'^{-1})dxdx'
\]

\[
= \int_{B \times B_0w'_i B_0} Y_{\xi, \Theta}(w \eta x'^{-1})dxdx'
\]

\[
= \text{vol}(B; d\gamma) \int_{B_0w'_i B_0} Y_{\xi, \Theta}(w \eta x')dx'.
\]
Lemma 4.2.8 For unramified characters easily check the following lemma, which is useful for our purpose. Hence it follows from Lemma 4.2.8 that

\[ \Omega_{\xi,z}(\phi_1 \otimes R(\eta w_1) \Phi_{w_1}) = \vol(B; dg) \int_{B w_1^{-1} B} Y_{\xi,z}(w_1 q x') dx' \]

\[ = q \cdot \vol(B; dg) \vol(B_0; dg') \int_0 Y_{\xi,z}(w_1 q x \beta \tau w^{-1}_1) d\tau, \]

we obtain the assertion (ii) by combining with Proposition 4.2.2.

Now we shall see Proposition 4.2.1. We set

\[ \kappa(g) := (\alpha_1(g^{-1}), \alpha_2(g^{-1}), \beta_1(g^{-1}), \beta_2(g^{-1})) \in F^4 \quad (\forall g \in G). \]

We note that \( \kappa(g) \in (F^\times)^4 \) if and only if \( g \in \text{P} w_1 \eta P_1 \). The next lemma is easily checked by direct calculation.

Lemma 4.2.7

i) \[ \kappa(w_i x_{\alpha_1}(\tau) w_1 \eta \eta) = \begin{cases} (-\tau, -1, 1 - \tau, -1) & \text{if } i = 1, \\ (1 - \tau, 1, 1 - \tau) & \text{if } i = 2. \end{cases} \]

ii) \[ \kappa(w_i \eta x_{\beta_1}(\tau) w^{-1}_1) = \begin{cases} (\tau, \tau - 1, 1, 1) & \text{if } i = 1, \\ (1, \tau, \tau + 1, 1) & \text{if } i = 2. \end{cases} \]

We extend an unramified character \( \chi \in X_{nr}(F^\times) \) to a function on \( F \) by setting \( \chi(0) = 0 \). We can easily check the following lemma, which is useful for our purpose.

Lemma 4.2.8 For unramified characters \( \chi, \chi' \in X_{nr}(F^\times) \) such that \( \|\chi(\varpi)\|, \|\chi'(\varpi)\| < q \), we have

\[ 1 + q \cdot \int_0 \chi(\tau) \chi'(1 + \tau) d\tau = (q - 1) \frac{1 - q^{-2}(\chi')'(\varpi)}{(1 - q^{-1}\chi(\varpi))(1 - q^{-1}\chi'(\varpi))}. \]

Proof of Proposition 4.2.1. We have the formula

\[ I_1 = 1 + q \cdot \int_0 Y_{\xi,z}(w_i x_{\alpha_1}(\tau) w_1 \eta \eta) d\tau \]

from Proposition 4.2.4. From Lemma 4.1.3 and Lemma 4.2.7, we have

\[ I_1 = 1 + q \cdot \int_0 (\Xi_1 \Xi_3 \xi_1 \xi_3 \cdot |^{-1/2}_2)(-\tau)(\Xi_2^{-1} \Xi_3^{-1} \xi_1^{-1} \xi_3^{-1} \cdot |^{-1/2}_2)(1 - \tau) d\tau \]

\[ = 1 + q \cdot \int_0 (\Xi_1 \Xi_3 \xi_1 \xi_3 \cdot |^{-1/2}_2)(\Xi_2^{-1} \Xi_3^{-1} \xi_1^{-1} \xi_3^{-1} \cdot |^{-1/2}_2)(1 + \tau) d\tau. \]

Hence it follows from Lemma 4.2.8 that

\[ I_1 = (q - 1) \frac{1 - q^{-1} \Xi_1 \Xi_2^{-1}}{(1 - q^{-1/2} \Xi_1 \Xi_3 \xi_1 \xi_3)(1 - q^{-1/2} \Xi_2^{-1} \Xi_3^{-1} \xi_1^{-1} \xi_3^{-1})}. \]

The assertions related to \( I_2, J_1 \) and \( J_2 \) follow also in the same way as \( I_1 \).
4.3 Uniqueness of Shintani functionals.

In this subsection, we shall prove that the dimension of the intertwining space \( \text{Hom}_{G_0}(\iota_P^G(\xi) \otimes \iota_P^G(\Xi), \mathbb{C}) \) is exactly one under the suitable assumption on \((\xi, \Xi)\) (Corollary 4.3.16). This result is indispensable for a meromorphic continuation of the intertwining operator \( \Omega_{\xi, \Xi} \in \text{Hom}_{G_0}(\iota_P^G(\xi) \otimes \iota_P^G(\Xi), \mathbb{C}) \) in the next subsection.

First we recall the definition of an \( l \)-space and some properties from \([BZ]\).

**Definition 4.3.1**

i) We call a locally compact Hausdorff topological space \( X \) an \( l \)-space if each point of \( X \) has a fundamental system consisting of compact open neighborhoods.

ii) We call a topological group which is an \( l \)-space a locally profinite group.

The following two results are well-known (see \([BZ, \text{Lemma 1.2, Proposition 1.5}]\)).

**Lemma 4.3.2** Let \( X \) be an \( l \)-space and \( Y \) a locally closed subset of \( X \). Then \( Y \) is an \( l \)-space in the induced topology.

**Proposition 4.3.3** Let \( X \) be an \( l \)-space and \( H \) a locally profinite group acting on \( X \) by a continuous left action. We assume that

i) \( H \) is countable at infinity, that is, \( H \) is covered by its countable compact subsets;

ii) The number of \( H \)-orbits in \( X \) is finite.

Then there exists an open \( H \)-orbit \( X_0 \subset X \) such that for any point \( x_0 \in X_0 \) the map \( H \to X_0 \) given by \( h \mapsto hx_0 \) is open.

Then we have the following proposition.

**Proposition 4.3.4** Along with the assumptions of Proposition 4.3.3, let the number of \( H \)-orbits in \( X \) be \( m + 1 \). Then there exist distinct \( H \)-orbits \( X_0, \ldots, X_m \) such that \( X_0 \cup \cdots \cup X_i \) is an open subset of \( X \) for any \( i = 0, \ldots, m \).

**Proof.** Let \( X_0, \ldots, X_{j-1} \subset X \) be distinct \( H \)-orbits such that \( X_0 \cup \cdots \cup X_i \) is an open subset of \( X \) for any \( i = 0, \ldots, j - 1 \). We put \( Y_{j-1} = X_0 \cup \cdots \cup X_{j-1} \). Since \( X - Y_{j-1} \) is a closed subset of \( X \), it is an \( l \)-space by Lemma 4.3.2. We note that \( H \) acts on \( X - Y_{j-1} \). Hence Proposition 4.3.3 implies that there exists an open \( H \)-orbit \( X_j \) in \( X - Y_{j-1} \). Then \( X_0 \cup \cdots \cup X_j \) is an open subset of \( X \). Indeed, since there exists an open subset \( U \) of \( X \) such that \( X_j = U \cap (X - Y_{j-1}) \), we have

\[
X_0 \cup \cdots \cup X_j = Y_{j-1} \cup (U \cap (X - Y_{j-1})) = Y_{j-1} \cup U.
\]

By induction, we have the assertion. \( \square \)

Let us return to our situation. We set

\[
K^{(n)} := \{ k \in K \mid k - 1_4 \in \varpi^n M_4(\mathfrak{o}) \} \quad (\forall n \geq 0).
\]

Then \( \{K^{(n)}\}_{n \geq 0} \) is a fundamental system consisting of open compact neighborhoods of the identity element of \( G \). Hence \( G \) is a locally profinite group. Let \( U \) be a locally closed subset of \( G \) invariant under the left and right translations by \( P \) and \( P_0 \), respectively. For \( \sigma \in X_{nr}(T) \), we denote by
$I_c^\infty(\sigma, U)$ the vector space consisting of $f \in C^\infty(U)$ with compact support modulo $P$, such that $f(px) = (\sigma^{1/2})(p)f(x)$ for $(p, x) \in P \times G$. Then $P_0$ acts on $I_c^\infty(\sigma, U)$ by the right translation. We set $O_0 := (P_0 \pi_\eta P)^{-1} = P w_\eta P_0$. Then $O_0$ is an open dense subset of $G$ by Lemma 3.3.7. We define an action of $P \times P_0$ on $G$ by $(P \times P_0) \times G \ni ((p, p_0), x) \mapsto pxP_0^{-1}$. From Lemma 3.3.5, the number of $(P \times P_0)$-orbits in $G$ is finite. If the number of $(P \times P_0)$-orbits in $G$ is $m + 1$, then Proposition 4.3.4 yields that there exist distinct $(P \times P_0)$-orbits $O_1, \ldots, O_m$ such that $\bar{O}_i := O_0 \cup \cdots \cup O_i$ is an open subset of $G$ for any $i = 0, \ldots, m$. We note that $\bar{O}_0 = O_0, \bar{O}_m = G$ and $O_i = \bar{O}_i \setminus \bar{O}_{i-1}$. In particular, the third equality implies that $O_i$ is a closed subset of $O_i$. Hence each orbit $O_i$ is a locally closed subset of $G$. By [C1, Lemma 6.1.1.], we have the following proposition.

**Proposition 4.3.5** Let $\sigma \in X_{nr}(T)$. For each $i = 1, \ldots, m$, the sequence of the $P_0$-modules

$$0 \to I_c^\infty(\sigma, \bar{O}_{i-1}) \to I_c^\infty(\sigma, \bar{O}_i) \to I_c^\infty(\sigma, O_i) \to 0$$

is exact.

We note that $I_c^\infty(\sigma, \bar{O}_m) = i_G(\sigma)$. For each $g \in G$, we set $O_g := PgP_0$ and denote by $\delta_g$ the modulus character of $P_0 \cap g^{-1}Pg$.

**Proposition 4.3.6** Let $\sigma \in X_{nr}(T)$ be an unramified character and $\rho : P_0 \to \mathbb{C}^\times$ a one-dimensional representation of $P_0$. Then we have

$$\dim_{\mathbb{C}} \text{Hom}_{P_0}(I_c^\infty(\sigma, O_g), \rho) \leq 1 \quad (\forall g \in G).$$

**Proof.** For $\sigma \in X_{nr}(T)$, it follows from definition that the $P_0$-module $I_c^\infty(\sigma, O_g)$ is isomorphic to the compact induction $c\text{-Ind}_{P_0 \cap g^{-1}P_0}^P(\sigma \otimes \delta^{1/2})$ via $I_c^\infty(\sigma, O_g) \ni f \mapsto L(g^{-1})f \in c\text{-Ind}_{P_0 \cap g^{-1}P_0}^P(\sigma \otimes \delta^{1/2})$, where $L(g^{-1})f(x) := f(gx)$ for any $g, x \in G$. Hence, for each $g \in G$, we have

$$\dim_{\mathbb{C}} \text{Hom}_{P_0}(I_c^\infty(\sigma, O_g), \rho) = \dim_{\mathbb{C}} \text{Hom}_{P_0}(I_c^\infty(\sigma, O_g) \otimes \rho^{-1}, \mathbb{C}) = \dim_{\mathbb{C}} \text{Hom}_{P_0}(c\text{-Ind}_{P_0 \cap g^{-1}P_0}^P(\sigma \delta^{1/2} \circ \text{Ad}(g)) \otimes \rho^{-1}, \mathbb{C}) = \dim_{\mathbb{C}} \text{Hom}_{P_0 \cap g^{-1}P_0}((\sigma \delta^{1/2} \circ \text{Ad}(g)) \otimes \rho^{-1} \otimes \delta_g^{-1}, \mathbb{C}) \leq 1.$$

**Corollary 4.3.7** Let $\sigma \in X_{nr}(T)$ be an unramified character and $\rho : P_0 \to \mathbb{C}^\times$ a one-dimensional representation of $P_0$. Then the condition $\dim_{\mathbb{C}} \text{Hom}_{P_0}(I_c^\infty(\sigma, O_g), \rho) = 1$ is equivalent to

$$(\sigma \delta^{1/2} \circ \text{Ad}(g)) \otimes \rho^{-1} \otimes \delta_g^{-1} \bigm|_{P_0 \cap g^{-1}P_0} (x) = 1 \quad (\forall x \in P_0 \cap g^{-1}P_0).$$

In particular, $(\sigma \rho^{-1})|_Z \equiv 1$ if and only if $\dim_{\mathbb{C}} \text{Hom}_{P_0}(I_c^\infty(\sigma, O_g), \rho) = 1$.

**Proof.** The first part of the assertion follows immediately from the proof of Proposition 4.3.6. We note that $P_0 \cap (w_\eta)^{-1}P w_\eta = Z$ and $\delta_{w_\eta} \equiv 1$. Then, for $z \in Z = P_0 \cap (w_\eta)^{-1}P w_\eta$, we have

$$\left(\sigma \delta^{1/2} \circ \text{Ad}(w_\eta)\right) \otimes \rho^{-1} \otimes \delta_{w_\eta}^{-1}(z) = \left(\sigma \delta^{1/2} \otimes \rho^{-1}\right)(z) = (\sigma \rho^{-1})(z).$$

Hence we have the assertion.

We have the following two lemmas by direct calculation.
Lemma 4.3.8 Let \( w_t \neq w \in W \). Then we have
\[
P_{\eta P_0} = \begin{cases} P_{wP_0} & (w = 14, w_2), \\ P_{\eta(1,0)P_0} & (w = w_1w_2w_1, w_1w_2w_1), \\ P_{\eta(0,1)P_0} & (w = w_1w_2, w_2w_1w_2). \end{cases}
\]

Lemma 4.3.9 For \( w \in W, \varepsilon_i = 0,1 (i = 1, 2) \), we put \( g = w\eta(\varepsilon_1, \varepsilon_2) \). Then we have
\[
T \cap g^{-1}Tg = \begin{cases} T & (\text{if } (\varepsilon_1, \varepsilon_2) = (0, 0)), \\ \{t(t_1, t_1, t_2) \in T\} & (\text{if } (\varepsilon_1, \varepsilon_2) = (1, 0)), \\ \{t(t_1, t_2, t_1t_2) \in T\} & (\text{if } (\varepsilon_1, \varepsilon_2) = (0, 1)), \\ Z & (\text{if } (\varepsilon_1, \varepsilon_2) = (1, 1)). \end{cases}
\]

Lemma 3.3.5 implies that
\[
G = \bigcup_{w \in W, \varepsilon_1, \varepsilon_2 = 0, 1} P_{\eta(\varepsilon_1, \varepsilon_2)P_0}.
\]

Let \( g = w\eta(\varepsilon_1, \varepsilon_2) \) and \( O_g = PgP_0 \). We assume that \( O_g \neq O_0 \). By Lemma 4.3.8, we may assume that \( \eta(\varepsilon_1, \varepsilon_2) \neq \eta \). Hence the subgroup \( T \cap g^{-1}Tg \) contains a torus which is properly larger than \( Z \) from Lemma 4.3.9. We set \( R := \{w\eta(\varepsilon_1, \varepsilon_2) w \in W, \varepsilon_1, \varepsilon_2 = 0, 1, \varepsilon_1\varepsilon_2 = 0\} \). Then the set \( R \) is the union of \( R_1 \) and \( R_2 \), where
\[
R_1 := \{g \in R \mid (T \cap g^{-1}Tg) - Z \text{ has an element of the form } t(t_1, t_1, t_2)\}, \\
R_2 := \{g \in R \mid (T \cap g^{-1}Tg) - Z \text{ has an element of the form } t(t_1, t_2, t_1t_2)\}.
\]

We set \( X := X_{nr}(T) \times X_{nr}(T) \simeq (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3 \) and, for each \( g \in R \),
\[
X_g := X - \{ (\xi, \Xi) \in X \mid (\Xi\delta^{1/2} \circ \Ad(g)) \otimes (\xi\delta^{-1/2} \otimes \delta^{-1}g) (x) = 1 \ (\forall x \in P_0 \cap g^{-1}Pg) \}.
\]

Then \( X_g \) is a Zariski open set of \( X \).

Theorem 4.3.10 Let \( (\xi, \Xi) \in \bigcap_{g \in R} X_g \). Then we have
\[
\dim_{\mathbb{C}} \Hom_{\mathcal{R}_0}(i_pG(\Xi), \xi^{-1}\delta^{1/2}) = \dim_{\mathbb{C}} \Hom_{\mathcal{O}_0}(i_pG(\xi) \otimes i_pG(\Xi), \mathbb{C}) \leq 1.
\]

In particular, if \( (\xi\Xi)|_Z = (\xi\Xi\delta^{-1/2})|_Z \neq 1 \), we have \( \Hom_{\mathcal{R}_0}(i_pG(\Xi), \xi^{-1}\delta^{1/2}) = \{0\} \).

Remark 4.3.11 We shall see that the Zariski open set \( \bigcap_{g \in R} X_g \) is nonempty (see Proposition 4.3.14).

Before proving Theorem 4.3.10, we shall see the following lemma.

Lemma 4.3.12 Let \( g \in R \) and \( (\xi, \Xi) \in X_g \). Then we have
\[
\dim_{\mathbb{C}} \Hom_{\mathcal{R}_0}(i_c\infty(\Xi, \mathcal{O}_g), \xi^{-1}\delta^{1/2}) = 0.
\]
Proof. From Corollary 4.3.7, the condition \( \dim_{\mathbb{C}} \text{Hom}_{R_0}(I_0^\infty(\Xi, O_0), \xi^{-1}(-1)/2) \neq 0 \) is equivalent to
\[
\left( (\sigma^1/2 \circ \text{Ad}(g)) \otimes \xi \otimes \delta \otimes \delta^{-1} \right)(x) = 1 \quad (\forall x \in P_0 \cap g^{-1}Pg).
\]

Proof of Theorem 4.3.10. Let \((\xi, \Xi) \in \bigcap_{g \in R} X_g\). We note that the Hom-functor \(\text{Hom}_{R_0}(\cdots, \xi^{-1}(-1)/2)\) is a left exact contravariant functor. We put \(m = \sharp(P \setminus G_0) - 1\). Then, from Proposition 4.3.5, we have an exact sequence
\[
0 \to \text{Hom}_{R_0}(I_0^\infty(\Xi, O_0), \xi^{-1}(-1)/2) \to \text{Hom}_{R_0}(I_0^\infty(\Xi, O_1), \xi^{-1}(-1)/2) \to \text{Hom}_{R_0}(I_0^\infty(\Xi, O_{-1}), \xi^{-1}(-1)/2)
\]
for each \(i = 1, \ldots, m\). Since \(I_0^\infty(\Xi, O_{-1}) = \iota^G_0(\Xi)\), we have an exact sequence
\[
0 \to \text{Hom}_{R_0}(I_0^G(\Xi), \xi^{-1}(-1)/2) \to \text{Hom}_{R_0}(I_0^G(\Xi, O_0), \xi^{-1}(-1)/2)
\]
by Lemma 4.3.12. We obtain the assertion from Lemma 4.3.6 and Corollary 4.3.7.

Corollary 4.3.13 Let \((\xi, \Xi) \in \bigcap_{g \in R} X_g\). Then we have
i) \(\dim_{\mathbb{C}} \text{Hom}_{G_0}(\iota_P^G(\Xi), \iota_P^G(\xi^{-1})) = \dim_{\mathbb{C}} \text{Hom}_{G_0}(\iota_P^G(\xi) \otimes \iota_P^G(\Xi), \mathbb{C}) \leq 1\).

ii) If \((\xi, \Xi) \neq 1\), we have \(\text{Hom}_{G_0}(\iota_P^G(\xi), \iota_P^G(\xi^{-1})) = \emptyset\).

Proof. It follows immediately from Frobenius reciprocity and Theorem 4.3.10.

For \((\xi, \Xi) \in X = (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^3\), the condition \((\xi, \Xi) \neq 1\) means that \((\xi, \Xi)\) is a zero of the polynomial \(Y_1 Y_2 Y_3 Z_1 Z_2 Z_3 - 1 \in \mathbb{C}[X] := \mathbb{C}[Y_1^+, Y_2^+, Y_3^+, Z_1^+, Z_2^+, Z_3^+]\). We consider the algebraic set \(X_0^2\) of \(X\) defined by
\[
X_0^2 = \{ (\xi, \Xi) \in X \mid \xi_1 \xi_2 \xi_3 \Xi_1 \Xi_2 \Xi_3 = 1 \} \simeq (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3.
\]
We note that modulus characters \(\delta \in X_{nr}(T)\) and \(\delta_0 \in X_{nr}(T_0)\) are identified with \((q^{-2}, q^{-2}, q^2)\) and \((q^{-2}, q^{-2}, q^2)\), respectively. We set \(U_0^2 := U_0^c \cap X_0^2\) and \(X_0^2 := X_g \cap X_0^2\). We note that \(U_0^2\) is a nonempty open subset of \(X_0^2\) in the Euclidean topology. Indeed, we have \((q^{-2}, q^{-2}, q^5), (q^{-2}, q^{-2}, 1) \in U_0^2\). We set \(\tilde{U}_0^2 := U_0^c \cap \bigcap_{g \in R} X_g^2\). Then \(\tilde{U}_0^2\) is an open subset of \(X_0^2\) in the Euclidean topology.

Proposition 4.3.14 The set \(\bigcap_{g \in R} X_g^2\) is an open dense subset of \(X_0^2\) in the Euclidean topology. In particular, we have \(\tilde{U}_0^2 = \emptyset\).

We use the following lemma, which is easily checked by direct computation, to prove Proposition 4.3.14.

Lemma 4.3.15 For \(g = \text{wq}(\varepsilon_1, \varepsilon_2)g \in R\), we put \(t' = g^{-1}Tg \in T \cap g^{-1}TG\).

i) If \(g \in R_1, t' = t(\varpi_{\lambda_1}, \varpi_{\lambda_1}, \varpi_{\lambda_2})\), we have
\[
t = \begin{cases} 
(\varpi_{\lambda_1}, \varpi_{\lambda_2}, \varpi_{\lambda_1 + \lambda_2}) & \text{if } w = w_4, w_1, w_2, w_4, w_1, w_2, w_1, w_4, w_2, w_1, w_1, w_2, w_1, w_4, w_2, w_1.
\end{cases}
\]
The unramified Shintani functions

ii) If \( g \in R_2, t' = t(\omega^{\lambda_1}, \omega^{\lambda_2}, \omega^{\lambda_1+\lambda_2}), \) we have

\[
t = \begin{cases}
  t' & (\text{if } w = 1, w_1, w_2w_1w_2, w_2), \\
  t(\omega^{\lambda_1}, \omega^{\lambda_2}) & (\text{if } w = 2, w_1w_2, w_2w_1, w_1w_2w_1).
\end{cases}
\]

Proof of Proposition 4.3.14. We note that for each \( g \in R \) there exist \( \gamma_i(g) \in Z(i = 1, 2, 3) \) such that \( \delta^{-1}_g \) is identified with \( (q^{\gamma_1(g)}, q^{\gamma_2(g)}, q^{\gamma_3(g)}) \in (\mathbb{C}^\times)^3 \). For any \( \lambda, \mu \in Z^3 \), we consider the element in \( \mathbb{C}[X] \) given by

\[
F_{\mu,\lambda}(Y_1^{\pm}, \cdots, Z_3^{\pm}) = 1 - (q^{-2}Z_1)^{\mu_1}(q^{-1}Z_2)^{\mu_2}(q^{\gamma_1(g)-1}Y_1)^{\lambda_1}(q^{\gamma_2(g)-1}Y_2)^{\lambda_2}(q^{\gamma_3(g)+5/2}Y_3Z_3)^{\lambda_3}
\]

and its image \( F^{\sharp}_{\mu,\lambda} \) of the canonical surjection \( \mathbb{C}[X] \to \mathbb{C}[X^2] \simeq \mathbb{C}[X]/(Y_1Y_2Y_3^2Z_1Z_2Z_3^2 - 1) \), that is,

\[
F^{\sharp}_{\mu,\lambda}(Y_1^{\pm}, \cdots, Z_3^{\pm}) = 1 - q^A Z_1^{\mu_1-\lambda_1}Z_2^{\mu_2-\lambda_1} Y_1^{\lambda_2-\lambda}(Y_3Z_3)^{-2\mu_1} \mod (Y_1Y_2Y_3^2Z_1Z_2Z_3^2 - 1),
\]

where \( A = A(g) \) is a certain rational number. Let \( g \in R \). We note that there exists \( t_0 \in T \cap g^{-1}Tg \) such that

\[
(4.1) \quad \left( (\Xi^{1/2} \circ \text{Ad}(g)) \otimes (\xi^{1/2} \circ \delta^{-1}_g) \right) (t_0) \neq 1
\]

if and only if there exist \( \lambda, \mu = \mu(\lambda) \in Z^3 \) and \( u \in T(g) \) such that

\[
t(\lambda) = g^{-1}t(\mu)ug, \quad F_{\lambda,\mu}(\xi, \Xi) \neq 0.
\]

For each \( g \in R \), we take \( \lambda(g), \mu(g) \in Z^3 \) such that \( t(\lambda(g)) = g^{-1}t(\mu(g))ug \in T \cap g^{-1}Tg \) and consider the algebraic set \( \mathcal{V}_g \) of \( X^2 \) given by

\[
\mathcal{V}_g = \{(\xi, \Xi) \in X^2 = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^3 | F^{\sharp}_{\mu(g),\lambda(g)}(\xi, \Xi) = 0\}.
\]

Here \( u \in T(g) \). We note that \( X^2 \setminus \mathcal{V}_g \subset X^2_g \).

We shall prove that \( X^2 \setminus \bigcup_{g \in R} \mathcal{V}_g = \bigcap_{g \in R} X^2 \setminus \mathcal{V}_g \subset \bigcap_{g \in R} X^2_g \) is a dense subset of \( X^2 \). By Proposition 4.3.17 described below in this subsection, it is enough to show that \( \mathcal{V}_g \neq X^2 \) for any \( g \in R \). For each \( g \in R \), we take \( \lambda(g) \in Z^3 \) so that

\[
\lambda(g) = \begin{cases}
  (0, 0, 1) & (\text{if } g \in R_1), \\
  (0, 1, 1) & (\text{if } g \in R_2).
\end{cases}
\]

Then Lemma 4.3.15 implies that for each \( g \in R \) we have \( F^{\sharp}_{\lambda(g),\mu(g)} \neq 0 \). This means that \( \mathcal{V}_g \neq X^2 \).

\[
\square
\]

The following corollary is crucial for our proof of a meromorphic continuation of the functional \( \Omega_{\xi,\Xi} \) (see Theorem 4.4.1).

Corollary 4.3.16 For \( (\xi, \Xi) \in \tilde{U}^2 \), we have

\[
\dim_C \text{Hom}_{G_0}(i^{G_0}_{\xi}(\Xi), i^{G_0}_{\xi}(\xi^{-1})) = \dim_C \text{Hom}_{G_0}(i^{G_0}_{\xi}(\xi) \otimes i^{G_0}_{\xi}(\Xi), \mathcal{O}) = 1.
\]
Proof. Since $\Omega_{\xi, \Xi} \in \text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_{P}^{G}(\Xi), \mathbb{C})$, Proposition 4.2.2 yields the assertion. \qed

Finally we prove the above-mentioned proposition, which are also repeatedly used in §4.4 and §4.5.

Proposition 4.3.17 Let $\mathcal{X}$ be an open subvariety of the affine variety $\mathbb{C}^n$ and $\{V_i\}_i$ a countable family consisting of proper Zariski closed subsets of $\mathcal{X}$. Then $\mathcal{X} - \bigcup_i V_i$ is a dense subset of $\mathcal{X}$ in the Euclidean topology.

Although Proposition 4.3.17 is a special case of Baire category theorem ([K, p.200, Theorem 34]), we will give a proof without using the theorem.

Lemma 4.3.18 Let $\mathcal{X}$ be a Hausdorff topological space and $\mathcal{B}$ the Borel algebra of $\mathcal{X}$. Also, let $\mu$ be a strictly positive measure on $(\mathcal{X}, \mathcal{B})$. If $C$ is an element of $\mathcal{B}$ such that $\mu(C) = 0$, then $\mathcal{X} - C$ is a dense subset of $\mathcal{X}$.

Proof. We put $Y = \mathcal{X} - C$. Then we have $\mathcal{X} - Y \subset C$. Since $\mu(C) = 0$ and $\mathcal{X} - Y$ is an open subset of $\mathcal{X}$, we have $\mathcal{X} - Y = \emptyset$. Namely, $\mathcal{X} = Y$. \qed

Lemma 4.3.19 Let $f$ be a nonzero holomorphic function on a domain $D \subset \mathbb{C}^n$. Then the set $\{z \in D \mid f(z) = 0\}$ has $2n$-dimensional Lebesgue measure zero.

Proof. See [GR, p.9, Corollary 10], for example. \qed

Proof of Proposition 4.3.17. We note that Lebesgue measure is strictly positive. Thus Proposition 4.3.17 immediately follows from Lemma 4.3.18, Lemma 4.3.19 and the countable additivity of Lebesgue measure. \qed

4.4 Continuation of Shintani functionals.

We put $V = C^\infty(P_0 \cap K_0 \backslash K_0) \otimes_\mathbb{C} C^\infty(P \cap K \backslash K)$. Then we have $V \simeq i_{P_0}^{G_0}(\xi) \otimes C i_{P}^{G}(\Xi)$. In this subsection, we shall give a “meromorphic continuation” of the functional $\Omega_{\xi, \Xi}$.

Theorem 4.4.1 i) For each $v \in V$, the function $\tilde{U}^{\xi}_{\mathcal{X}} \ni (\xi, \Xi) \mapsto \Omega_{\xi, \Xi}(v) \in \mathbb{C}$ extends to a rational function on $X^\mathcal{X}^\sharp$;

ii) There exists a subset $X^\dagger \subset X^\sharp$, which is the complement of a countable union of hypersurfaces, such that the rational function $X^\dagger \ni (\xi, \Xi) \mapsto \Omega_{\xi, \Xi}(v) \in P^1(\mathbb{C})$ has no poles on $X^\dagger$ for every $v \in V$;

iii) For $(\xi, \Xi) \in X^\dagger$, the above extended map $\Omega_{\xi, \Xi} : V \to \mathbb{C}$ induces an intertwining operator $\Omega_{\xi, \Xi} \in \text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_{P}^{G}(\Xi), \mathbb{C})$ which satisfies $\Omega_{\xi, \Xi}(\phi_{14} \otimes R(w\eta)\Phi_{14}) = \text{vol}(B)\text{vol}(B_0)$.

Corollary 4.4.2 There exists a dense subset $D^\sharp \subset X^\sharp$ which satisfies the following two properties:

i) The set $D^\sharp$ is the complement of a countable union of proper Zariski closed subsets of $X^\sharp$ and stable under the action of the Weyl group $W_0 \times W$;

ii) For any element $(\xi, \Xi) \in D^\sharp$, we have

$$\text{Hom}_{G_0}(i_{P_0}^{G_0}(\xi) \otimes i_{P}^{G}(\Xi), \mathbb{C}) = \mathbb{C} \cdot \Omega_{\xi, \Xi}.$$
There are several ways to give a meromorphic continuation of generalized spherical functions, like Whittaker functions and Bessel functions, or the intertwining operators associated with such functions. For example, Bump–Friedberg–Furusawa applied Hartogs’ theorem to give the analytic continuation of the unramified Bessel functions on the split special orthogonal group of odd degree (see [BFF, p.153]). Also, Kato–Murase–Sugano introduced a new rationality argument to give a meromorphic continuation of an intertwining operator attached to generalized spherical functions on connected split reductive groups (see [KMS, section 2]).

We take a subset $\Theta$ of $\mathcal{V}$ and consider $\chi \in \mathcal{X}$. The system of regular functions on $X$ is a dual space. We take a subset $\Theta$ of $\mathcal{V}$ and consider $\chi \in \mathcal{X}$. Hence such a map $\lambda \in V^* \rightarrow \lambda(v) = s$ for any $(v, s) \in \Theta$.

We shall recall Bernstein’s rationality theorem. Let $k$ be a field, $V$ a $k$-vector space and $V^*$ its dual space. We take a subset $\Theta$ of $\mathcal{V}$ and consider $\chi \in \mathcal{X}$. For each $\chi \in \mathcal{X}$, we take a system $\Theta_\chi = \{v_r, r \in \mathcal{R}\}$, where $\mathcal{R}$ is a common indexing set for any $\chi$. We consider the family of the systems $\Theta_\chi = \{\Theta_\chi | \chi \in \mathcal{X}\}$. We say that $\Theta_\chi$ is regular if both $\chi \mapsto v_r$ and $\chi \mapsto s_r$ are regular for all $r \in \mathcal{R}$. We denote by $\mathcal{C}(\mathcal{X})$ the field of fractions of $\mathcal{C}[\mathcal{X}]$. Let $V_{\mathcal{C}(\mathcal{X})} := \mathcal{C}(\mathcal{X}) \otimes_{\mathcal{C}} V$ and $V^*_{\mathcal{C}(\mathcal{X})} := \text{Hom}_{\mathcal{C}(\mathcal{X})}(V_{\mathcal{C}(\mathcal{X})}, \mathcal{C}(\mathcal{X}))$. If a family of systems $\Theta_\chi$ in $V \times \mathcal{C}$ is regular, for each $r \in \mathcal{R}$ there exist $r_r \in V_{\mathcal{C}(\mathcal{X})}$ and $s_r \in \mathcal{C}[\mathcal{X}]$ such that

$$\chi \mapsto \chi(v_r) = v_{r, \chi}, \quad \chi \mapsto s_r(\chi) = s_{r, \chi}.$$ 

Hence we can regard such a family of systems $\Theta_\chi$ as a single system $\Theta$ in $V_{\mathcal{C}(\mathcal{X})} \times \mathcal{C}(\mathcal{X})$:

$$\Theta := \{(v_r, s_r) \in V_{\mathcal{C}(\mathcal{X})} \times \mathcal{C}(\mathcal{X}) | r \in \mathcal{R}\}.$$ 

We note that $\lambda \in V^*_{\mathcal{C}(\mathcal{X})}$ is determined by its values on elements of the form $1 \otimes v(v \in V)$. For $\lambda \in V^*_{\mathcal{C}(\mathcal{X})}$ and $\chi \in \mathcal{X}$, we define $\lambda_\chi \in V^*$ by $\lambda_\chi(v) = \lambda(1 \otimes v)\chi(\chi \in \mathcal{X})$ (Wv \in V$). We note that $\lambda(\chi) \in \mathcal{C}(\mathcal{X})$ has no poles at $\chi$ for all $\chi \in \mathcal{X}$ if $\lambda(1 \otimes v) \in \mathcal{C}(\mathcal{X})$ has no poles at $\chi$ for all $v \in V$.

Lemma 4.4.3 For $\chi \in \mathcal{X}$, we have $\lambda_\chi(\chi(v)) = \lambda(v)(\chi)$. In particular, if $\lambda \in V^*_{\mathcal{C}(\mathcal{X})}$ is a solution to the system $\Theta$, then $\lambda_\chi$ is a solution to the system $\Theta_{\chi}$ for all $\chi \in \mathcal{X}$.

With the notation above, we state Bernstein’s rationality theorem. For the proof, refer to [Ba] or [G2, Appendix].

Theorem 4.4.4 (Bernstein[Be]) Along with the assumptions above, suppose that $V$ has a countable basis over $\mathcal{C}$, and there exists a nonempty subset $\mathcal{X}_0 \subset \mathcal{X}$, which is open in the Euclidean topology, such that $\Theta_{\chi}$ has a unique solution for all $\chi \in \mathcal{X}_0$. Then

i) $\Theta$ has a unique solution $\lambda \in V^*_{\mathcal{C}(\mathcal{X})}$.
ii) There exists a subset $X^\dagger \subset X$, which is the complement of a countable union of hypersurfaces, such that, for any $X \in X^\dagger$, $\lambda_X \in V^*$ is defined and is the unique solution to $\Theta_X$.

**Remark 4.4.5** It seems that Proposition 4.3.17 is implicitly used to prove the uniqueness of the solution $\lambda \in V_{\mathbb{C}(X)}^*$ in Bernstein’s rationality theorem in [Ba]. We also note that it is also possible to prove it without using Proposition 4.3.17 (see [G2, Appendix]).

Let us return to our situation. We put $V = C^\infty(P_0 \cap K_0 \setminus K_0) \otimes_{\mathbb{C}} C^\infty(P \cap K \setminus K)$. First we see the following lemma to prove Theorem 4.4.1.

**Lemma 4.4.6** For $f' \otimes f \in V$ and $(g', g) \in G_0 \times G$, the map $X \to V, (\xi, \Xi) \mapsto R_\xi(g) f' \otimes R_\Xi(g) f$ is regular in the sense of the above.

**Proof.** For any $(f, g) \in C^\infty(P \cap K \setminus K) \times G$ and $(f', g') \in C^\infty(P_0 \cap K_0 \setminus K_0) \times G_0$, it is enough to show that the maps $X_{nr}(T) \ni \Xi \mapsto R_\Xi(g) f \in C^\infty(P \cap K \setminus K)$ and $X_{nr}(T_0) \ni \xi \mapsto R_{\xi}(g') f' \in C^\infty(P_0 \cap K_0 \setminus K_0)$ are regular, respectively. We see that $X_{nr}(T) \ni \Xi \mapsto R_{\Xi}(g) f \in C^\infty(P \cap K \setminus K)$ is regular. Let $g \in G$. We note that

$$C^\infty(P \cap K \setminus K) = \bigcup_{l \geq 0} C^\infty(P \cap K \setminus K)^{(l)}.$$ 

Since $\{K^{(l)}\}_{l \geq 0}$ is the fundamental system of the identity element of $G$, for any $l \geq 0$ there exists $l_0 > l$ such that $K^{(l_0)} \subset g K^{(l)} g^{-1} \cap K^{(l)}$. Hence, for $f \in C^\infty(P \cap K \setminus K)^{(l_0)}$ and $k \in K^{(l_0)}$, we have

$$[R_\Xi(k) R_\Xi(g) f](x) = f_\Xi(x k g) = f_\Xi(x g g^{-1} k g) = f_\xi(x g) = [R_\Xi(g) f](x) \quad (\forall x \in K).$$

Namely, we have $R_\Xi(g) f \in C^\infty(P \cap K \setminus K)^{(l_0)}$. We note that $m(l_0) := m(P \cap K \setminus K)^{(l_0)} < \infty$. This means that $C^\infty(P \cap K \setminus K)^{(l_0)}$ is an $m(l_0)$-dimensional vector space over $\mathbb{C}$. We consider the basis of $C^\infty(P \cap K \setminus K)^{(l_0)}$ given by $f_i^{(l_0)}(u_j) = \delta_{ij}(i, j = 1, \cdots, m(l_0))$. Then we can write $R_\Xi(g) f = \sum_{n=1}^m c_n(\Xi) f_n^{(l_0)}$. We shall see that $c_i \in \mathbb{C}[X_{nr}(T)]$ for all $i = 1, \cdots, m(l_0)$. Let $u_1, \cdots, u_{m(l_0)} \in K$ be a complete system of representatives of $P \setminus G/K^{(l_0)}$. Then, for each $u_i$, there exist $p_i \in P, k_i \in K^{(l_0)} \subset K^{(l)}$ and $j(i) = 1, \cdots, m(l_0)$ such that $u_i g = p_i u_{j(i)} k_i$. Hence we have

$$[R_\Xi(g) f](u_i) = f_\Xi(p_i u_{j(i)} k_i) = (\Xi \delta^{1/2})(p_i) f(u_{j(i)}),$$

$$= \sum_{n=1}^m c_n(\Xi) f_n^{(l_0)}(u_i) = c_i(\Xi).$$

If $p_i = t(\mu^{(i)})$ mod $(T \cap K) N$ for some $\mu^{(i)} \in \mathbb{Z}^3$, the above equality yields that

$$c_i(\Xi) = (\Xi \delta^{1/2})(p_i) = q^{2 \mu_1^{(i)} - 2 \mu_2^{(i)} + (3/2) \mu_3^{(i)}} \Xi^{\mu_1^{(i)}} \Xi^{\mu_2^{(i)}} \Xi^{\mu_3^{(i)}}$$

for all $i = 1, \cdots, m(l_0)$, that is, $c_i \in \mathbb{C}[X_{nr}(T)]$. Therefore the map $\Xi \mapsto R_\Xi(g) f$ is regular. It follows that the map $X_{nr}(T_0) \ni \xi \mapsto R_{\xi}(g') f' \in C^\infty(P_0 \cap K_0 \setminus K_0)$ is regular in the same way as above. 

Now we prove Theorem 4.4.1 as an application of Theorem 4.4.4.
Proof of Theorem 4.4.1. Since \( i^G_\ell(\Xi) \) (resp. \( i^G_0(\xi) \)) is an admissible representation of \( G \) (resp. \( G_0 \)), \( i^G_\ell(\Xi) \) (resp. \( i^G_0(\xi) \)) has a countable basis \( \{ F_i \}_{i \in \mathbb{Z}} \) (resp. \( \{ f_j \}_{j \in \mathbb{Z}} \)) over \( \mathbb{C} \). Then \( \{ f_i \otimes F_j \}_{i,j \in \mathbb{Z}} \) is a countable basis of \( V \). For any \( (\xi, \Xi) \in X^2 \), we define a system \( \Theta_{\xi, \Xi} = \Theta(\xi, \Xi) \subset V \times \mathbb{C} \) by

\[
\Theta_{\xi, \Xi} := \{(R_\xi(g')f_i \otimes R_\Xi(g')F_j - f_i \otimes F_j, 0) \mid g' \in G_0, i, j \in \mathbb{Z}\} \cup \{(\phi_{14} \otimes R(w\eta))\Phi_{14}, \text{vol}(B)\text{vol}(B_0)\}.
\]

Let \( \Theta = \Theta_X \) be the family of the systems. Then \( \Theta \) is regular from Lemma 4.4.6. For all \( (\xi, \Xi) \in \tilde{U}^\sharp_\ell \), Proposition 4.2.2 and Corollary 4.3.16 imply that the functional \( \Omega_{\xi, \Xi} \in V^* \) is a unique solution to \( \Theta_{\xi, \Xi} \). Since \( \tilde{U}^\sharp_\ell \) is the nonempty open subset of \( X^2 \) in the Euclidean topology, Theorem 4.4.4 yields that

a) \( \Theta \) has a unique solution \( \Omega \in V^*_C(X^2) \);

b) There exists a subset \( X^\dagger \subset X^2 \), which is the complement of a countable union of hypersurfaces, such that, for any \( (\xi, \Xi) \in X^\dagger \), \( \Omega_{\xi, \Xi} \in V^* \) is defined and is the unique solution to \( \Theta_{\xi, \Xi} \). In particular, we have \( \Omega_{\xi, \Xi} = \Omega_{\xi, \Xi} \) for \( (\xi, \Xi) \in X^\dagger \cap \tilde{U}^\sharp_\ell \).

For \( (\xi, \Xi) \in X^\dagger \), we set \( \Omega_{\xi, \Xi}(v) = \Omega_{\xi, \Xi}(v) \) for all \( v \in V \). Then we have a family \( \{\Omega_{\xi, \Xi}\}_{(\xi, \Xi) \in X^\dagger} \) of intertwining operators. Hence we obtain the assertions. \( \square \)

Finally we construct a subset \( D^\sharp \subset X^2 \) which satisfies the properties in Corollary 4.4.2. We take and fix the set \( X^\dagger \) considered in Theorem 4.4.1. For each subset \( Y \subset X^2 \), we set

\[
Y_w := \{\chi \in Y \mid \varphi \in Y \} = Y \cap w^{-1}Y \quad (w \in W_0 \times W),
\]

\[
Y_{W_0 \times W} := \bigcap_{w \in W_0 \times W} Y_w = \bigcap_{w \in W_0 \times W} w^{-1}Y.
\]

We note that the Weyl group \( W_0 \times W \) acts on \( X^2 \). Indeed, if \( (\xi, \Xi) \in X^2 \), then

\[
(w'\xi)(z)(w\Xi)(z) = \xi(w'^{-1}zw)\Xi(w^{-1}zw) = \xi(z)\Xi(z) = 1
\]

for \( (w', w) \in W_0 \times W \) and \( z \in Z \). The following lemma, which is easily checked, is useful in §5.3.

Lemma 4.4.7 Let \( \Xi = (\Xi_1, \Xi_2, \Xi_3) \in X_{nr}(T), \xi = (\xi_1, \xi_2, \xi_3) \in X_{nr}(T_0) \). Then we have

\[
w\Xi = \begin{cases}
(\Xi_2, \Xi_1, \Xi_3) & \text{if } w = w_1, \\
(\Xi_1, \Xi_2^{-1}, \Xi_3\Xi_3) & \text{if } w = w_2, \\
(\Xi_2^{-1}, \Xi_1, \Xi_3) & \text{if } w = w_1w_2, \\
(\Xi_2, \Xi_1^{-1}, \Xi_1\Xi_3) & \text{if } w = w_2w_1, \\
(\Xi_1^{-1}, \Xi_2, \Xi_3) & \text{if } w = w_1w_2w_1, \\
(\Xi_2^{-1}, \Xi_1^{-1}, \Xi_1\Xi_2\Xi_3) & \text{if } w = w_2w_1w_2, \\
(\Xi_1^{-1}, \Xi_2^{-1}, \Xi_1\Xi_2\Xi_3) & \text{if } w = w_1w_2w_1, \\
(\Xi_1^{-1}, \Xi_2^{-1}, \Xi_1\Xi_2\Xi_3) & \text{if } w = w_1.
\end{cases}
\]

We set

\[
D^\sharp := X^\dagger_0 \cap_{g \in R} X^\dagger_0 \cap X_{nr}^{reg}(T_0) \times X_{nr}^{reg}(T) \subset X^2.
\]

We note that \( D^\sharp \) is stable under the action of the Weyl group \( W_0 \times W \). In order to prove Corollary 4.4.2, it is enough to show the following lemma. Indeed, Corollary 4.4.2 is immediately obtained from Theorem 4.4.1.
Lemma 4.4.8 The set $D^\sharp$ is the complement of a countable union of proper Zariski closed subsets of $X^\sharp$. In particular, $D^\sharp$ is a dense subset of $X^\sharp$ in the Euclidean topology.

Proof. We note that $X_{W_0 \times W}^\dagger = \bigcap_{w \in W_0 \times W} w \cdot X^\dagger$ is the complement of a countable union of hypersurfaces. Since the set $X^\dagger \cap X_{nr}^{reg}(T_0) \times X_{nr}^{reg}(T)$ is the complement of a finite union of proper Zariski closed subsets of $X^\sharp$, the set $D^\sharp = X_{W_0 \times W}^\dagger \cap (X^\sharp \cap X_{nr}^{reg}(T_0) \times X_{nr}^{reg}(T))$ is the complement of a countable union of proper Zariski closed subsets of $X^\sharp$. Hence we have the assertion by Proposition 4.3.17. \square

4.5 An explicit formula of Shintani functions.

In this subsection, we shall prove an explicit formula of the Shintani function. For any $(\xi, \Xi) = ((\xi_1, \xi_2, \xi_3), (\Xi_1, \Xi_2, \Xi_3)) \in X^\sharp$, we set

$$\zeta(\xi, \Xi) := \frac{e'(\xi) e(\Xi)}{b(\xi, \Xi)},$$

where

$$e'(\xi) := (1 - q^{-1} \xi_1)(1 - q^{-1} \xi_2), \quad e(\Xi) := (1 - q^{-1} \Xi_1 \Xi_2)(1 - q^{-1} \Xi_1 \Xi_3^{-1})(1 - q^{-1} \Xi_2)(1 - q^{-1} \Xi_3),$$

$$b(\xi, \Xi) := (1 - q^{-1/2} \Xi_1 \Xi_3 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_3 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_2 \Xi_3 \xi_2 \xi_3) \times (1 - q^{-1/2} \Xi_2 \Xi_3 \Xi_1 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \Xi_3 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \Xi_3 \xi_2 \xi_3)(1 - q^{-1/2} \Xi_1 \Xi_2 \Xi_3 \xi_2 \xi_3).$$

First we shall prove the next proposition, which is an analogue of Jacquet’s functional equation for the unramified Whittaker function in [J] (see also [CS]).

Proposition 4.5.1 Let $g \in G$. For $(\xi, \Xi) \in D^\sharp$ such that $\zeta(\xi, \Xi) \neq 0$, the value $S_{\xi, \Xi}(g) / \zeta(\xi, \Xi)$ is $W_0 \times W$-invariant as a function of $(\xi, \Xi)$.

We take an element $(w', w) \in W_0 \times W$. Then we have

$$(T_{w', \xi} \otimes T_{w, \Xi})^* \Omega_{w', \xi, w \Xi} := \Omega_{w', \xi, w \Xi} \circ (T_{w', \xi} \otimes T_{w, \Xi}) \in \text{Hom}_{G_0}(i_{P_0}^C(\xi) \otimes i_{P_0}^C(\Xi), \mathbb{C})$$

for any $(\xi, \Xi) \in D^\sharp$. Hence it follows from Corollary 4.4.2 that there exists a scalar factor $a_{w', w}(\xi, \Xi) \in \mathbb{C}$ such that

$$(T_{w', \xi} \otimes T_{w, \Xi})^* \Omega_{w', \xi, w \Xi} = c_{w'}(\xi) c_w(\Xi) a_{w', w}(\xi, \Xi) \Omega_{\xi, \Xi}$$

for any $(\xi, \Xi) \in D^\sharp$ which satisfies $c_{w'}(\xi) c_w(\Xi) \neq 0$. Here

$$c_{w'}(\xi) = \prod_{\beta \in \Sigma_0^+ \beta \not\in \Sigma^w} c_{w'}(\xi), \quad c_w(\Xi) = \prod_{\alpha \in \Sigma^{+w} \alpha \not\in \Sigma^w} c_{w}(\Xi).$$

First we consider the case where $w = w_{\alpha_i} = w_i$ and $w' = 1$. Then, since $T_{w_i, \Xi}(\Phi_1 + \Phi_{w_i}) = c_{\alpha_i}(\Xi)(\Phi_1 + \Phi_{w_i})$, we have

$$c_{\alpha_i}(\Xi)a_{14, w_i}(\xi, \Xi) \Omega_{\xi, \Xi}(\phi_{14} \otimes R(\eta w_i)(\Phi_1 + \Phi_{w_i}))$$

$$= (id \otimes T_{w_i, \Xi})^* \Omega_{\xi, w_i, \Xi}(\phi_{14} \otimes R(\eta w_i)(\Phi_1 + \Phi_{w_i}))$$

$$= c_{\alpha_i}(\Xi)\Omega_{\xi, w_i, \Xi}(\phi_{14} \otimes R(\eta w_i)(\Phi_1 + \Phi_{w_i})).$$
that is,
\[ a_{14,w_1}(\xi, \Xi) = \frac{\Omega_{\xi,w_1,\Xi}(\phi_{14} \otimes R(\eta w_1)(\Phi_{14} + \Phi_{w_1}))}{\Omega_{\xi,\Xi}(\phi_{14} \otimes R(\eta w_1)(\Phi_{14} + \Phi_{w_1}))}. \]

Next we consider the case where \( w = 14 \) and \( w' = w_{31} = w'_1 \). Then, since \( T_{w_1}(\phi_{14} + \phi_{w'_1}) = c_{\beta}(\xi)(\phi_{14} + \phi_{w'_1}) \), we have
\[ a_{w_1,14}(\xi, \Xi) = \frac{\Omega_{w_1,\xi,\Xi}(\phi_{14} + \phi_{w'_1}) \otimes R(\eta w_1)\Phi_{14})}{\Omega_{\xi,\Xi}(\phi_{14} + \phi_{w'_1}) \otimes R(\eta w_1)\Phi_{14})} \]
in the same way as the first case. In order to prove Proposition 4.5.1, we prove the following lemma.

**Lemma 4.5.2** For \( i = 1, 2 \), we have
\[ \zeta(\xi, w_i \Xi) = a_{14, w_i}(\xi, \Xi) \zeta(\xi, \Xi) = a_{w_i, 14}(\xi, \Xi) \zeta(\xi, \Xi). \]

**Proof.** From Proposition 4.2.1, we have
\[ a_{14, w_1}(\xi, \Xi) = \frac{\Omega_{\xi, w_1, \Xi}(\phi_{14} \otimes R(\eta w_1)(\Phi_{14} + \Phi_{w_1}))}{\Omega_{\xi, \Xi}(\phi_{14} \otimes R(\eta w_1)(\Phi_{14} + \Phi_{w_1}))} \]
\[ = \frac{(1 - q^{-1} \Xi^{-1}_1 \Xi^{-1}_2)(1 - q^{-1/2} \Xi_2 \Xi_3)(1 - q^{-1/2} \Xi_1 \Xi_3 \Xi_3)}{(1 - q^{-1} \Xi^{-1}_2 \Xi^{-1}_1)(1 - q^{-1/2} \Xi_2 \Xi_3)(1 - q^{-1/2} \Xi_1 \Xi_3 \Xi_3)} \]
\[ = \frac{\zeta(\xi, w_1 \Xi)}{\zeta(\xi, \Xi)}. \]
The other cases are obtained in the same way as above. \( \square \)

**Proof of Proposition 4.5.1.** Since \( T_{w_i, \Xi}(\phi_{K, \Xi}) = c_{w_i}(\Xi)\phi_{K, \Xi} \) holds for \( i = 1, 2 \), we have
\[ \frac{S_{\xi,w_i,\Xi}(g)}{\zeta(\xi, w_i \Xi)} = \frac{\Omega_{\xi, w_i, \Xi}(\phi_{K_{0, \xi}} \otimes R(g)\phi_{K, \Xi})}{\zeta(\xi, w_i \Xi)} \]
\[ = c_{w_i}(\Xi)^{-1} \frac{\Omega_{\xi, w_i, \Xi}(\phi_{K_{0, \xi}} \otimes T_{w_i, \Xi}(R(g)\phi_{K, \Xi}))}{\zeta(\xi, w_i \Xi)} \]
\[ = c_{w_i}(\Xi)^{-1} (\text{id} \otimes T_{w_i, \Xi})^* \frac{\Omega_{\xi, w_i, \Xi}(\phi_{K_{0, \xi}} \otimes R(g)\phi_{K, \Xi})}{\zeta(\xi, w_i \Xi)} \]
\[ = a_{14, w_i}(\xi, \Xi) \frac{\Omega_{\xi, \Xi}(\phi_{K_{0, \xi}} \otimes R(g)\phi_{K, \Xi})}{\zeta(\xi, w_i \Xi)} \]
\[ = \frac{\Omega_{\xi, \Xi}(\phi_{K_{0, \xi}} \otimes R(g)\phi_{K, \Xi})}{\zeta(\xi, \Xi)} \]
\[ = \frac{S_{\xi, \Xi}(g)}{\zeta(\xi, \Xi)}. \]
This means that the value \( S_{\xi, \Xi}(g)/\zeta(\xi, \Xi) \) is \( W \)-invariant as a function of \( \Xi \). The \( W_0 \)-invariance follows in the same manner. \( \square \)

Next we prove an explicit formula of the Shintani function \( S_{\xi, \Xi} \) for \( (\xi, \Xi) \in X^2 \) such that \( \zeta(\xi, \Xi) \neq 0 \). By Theorem 3.2.1, it suffices to know the value
\[ S_{\xi, \Xi}(t(\lambda')\eta t(\lambda)) = S_{\xi, \Xi}(t(\lambda')\eta t(\lambda)^{-1}t(\lambda_3)) \]
for \((\lambda, \lambda') \in \Lambda^+ \times \Lambda^+_0\). For \((\xi, \Xi) \in X^2\), we set
\[
c_{S}(\xi, \Xi) := \frac{\prod_{\beta \in \Sigma^+} c_{\beta}(\xi) \prod_{\alpha \in \Sigma^+} c_{\alpha}(\Xi)}{\zeta(\xi, \Xi)} = \frac{b(\xi, \Xi)}{d'(\xi)d(\Xi)},
\]
where
\[
d'(\xi) := (1 - \xi_1)(1 - \xi_2), \quad d(\Xi) := (1 - \Xi_1\Xi_2)(1 - \Xi_1\Xi_2^{-1})(1 - \Xi_1)(1 - \Xi_2).
\]

**Theorem 4.5.3** For any element \((\xi, \Xi) \in X^2\) such that \(\zeta(\xi, \Xi) \neq 0\), we have
\[
S_{\xi, \Xi}(t(\lambda')\eta w_t t(\lambda)^{-1}) = q^6 \cdot \text{vol}(B) \text{vol}(B_0) \sum_{w, w' \in W} c_{S}(w' \xi, w \Xi) \left((w \Xi)^{-1} \delta_{1/2}\right)(t(\lambda))(w' \xi)^{-1} \delta_{1/2}^{0}(t(\lambda')).
\]

**Proof.** We can see that
\[
B_0 t(\lambda') B_0 \eta w_t B t(\lambda)^{-1} B \subset K_0 t(\lambda') \eta w_t t(\lambda)^{-1} K
\]
in the same way as Lemma 4.2.3. Hence we have
\[
\begin{align*}
L(\chi_{B_0 t(\lambda')^{-1} B_0}) R(\chi_{B t(\lambda)^{-1} B}) S_{\xi, \Xi}(\eta w_t) &= \int_G dg \int_{G_0} dg \chi_{B_0 t(\lambda')^{-1} B_0}(g') S_{\xi, \Xi}(g' \eta w_t g) \chi_{B t(\lambda)^{-1} B}(g) \\
&= \int_G dg \int_{G_0} dg \chi_{B_0 t(\lambda')^{-1} B_0}(g') S_{\xi, \Xi}(g' \eta w_t g) \chi_{B t(\lambda)^{-1} B}(g) \\
&= \int_{B t(\lambda)^{-1} B} \int_{B_0 t(\lambda') B_0} dg' S_{\xi, \Xi}(g' \eta w_t g) \\
&= \text{vol}(B t(\lambda)^{-1} B) \text{vol}(B_0 t(\lambda') B_0) \text{vol}(B_0) \text{vol}(B t(\lambda') B_0) \text{vol}(B_0) S_{\xi, \Xi}(t(\lambda') \eta w_t t(\lambda)^{-1}) \\
&= \text{vol}(B t(\lambda) B) \text{vol}(B_0 t(\lambda') B_0) \text{vol}(B_0) S_{\xi, \Xi}(t(\lambda') \eta w_t t(\lambda)^{-1}).
\end{align*}
\]

Let \(\{g_w\}_{w \in W}\) and \(\{g'_{w'}\}_{w' \in W_0}\) be the bases of \(i_{B_0}^{G}(\Xi)^B\) and \(i_{B_0}^{G_0}(\xi)^{B_0}\) obtained by Proposition 2.1.3, respectively. We note that these bases satisfy
\[
\phi_{K, \Xi} = q^4 \sum_{w \in W} \tau_w(\Xi) g_w, \quad \phi_{K_0, \xi} = q^2 \sum_{w' \in W_0} \tau'_{w'}(\xi) g'_{w'},
\]
where
\[
\tau_w(\Xi) := \prod_{\alpha \in \Sigma^+} c_{\alpha}(\Xi), \quad \tau'_{w'}(\xi) := \prod_{\beta \in \Sigma^+} c'_{\beta}(\xi).
\]
Hence \(S_{\xi, \Xi}(t(\lambda') \eta w_t t(\lambda)^{-1})\) can be expressed as
\[
S_{\xi, \Xi}(t(\lambda') \eta w_t t(\lambda)^{-1})
\]
The unramified Shintani functions

\[ = \text{vol}(Bt(\lambda)B; dg)^{-1}\text{vol}(B_{0t}(\lambda')B_{0}; dg')^{-1}L(\text{ch}_{B_{0t}(\lambda')^{-1}B_{0}})R(\text{ch}_{Bt(\lambda)^{-1}B})S_{\zeta, \Xi}(\eta w_{t}) \]
\[ = \text{vol}(Bt(\lambda)B; dg)^{-1}\text{vol}(B_{0t}(\lambda')B_{0}; dg')^{-1} \]
\[ \times L(\text{ch}_{B_{0t}(\lambda')^{-1}B_{0}})R(\text{ch}_{Bt(\lambda)^{-1}B})\Omega_{\zeta, \Xi}(\phi_{K_{0}, \xi} \otimes R(\eta w_{t})\phi_{K, \Xi}) \]
\[ = q^{6} \cdot \text{vol}(Bt(\lambda)B; dg)^{-1}\text{vol}(B_{0t}(\lambda')B_{0}; dg')^{-1} \]
\[ \times \sum_{w' \in W_{0}}(\Xi)\Omega_{\zeta, \Xi}(R(\text{ch}_{B_{0t}(\lambda')^{-1}B_{0}})g_{w'} \otimes R(\eta w_{t})g_{w}) \]
\[ = q^{6} \sum_{w \in W} \sum_{w' \in W_{0}}(\Xi)\Omega_{\zeta, \Xi}(R(\text{ch}_{B_{0t}(\lambda')^{-1}B_{0}})g_{w'} \otimes R(\eta w_{t})g_{w}). \]

Since \( \Phi_{14} = g_{14} \) and \( \phi_{14} = g'_{14} \), we have
\[ \Omega_{\zeta, \Xi}(g'_{14} \otimes R(\eta w_{t})g_{14}) = \text{vol}(B)\text{vol}(B_{0}) \]
by Proposition 4.2.2. Therefore we have

\[ \frac{S_{\zeta, \Xi}(t(\lambda')\eta w_{t}(\lambda)^{-1})}{\zeta(\xi, \Xi)} = q^{6} \cdot \text{vol}(B; dg)\text{vol}(B_{0}; dg) \overline{c}_{\Xi}(\zeta(\xi, \Xi)) \left( \Xi^{-1}\delta^{1/2} \right) (t(\lambda)) \left( \xi^{-1}\delta_{0}^{1/2} \right) (t(\lambda')) \]
\[ + q^{6} \sum_{(w, w') \in W_{0} \times W_{0} - \{(1_{14}, 1_{4})\}} \overline{c}_{\Xi}(\zeta(\xi, \Xi)) \left( \Xi^{-1}\delta^{1/2} \right) (t(\lambda)) \left( (w')^{-1}\delta_{0}^{1/2} \right) (t(\lambda')) \]
\[ \times \Omega_{\zeta, \Xi}(g_{w'} \otimes R(\eta w_{t})g_{w}) \]
\[ = q^{6} \cdot \text{vol}(B; dg)\text{vol}(B_{0}; dg') \overline{c}_{\Xi}(\zeta(\xi, \Xi)) \left( \Xi^{-1}\delta^{1/2} \right) (t(\lambda)) \left( \xi^{-1}\delta_{0}^{1/2} \right) (t(\lambda')) \]
\[ + q^{6} \sum_{(w, w') \in W_{0} \times W_{0} - \{(1_{14}, 1_{4})\}} \overline{c}_{\Xi}(\zeta(\xi, \Xi)) \left( \Xi^{-1}\delta^{1/2} \right) (t(\lambda)) \left( (w')^{-1}\delta_{0}^{1/2} \right) (t(\lambda')) \]
\[ \times \Omega_{\zeta, \Xi}(g_{w'} \otimes R(\eta w_{t})g_{w}). \]

Since \( \Xi \) and \( \xi \) are regular, the set \( \{(w\Xi)^{-1}\delta^{1/2} \otimes (w'\xi)^{-1}\delta_{0}^{1/2} \mid (w', w) \in W_{0} \times W\} \) of characters of \( T_{0} \times T \) is linearly independent. Hence Proposition 4.5.1 implies that

\[ \frac{S_{\zeta, \Xi}(t(\lambda')\eta w_{t}(\lambda)^{-1})}{\zeta(\xi, \Xi)} = q^{6} \cdot \text{vol}(B; dg)\text{vol}(B_{0}; dg') \]
\[ \sum_{w \in W} \sum_{w' \in W_{0}} \overline{c}_{\Xi}(w'\xi, w\Xi) \left( (w\Xi)^{-1}\delta^{1/2} \right) (t(\lambda)) \left( (w'\xi)^{-1}\delta_{0}^{1/2} \right) (t(\lambda')). \]
Finally we prove our main theorem, which is an explicit formula of the Shintani function of type $(\xi, \Xi)$ for any $(\xi, \Xi) \in X^d$.

**Theorem 4.5.4** Let $(\xi, \Xi)$ be any element of $X_{nr}(T_0) \times X_{nr}(T)$. Then we have

$$\dim_{\mathbb{C}} S(\xi, \Xi) = \begin{cases} 
1 & \text{if } (\xi, \Xi) \in X^d, \\
0 & \text{otherwise}.
\end{cases}$$

If $(\xi, \Xi) \in X^d$, for any nonzero element $S \in S(\xi, \Xi)$ we have $S(1_4) \neq 0$, and the Shintani function $W_{\xi, \Xi} \in S(\xi, \Xi)$ with $W_{\xi, \Xi}(1_4) = 1$ is given by

$$W_{\xi, \Xi}(g(\lambda', \lambda)) = (\Xi_1 \Xi_2 \Xi_3^2)^{\lambda_3} W_{\xi, \Xi}(t(\lambda') r(\lambda t(\lambda)^{-1})$$

$$= \frac{(\Xi_1 \Xi_2 \Xi_3^2)^{\lambda_3}}{(1 - q^{-2})^2} \sum_{w \in W, w' \in W_0} c_S(w', w_\Xi) \left( (w_\Xi)^{-1} \delta^{1/2} \right) (t(\lambda)) \left( (w' \xi)^{-1} \delta_{0}^{1/2} \right) (t(\lambda'))$$

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$.

In order to prove Theorem 4.5.4, we first compute the value

$$S_{\xi, \Xi}(1_4) = S_{\xi, \Xi}(\eta w_t) = q^6 \cdot \zeta(\xi, \Xi) \text{vol}(B; dg) \text{vol}(B_0; dg') \sum_{w \in W, w' \in W_0} c_S(w', w_\Xi)$$

at the identity and normalize the Shintani function. The following lemma can be easily checked by direct computation.

**Lemma 4.5.5**

$$S_{\xi, \Xi}(1_4) = S_{\xi, \Xi}(\eta w_t)$$

$$= q^6 (1 - q^{-2})^2 \zeta(\xi, \Xi) \text{vol}(B; dg) \text{vol}(B_0; dg').$$

Hence we have immediately the following proposition.

**Proposition 4.5.6** For any $(\xi, \Xi) \in D^d$ such that $\zeta(\xi, \Xi) \neq 0$, the basis of $S(\xi, \Xi)$, $W_{\xi, \Xi} \in S(\xi, \Xi)$ with $W_{\xi, \Xi}(1_4) = 1$, is given by

$$W_{\xi, \Xi}(g(\lambda', \lambda)) = (\Xi_1 \Xi_2 \Xi_3^2)^{\lambda_3} S_{\xi, \Xi}(t(\lambda') \eta w_t (\lambda t)^{-1}) / S_{\xi, \Xi}(1_4)$$

$$= \frac{(\Xi_1 \Xi_2 \Xi_3^2)^{\lambda_3}}{(1 - q^{-2})^2} \sum_{w \in W, w' \in W_0} c_S(w', w_\Xi) \left( (w_\Xi)^{-1} \delta^{1/2} \right) (t(\lambda)) \left( (w' \xi)^{-1} \delta_{0}^{1/2} \right) (t(\lambda'))$$

for all $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$.

For any $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$, we denote by $F_{\xi, \Xi}(\lambda', \lambda)$ a rational function on $X^d$ given by the right hand side of the formula in Proposition 4.5.6. We set $F_{\xi, \Xi}(k_0 g(\lambda', \lambda) k) = F_{\xi, \Xi}(\lambda', \lambda)$ for any $(k_0, \lambda', \lambda, k) \in K_0 \times \Lambda_0^{++} \times \Lambda^+ \times K$. Then, for $(\xi, \Xi) \in D^d$ such that $\zeta(\xi, \Xi) \neq 0$, the function $F_{\xi, \Xi}$ is a Shintani function with $F_{\xi, \Xi}(1_4) = 1$. Theorem 4.5.4 follow from Theorem 3.3.1, Proposition 4.5.6 and the following proposition.
Proposition 4.5.7  

i) For any $x \in G$, the rational function $F_{\xi, \mu}(x)$ is regular on the whole $X^\sharp$;

ii) For any $(\xi, \Xi) \in X^\sharp$, the function $F_{\xi, \mu}$ on $G$ is the Shintani function of type $(\xi, \Xi)$ with $F_{\xi, \Xi}(1_4) = 1$.

Proof. From Theorem 3.2.1, it is enough to show that $F_{\xi, \mu}(\mu', \mu)$ is regular on $X^\sharp$ for any $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$. By induction on $m(\mu', \mu) \geq 1$, we shall see the assertion (i). We put

$$U_0 = \{ (\xi, \Xi) \in D^\sharp \mid \zeta(\xi, \Xi) \neq 0 \}.$$ 

Then $X^\sharp - U_0$ is a countable union of proper Zariski closed subsets in $X^\sharp$ (see Corollary 4.4.2). In particular, $U_0$ is a dense subset of $X^\sharp$ by Proposition 4.3.17.

First we assume that $m(\mu', \mu) = 1$. Then, since

$$K_0(\mu')K = K_0g(\mu', \mu)K,$$

for all $(\xi, \Xi) \in U_0$ we have

$$\omega_{\xi}(\ch_{K_0}(\mu')^{-1}K_0)\omega_{\Xi}(\ch_{K_0}K) = C_{\mu', \mu}\vol(K_0g(\mu', \mu)K; dg) F_{\xi, \Xi}(\mu', \mu),$$

where $C_{\mu', \mu}$ is a certain positive integer, which does not depend on $(\xi, \Xi)$. Hence $F_{\xi, \Xi}(\mu', \mu)$ is regular on $X^\sharp$.

Next we assume that $F_{\xi, \Xi}(\lambda', \lambda)$ is regular on $X^\sharp$ for any $(\lambda', \lambda) \in \Lambda_0^{++} \times \Lambda^+$ which satisfies $m(\mu, \mu) > m(\lambda', \lambda)$. If

$$K_0(\mu')K = \bigcup_{i=0}^M K_0g(\lambda'_i, \lambda_i)K, \quad (\lambda'_{(0)}, \lambda_{(0)}) = (\mu', \mu),$$

then we have an equality

$$\omega_{\xi}(\ch_{K_0}(\mu')^{-1}K_0)\omega_{\Xi}(\ch_{K_0}K) = C_{\mu', \mu}\vol(K_0g(\mu', \mu)K; dg) F_{\xi, \Xi}(\mu', \mu) + \sum_{i=1}^M C_{\mu', \mu}^{(i)}\vol(K_0g(\lambda'_i, \lambda_i)K; dg) F_{\xi, \Xi}(\lambda'_i, \lambda_i)$$

for all $(\xi, \Xi) \in U_0$ (see (3.3)). Here $C_{\mu', \mu}^{(i)}$ is a certain positive integer which does not depend on $(\xi, \Xi)$ for all $i = 0, \cdots, M$. Since $m(\mu', \mu) > m(\lambda'_i, \lambda_i)$ for $i = 1, \cdots, M$ by Lemma 3.3.13, each $F_{\xi, \Xi}(\lambda'_i, \lambda_i)$ is regular on $X^\sharp$ by the induction hypothesis. Hence $F_{\xi, \Xi}(\mu', \mu)$ is regular on $X^\sharp$.

We show the assertion (ii). The assertion (i) implies that, for all $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$ and all $(\phi, \Phi) \in \mathcal{H}(G_0, K_0) \times \mathcal{H}(G, K)$, the function $X^\sharp \ni (\xi, \Xi) \mapsto \langle L(\phi)R(\Phi)F_{\xi, \Xi}(g(\mu', \mu)) \rangle \in \mathbb{C}$ is regular on $X^\sharp$. If for any $(\mu', \mu) \in \Lambda_0^{++} \times \Lambda^+$ we put $f_{\mu', \mu}(\xi, \Xi) = \langle L(\phi)R(\Phi)F_{\xi, \Xi}(g(\mu', \mu)) - \omega_\xi(\phi)\omega_{\Xi}(\Phi)F_{\xi, \Xi}(g(\mu', \mu)) \rangle$, then $f_{\mu', \mu}$ is regular on $X^\sharp$. Since $f_{\mu', \mu}$ is zero on the dense subset $U_0$ of $X^\sharp$, we have $f_{\mu', \mu} = 0$. Namely, for any $(\xi, \Xi) \in X^\sharp$, the function $F_{\xi, \Xi}$ is the Shintani function of type $(\xi, \Xi)$ with $F_{\xi, \Xi}(1_4) = 1$. □

5 Evaluation of a local zeta integral of Murase–Sugano type

In this section, we introduce a local zeta integral of Murase–Sugano type (cf. [MS]) and evaluate it as an application of our explicit formula of the Shintani function on $G$. 

5.1 A local zeta integral of Murase–Sugano type.

In this subsection, we define a local zeta integral of Murase–Sugano type and state that the zeta integral represents the local spin $L$-factor of $GSp_4$. Let $G_1$ be the subgroup of $GL_4(F)$ defined by

$$G_1 := \{ g \in GL_4(F) | \det(g) \in (F^\times)^2 \}$$

and $P_{22}$ the maximal parabolic subgroup of $G_1$ given by

$$P_{22} = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \in G_1 \right\}.$$

We note that $G \subset G_1$. We put

$$\kappa = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in G_1.$$

Let $P_{22} = M_{22}N_{22}$ be the Levi decomposition of $P_{22}$, where

$$M_{22} = \left\{ m_1(a,b) := \kappa \begin{pmatrix} a \\ b \end{pmatrix} \kappa^{-1} | a,b \in GL_2(F), \det(ab) \in (F^\times)^2 \right\},$$

$$N_{22} = \left\{ \begin{pmatrix} 1 & * & * \\ 1 & * & 1 \\ * & 1 \end{pmatrix} \in G_1 \right\}.$$

We note that every $m_1(a,b) \in M_{22}$ has a factorization

$$m_1(a,b) = \kappa \begin{pmatrix} \alpha^{-1} \cdot a \\ b \end{pmatrix} \begin{pmatrix} \alpha \cdot 1_2 \\ 1_2 \end{pmatrix} \kappa^{-1}, \quad \alpha^2 = \frac{\det(ab)}{\det(b)^2} = \frac{\det(a)}{\det(b)}.$$

Namely, for any $m_1 \in M_{22}$ we have a factorization $m_1 = \beta(m_1) \text{diag}(\alpha(m_1), 1, \alpha(m_1), 1)$ for some $(\beta(m_1), \alpha(m_1)) \in G_0 \times F^\times$. We note that such a factorization of $m_1$ is not unique. We set $K_1 := G_1 \cap GL_4(o)$. Then every $g \in G_1$ has an Iwasawa decomposition

$$g = m_1(g)n_1(g)k_1(g)$$

$$= \beta(m_1(g)) \text{diag}(\alpha(m_1(g)), 1, \alpha(m_1(g)), 1)n_1(g)k_1(g)$$

for some $(m_1(g), n_1(g), k_1(g)) \in M_{22} \times N_{22} \times K_1$. For every $g \in G_1$, we fix such a factorization and set $\beta(g) := \beta(m_1(g))$ and $\alpha(g) := \alpha(m_1(g))$. The following lemma is easily checked by direct calculation.

**Lemma 5.1.1**

$$P_{22} \cap K_1 = \kappa \begin{pmatrix} GL_2(o) & M_2(o) \\ GL_2(o) \end{pmatrix} \kappa^{-1} \cap K_1.$$
Let \((\xi, \Xi) \in X^\xi\). For any Shintani function \(S \in S(\xi, \Xi)\), we define a local zeta integral of Murase-Sugano type by

\[
Z_{MS}(s; S) := \int_{G_0 \backslash G} W(\beta(g)^{-1}g) |\alpha(g)|^s \, dg \quad (s \in \mathbb{C}),
\]

where \(dg\) is the right invariant measure of \(G_0 \backslash G\). Since a Shintani function \(S \in S(\xi, \Xi)\) can be regarded as a function on \(K_0 \backslash G/K\), it follows from Lemma 5.2.3, described in §5.2, that the value \(S(\beta(g)^{-1}g)|\alpha(g)|^s\) is independent of a choice of an Iwasawa decomposition of \(g \in G \subset G_1\). For \(\chi = (\chi_1, \chi_2, \chi_3) \in (\mathbb{C}^\times)^3, s \in \mathbb{C}\), we set

\[
L(\chi; s) := (1 - \chi_3q^{-s})^{-1}(1 - \chi_1\chi_3q^{-s})^{-1}(1 - \chi_2\chi_3q^{-s})^{-1}(1 - \chi_1\chi_2\chi_3q^{-s})^{-1}.
\]

As an application of the explicit formula of Shintani functions, we shall prove the following theorem in §5.3.

**Theorem 5.1.2** Let \((\xi, \Xi) \in X^\xi\). For the Shintani function \(S \in S(\xi, \Xi)\) such that \(S(1_4) = 1\), the zeta integral (5.1) is absolutely convergent if

\[
\Re(s) > s_\Xi := \max \{ \log_2 \|\Xi_3\|, \log_2 \|\Xi_1\Xi_3\|, \log_2 \|\Xi_2\Xi_3\|, \log_2 \|\Xi_1\Xi_2\Xi_3\| \}.
\]

Then the zeta integral (5.1) can be evaluated as

\[
Z_{MS}(s; S) = \frac{L(\Xi; s)}{L(\xi; s + 1/2)} \quad (\Re(s) > s_\Xi).
\]

**Remark 5.1.3** Theorem 5.1.2 is generalization of [MS, Theorem 1.6] for the pair \((SO_5(F_0), SO_4(F_0))\) of split special orthogonal groups. Here \(F_0\) is a non-archimedean local field of characteristic 0. While they proved their result without using the explicit formula of Shintani functions for \((SO_5(F_0), SO_4(F_0))\), we compute the local zeta integral (5.1) using that for \((G, G_0)\).

### 5.2 A double coset decomposition.

In this subsection, we prove the following theorem, which is necessary for our computation of the zeta integral \(Z_{MS}(s; S)\).

**Theorem 5.2.1** We have

\[
G = \bigsqcup_{l \geq 0} G_0 a(l) K, \quad a(l) := g(0, (l, l, l)).
\]

**Remark 5.2.2** An Iwasawa decomposition of \(a(l)\) is given by

\[
a(l) = \begin{pmatrix} -\omega^l & -\omega^l - 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \omega^l & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \omega^{-l} \\ -\omega^{-l} & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -\omega^l & -\omega^l - 1 \\ -1 & 1 \end{pmatrix} \in K_0.
\]

We note that

\[
\begin{pmatrix} -\omega^l & -\omega^l - 1 \\ -1 & 1 \end{pmatrix} \in K_0.
\]

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First we prove the following lemma.

**Lemma 5.2.3** Let \( g \in G \). We assume that \( g \in G_0 a(l) K \) for some \( l \geq 0 \). Then the following assertions hold:

i) \( |\alpha(g)| = q^{-l} \);

ii) \( \beta(g)^{-1} g \in K_0 a(l) K \).

**Proof.** Let \( g = h a(l) k \) for some \( h, k \in G_0 \times K \). We note that \( a(l) \in K_0 \text{diag}(\varpi^l, 1, \varpi^l, 1) N_{22} K_1 \) (see Remark 5.2.2). Hence we have

\[
\beta(g)^{-1} g = \text{diag}(\alpha(g), 1, \alpha(g), 1) n_1(g) k_1(g) \in \beta(g)^{-1} h K_0 \text{diag}(\varpi^l, 1, \varpi^l, 1) N_{22} K_1.
\]

In particular, \( \beta(g)^{-1} h \in K_0 \) and \( \alpha(g) \in \varpi^l p^\times \) from Lemma 5.1.1. \qed

Next we show the following lemma to prove the disjointness of Theorem 5.2.1.

**Lemma 5.2.4** Let \( g \in G \). If \( |\alpha(g)| = q^{-l} \) and \( g = h a(l') k \) for some \( h, k \in G_0 \times K \), then \( l = l' \).

**Proof.** We have

\[
\text{diag}(\alpha(g), 1, \alpha(g), 1) n_1(g) k_1(g) \in K_0 a(l') K
\]

by Lemma 5.2.3 (ii). By comparing the determinants of both sides, we have \( q^{-2l} = |\alpha(g)|^2 = q^{-2l'} \). \qed

**Proof of Theorem 5.2.1.** By the Iwasawa decomposition of \( G \), \( G \) can be written as

\[
G = G_0 N K
\]

where

\[
G = G_0 K \cup G_0 \left\{ \begin{pmatrix} 1 & x_0 \\ 1 & 1 \\ 1 & -x_0 \\ 1 \end{pmatrix} \bigg| x_0 \neq 0 \right\} K
\]

\[
\cup G_0 \left\{ \begin{pmatrix} 1 & x_2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \bigg| x_2 \neq 0 \right\} K \cup G_0 \left\{ \begin{pmatrix} 1 & x_0 \\ 1 & x_2 \\ 1 & x_2 \\ 1 \end{pmatrix} \bigg| x_0, x_2 \neq 0 \right\} K.
\]

Since \( w_2 = w_2' \in G_0 \cap K = K_0 \), we have

\[
G_0 \begin{pmatrix} 1 & x_0 & x_2 \\ 1 & 1 & 1 \\ 1 & -x_0 & 1 \end{pmatrix} K = G_0 \begin{pmatrix} 1 & x_2 & x_0 \\ 1 & 1 & 1 \\ 1 & -x_2 & 1 \end{pmatrix} K.
\]

Hence we have

\[
G = G_0 K \cup G_0 \left\{ \begin{pmatrix} 1 & x_0 \\ 1 & 1 \\ 1 & -x_0 \\ 1 \end{pmatrix} \bigg| x_0 \neq 0 \right\} K \cup G_0 \left\{ \begin{pmatrix} 1 & x_0 \\ 1 & x_2 \\ 1 & x_2 \\ 1 \end{pmatrix} \bigg| x_0, x_2 \neq 0 \right\} K
\]
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\[ G_0 K \cup \bigcup_{k \in \mathbb{Z}} G_0 \begin{pmatrix} 1 & \varpi^k \\ -1 & 1 \end{pmatrix} K \cup \bigcup_{k, l \in \mathbb{Z}} G_0 \begin{pmatrix} 1 & \varpi^l \\ -1 & 1 \end{pmatrix} K \]

\[ = G_0 K \cup \bigcup_{k > 0} G_0 \begin{pmatrix} 1 & \varpi^{-k} \\ -1 & 1 \end{pmatrix} K \cup \bigcup_{k, l > 0} G_0 \begin{pmatrix} 1 & \varpi^{-l} \\ -1 & 1 \end{pmatrix} K \]

Since

\[ G_0 \begin{pmatrix} 1 & \varpi^{-k} \\ -1 & 1 \end{pmatrix} K = G_0 \begin{pmatrix} 1 & \varpi^{-l} \\ -1 & 1 \end{pmatrix} K, \]

we have

\[ G = \bigcup_{k \geq l \geq 0} G_0 \begin{pmatrix} 1 & \varpi^{-k} & \varpi^{-l} \\ 1 & -1 & 1 \end{pmatrix} K. \]

If \( k \geq l \geq 0 \), we have

\[ a(k) = \begin{pmatrix} \varpi^k & -1 - \varpi^k & \varpi^k \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \varpi^{-k} & \varpi^{-l} \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}. \]

Hence we obtain

\[ G = \bigcup_{l \geq 0} G_0 a(l) K. \]

From Remark 5.2.2 and Lemma 5.2.4, if \( a(l') \in G_0 a(l) K \), then \( l = l' \). This means that the above union is disjoint. We have completed the proof of Theorem 5.2.1.

5.3 Evaluation of the local zeta integral.

In this subsection, we evaluate the zeta integral \( Z_{MS}(s; S) \) by using our explicit formula of the Shintani function. Theorem 5.2.1 yields that

\[ \int_{G_0 \backslash G} F(g) dg = \sum_{l=0}^{\infty} F(a(l)) v_1, \quad v_1 := \text{vol}(G_0 \cap a(l) K a(l)^{-1}; dg)^{-1} \]
for any function $F : G_0 \setminus G/K \to \mathbb{C}$. We note that the integrand $S(\beta(g)^{-1}g)|\alpha(g)|^s$ of the zeta integral is a function on $G_0 \setminus G/K$. Hence we have

$$Z_{MS}(s; S) = \sum_{l=0}^{\infty} S(\beta(a(l))^{-1}a(l))|\alpha(a(l))|^s v_l$$

$$= \sum_{l=0}^{\infty} S(\beta(a(l))^{-1}a(l))v_l q^{-ls}.$$ 

Since $\beta(a(l))^{-1}a(l) \in K_0 a(l)K$, it is enough to compute the volume $v_l$ and the value

$$S(\beta(a(l))^{-1}a(l)) = S(a(l)).$$

First we calculate the volume $v_l$. From the definition of the measure $dg'$ of $G_0$, we have $v_0^{-1} = \text{vol}(G_0 \cap \eta K \eta^{-1}; dg') = \text{vol}(K_0; dg') = 1$. Let $l \geq 0$. We put

$$\tilde{a}(l) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} a(l) \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \omega^l & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

We note that $\tilde{a}(l) \in K_0 a(l)K$. By the invariance of the Haar measure $dg'$, we have

$$v_l^{-1} = \text{vol}(G_0 \cap \tilde{a}(l) K \tilde{a}(l)^{-1}; dg') = \text{vol}(K_0^{(l)}; dg'),$$

where $K_0^{(l)} := G_0 \cap \tilde{a}(l) w_2 K w_2^{-1} \tilde{a}(l)^{-1}$. Then $K_0^{(l)}$ is the group consisting of elements of $K_0$ which satisfy the following four congruences:

$$(5.2) \quad h = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right) \pmod{p^l}$$

$$(5.3) \quad \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) \equiv \left( \begin{array}{cc} a_2 & -b_2 \\ -c_2 & d_2 \end{array} \right) \pmod{p^l}$$

We note that $K_0^{(0)} = K_0$ and $K_0^{(l)} \supset K_0^{(l+1)}$ $(l \geq 0)$. In order to compute the volume $v_l$, it is enough to compute the index $[K_0^{(l)} : K_0^{(l+1)}]$ for all $l \geq 0$.

**Proposition 5.3.1** We have disjoint unions

$$K_0 = \bigsqcup_{u \equiv 0 \pmod{p}} K_0^{(1)} \left( \begin{pmatrix} u & x \\ u^{-1} & 1 \end{pmatrix}, 1 \right) \bigsqcup_{u, v \equiv 0 \pmod{p}} K_0^{(1)} \left( \begin{pmatrix} u & v \\ u^{-1} & 1 \end{pmatrix}, 1 \right)$$

$$\bigsqcup_{u \equiv 0 \pmod{p}} K_0^{(1)} \left( \begin{pmatrix} x & u \\ -u^{-1} & 1 \end{pmatrix}, 1 \right) \bigsqcup_{u \equiv 0 \pmod{p}} K_0^{(1)} \left( \begin{pmatrix} -u & 0 \\ u^{-1} & 1 \end{pmatrix}, 1 \right)$$
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\[ \bigcup_{u_1, u_2 \not\equiv 0 \mod p} K^{(1)}_0 \left( \begin{array}{c} u_1 \\ v_1 \\ u_2 \end{array} \right), 1_2 \]

and

\[ K^{(l)}_0 = \bigcup_{x, y \mod p} K^{(l+1)}_0 \left( \begin{array}{c} 1 + \omega^lx \\ \omega^ly \\ (1 + \omega^lx)^{-1} \end{array} \right), 1_2 \]

for any \( l > 0. \)

Before proving Proposition 5.3.1, we calculate the volume \( v_l \) for all \( l \geq 0 \) and prove Theorem 5.1.2.

**Corollary 5.3.2** For \( l \geq 0, \) we have

\[ v_l = \begin{cases} 1 & \text{(if } l = 0), \\ q^{3l(1-q^{-2})} & \text{(if } l > 0). \end{cases} \]

In particular, the generating function for the sequence \( \{v_l\}_{l \geq 0} \) is given by

\[ \sum_{l=0}^{\infty} v_l t^l = \frac{1 - qt}{1 - q^3t}. \]

The domain of convergence of the above power series is \( ||t|| < q^{-3}. \)

**Proof.** Since

\[ [K_0 : K^{(1)}_0] = q(q^2 - 1) = q^3(1 - q^{-2}), \quad [K^{(l)}_0 : K^{(l+1)}_0] = q^3 (l > 0) \]

from Proposition 5.3.1, we obtain \( v_l^{-1} = \text{vol}(K^{(l)}_0; dh') = q^{-3l(1-q^{-2})^{-1}} \) for all \( l > 0. \)

**Remark 5.3.3** Murase–Sugano proved Corollary 5.3.2 as a corollary of computation of their local zeta integral (see [MS, Lemma 1.12]).

**Proof of Theorem 5.1.2.** Let \( S \in S(\xi, \Xi) \) be the Shintani function such that \( S(1_4) = 1. \) From Theorem 4.5.4, we have

\[ S(a(l)) = \left( \Xi_1 \Xi_2 \Xi_3^2 \right)^l \sum_{w \in W} c_S(w', w) \left( (w \Xi)^{-1} \delta^{1/2} \right)(a((l, l, l))) \]
We note that $(\Xi_1 \Xi_2^2 \Xi_3^2)^l = \frac{q^{-3l}(\Xi_1 \Xi_2^2 \Xi_3^2)^l}{(1 - q^{-2})^2} \sum_{w \in W} c_s(w', \xi, w) w^{-1}((l, l, l))).$

Hence we have

$$Z_{MS}(s; S) = \sum_{l=0}^{\infty} S(a(l)) q^{-ls} v_l$$

$$= (1 - q^{-2})^{-2} \sum_{w \in W} c_s(w', \xi, w) \zeta_w(s),$$

where

$$\zeta_w(s) = \sum_{l=0}^{\infty} (\Xi_1 \Xi_2^2 \Xi_3^2)^l (w \Xi)^{-1}((l, l, l))) q^{-l(s+3)} v_l$$

for any $w \in W$. From Corollary 5.3.2, we have

$$\zeta_w(s) = \begin{cases} 
1 - q^{-s+2} \Xi_3 & \text{(if } w = 1_4, w_1), \\
1 - q^{-s-2} \Xi_2^3 & \text{(if } w = w_2, w_1 w_2), \\
1 - q^{-s} \Xi_2 \Xi_3 & \text{(if } w = w_2 w_1, w_1 w_2 w_1), \\
1 - q^{-s} \Xi_1 \Xi_2^2 \Xi_3 & \text{(if } w = w_2 w_1 w_2, w_2) \\
1 - q^{-s} \Xi_1 \Xi_2 \Xi_3 & \text{(if } w = w_2 w_1 w_2, w_2) 
\end{cases}$$

for $\Re(s) > s_\Xi$. Hence we obtain

$$Z_{MS}(s; S) = (1 - q^{-2})^{-2} \sum_{w \in W} c_s(w', \xi, w) \zeta_w(s)$$

$$= (1 - q^{-s-1/2} \xi_1^{-1} \xi_3^{-1})(1 - q^{-s-1/2} \xi_2^{-1} \xi_3^{-1})(1 - q^{-s-1/2} \xi_1^{-1} \xi_2^{-1} \xi_3^{-1}) (1 - q^{-s-1/2} \xi_3^{-1})$$

$$= \frac{L(\xi; s)}{L(\xi^{-1}; s + 1/2)}$$

for $\Re(s) > s_\Xi$. □

Finally we prove Proposition 5.3.1.

**Proof of Proposition 5.3.1.** For $g \in GL_2(\mathfrak{g})$, we put

$$g^* = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in GL_2(\mathfrak{g}).$$

We note that $(g^*)^{-1} = (g^{-1})^*$ and $(g^*)^* = g$. For any $l > 0$, every element of $K_0^{(l)}$ can be written in the form $(g + \xi^l X, g^*)$ for some $(g, X) \in GL_2(\mathfrak{g}) \times M_2(\mathfrak{g})$. In particular, $(g, g^*) \in K_0^{(l)}$ for any $g \in GL_2(\mathfrak{g})$. Since for any $(k_1, k_2) \in K_0$ the equality

$$K_0^{(l)}(k_1, k_2) = K_0^{(l)}((k_2^{-1})^*, k_2^{-1})(k_1, k_2)$$
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\[ K_0^{(l)}((k_2^{-1})^*k_1, 1_2) \]

holds, we have

\[ K_0^{(l)} = \bigcup_{k \in SL_2(\mathfrak{o}) \atop k \equiv 1_2 \text{ mod } p^l} K_0^{(l+1)}(k, 1_2) \]

for any \( l \geq 0 \). Let \( l \geq 0 \). We take \( k = (k_{ij}) \in SL_2(\mathfrak{o}) \) such that \( k \equiv 1_2 \text{ mod } p^l \).

i) We handle the case where \( k_{21} = 0 \). We note that \( k_{11}, k_{22} \in \mathfrak{o}^\times \). First we assume that \( l > 0 \). Since \( k_{11} = k_{22}^{-1} \), we have

\[
K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}\left(\begin{pmatrix} 1 & k_{11}k_{12} \\ k_{11} & k_{11}^{-1} \end{pmatrix}, 1_2\right) \\
= K_0^{(l+1)}\left(\begin{pmatrix} 1 & \varpi^l b \\ 1 & k_{11}^{-1} \end{pmatrix}, 1_2\right) \pmod{p^l} \\
= K_0^{(l+1)}\left(\begin{pmatrix} k_{11} \\ k_{11}^{-1} \end{pmatrix}, 1_2\right) \pmod{p^l} \\
= K_0^{(l+1)}\left(1 + \varpi^l a \\ (1 + \varpi^l a)^{-1}\right) \pmod{p^l}.
\]

We note that the last equality follows from \((1+p^l)/(1+p^{l+1}) \simeq \mathfrak{o}/p \). Hence there exist \( a, b \text{ mod } p \) such that

\[
K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}\left(1 + \varpi^l a \\ (1 + \varpi^l a)^{-1}\right), 1_2) \pmod{p^l}.
\]

Next we assume that \( l = 0 \). Then we have

\[
K_0^{(1)}(k, 1_2) = K_0^{(1)}\left(\begin{pmatrix} 1 & k_{11}k_{12} \\ k_{11} & k_{11}^{-1} \end{pmatrix}, 1_2\right) \\
= K_0^{(1)}\left(\begin{pmatrix} 1 & \varpi^l b \\ 1 & k_{11}^{-1} \end{pmatrix}, 1_2\right) \pmod{p^l} \\
= K_0^{(1)}\left(\begin{pmatrix} k_{11} \\ k_{11}^{-1} \end{pmatrix}, 1_2\right) \pmod{p^l} \\
= K_0^{(1)}\left(\begin{pmatrix} a \\ a^{-1} \end{pmatrix}, 1_2\right) \pmod{p^l}.
\]

We note that the last equality follows from \( \mathfrak{o}^\times / (1+p) \simeq (\mathfrak{o}/p)^\times \). Hence there exist \( a \neq 0, b \text{ mod } p \) such that

\[
K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}\left(\begin{pmatrix} a \\ a^{-1} \end{pmatrix}, 1_2\right) \pmod{p^l}.
\]

ii) We handle the case where \( k_{21} \neq 0 \). Since \( k_{21} \equiv 0 \text{ mod } p^l \), there exist \( u \in \mathfrak{o}^\times \) and \( n \geq 0 \) such that \( k_{21} = \varpi^{l+n} u \). If \( n \geq 1 \), then

\[
K_0^{(l+1)}(k, 1_2) = K_0^{(l+1)}\left(\begin{pmatrix} k_{11} \\ \varpi^{l+n} u \end{pmatrix}, 1_2\right) \pmod{p^l}.
\]
We assume that $n = 0$. Then we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{12} \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2),
\]
This case is reduced to the case (i). We may assume that $n = 0$. Then we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{12} \\ \varpi^l u & k_{22} \end{array}\right), 1_2)
\]
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{12} \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2) \quad (\equiv v_1 \not\equiv 0 \mod p).
\]
We now consider the case (ii-i) $k_{12} = 0$ and the case (ii-ii) $k_{12} \not\equiv 0$, separately.

ii-i) We assume that $k_{12} = 0$. If $l > 0$, we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} 1 + \varpi^l a \\ \varpi^l v_1 & (1 + \varpi^l a)^{-1} \end{array}\right), 1_2) \quad (\equiv v_1' \not\equiv 0 \mod p, a \mod p).
\]
If $l = 0$, we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} a \\ \varpi^l v_1 & a^{-1} \end{array}\right), 1_2) \quad (\equiv v_1', a \not\equiv 0 \mod p).
\]
We note that $\varpi^\times/(1 + p) \simeq (\varpi/p)^\times$.

ii-ii) We assume that $k_{12} \not\equiv 0$. Since $k_{12} \equiv 0 \mod p'$, there exist $u \in \varpi^\times$ and $n \geq 0$ such that
\[
k_{12} = \varpi^{l+n} u, \text{ that is,}
\]
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & \varpi^{l+n} u \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2).
\]
If $n \geq 1$, we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{22}^{-1} \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2).
\]
This case is reduced to the case (ii-i). We may assume that $n = 0$. Then we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & \varpi^l u \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2)
\]
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{12} \varpi^l v_2 \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2) \quad (\equiv v_2 \not\equiv 0 \mod p).
\]
Let $l > 0$. Then, since $k_{11}, k_{22} \equiv 1 \mod p'$, we have
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{12} \varpi^l v_2 \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2)
\]
\[
K_0^{(l+1)}(k, 1) = K_0^{(l+1)}\left(\begin{array}{cc} k_{11} & k_{12} \varpi^l v_2 \\ \varpi^l v_1 & k_{22} \end{array}\right), 1_2) \quad (\equiv v_1', v_2' \not\equiv 0 \mod p)
\]
\[ K_0^{(l+1)}(1 + \varpi^l a \left( \begin{array}{cc} 1 & \varpi^l v' \\ \varpi^l v' & 1 \end{array} \right), 1, 2) (\exists a, d \mod p). \]

Let \( l = 0 \). We note that

\[ k = \left( \begin{array}{cc} k_{11} & v_2 \\ v_1 & k_{22} \end{array} \right) = \left( \begin{array}{cc} v_2 & -k_{11} \\ k_{22} & -v_1 \end{array} \right) \left( \begin{array}{c} -1 \\ 1 \end{array} \right). \]

If \( k_{11}k_{22} = 0 \), we have

\[
K_0^{(1)}(k, 1, 2) = \begin{cases} 
K_0^{(l+1)}(a \left( \begin{array}{cc} 1 & b \\ a^{-1} & 1 \end{array} \right), 1, 2) (\exists a \neq 0 \mod p, b \mod p) & (\text{if } k_{22} = 0), \\
K_0^{(1)}(a \left( \begin{array}{cc} a & v'_1 \\ v'_1 & a^{-1} \end{array} \right), 1, 2) (\exists v'_1, a \neq 0 \mod p) & (\text{if } k_{11} = 0 \text{ and } k_{22} \neq 0), \\
K_0^{(l+1)}(b' \left( \begin{array}{cc} a & b' \\ -a^{-1} & a \\ \end{array} \right), 1, 2) (\exists b' \mod p) & (\text{if } k_{22} = 0), \\
K_0^{(1)}(a \left( \begin{array}{cc} a & v'_1 \\ -v'_1 & a^{-1} \end{array} \right), 1, 2) & (\text{if } k_{11} = 0 \text{ and } k_{22} \neq 0).
\end{cases}
\]

There remains only the case where \( k_{11}k_{22} \neq 0 \). Then there exist \( m, n \geq 0, u, u' \in \mathfrak{o}^\times \) such that

\[ K_0^{(1)}(k, 1, 2) = K_0^{(1)}(v_2 \left( \begin{array}{cc} u & -\varpi^m u \\ \varpi^m u & -v_1 \end{array} \right), 1, 2). \]

If \( m > 0 \) (resp. \( n > 0 \)), the case is reduced to \( k_{11} = 0 \) (resp. \( k_{22} = 0 \)). We may assume that \( m = n = 0 \). Then we have

\[
K_0^{(1)}(k, 1, 2) = K_0^{(1)}(v_2 \left( \begin{array}{cc} u & -u \\ u & -v_1 \end{array} \right), 1, 2) \\
= K_0^{(1)}(u \left( \begin{array}{cc} 1 & v_2 u^{-1} \\ v_1 u^{-1} & 1 \end{array} \right), 1, 2) \\
= K_0^{(1)}(u \left( \begin{array}{cc} u_1 & v_2' \\ v'_1 & u_2 \end{array} \right), 1, 2) (\exists u_1, u_2, v'_1, v'_2 \neq 0 \mod p).
\]

We note that \( u_1u_2 - v'_1v'_2 = 1 \).

Therefore the results for the cases (i) and (ii) yield that

\[ K_0 = \bigcup_{u \neq 0 \mod p, x \mod p} K_0^{(1)}(u \left( \begin{array}{cc} x & \varpi^{-1} \\ u^{-1} & 1 \end{array} \right), 1, 2) \cup \bigcup_{u, v \neq 0 \mod p} K_0^{(1)}(u \left( \begin{array}{cc} u & \varpi^{-1} \\ v & u^{-1} \end{array} \right), 1, 2) \]
and, for $l > 0$,

\[
K_0^{(l)} = \bigcup_{x, y \mod p} K_0^{(l+1)} \left( \begin{pmatrix} 1 + \omega^l x & \omega^l y \\ \omega^l u & \omega^l v \end{pmatrix}, 1_2 \right) \\
\bigcup_{u \not\equiv 0 \mod p} K_0^{(l+1)} \left( \begin{pmatrix} 1 + \omega^l x \\ \omega^l u \end{pmatrix}, 1_2 \right) \\
\bigcup_{x, y \mod p} K_0^{(l+1)} \left( \begin{pmatrix} 1 + \omega^l x & \omega^l u \\ \omega^l v & 1 + \omega^l y \end{pmatrix}, 1_2 \right) \\
\bigcup_{x + y + \omega^l (xy - uv) = 0} K_0^{(l+1)} \left( \begin{pmatrix} 1 + \omega^l x \\ \omega^l u \end{pmatrix}, 1_2 \right) \\
\bigcup_{u \not\equiv 0 \mod p} K_0^{(l+1)} \left( \begin{pmatrix} 1 + \omega^l x \\ \omega^l u \end{pmatrix}, 1_2 \right) \\
\bigcup_{x \mod p} K_0^{(l+1)} \left( \begin{pmatrix} 1 + \omega^l x & \omega^l u \\ \omega^l v & 1 + \omega^l y \end{pmatrix}, 1_2 \right).
\]

We can easily check that these union is disjoint. □
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References


