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Ph.D. Thesis

Non-perturbative topological string, Spectral theory, and Condensed matter physics

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Abstract

We study non-perturbative aspects of topological string theory. The non-perturbative effects can be obtained from two methods: one is the duality between the topological string theory and the ABJM theory, the other is the quantization of the mirror curves in the mirror Calabi–Yau manifolds. These non-perturbative effects are closely related by the quantum mirror map. We call the former non-perturbative completion as the A-model, and the latter one as the B-model. In the B-model side, the quantization of the mirror curve is almost the same as the Hofstadter model describing a dynamics of electrons on a lattice under a magnetic field. In this thesis, first we explain how to define two kinds of the non-perturbative completion, and how to be related. Next, we define the Hofstadter model, and show some known results. Then, we show that the B-model topological string theory describes the Hofstadter model. These results show that one can study the non-perturbative topological string theory from the well-known condensed matter physics.

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1 Introduction

1.1 Why topological string theory ?

It has been thirty years when the topological string theory was proposed by E.Witten [1]. The topological string theory has been studied from the point of view of physics and mathematics. The physical motivation is to study string theory.

Our world is well described by the Standard Model. However, the Standard Model has several problems: lack of gravity, too many input parameters, hierarchy problem, and so on. Especially, the lack of the gravity is crucial problem; we cannot construct quantum gravity. To overcome these problems, the superstring theory has been studied. The superstring theory is one of the candidate to describe the quantum gravity, and defined in ten-dimensional space-time. In order to consider the superstring theory as a model in our world, the six-dimensional space has to be compactified on some manifold $\mathcal{M}_6$ whose size is too tiny to see at low energy.

To be consistent with the Standard Model, we choose $\mathcal{M}_6$ as a Calabi–Yau manifold. Thus, to calculate the effects of the strings in the four-dimensional space-time, we have to consider the dynamics of the strings propagating in $\mathcal{M}_6$. In general, the actual calculation of the effects is difficult.

Then, we can use the topological string theory to calculate some parts of the effects. In general, in order to calculate the quantum corrections, we need to count the all maps from the worldsheet to the target space, according to the path integral formalism. In the topological string theory, the path integral is restricted to to the holomorphic maps by the supersymmetry. This restriction simplifies the problem drastically, however, the topological string still has the non-trivial information. For example, the free energy of the topological string theory gives a certain term of the effective theory in the superstring theory compactified on the Calabi–Yau manifold [2, 3]. This is the motivation of studying the topological string theory in the physical sense.

One of the mathematical reason is to study the mirror symmetry [4, 5]. Mathematically, the mirror symmetry claims that the moduli space of the the Kähler structure in the Calabi–Yau manifold CY_A agrees with the moduli space of the complex structure in the corresponding Calabi–Yau manifold CY_B.

In the view point of the topological string theory, the mirror symmetry is the relationship between the topological string theories on CY_A and CY_B. We call the topological string theories on CY_A and CY_B as the A-mode topological string theory and the B-model topological string theory, respectively. The relationship means that the genus-g free energy $F_{g,CY_A}(\boldsymbol{t})$ of the topological string theory on CY_A agrees with the free energy $F_{g,CY_B}(\boldsymbol{z})$ of the topological string theory on CY_B,

$$F_{g,CY_A}(\boldsymbol{t}) = F_{g,CY_B}(\boldsymbol{z}), \tag{1.1}$$

where $\boldsymbol{t}$ and $\boldsymbol{z}$ are the Kähler moduli of CY_A and the complex moduli of CY_B, respectively.

In order to study the mirror symmetry, we have to know the "coordinate transformation" between the Kähler moduli $\boldsymbol{t}$ and the complex moduli $\boldsymbol{z}$,

$$t_i = t_i(\boldsymbol{z}). \tag{1.2}$$

This function is called as the mirror map. In section 2.1 and 2.2, we will explain how to calculate the mirror map, and check the mirror symmetry for the genus-zero case.

1.2 Toward Non-perturbative topological string: A-model side

String theory is defined as the perturbative expansion of the string coupling constant. To define the non-perturbative string theory, various attempts have been made: dualities point of view, string field theories, matrix models, and so on. In recent years, there was remarkable development in the non-perturbative definition of the topological string theory on a certain class of the Calabi–Yau manifolds: non-compact toric Calabi–Yau manifolds. The key idea is the correspondence between the ABJM theory and the A-model topological string theory on the non-compact toric Calabi–Yau manifold [6, 7, 8].

The ABJM theory is the low-energy effective theory of M2-branes, and defined as the $\mathcal{N} = 6$ supersymmetric $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern–Simons-matter theory [6]. Thanks to the supersymmetry, we can use the localization techniques to calculate the partition function [9].

After some calculations, the path integral reduces to the matrix integral. Furthermore, we can show that the matrix integral reduces to the canonical ensemble for the fermions.[1] Therefore, we can use some techniques of statistical mechanics. This method is called as "Fermi gas formalism." From this expression, we can evaluate two kinds of the non-perturbative contributions. These differences come from where the M2-branes wrap in the gravity dual theory predicted by the AdS/CFT correspondence [10, 11]; when the M2-branes wrap on the M theory circle, in the weak string coupling limit the M2 branes reduce to the fundamental strings. In this sense, the non-perturbative contributions of these M2-branes are called as worldsheet instantons. On the other hand, when the M2-branes do not wrap on the M theory circle, in the string theory the M2-branes reduce to the D2-branes. These contributions are called as membrane instantons.

The duality between the A-model topological string theory and the ABJM theory claims that the free energy of the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ agrees with the free energy of the ABJM theory,[2] where local $\mathbb{P}^1 \times \mathbb{P}^1$ is one of the non-compact toric Calabi–Yau manifold. This duality is proven partially since the topological string theory is defined

[1] This is the calculation result, and we do not know why the fermion system occurs.

[2] This duality is mathematical fact, and it has been not known physical reason.

perturbatively. Then, if the duality is true for all contributions, we can provide the novel contributions of the topological string theory from the ABJM theory. The remarkable point is that the predicted contribution is given by $1/g_s$. These contributions are suppressed in the perturbative expansion of the string coupling constant. In this sense we call these contributions as the non-perturbative contributions.

This result is valid for the case of local $\mathbb{P}^1 \times \mathbb{P}^1$, however, the authors in [8] proposed the free energy of the non-perturbative topological string theory on general non-compact toric Calabi–Yau manifolds. We will explain above process in section 2.3.

1.3 Toward Non-perturbative topological string: B-model side

However, this is not the end of the story; there was also remarkable progress in the B-model topological string theory [12]. In this theory, the perturbative information is encoded to the mirror curve which is the complex one-dimensional manifold and defined as a subspace of the Calabi–Yau manifolds [13]. More concretely, let us consider the case of the genus-one mirror curve whose definition is given by

$$W_X(\mathrm{e}^x, \mathrm{e}^y, \boldsymbol{m}) = \mathcal{E}, \qquad \mathrm{e}^x, \mathrm{e}^y \in \mathbb{C}^*. \tag{1.3}$$

where $\mathcal{E}$ is the true modulus, and $\boldsymbol{m}$ are the tunable parameters [14]. The important point is that the free energy of the B-model non-perturbative topological string theory is given by the quantization of the mirror curve [12] defined as[3]

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar, \qquad \hbar = \frac{4\pi^2}{g_s}, \tag{1.4}$$

where we denote the operator as Sans-serif, such as x, y. In this quantum theory, the true modulus promoted to the eigenvalue of the operator $W_X(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}}, \boldsymbol{m})$, and this gives the non-perturbative contribution of the B-model topological string theory.

Then, as a naive guess based on the mirror symmetry, this eigenvalue would be related to the A-model non-perturbative topological string theory. Indeed, this conjecture is investigated in some mirror curves [12, 15]. Note that, as we mentioned before, we have to calculate the mirror map to investigate the mirror symmetry. We call the mirror map in the quantum theory as the quantum mirror map $t(\mathcal{E}, \boldsymbol{m}, \hbar)$.[4] By using the prescription in [16], we can calculate the quantum mirror map order by order. We will explain how to calculate the non-perturbative free energy in section 2.4, and how to connect the A-model with the B-model through the quantum mirror map in 2.5.

[3] In [13] $\hbar$ is defined as $\hbar = g_s$. However, in the context of the non-perturbative topological string theory [12], $\hbar$ is defined as $\hbar = 4\pi^2/g_s$. In this thesis, we adopt the latter definition.

[4] In contrast with the quantum mirror map, the mirror map in the classical limit $t(\mathcal{E}, \boldsymbol{m}, \hbar = 0)$ is called as classical mirror map.

In summary, we propose the non-perturbative definitions of the A-model and B-model topological string theories, and these definitions are related through the quantum mirror map. From the definitions, the duality web is extended drastically; the non-perturbative topological string theory is related to the relativistic Toda lattice [17], supersymmetric gauge theories, the ABJM theories [18], and condensed matter physics [19]. The last one is the main theme in this thesis.

1.4 Topological string-Condensed matter physics correspondence(TS-CMP correspondence)

In this subsection, we propose the TS-CMP correspondence in accordance with [19] which is the first paper proposed the correspondence. The starting point is a naive observation as following: we consider the mirror curve of local $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$W_{\mathbb{P}^1\times\mathbb{P}^1}(\mathrm{e}^x, \mathrm{e}^y) = \mathrm{e}^x + \mathrm{e}^{-x} + m(\mathrm{e}^y + \mathrm{e}^{-y}) = \mathcal{E}. \tag{1.5}$$

By quantizing the mirror curve, we define the quantum theory whose “Hamiltonian” is given by the operator $W_{\mathbb{P}^1\times\mathbb{P}^1}(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}})$.

Next, we consider the Hofstadter model which describes a dynamics of electrons on a lattice under a uniform magnetic field perpendicular to the lattice [20]. The authors in [19] consider the Hofstadter model defined on the square lattice. Its Hamiltonian in the tight binding approximation is given by

$$H_{\mathrm{sq}} = T_x + T_x^\dagger + \lambda(T_y + T_y^\dagger), \tag{1.6}$$

where $T_{x,y}$ are the hopping operators of the electrons along x- and y-directions, called as the magnetic translation operators. Since there is the magnetic flux, the operators T_x and T_y do not commute, and the commutation relation of these operators is given by

$$T_x T_y = \mathrm{e}^{\mathrm{i}\phi} T_y T_x, \tag{1.7}$$

where ϕ is the magnetic flux.

At this stage, we find that the quantization of local $\mathbb{P}^1 \times \mathbb{P}^1$ is precisely the same as the Hofstadter model under $\lambda = m,\ \phi = -\hbar$ and redefining $\mathsf{x}, \mathsf{y} \to \mathrm{i}\mathsf{x}, \mathrm{i}\mathsf{y}$. Based on this observation, the authors in [19] expect that there might be further correspondence between two theories. Then, they find that the branch cut of the quantum mirror map describes the energy band called as the Hofstadter butterfly, and the imaginary part of the quantum mirror map gives the density of state. The key point of the correspondence is the S-duality in the non-perturbative topological string theory. Let us explain it briefly. The detail explanation will be presented in section 2.6.

The quantum theory defined by $W_{\mathbb{P}^1\times\mathbb{P}^1}(\mathrm{e}^{\mathsf{x}},\mathrm{e}^{\mathsf{y}})$ is quantum mechanics, so that we can consider the Bohr–Sommerfeld quantization condition [21]. The quantization condition is given by

$$\oint_B \mathrm{d}x\, y(x,\mathcal{E}) = 2\pi\hbar\left(n+\frac{1}{2}\right), \qquad n\in\mathbb{Z}_{>0}, \tag{1.8}$$

where B is a one-cycle defined in the mirror curve, $W_{\mathbb{P}^1\times\mathbb{P}^1}(\mathrm{e}^{x},\mathrm{e}^{y})=\mathcal{E}$. On the other hand, this integral is related to the genus-zero free energy $F_0(t)$ [2],

$$\frac{\partial F_0(t)}{\partial t} = \oint_B \mathrm{d}x\ y(x,\mathcal{E}), \qquad t=t(\mathcal{E}). \tag{1.9}$$

By combining these facts, $F_0(t)$ is quantized. By substituting the classical mirror map $t(\mathcal{E})$ to (1.9), we can solve the eigenvalue problem for the Hamiltonian $W_{\mathbb{P}^1\times\mathbb{P}^1}(\mathrm{e}^{\mathsf{x}},\mathrm{e}^{\mathsf{y}})$.

The Bohr–Sommerfeld quantization condition is the first order correction in the semi-classical limit. Then, the natural question is how we calculate the quantum corrections. Then, Nekrasov and Shatashvili proposed that the quantization condition including the quantum correction is given by [22]

$$\frac{\partial F_{\mathrm{NS}}(t,\hbar)}{\partial t} = 2\pi\left(n+\frac{1}{2}\right), \tag{1.10}$$

where $F_{\mathrm{NS}}(t,\hbar)$ is the one-parameter deformation of the free energy called as Nekrasov–Shatashvili free energy that we will define in section 2.3. However, this expression has a crucial problem; for $\hbar=2\pi a/b,\ a,b\in\mathbb{Z}_{>0}$, the left hand side diverges even if the quantization condition should be valid in arbitrary values of $\hbar$. This is due to resum the quantum corrections.

To resolve this problem, in [21], they proposed that the quantization condition is finite for any values of $\hbar$ by adding the non-perturbative part as follows,

$$\frac{\partial F_{\mathrm{NS}}(t,\hbar)}{\partial t} + \frac{\partial F_{\mathrm{NS}}(\tilde{t},\tilde{\hbar})}{\partial \tilde{t}} = 2\pi\left(n+\frac{1}{2}\right), \qquad (\tilde{t},\tilde{\hbar}) = \left(\frac{2\pi t}{\hbar},\frac{4\pi^2}{\hbar}\right), \tag{1.11}$$

which is called as the exact quantization condition. The first term and the second term in (1.11) have the poles at $\hbar=2\pi a/b$, however, these poles are precisely canceled. Therefore, the summation is finite for arbitrary values of $\hbar$. The pole cancellation phenomenon is called as the HMO cancellation mechanism [7]. The exact quantization condition has been passed several consistency checks.

This expression is clearly invariant under the following S-transformation,

$$(t,\hbar)\leftrightarrow\left(\frac{2\pi t}{\hbar},\frac{4\pi^2}{\hbar}\right), \tag{1.12}$$

called as the S-duality. This means that there is the dual theory for the $W_{\mathbb{P}^1\times\mathbb{P}^1}$-system,

$$\begin{aligned}
&\widetilde{W}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathrm{e}^{\tilde{x}} + \mathrm{e}^{-\tilde{x}} + \tilde{m}(\mathrm{e}^{\tilde{y}} + \mathrm{e}^{-\tilde{y}}) = \tilde{\mathcal{E}}, \\
&\tilde{x} = \frac{2\pi}{\hbar}x,\ \tilde{y} = \frac{2\pi}{\hbar}y, \\
&\tilde{m} = m^{\frac{2\pi}{\hbar}},
\end{aligned} \tag{1.13}$$

and the true modulus $\tilde{\mathcal{E}}$ in the $\widetilde{W}_{\mathbb{P}^1\times\mathbb{P}^1}$-system is related to the one $\mathcal{E}$ in the $W_{\mathbb{P}^1\times\mathbb{P}^1}$-system via the quantum mirror map,

$$\begin{aligned}
&\frac{2\pi}{\hbar}t(\tilde{\mathcal{E}}, \tilde{m}, \tilde{\hbar}) = t(\mathcal{E}, m, \hbar), \\
&\tilde{\hbar} = \frac{4\pi^2}{\hbar}.
\end{aligned} \tag{1.14}$$

The quantum mirror map in this case is also invariant under the following T-transformation,

$$t(\mathcal{E}, m, \hbar \pm 2\pi) = t(\mathcal{E}, m, \hbar). \tag{1.15}$$

By utilizing these transformations, and setting $\hbar = 2\pi a/b$, we can obtain the quantum mirror map in a closed form. From the expression, we can see where the branch cuts are, and calculate the imaginary part of the quantum mirror map. We remark that the S- and the T-transformation correspond to the maps $(\phi, E_{\mathrm{sq}}) \mapsto (4\pi^2/\phi, \tilde{E}_{\mathrm{sq}})$ and $(\phi, E_{\mathrm{sq}}) \mapsto (\phi \pm 2\pi, E_{\mathrm{sq}})$ in the condensed matter physics side, where E_{sq} is the eigenvalue of H_{sq} .

1.5 Motivation and Summary of our work

In the previous subsections, we explained the TS-CMP correspondence. Since the Hofstadter model has been studied actively, we can study the non-perturbative topological string theory through this correspondence. However, the TS-CMP correspondence is studied only in one case. Therefore, it is worth investigating the TS-CMP correspondence in general cases. In this thesis, we consider the case of local $\mathcal{B}_3$ which is the general case of the local $\mathbb{P}^1 \times \mathbb{P}^1$.

The mirror curve is given by

$$W_{\mathcal{B}_3}(\mathrm{e}^x, \mathrm{e}^y, \mathcal{E}) = \mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{-x-y} + m_1\mathrm{e}^{-x} + m_2\mathrm{e}^{-y} + m_3\mathrm{e}^{x+y} = \mathcal{E}, \tag{1.16}$$

where $m_{1,2,3}$ are the tunable parameters. By shifting $x \to x + \log(m_1)/2$ and $y \to y + \log(m_2)/2$, and dividing $\sqrt{m_1}$ in both side, the mirror curve becomes

$$\mathrm{e}^x + \mathrm{e}^{-x} + \frac{\sqrt{m_2}}{\sqrt{m_1}}(\mathrm{e}^y + \mathrm{e}^{-y}) + \sqrt{m_1 m_2}m_3\mathrm{e}^{x+y} + \frac{1}{\sqrt{m_1 m_2}}\mathrm{e}^{-x-y} = \frac{\mathcal{E}}{\sqrt{m_1}}. \tag{1.17}$$

Then, by imposing $m_1m_2m_3 = 1$, we expect the quantum theory of local $\mathcal{B}_3$ to correspond to the Hofstadter model on the triangular lattice [23, 24, 25] whose Hamiltonian is given by

$$H_{\mathrm{tri.}} = T_x + T_x^\dagger + \lambda_1(T_y + T_y^\dagger) + \lambda_2(\mathrm{e}^{-\mathrm{i}\phi/2}T_xT_y + \mathrm{e}^{\mathrm{i}\phi/2}T_y^\dagger T_x^\dagger). \tag{1.18}$$

Since the band spectrum and the density of state are known, the rest task is to calculate the quantum mirror map. First, we check that the exact quantization condition is valid in this case, and this system has the S-dual structure. Then, we calculate the quantum mirror map by utilizing the S- and the T-transformations. However, the direct calculation from the quantum mirror curve is difficult. Then, we use the knowledge of the condensed matter physics side to calculate the quantum mirror map. Finally, we find that the TS-CMP correspondence is valid in this case. We will explain the calculation process in section 4. We summarize the correspondence in table 1

Topological String	Condensed Matter Physics
Quantized mirror curve $W_{\mathcal{B}_3}(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}}, \mathcal{E}) = \mathcal{E}$	Hamiltonian $H_{\text{tri.}}$
Planck constant $\hbar = 2\pi a/b$	Magnetic flux $\phi = 2\pi a/b$
complex moduli $\mathcal{E}/\sqrt{m_1}$	Eigenvalue of Hamiltonian E
Mass parameters $\boldsymbol{m},\ m_1 m_2 m_3 = 1$	Hopping parameters $\boldsymbol{\lambda}$
Branch cuts of quantum mirror map $t(\mathcal{E}, \boldsymbol{m}, \hbar = 2\pi a/b)$	Hofstadter butterfly
Imaginary part of quantum mirror map $\partial t(\mathcal{E}, \boldsymbol{m}, \hbar = 2\pi a/b)/\partial \mathcal{E}$	Density of state $\rho(E, \boldsymbol{\lambda}, \phi = 2\pi a/b)$

Table 1: The correspondence between the topological string theory and the condensed matter physics.

1.6 Organization of this paper

This thesis is organized as follows. In section 2, we review some basic concepts of the perturbative and the non-perturbative topological string theory. Here we provide some conjectures, and check them by giving some specific examples. In section 3, we review the Hofstadter model in the condensed matter physics. In this thesis, we consider the Hofstadter model on the triangular lattice. In section 4, we present how to obtain the Hofstadter butterfly and the density of state in the topological string theory on local $\mathcal{B}_3$. Finally, we summarize our results and discuss further research in section 5.

2 Topological string theory and Non-perturbative effect

In this section, we discuss the definition of the perturbative topological string theory and how to include the non-perturbative effects. There are two types of the topological string theory: the A-model topological string theory and the B-model topological string theory. These theories are related via the mirror symmetry.

Even if we consider the non-perturbative effects, the relation like the mirror symmetry exists. In the A-model side, we propose the non-perturbative effects from the ABJM theory. In the B-model side, based on the Fermi gas formalism in the ABJM theory, the quantization of the mirror curves give the grand partition function including both the perturbative and the non-perturbative effects.

First we explain the perturbative definition of these theories, and how to relate. Then, we discuss the non-perturbative effects.

2.1 Perturbative topological string theory

Here we define the topological string theory. We start with $\mathcal{N} = (2,2)$ two-dimensional supersymmetric Non-Linear Sigma Model (susy-NLSM) whose action is given by

$$\begin{aligned} S = \int d^2z \{ & G_{I\bar{J}}(4\partial_z\phi^I\partial_{\bar{z}}\bar{\phi}^{\bar{J}} + 2\mathrm{i}\psi_+^{\bar{J}}D_z\psi_+^I + 2\mathrm{i}\psi_-^{\bar{J}}D_{\bar{z}}\psi_-^I) \\ & + R_{I\bar{J}K\bar{L}}\psi_+^I\psi_+^{\bar{J}}\psi_-^K\psi_-^{\bar{L}} + G_{I\bar{J}}(F^I - \Gamma^I_{JK}\psi_+^J\psi_-^K)(F^{\bar{J}} - \Gamma^{\bar{J}}_{\bar{K}\bar{L}}\psi_+^{\bar{L}}\psi_-^{\bar{K}})\}, \end{aligned} \tag{2.1}$$

where ϕ^I are the bosonic fields, $\psi^I_\pm$ are the fermionic fields, F^I are the non-dynamical fields, and $G_{I\bar{J}}$ is the metric of the target space. The bar " $^-$ " denotes the complex conjugate.

To consider the string theory, we have to couple susy-NLSM to the two-dimensional gravity. However, naively we cannot consider the gravity while preserving the supersymmetry as the following reason [26][5]: the supersymmetry is defined in the worldsheet theory. When we consider the supersymmetry on the curved space, we have to find an infinitesimal fermionic parameter ϵ which is satisfied with $\nabla\epsilon = 0$. However, this condition cannot be satisfied in general metric since ϵ varies covariantly under a parallel transport, and the requirement of the covariantly constant for ϵ is very special requirement.

To resolve this problem, we perform the topological twist [1]. In susy-NLSM, there are two types of the U(1) symmetry associated with the supersymmetry; axial U(1) symmetry and vector U(1) symmetry,

$$\begin{aligned} \text{axial U}(1) : \psi_\pm \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\psi_\pm, \\ \text{vector U}(1) : \psi_\pm \mapsto \mathrm{e}^{\mp\mathrm{i}\alpha}\psi_\pm, \end{aligned} \tag{2.2}$$

where α is a parameter. Then, we can construct the Noether current associated with these symmetries, J_A and J_V. For these symmetries, we consider the two types of interactions

[5] This paper is review paper, however, we think that this explanation is very clear.

between the current and the spin connection ω_a, so-called gauging,[6] which correspond to the A-model and the B-model,

$$\begin{aligned} &\text{A-model: } S_{\text{int}} = -2\mathrm{i}\int d^2z\sqrt{g}\omega_a J_A^a, \\ &\text{B-model: } S_{\text{int}} = -2\mathrm{i}\int d^2z\sqrt{g}\omega_a J_V^a, \end{aligned} \tag{2.3}$$

where $\sqrt{g}$ is the determinant of the worldsheet metric g_{ab}. These interactions change the spins for the fermions as follows,

- A-model: ψ_+^I:+1/2 $\to$ 0, ψ_-^I:-1/2 $\to$ -1, $\psi_-^{\bar{I}}$:+1/2 $\to$ +1, $\psi_-^{\bar{I}}$:-1/2 $\to$ -1,
- B-model: ψ_+^I:+1/2 $\to$ +1, ψ_-^I:-1/2 $\to$ -1, $\psi_-^{\bar{I}}$:+1/2 $\to$ 0, $\psi_-^{\bar{I}}$:-1/2 $\to$ 0.

Then, we can couple susy-NLSM to the two-dimensional gravity while preserving the supersymmetry since we have the parameter ϵ with spin zero. In the construction we obtain a scalar and nilpotent charge from four supercharges $Q^{\pm}$ and $\bar{Q}^{\pm}$,

- A-model: $Q_A = \bar{Q}^+ + Q^-,\ Q_A^2 = 0$
- B-model: $Q_B = \bar{Q}^+ + \bar{Q}^-,\ Q_B^2 = 0$

Therefore, the physical observables are restricted to the $Q_{A,B}$-invariant quantity. This restriction simplifies the theory drastically; in the path integral formalism, we count only holomorphic maps $\partial_{\bar{z}}\phi^I = \partial_z\phi^{\bar{I}} = 0$, in the A-model side, and only constant maps $\partial_z\phi^I = \partial_{\bar{z}}\phi^I = 0$ in the B-model side. Furthermore, one can show that the resulting theory which couples to the two-dimensional gravity is topological theory. In this sense, the theory is called as the topological string theory. "Topological" means that the physical observables do not depend on the two-dimensional metric. This can be seen from the fact that the action is given by Q-exact form,

$$S^{\text{top.}} = \{Q_{A,B}, V\} + (\text{topological term}), \tag{2.4}$$

where V is the functional of the boson, fermion, and two-dimensional metric.

Now let us define the free energy of the topological string theory. For instance, we consider the A-model topological string theory. Notice that the mathematical structure of the topological string theory is precisely the same as usual bosonic string theory under the appropriate field and charge correspondence. For example, the energy-momentum tensors in

[6] Precisely speaking, we have to be care about the anomaly since we perform the gauging for the global symmetry. However, in this thesis we consider the Calabi–Yau manifold as a target space. In this case, we can construct the theory without the anomaly.

the topological string theory and the bosonic string theory are given by

$$\begin{aligned} T^{ab}_{\text{top.}} &= \left\{ Q_A, \frac{2}{\sqrt{g}} \frac{\delta V}{\delta g_{ab}} \right\}, \\ T^{ab}_{\text{bosonic}} &= \{ Q_{\text{BRST}}, b^{ab} \}, \end{aligned} \tag{2.5}$$

where b^{ab} are the ghost fields, and Q_{BRST} is the BRST charge.

Therefore we can calculate the free energy of the topological string theory of genus-g worldsheet Σ_g in the same manner as the calculation in the bosonic string theory,

$$\begin{aligned} F_1(\boldsymbol{t}) &= \int_{\mathcal{M}_1} \frac{\mathrm{d}^2\tau}{\mathrm{Im}\tau} \mathrm{Tr}\big[(-1)^{F_L+F_R} F_L F_R q^{L_0} \bar{q}^{\bar{L}_0}\big], \ q = \mathrm{e}^{2\pi \mathrm{i}\tau}, \\ F_{g>2}(\boldsymbol{t}) &= \int_{\mathcal{M}_g} \mathrm{d}^{3g-3}m \mathrm{d}^{3g-3}\bar{m} \left\langle \prod_{i=1}^{3g-3} \int_{\Sigma_g} \mu_k G^- \int_{\Sigma_g} \bar{\mu}_k \bar{G}^-) \right\rangle, \ \mu_k = \frac{1}{2} g^{bc} \frac{\partial g_{ac}}{\partial m_k}, \end{aligned} \tag{2.6}$$

where G^- and $\bar{G}^-$ are the supercurrents, τ, m_k are the moduli of the worldsheet Σ_g, $F_{L,R}$ are the Fermion numbers of the left and the right moving fermions, and $\boldsymbol{t}$ are the Kähler moduli. The integral is over the moduli space $\mathcal{M}_g$ of Σ_g, and the expectation value is defined in the conformal field theory. The genus-zero free energy is determined by the Yukawa coupling C_{ijk},

$$\frac{\partial^3 F_0(\boldsymbol{t})}{\partial t_i \partial t_j \partial_k} = C_{ijk}. \tag{2.7}$$

The total free energy sums $F_g(\boldsymbol{t})$ to all genera,

$$F(\boldsymbol{t}, g_s) = \sum_{g\geq 0} g_s^{2g-2} F_g(\boldsymbol{t}) \tag{2.8}$$

From now on, we call $F(\boldsymbol{t}, g_s)$ as the free energy.

The free energy can be expressed by the perturbative parts of $\boldsymbol{t}$ and the instanton summation. Here we focus on the instanton part $F_{\text{inst}}(\boldsymbol{t}, g_s)$,

$$F_{\text{inst}}(\boldsymbol{t}, g_s) = \sum_{g\geq 0} g_s^{2g-2} \sum_{\boldsymbol{d}} N_{g,\boldsymbol{d}} \ \mathrm{e}^{-\boldsymbol{d}\cdot\boldsymbol{t}}, \tag{2.9}$$

where $N_{g,\boldsymbol{d}}$ are the Gromov–Witten invariants which are topological invariant quantities, and $\boldsymbol{d}$ is the integer vector which has the information which 2-cycles the strings wrap in the Calabi–Yau manifold. We can also have the another expression by taking the resummation,

$$F_{\text{inst}}(\boldsymbol{t}, g_s) = \sum_{g\geq 0} \sum_{w=1}^{\infty} \sum_{\boldsymbol{d}} \frac{n_g^{\boldsymbol{d}}}{w} \left(2\sin\left[\frac{w g_s}{2}\right]\right)^{2g-2} \mathrm{e}^{-w\boldsymbol{d}\cdot\boldsymbol{t}}, \tag{2.10}$$

where $n_g^{\boldsymbol{d}}$ are the Gopakumar–Vafa invariants which are another topological invariant quantities [27, 28].

The Gopakumar–Vafa invariant can be interpreted as counting the BPS states which come from M2-branes wrapping on the Calabi–Yau manifold in the M theory. These BPS states are labelled by $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The Gopakumar–Vafa invariant counts the BPS states under the self-dual field strength of the graviphoton, $\epsilon_1 = -\epsilon_2$. Since the self-dual graviphoton does not couple to the right spin, $n_g^{\boldsymbol{d}}$ count only the left spins. The BPS state counting under the non-self-dual background, $n_{g_L,g_R}^{\boldsymbol{d}}$, are related to the refined topological string theory as we will explain later.

e.g.) Resolved Conifold

As an example, let us consider the A-model topological string theory on the resolved conifold which is one of the non-compact toric Calabi–Yau manifold. We present the definition of the Calabi–Yau manifold in appendix B. The free energy is given by

$$\begin{aligned} F(t,g_s) &= \sum_{w=1}^{\infty} \frac{1}{w} \left(2\sin\left[\frac{w g_s}{2} \right] \right)^{-2} \mathrm{e}^{-wt} \\ &= \sum_{w=1}^{\infty} \frac{1}{w} \left(\frac{1}{w^2 g_s^2} + \frac{1}{12} + \frac{w^2 g_s^2}{240} + \frac{w^4 g_s^4}{6048} + \cdots \right) \mathrm{e}^{-wt} \\ &= \sum_{g\geq 0} g_s^{2g-2} \sum_{w=1}^{\infty} \frac{|B_{2g}|}{2g(2g-2)!} \frac{1}{w^{3-2g}} \mathrm{e}^{-dt}, \end{aligned} \tag{2.11}$$

where B_n are the Bernoulli numbers. Then we find that

$$n_g^1 = 1, \quad n_g^{d\geq 2} = 0, \tag{2.12}$$

$$N_{g,d} = \frac{|B_{2g}|}{2g(2g-2)!} \frac{1}{d^{3-2g}}. \tag{2.13}$$

Note that the Gromov–Witten invariants are non-integer, and we cannot interpret as the counting of the BPS states. □

2.2 Calculation method

In this subsection, we explain how to calculate the free energy. We choose the target space to be the non-compact toric Calabi–Yau manifold. In the A-model side, we use the (refined) topological vertex formalism. In the B-model side, we can calculate the prepotential and the mirror map from the period integrals for the Calabi–Yau manifold.

2.2.1 Calculation in A-model

Here we introduce the topological vertex formalism [29, 30, 31, 32, 33] which is a systematic way to calculate the partition function of the topological string theory on the non-compact toric Calabi–Yau manifold,

$$\mathcal{Z}(\boldsymbol{t}, g_s) = \mathrm{e}^{-F(\boldsymbol{t}, g_s)}. \tag{2.14}$$

The formalism is very similar with the Feynman rules in quantum field theories. Here we just provide the rule to calculate the partition function without the derivation.

We start with the topological string theory with the toric A-branes on three-dimensional complex space, $\mathbb{C}^3$. The toric A-brane is the object which the ends of the open topological strings attach, and is labelled by the Young diagrams $\mu,\ \nu,\ \lambda$ as in figure 1, where we represent $\mathbb{C}^3$ in the web diagram that we will explain in appendix B.

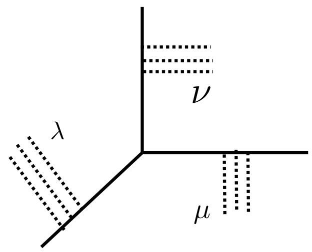

Figure 1: The web diagram description of $\mathbb{C}^3$ called as the topological vertex, where the dashed lines denote the toric A-branes.

The partition function $C_{\mu\nu\lambda}(q)$ of the topological string theory on $\mathbb{C}^3$ is given by[7]

$$\begin{aligned} C_{\mu\nu\lambda}(q) &= q^{\frac{\kappa_\mu+\kappa_\nu}{2}} s_\nu(q^{-\rho}) \sum_\eta s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\rho-\nu^t}), \\ \kappa_\mu &= ||\mu||^2 - ||\mu^t||^2, \quad q = \mathrm{e}^{\mathrm{i} g_s}, \end{aligned} \tag{2.15}$$

where $s_\mu(\boldsymbol{x})$ is the Schur function, and $|\mu|$, $||\mu||$, and ρ are defined as follows,

$$|\mu| = \sum_j \mu_j, \quad ||\mu|| = \sum_j \mu_j^2, \quad \rho = \frac{1}{2}, \frac{3}{2}, \cdots . \tag{2.16}$$

From this expression, we can calculate the partition function of the topological string theory on general non-compact toric Calabi–Yau manifolds by putting the topological vertex and the internal line called as "propagator" (see figure 2) in accordance with the following procedure:

1 First we express the non-compact toric Calabi-Yau manifold as the web diagram.

2 We put the factor $C_{\mu\nu\lambda}(q)$ for each vertices, and $(-\mathrm{e}^{-t})^{|\mu|}$ for each internal lines labelled by the Young diagrams.

3 We set the Young diagram to empty in the external leg of the web diagram. When the toric A-branes are inserted, put the non-trivial Young diagrams on corresponding legs like $\nu,\ \lambda,\ \mu'$, and λ'. The legs of two vertices gluing each other need to be inserted the Young diagrams μ and μ^t, where "t" denotes the transpose of the Young diagram.

[7] Some notations, definitions, and formulae are summarized in appendix A.

4 If the line of the counterclockwise seen from the glued leg (red line in the figure 2) is not parallel to the one of the counterclockwise seen from the another glued leg (blue line in the figure 2), we have to put additional factor $f(q) = (-1)^{|\mu|} q^{-\frac{\kappa_\mu}{2}}$ called as the gluing factor. For example, when the vector made by the red line as the origin O and blue line as the origin O' denote v and v', we have to put the factor $f^{v \wedge v'}(q)$, where the wedge product "∧" is defined as

$$v \wedge v' = v_1 v_2' - v_2 v_1', \quad v = (v_1, v_2),\ v' = (v_1', v_2') \tag{2.17}$$

5 By taking the summation in all Young diagrams inserted on the internal legs, we obtain the partition function.

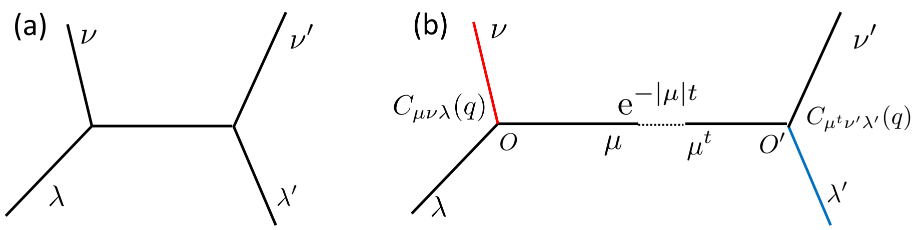

Figure 2: (a)The web diagram and (b)its decomposition into the vertices and the propagator. Here we omit the dashed lines which denote the toric A-branes.

Note that the topological vertex has the cyclic symmetry corresponding to the symmetry of the web diagram in the figure 1,

$$C_{\mu\nu\lambda}(q) = C_{\lambda\mu\nu}(q) = C_{\nu\lambda\mu}(q). \tag{2.18}$$

Let us demonstrate the above prescription.

e.g.) Resolved Conifold

Let us consider the topological string theory on the resolved conifold in figure 3. Since $v = (0, 1)$ and $v' = (0, -1)$, the wedge product of these vectors vanishes, and we need not put the gluing factor $f(q)$. The partition function is given by

$$\begin{aligned} \mathcal{Z}_{\text{coni.}}(t, g_s) &= \sum_\mu (-e^{-t})^{|\mu|} C_{\mu\emptyset\emptyset}(q) C_{\mu^t\emptyset\emptyset}(q) \\ &= \sum_\mu (-e^{-t})^{|\mu|} s_\mu(q^{-\rho}) s_{\mu^t}(q^{-\rho}) \\ &= \prod_{i,j=1}^{\infty} (1 - Q q^{i+j-1}), \quad Q = e^{-t}. \end{aligned} \tag{2.19}$$

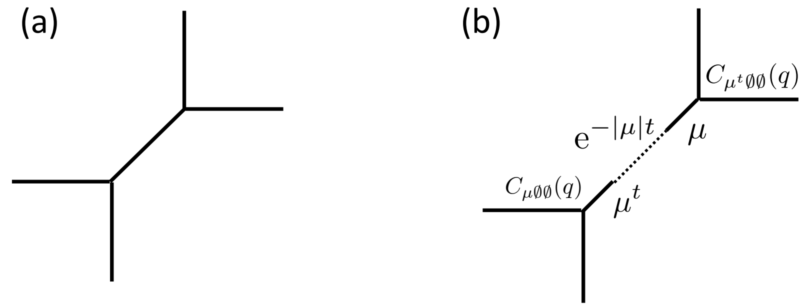

Figure 3: (a)The web diagram of the resolved conifold, and (b)its decomposition into the vertices and the propagator.

The free energy is then,

$$F_{\text{coni.}}(t,g_s) = -\log[\mathcal{Z}_{\text{coni.}}(t,g_s)] = \sum_{d=1}^{\infty} \frac{1}{d}\left(2\sin\left[\frac{dg_s}{2}\right]\right)^{-2} \mathrm{e}^{-dt}. \tag{2.20}$$

This result is the same as (2.11). □

We can consider the refinement of the topological vertex formalism [32]. As we mentioned in the section 2.1, the free energy of the topological string theory can be interpreted as the BPS state counting in the self-dual field strength of the graviphoton, $\epsilon_1 = -\epsilon_2$. Moreover, the partition function of the topological string theory gives the Nekrasov partition functions of $\mathcal{N} = 2$ five-dimensional supersymmetric gauge theories in the self-dual omega-background [34, 35]. Then, the natural desire is to formulate the topological string theory counting the BPS states in $\epsilon_1 \neq -\epsilon_2$, and giving the Nekrasov partition functions of the supersymmetric gauge theories in the general omega-background. This theory is called as the refined topological string theory.

There are several attempts to define the refined topological string theory [36, 37, 38, 39]. In this thesis, we adopt the refined topological vertex formalism [32] to define the refined topological string theory. The topological vertex $C_{\mu\nu\lambda}$ and the gluing factor f_μ are generalized as following,[8]

$$
\begin{aligned}
C_{\lambda\mu\nu}(q_1,q_2) &= q_1^{-\frac{||\mu^t||^2}{2}} q_2^{\frac{||\mu||^2+||\nu||^2}{2}} \tilde{Z}_\nu(q_1,q_2) \sum_\eta \left(\frac{q_2}{q_1}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(q_1^{-\rho} q_2^{-\nu}) s_{\mu/\eta}(q_1^{-\nu^t} q_2^{-\rho}),\\
\tilde{Z}_\nu(q_1,q_2) &= \prod_{(i,j)\in\nu} (1 - q_1^{\nu_j^t - i+1} q_2^{\nu_i - j})^{-1},\\
f_\mu(q_1,q_2) &= (-1)^{|\mu|} q_1^{\frac{||\mu^t||^2}{2}} q_2^{-\frac{||\mu||^2}{2}},\ \ \tilde{f}_\mu(q_1,q_2) = (-1)^{|\mu|} \left(\frac{q_2}{q_1}\right)^{\frac{|\mu|}{2}} q_1^{\frac{||\mu^t||^2}{2}} q_2^{-\frac{||\mu||^2}{2}},
\end{aligned}
\tag{2.21}
$$

where $q_1 = \mathrm{e}^{-\mathrm{i}\epsilon_1}$, $q_2 = \mathrm{e}^{\mathrm{i}\epsilon_2}$. As a remarkable difference, there are two kinds of the gluing factors $f_\mu(q_1,q_2)$ and $\tilde{f}_\mu(q_1,q_2)$ because of the preferred direction [40, 41].

In the usual topological vertex formalism, the Young diagrams λ, μ, and ν correspond to the slicing of a 3D partition as in figure 4. The cyclic symmetry reflects the invariance of

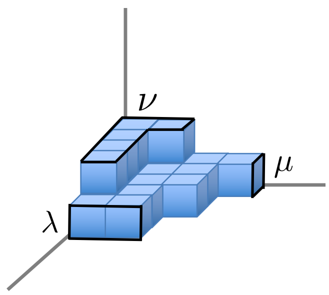

Figure 4: The 3D partition. The boxes enclosed by the black lines correspond to the Young diagrams λ, μ, and ν.

the circular permutation of the 3D partition.

However, the situation is different in the refinement case since two kinds of the string coupling constants give two kinds of the branes called as the ϵ_1-brane and the ϵ_2-brane. To define the refined topological vertex, we have to put the ϵ_1-brane and ϵ_2-brane in the λ-direction and the μ-direction, respectively. The remaining direction, ν-direction, is called as the preferred direction. The putting two kinds of the branes clearly breaks the cyclic symmetry,

$$
C_{\lambda\mu\nu}(q_1,q_2) \neq C_{\nu\lambda\mu}(q_1,q_2). \tag{2.22}
$$

The partition function should not depend on the preferred direction since the choice of the preferred direction is the artificial, so that we can obtain the non-trivial formula from the partition functions calculated in two ways as we will see in an example.

[8]The notation of the refined topological vertex is different from the usual one. However, in order to avoid confusing the Kähler moduli with the refined topological string coupling, we use this notation.

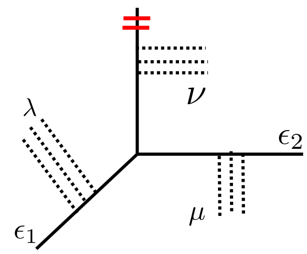

Figure 5: The refined topological vertex. There are three legs: the leg with the ϵ_1-branes, the leg with the ϵ_2-branes, and the preferred direction. The preferred direction denotes the double red line.

In order to glue the internal legs, we have to insert the ϵ_1-branes on either of the internal legs, and ϵ_2-branes on the remaining one, so that we need to add the following rules to the rule 3 and 4 to calculate the partition function of the refined topological string theory,

2' To glue the legs, we have to insert the vertex factors $C_{\mu\nu\lambda}(q_1, q_2)$ for either of the vertex, and $C_{\mu^t\nu'\lambda'}(q_2, q_1)$ for the remaining one.

4' If the gluing legs are the preferred direction, we use the gluing factor $f_\mu(q_1, q_2)$. If not, we use the gluing factor $\tilde{f}_\mu(q_1, q_2)$.

e.g.) Resolved Conifold

We consider the refined topological string theory on the resolved conifold. We demonstrate two calculations; the difference is the choice of the preferred direction as in figure 6. The partition function in the case (a) is given by

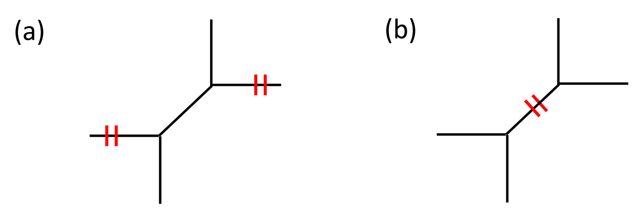

Figure 6: The resolved conifold. We set the preferred direction as (a) the horizontal direction and (b) the diagonal direction.

$$\mathcal{Z}_{(\mathrm{a})}(t, q_1, q_2) = \sum_\mu (-Q)^\mu C_{\emptyset\mu\emptyset}(q_1, q_2) C_{\emptyset\mu^t\emptyset}(q_2, q_1) = \prod_{i,j=1}^{\infty} (1 - Q q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}}),\ Q = \mathrm{e}^{-t}. \tag{2.23}$$

The partition function in the case (b) is given by

$$
\begin{aligned}
\mathcal{Z}_{(\mathrm{b})}(t,q_1,q_2) &= \sum_{\mu}(-Q)^{\mu}C_{\emptyset\emptyset\mu}(q_1,q_2)C_{\emptyset\emptyset\mu^t}(q_2,q_1) \\
&= \sum_{\mu}(-Q)^{\mu}q_1^{\frac{||\mu^t||^2}{2}}q_2^{\frac{||\mu||^2}{2}}\prod_{(i,j)\in\mu}\frac{1}{(1-q_1^{\mu_j^t-i+1}q_2^{\mu_i-j})(1-q_1^{\mu_j^t-i}q_2^{\mu_i-j+1})}.
\end{aligned}
\tag{2.24}
$$

Since two expressions should agree, we find the following formula,

$$
\sum_{\mu}(-Q)^{\mu}t^{\frac{||\mu^t||^2}{2}}q^{\frac{||\mu||^2}{2}}\prod_{(i,j)\in\mu}\frac{1}{(1-q^{\mu_i-j}t^{\mu_j^t-i+1})(1-q^{\mu_i-j+1}t^{\mu_j^t-i})} = \prod_{i,j=1}^{\infty}(1-Qt^{i-\frac{1}{2}}q^{j-\frac{1}{2}})
\tag{2.25}
$$

This identity is shown in [42]. □

In general, the free energy of the refined topological string theory is given by the following expression,

$$
F_{\mathrm{ref.}}(\boldsymbol{t},\epsilon_1,\epsilon_2) = \sum_{g_L,g_R\geq 0}\sum_{w\geq 1}\sum_{\mathbf{d}}\frac{n^{\boldsymbol{d}}_{g_L,g_R}}{w}\frac{\left(2\mathrm{sin}\left[\frac{w(\epsilon_1-\epsilon_2)}{4}\right]\right)^{2g_L}\left(2\mathrm{sin}\left[\frac{w(\epsilon_1+\epsilon_2)}{4}\right]\right)^{2g_R}}{\left(2\mathrm{sin}\left[\frac{w\epsilon_1}{2}\right]\right)\left(2\mathrm{sin}\left[\frac{w\epsilon_2}{2}\right]\right)}\mathrm{e}^{w\boldsymbol{d}\cdot\boldsymbol{t}}.
\tag{2.26}
$$

This free energy reproduces the free energy of the usual topological string theory by setting $\epsilon_1 = -\epsilon_2$ and $n^{\boldsymbol{d}}_{g,0} = n^{\boldsymbol{d}}_{g}$.

2.2.2 Calculation in B-model

In general, there is uniquely determined holomorphic 3-form $\Omega = \Omega(u_i,\boldsymbol{z})\mathrm{d}u_1\wedge\mathrm{d}u_2\wedge\mathrm{d}u_3 \in H^{(3,0)}(CY)$ in the Calabi–Yau manifold, where $\boldsymbol{z}$ are the complex moduli and $H^{(3,0)}(CY)$ is the Dolbeault cohomology. Then, by expanding Ω as a basis of the cohomology α_i and β^j,

$$
\begin{aligned}
&\Omega = \alpha_i X^i - \beta^j F_j, \\
&\qquad\int_{C^i}\alpha_j = \int_{\hat{C}_j}\beta^i = \delta^i_j, \quad \int_{C^i}\beta_j = \int_{\hat{C}_j}\alpha^i = 0, \\
&\qquad C^i\cap C^j = 0, \quad \hat{C}_i\cap\hat{C}_j = 0, \quad C^i\cap\hat{C}_j = \delta^i_j.
\end{aligned}
\tag{2.27}
$$

we define the coordinates of the complex moduli,

$$
X^i = \int_{C^i}\Omega, \quad i = 1,2,...,1+\dim(\mathcal{M}_{CY})
\tag{2.28}
$$

where C^i and $\hat{C}^i$ are the 3-cycles in the Calabi–Yau manifold. However, there are too many coordinates due to the ambiguity of an overall complex factor. In order to be well-defined, we rescale the coordinates X^i as following,

$$
t^j(\boldsymbol{z}) = \frac{X^j}{X^1}, \quad j = 1,...,\dim(\mathcal{M}_{CY}).
\tag{2.29}
$$

The mirror symmetry claims that $\boldsymbol{t}(\boldsymbol{z})$ are the mirror maps from the B-model to the A-model.

We also define another 3-cycle integral,

$$F_i = \int_{\hat{C}_i} \Omega. \tag{2.30}$$

Since F_i are the functions of X_i through the holomorphic 3-form Ω, one can show that the F_i satisfy the integrability conditions,

$$\frac{\partial F_j}{\partial X_i} = \frac{\partial F_i}{\partial X_j}, \tag{2.31}$$

and there is a function F_0 given by

$$F_0 = \frac{1}{2} X^i F_i. \tag{2.32}$$

The function F_0 is called as the prepotential which is the genus-zero free energy of the B-model topological string theory.

The rest task is how to calculate $t^j(z)$ and F_0 explicitly. Let us consider the case of the non-compact toric Calabi–Yau manifold. The algebraic definition is given by

$$vw - H(\mathrm{e}^x, \mathrm{e}^y; \boldsymbol{z}) = 0, \quad v, w \in \mathbb{C}. \tag{2.33}$$

It is known that the periods $X_i(\boldsymbol{z})$ satisfy the following differential equations called as the Picard–Fuchs equations,

$$\begin{aligned} &\Bigg(\prod_{i, Q_i^\alpha > 0} \partial_{x_i}^{Q_i^\alpha} - \prod_{i, Q_i^\alpha < 0} \partial_{x_i}^{-Q_i^\alpha} \Bigg) X_i(z_\alpha) = 0, \\ &\prod_{i=1}^{k+3} x_i^{Q_i^\alpha} = z_\alpha, \end{aligned} \tag{2.34}$$

where Q^α are the charge vectors that we will explain in appendix B. By solving these differential equations, we can obtain the mirror maps and the genus-zero free energy.

e.g.) Local $\mathbb{P}^2$

Let us consider local $\mathbb{P}^2$ as a non-compact toric Calabi–Yau manifold with a complex modulus. The charge vector is $Q = (-3, 1, 1, 1)$, and the complex modulus is

$$z = \frac{x_2 x_3 x_4}{x_1^3}, \quad \partial_{x_i} = (-3)^{\delta_{1,i}} x_i^{-1} z \partial_z, \ i = 1, 2, 3, 4. \tag{2.35}$$

Then, the Picard–Fuchs equation is

$$\big(\theta_z^3 + z(3\theta_z + 2)(3\theta_z + 1)3\theta_z\big) X_i(z) = 0, \quad \theta_z = z\partial_z. \tag{2.36}$$

By using the Frobenius method, we can solve the difference equation. First, we set

$$\pi(\epsilon, z) = \sum_{n=0}^{\infty} a_n(\epsilon) z^{n+\epsilon}. \tag{2.37}$$

Substituting (2.37) to the Picard–Fuchs equation, we obtain the recursion relation,

$$a_n(\epsilon)(n+\epsilon)^3 + a_{n-1}(\epsilon)\big\{3(n+\epsilon)-1\big\}\big\{3(n+\epsilon)-2\big\}\big\{3(n+\epsilon)-3\big\} = 0$$
$$\Rightarrow a_n(\epsilon) = \frac{(3n+3\epsilon)!}{\{(n+\epsilon)!\}^3}. \tag{2.38}$$

Then we can construct the solutions as following,

$$X_i(z) = \left.\frac{\partial^j \pi(\epsilon, z)}{\partial \epsilon^j}\right|_{\epsilon=0}, \quad j = 0, 1, 2. \tag{2.39}$$

Therefore, the mirror map is given by

$$-t(z) = \frac{X_1(z)}{X_0(z)} = \log[z] + 6z\,{}_4F_3\left(1, 1, \frac{4}{3}, \frac{5}{3}; 2, 2, 2; 27z\right)$$
$$= \log[z] + 6z + 45z^2 + 560z^3 + \frac{17325z^4}{2} + \frac{756756z^5}{5} + \mathcal{O}(z^6). \tag{2.40}$$

Its inverse function is

$$z = Q - 6Q^2 + 9Q^3 - 56Q^4 - 300Q^5 - 3942Q^6 - 48412Q^7 + \mathcal{O}(Q^8), \quad Q = \mathrm{e}^{-t}. \tag{2.41}$$

In this method, the prepotential can be calculated as follows,

$$\frac{\partial F_0(z(t))}{\partial t} = \frac{X_2(z(t))}{X_0(z(t))} = \frac{t^2}{6} + 3Q + \frac{45Q^2}{4} + \frac{244Q^3}{3} + \frac{12333Q^4}{16} + \frac{211878Q^5}{25} + \mathcal{O}(Q^6). \tag{2.42}$$

After integrating over t, we obtain

$$F_0(z(t)) = \frac{t^3}{18} - 3Q - \frac{45Q^2}{8} - \frac{244Q^3}{9} - \frac{12333Q^4}{64} - \frac{211878Q^5}{125} + O\left(Q^6\right). \tag{2.43}$$

The instanton part of this free energy agrees with the genus-zero free energy calculated in the A-model topological string theory. □

In the non-compact toric Calabi–Yau manifold, the holomorphic 3-form Ω is

$$\Omega = \frac{\mathrm{d}v \wedge \mathrm{d}x \wedge \mathrm{d}y}{v}. \tag{2.44}$$

Then, by the Cauchy's theorem and the Stokes' theorem, the 3-cycle integrals reduce to the 1-cycle integrals,

$$X_i(\boldsymbol{z}) = \int_{C^i} \Omega = \int_{D^i} \mathrm{d}y \wedge \mathrm{d}x = \oint_{A^i} \mathrm{d}x\, y(x), \tag{2.45}$$

where D^i is the (x, y)-plane defined by $H(\mathrm{e}^x, \mathrm{e}^y; \boldsymbol{z}) = 0$ called as the mirror curve,[9] and A^i are the 1-cycles along the mirror curve. Similarly, $\hat{C}_i$ integrals reduce to another 1-cycles B^i along the mirror curve. For example, the mirror curve of local $\mathbb{P}^2$ is given by

$$\mathrm{e}^x + \mathrm{e}^y + z\mathrm{e}^{-x-y} - 1 = 0, \tag{2.46}$$

and the A- and the B- cycles are defined in figure 7.

[9] When we define $H(\mathrm{e}^x, \mathrm{e}^y; z) = W_x(\mathrm{e}^x, \mathrm{e}^y) - \mathcal{E}$ in the case of the genus-one mirror curve, this notation is consistent with the mirror curve defined in the introduction.

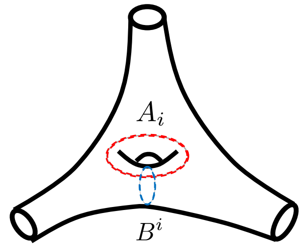

Figure 7: The mirror curve of local $\mathbb{P}^2$.

2.3 Non-perturbative effect: A-model side

Here we discuss how to include the non-perturbative contributions. These contributions are predicted by the ABJM theory [6, 8] which describes the dynamics of the M2-branes in the M theory. The partition function of the ABJM theory is given by the matrix integral [43],

$$\mathcal{Z}_{\text{ABJM}}(N,k) = \frac{1}{(N!)^2}\int \frac{\mathrm{d}^n\mu}{(2\pi)^N}\frac{\mathrm{d}^n\nu}{(2\pi)^N}\frac{\prod_{i<j}\left\{2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right\}^2\left\{2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right\}^2}{\prod_{i,j}\left\{2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right\}^2}\exp\left[\frac{\mathrm{i}k}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right], \tag{2.47}$$

where k is the Chern–Simons level, and N is the gauge rank. By using the Cauchy's formula,

$$\frac{\prod_{i<j}\left\{2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right\}^2\left\{2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right\}^2}{\prod_{i,j}\left\{2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right\}^2} = \sum_{\sigma\in S_N}\prod_i\frac{1}{2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)}, \tag{2.48}$$

the partition function can be written as

$$\mathcal{Z}_{\text{ABJM}}(N,k) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}\int\frac{\mathrm{d}^N x}{(2\pi k)^N}\frac{1}{\prod_{i=1}^N 2\cosh\left[\frac{x_i}{2}\right]2\cosh\left[\frac{x_i-x_{\sigma(i)}}{2k}\right]} \tag{2.49}$$

At this point, when we define the quantum density matrix $\rho(\mathsf{p},\mathsf{q})$ as

$$\langle x_i|\rho(\mathsf{p},\mathsf{q})|x_j\rangle = \frac{1}{2\pi k}\frac{1}{(2\cosh\left[\frac{x_i}{2}\right])^{\frac{1}{2}}}\frac{1}{(2\cosh\left[\frac{x_i-x_j}{2}\right])}\frac{1}{(2\cosh\left[\frac{x_j}{2}\right])^{\frac{1}{2}}}, \tag{2.50}$$

the partition function reduces to the canonical partition function of the Fermi gas with N particles [44],

$$\mathcal{Z}_{\text{ABJM}}(N,k) = \int \mathrm{d}^N x\sum_{\sigma\in S_N}\frac{(-1)^{\epsilon(\sigma)}}{N!}\prod_{i=1}^N\langle x_i|\rho(\mathsf{p},\mathsf{q})|x_{\sigma(i)}\rangle. \tag{2.51}$$

Therefore, we can use the knowledge of statistical mechanics to calculate the partition function of the ABJM theory.

It is convenient to define the grand partition function,

$$\Xi(\mu,k) = 1+\sum_{N=1}^{\infty}Z(N,k)\mathrm{e}^{N\mu}, \tag{2.52}$$

and the grand potential,

$$J(\mu, k) = \log[\Xi(\mu, k)]. \tag{2.53}$$

In [45], they calculated the grand potential numerically by utilizing the method in [46]. On the other hand, the instanton behavior is investigated in the large N limit [10, 11]. Based on these results, the authors in [47] proposed an analytical expression of the grand potential,

$$J(\mu, k) = \frac{2}{3\pi^2 k}\mu^3 + \left(\frac{k}{24} + \frac{1}{3k}\right)\mu + C(k) + \sum_{l,m=0,(l,m)\neq(0,0)}^{\infty} f_{l,m}(\mu, k)\exp\left[-\left(2l + \frac{4m}{k}\right)\mu\right], \tag{2.54}$$

where the coefficients $f_{l,m}$ are determined by comparing the numerical result in [45], and $C(k)$ is the function which is independent of μ.

Let us interpret the non-perturbative contributions from the point of view of the gravity dual theory which is the M theory on $AdS_4 \times S^7/\mathbb{Z}_k$. The label (l, m) denotes how many times the M2-branes wrap, and which the M2-branes wrap on M theory-circle or not. In the language of the type IIA superstring theory, the contributions l and m correspond to l D2-brane instantons and m worldsheet instantons, respectively. The contributions of non-zero values of l and m correspond to the bound states of the D2-branes and the fundamental strings.

The grand potential should be related to the topological string theory as following reason: as is well known, the topological string theory on the resolved conifold is dual to the U(N) Chern–Simons theory on S^3 [48]. This duality is called as the geometric transition. The conifold in figure 8(b) has a singularity in the two-intersection. Algebraically, the conifold is given by

$$s_+ s_- = w_+ w_-, \;\; s_\pm, w_\pm \in \mathbb{C}. \tag{2.55}$$

To avoid the singularity, we can take two methods; one is inserting S^2 structure with size t as in figure 8(a) as we will explain in appendix B. The other is inserting the S^3 structure as in figure 8(c) by turning on the parameter $R^2 \in \mathbb{R}$ in the right hand side of (2.55),

$$\begin{aligned} &s_+ s_- = w_+ w_- + R^2 \Leftrightarrow a_1^2 + a_2^2 + b_1^2 + b_2^2 = R^2 \\ &\Rightarrow \sum_{i=1}^{2}\left\{\left(\mathrm{Re}(a_i)^2 + \mathrm{Re}(b_i)^2\right) - \left(\mathrm{Im}(a_i)^2 + \mathrm{Im}(b_i)^2\right)\right\} = R^2 \\ &s_\pm = a_1 \pm \mathrm{i}a_2, \;\; w_\pm = \mathrm{i}(b_1 \pm \mathrm{i}b_2), \quad a_i, b_i \in \mathbb{C}, \end{aligned} \tag{2.56}$$

which is called as deformed conifold.

The remarkable difference is that we cannot wrap the brane while preserving the supersymmetry in the manifold (a), so that the resulting theory is closed topological string

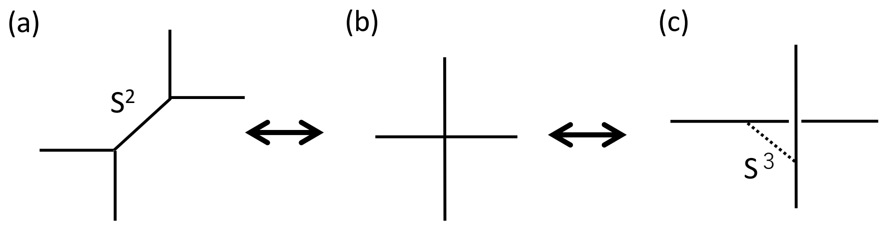

Figure 8: The geometric transition. The figure (a), (b), and (c) correspond to the resolved conifold, the singular conifold, and the deformed conifold.

theory. However, we can wrap the branes on S^3 without breaking the supersymmetry in the manifold (c), and the worldvolume theory on them is U(N) Chern–Simons theory [49].

Since the manifolds (a) and (c) are related via the singular manifold, the theories on these manifolds might be related, and explicit calculations of the free energies show that the closed topological string theory on the resolved conifold is dual to the U(N) Chern–Simons theory on S^3 under the following parameter correspondence,

$$t = \mathrm{i}g_s N, \;\; g_s = \frac{2\pi\mathrm{i}}{k+N}. \tag{2.57}$$

The geometric transition can apply to this situation as in figure 9. In [50], they pointed out that the ABJM theory is related to the U(N) Chern–Simons theory on $S^3/\mathbb{Z}_2$ under the analytic continuation. Furthermore, this theory is dual to the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [51] by the geometric transition.

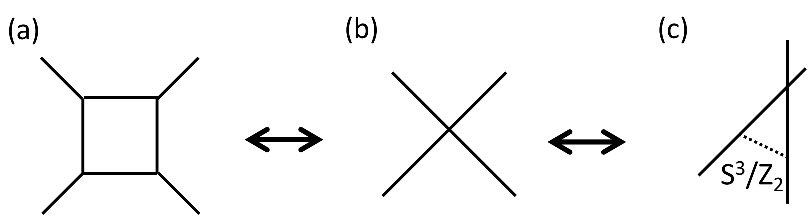

Figure 9: The another geometric transition. The topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ in the panel (a) is dual to the U(N) Chern–Simons theory on $S^3/\mathbb{Z}_2$ in the panel (c) via the singular manifold in the panel (b).

Keeping this fact in mind, let us back to the free energy of the ABJM theory. In [45], they showed that the worldsheet instanton summation in (2.58) agrees with the free energy of the topological string theory. The D2-bane contributions are fixed numerically [45], and the bound states contributions are fixed by imposing the finiteness for arbitrary Chern–Simons level k; the authors in [45] found that the worldsheet and the D2-branes contributions have the poles for $k = 2\pi a/b,\ a, b \in \mathbb{Z}$. However, the partition function of the ABJM theory is finite for arbitrary k. Therefore, to be finite in the ABJM partition function, the poles of all contributions are precisely canceled.

The D2-brane and the bound state contributions can not find in the topological string theory side. If the geometric transition can be applied in the non-perturbative regime, their contributions should be given in the language of the topological string theory. Based on this consideration, the authors in [8] proposed that the grand potential of the ABJM theory is given by the topological string and the refined topological string theory,

$$J(\mu, k) = J^{(\mathrm{W.S.})}(\mu_{\mathrm{eff}}, k) + \mu_{\mathrm{eff}} J_b(\mu_{\mathrm{eff}}, k) + J_c(\mu_{\mathrm{eff}}, k), \tag{2.58}$$

where we define several functions as following,

$$J^{(\mathrm{W.S.})}(\mu, k) = \sum_{g\geq 0} \sum_{w,d\geq 1} (-1)^{dw} \frac{n_g^d}{w} \left(2\sin\left[\frac{2\pi w}{k}\right]\right)^{2g-2} \mathrm{e}^{-\frac{4dw}{k}\mu}, \tag{2.59}$$

$$\mu J_b(\mu, k) + J_c(\mu, k) = -\frac{k^2}{2\pi\mathrm{i}} \frac{\partial}{\partial k} \left[\frac{1}{k} F_{\mathrm{NS}}\left(2\mu - \frac{\mathrm{i}\pi k}{2}, 2\mu + \frac{\mathrm{i}\pi k}{2}, \frac{k}{2}\right)\right], \tag{2.60}$$

$$\mu_{\mathrm{eff}} = \mu + 2(-1)^{n-1}\mathrm{e}^{-2\mu}\ {}_4F_2\left(1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2, 2; 16(-1)^n \mathrm{e}^{-2\mu}\right),\ k = 2n\ (k : \mathrm{even}), \tag{2.61}$$

$$\mu_{\mathrm{eff}} = \mu + \mathrm{e}^{-4\mu}\ {}_4F_2\left(1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2, 2; -16\mathrm{e}^{-4\mu}\right),\ k : \mathrm{odd}, \tag{2.62}$$

and F_{NS} is defined as

$$F_{\mathrm{NS}}(\boldsymbol{t}, \hbar) = \lim_{\epsilon_2 \to 0} \epsilon_2 F_{\mathrm{ref.}}(\boldsymbol{t}, \epsilon_1 = \hbar, \epsilon_2), \tag{2.63}$$

called as the Nekrasov–Shatashvili free energy.

By substituting the following correspondence between the ABJM theory and the topological string theory,

$$\frac{4\mu}{k} \pm \mathrm{i} = t_{1,2},\ k = \frac{2}{g_s}, \tag{2.64}$$

we can obtain the free energy of the topological string theory including the non-perturbative effects. This duality is valid in the case of local $\mathbb{P}^1 \times \mathbb{P}^1$, howerver, the authors in [8] generalize the above result, and propose the non-perturbative free energy of the topological string theory on general non-compact toric Calabi–Yau manifold,

$$\begin{aligned} F^{(\mathrm{n.p.})}(\boldsymbol{t}_{\mathrm{eff}}, g_s) &= F\left(\boldsymbol{t}_{\mathrm{eff}} + \pi\mathrm{i}\boldsymbol{B}, g_s\right) + \frac{1}{2\pi\mathrm{i}} \frac{\partial}{\partial g_s} \left[g_s F_{\mathrm{NS}}\left(\frac{\boldsymbol{t}_{\mathrm{eff}}}{g_s}, \frac{1}{g_s}\right)\right], \\ t_{\mathrm{eff},i} &= t_i + 2\pi\mathrm{i}n_i - g_s \Pi_{A_i}\left(\frac{t_i + 2\pi\mathrm{i}n_i}{g_s}, \frac{2\pi\mathrm{i}}{g_s}\right),\ n_i \in \mathbb{Z}_{>0}, \end{aligned} \tag{2.65}$$

where $\boldsymbol{B}$ is the B-field which has a integer value, and Π_{A_i} are the quantum A-periods that we will explain in next subsection. Of course this expression is finite for arbitrary values of g_s.

We have a comment on this non-perturbative completion. In [52], a non-perturbative free energy of the A-model refined topological string is proposed from the M theory analysis,

$$F^{\text{n.p.}}_{\text{ref.}}(\boldsymbol{t},\epsilon_1,\epsilon_2)=F_{\text{ref.}}(\boldsymbol{t},\epsilon_1,\epsilon_2)+F_{\text{ref.}}\left(\frac{\boldsymbol{t}}{\epsilon_1},\frac{1}{\epsilon_1},\frac{\epsilon_2}{\epsilon_1}+2\pi\mathrm{i}\right)+F_{\text{ref.}}\left(\frac{\boldsymbol{t}}{\epsilon_1},\frac{\epsilon_1}{\epsilon_2}+2\pi\mathrm{i},\frac{1}{\epsilon_2}\right) \tag{2.66}$$

Then, if the non-perturbative topological string theory is uniquely determined, $F^{\text{n.p.}}_{\text{ref.}}(\boldsymbol{t},\epsilon_1,-\epsilon_1)$ should agree with (2.65). However, these two expressions are slightly different. We do not know which there are several non-perturbative definitions or not.

2.4 Non-perturbative effect: B-model side

Let us discuss the B-model side. Again we consider the ABJM theory. The partition function of the ABJM theory, which is related to the topological string theory on local $\mathbb{P}^1\times\mathbb{P}^1$, can be treated as the Fermi gas. By using the Wigner transformation, we can read off the quantum density matrix $\hat{\rho}$ from the expectation value of the operator and define the quantum Hamiltonian $\hat{H}$,

$$\rho(\mathsf{p},\mathsf{q})=\frac{1}{\left(2\cosh\left[\frac{\mathsf{q}}{2}\right]\right)^{\frac{1}{2}}}\frac{1}{\left(2\cosh\left[\frac{\mathsf{p}}{2}\right]\right)}\frac{1}{\left(2\cosh\left[\frac{\mathsf{q}}{2}\right]\right)^{\frac{1}{2}}}, \tag{2.67}$$

$$\mathrm{e}^{-H(\mathsf{p},\mathsf{q})}=\rho, \tag{2.68}$$

where

$$[\mathsf{q},\mathsf{p}]=\mathrm{i}\hbar,\quad \hbar=2\pi k. \tag{2.69}$$

In the classical limit $\hbar=0$, the density matrix operator is written as

$$\rho_{\text{cl.}}=\mathrm{e}^{-E}=\frac{1}{\mathrm{e}^{\frac{q+p}{2}}+\mathrm{e}^{\frac{-q-p}{2}}+\mathrm{e}^{\frac{q-p}{2}}+\mathrm{e}^{\frac{-q+p}{2}}}$$
$$=\frac{1}{\mathrm{e}^{x}+\mathrm{e}^{-x}+\mathrm{e}^{y}+\mathrm{e}^{-y}}, \tag{2.70}$$
$$\left(x=\frac{q+p}{2},\quad y=\frac{q-p}{2}\right). \tag{2.71}$$

At this point, this expression is exactly the same as the inverse of the mirror curve of local $\mathbb{P}^1\times\mathbb{P}^1$ defined in section 1.4,

$$W_{\mathbb{P}^1\times\mathbb{P}^1}(\mathrm{e}^{x},\mathrm{e}^{y})=\mathrm{e}^{x}+\mathrm{e}^{-x}+\mathrm{e}^{y}+\mathrm{e}^{-y}=\mathrm{e}^{E}, \tag{2.72}$$

where we set $m=1$ and $\mathrm{e}^{E}=\mathcal{E}$. Then, as a naive guess we might be able to define the B-model non-perturbative topological string theory on general non-compact toric Calabi–Yau manifold by the quantization of the mirror curve. In order to compare with the A-model side like the mirror symmetry, we need to define the mirror map in the quantum mirror curve. This mirror map is called as the quantum mirror map [16, 53, 54].

To define the quantum mirror map, we consider the following Schrödinger equation for the quantum mirror curve H,[10]

$$H(\mathrm{e}^{\mathsf{x}},\mathrm{e}^{\mathsf{y}};\boldsymbol{z})\Psi(x)=0,\ \ [\mathsf{x},\mathsf{y}]=\mathrm{i}\hbar,$$

In the semi-classical limit, we can use the WKB method, and the wave function can be expressed as

$$\Psi(x)=\exp\left[\frac{1}{\hbar}S(x)\right], \tag{2.73}$$

$$S(x)=\sum_{n=0}^{\infty}S_n(x)\hbar^n. \tag{2.74}$$

The leading term is given by [13],

$$\Psi_{\text{leading}}(x)=\exp\left[\frac{1}{\hbar}\int^{x}\mathrm{d}x\,y(x)\right], \tag{2.75}$$

so that the derivative of $S_0(x)$ is determined as

$$\frac{\partial S_0(x)}{\partial x}=y(x). \tag{2.76}$$

Recall that we integrate the function $y(x)$ over x to obtain the classical mirror map. In order to be consistent with the classical limit, we define the quantum A- and B-period as

$$\Pi_{A_i}(\boldsymbol{z},\hbar)=\int_{A^i}\mathrm{d}x\,\frac{\partial S(x)}{\partial x},\quad \Pi_{B_i}(\boldsymbol{z},\hbar)=\int_{B^i}\mathrm{d}x\,\frac{\partial S(x)}{\partial x}. \tag{2.77}$$

For the convenience of calculating the quantum periods, we define

$$V(x)=\frac{\Psi(x+\hbar)}{\Psi(x)}. \tag{2.78}$$

Then,

$$\begin{aligned}\oint\log[V(x)]\,\mathrm{d}x&=\oint\mathrm{d}x\,\frac{S(x+\hbar)-S(x)}{\hbar}\\&=\oint\mathrm{d}x\,\frac{\partial S(x)}{\partial x}+\oint\sum_{n=2}^{\infty}\frac{\partial^n S(x)}{\partial x^n}\frac{\hbar^{n-1}}{n!}\\&=\oint\mathrm{d}x\,\frac{\partial S(x)}{\partial x},\end{aligned} \tag{2.79}$$

where we impose that the wave function should be single-valued in order to be well-defined. This means that $\oint\mathrm{d}x\,\partial_x^{n\geq 2}S(x)$ has to be vanished. As a result, the quantum mirror map is given by the residues of $\log[V(x)]$. Let us demonstrate the calculation of the quantum period and the partition function.

[10]In order to discuss the quantum mirror map of genus-g mirror curve, we use this notation defined in the section 2.2.2.

e.g.) Local $\mathbb{P}^2$

We calculate the quantum A-period of local $\mathbb{P}^2$. By quantizing the mirror curve (2.46), the Schrödinger equation is rewritten as

$$\mathrm{e}^x\Psi(x) + \Psi(x-\mathrm{i}\hbar) + z\mathrm{e}^{-x-\frac{\mathrm{i}\hbar}{2}}\Psi(x+\mathrm{i}\hbar) - \Psi(x) = 0. \tag{2.80}$$

By dividing the both sides by $\Psi(x)$, (2.80) becomes

$$X + \frac{1}{V(q^{-1}X)} + zq^{-\frac{1}{2}}X^{-1}V(x) - 1 = 0, \quad q = \mathrm{e}^{\mathrm{i}\hbar},\ X = \mathrm{e}^x. \tag{2.81}$$

We can solve the difference equation recursively,

$$V(X) = \sum_{n=0}^{\infty} v_n(X)z^n, \tag{2.82}$$

$$\begin{aligned} v_0(X) &= \frac{1}{1-qX},\ v_1(X) = \frac{1}{q^{\frac{3}{2}}X(1-qX)^2(1-q^2X)}, \\ v_3(X) &= \frac{1+q-qX-q^4X}{q^4X^2(1-qX)^3(1-q^2X)(1-q^3X)}, \cdots . \end{aligned} \tag{2.83}$$

Thus, the quantum mirror map is

$$\begin{aligned} -t(z,\hbar) &= \log[z] + 3\Pi_A(z,\hbar) \\ &= \log[z] + 3(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) + \frac{3}{2}(2q^2 + 7q + 12 + 7q^{-1} + 2q^{-2}) + \cdots , \end{aligned} \tag{2.84}$$

where we multiply 3 in front of the quantum A-period. This factor is determined from the geometries. In the genus-one mirror curve, the factor is the anti-canonical class of the del Pezzo surface. Note that we cannot obtain the logarithmic term in this calculation since $V(x)$ becomes 1 in the classical limit.

Next we consider the inverse of the quantum mirror curve as is the case in the Fermi gas formalism. The calculation is mainly based on [55]. We shift the variables $\mathsf{x} \to \mathsf{x} - \log[u]$ and $\mathsf{y} \to \mathsf{y} - \log[u]$, where $z = u^{-3}$. Then the quantum mirror curve is

$$H(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}}; z) = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{\mathsf{x}+\mathsf{y}} + u =: W_{\mathbb{P}^2}(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}}) + u. \tag{2.85}$$

After some calculations, we obtain

$$\begin{aligned} \rho_{\mathbb{P}^2} &= W_X(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}})^{-1} = A^*A, \\ A &= \Phi_b^*(\mathsf{q} - \mathrm{i}b/2)\Phi_b^*(\mathsf{p})\exp\left[\frac{2\pi b}{3}\mathsf{q}\right]\exp\left[\frac{\pi b}{3}\mathsf{p}\right], \end{aligned} \tag{2.86}$$

where Φ is the quantum dilogarithm that we will define in appendix A, and we define some operators and quantities as following,

$$\begin{aligned} &\mathsf{x} = \frac{2\pi b}{3}(2\mathsf{p} + \mathsf{q}),\ \mathsf{y} = \frac{2\pi b}{3}(\mathsf{p} + 2\mathsf{q}),\ [\mathsf{p}, \mathsf{q}] = \frac{1}{2\pi \mathrm{i}}, \\ &\hbar = \frac{2\pi b^2}{3},\ b \in \mathbb{R}_{>0}. \end{aligned} \tag{2.87}$$

From this expression, the density matrix operator is the trace class operator, so that the eigenvalues of $W_{\mathbb{P}^2}(\mathrm{e}^{\mathsf{x}}, \mathrm{e}^{\mathsf{y}})$ are discrete. By defining the grand partition function in terms of the density of state,[11]

$$\Xi_{\mathbb{P}^2}(u, \hbar) = \det[1 + u\rho_{\mathbb{P}^2}], \;\; u = \mathrm{e}^{\mu}, \tag{2.88}$$

and by expanding around $u = 0$, we can obtain the generating function of the trace of $\rho_{\mathbb{P}^2}$,

$$\log[\Xi_{\mathbb{P}^2}(u, \hbar)] = -\sum_{L=0}^{\infty} \frac{(-u)^L}{L} \mathrm{Tr}[\rho_{\mathbb{P}^2}^L] \tag{2.89}$$

By combining this result with (2.52), we can determine the partition function $Z_{\mathbb{P}^2}(N, \hbar)$ sequentially,

$$\begin{aligned} &Z_{\mathbb{P}^2}(1, \hbar) = \mathrm{Tr}[\rho_{\mathbb{P}^2}], \\ &Z_{\mathbb{P}^2}(2, \hbar) = \frac{1}{2}\big\{(\mathrm{Tr}[\rho_{\mathbb{P}^2}])^2 - \mathrm{Tr}[\rho_{\mathbb{P}^2}]^2]\big\}, \\ &\cdots . \end{aligned} \tag{2.90}$$

For example, when we set $\hbar = 2\pi$, the grand partition function is

$$\begin{aligned} \Xi_{\mathbb{P}^2}(u, \hbar = 2\pi) &= 1 + \sum_{N=1}^{\infty} Z_{\mathbb{P}^2}(N, \hbar = 2\pi) u^N \\ &= 1 + \frac{1}{9}u + \left(\frac{1}{12\sqrt{3}\pi} - \frac{1}{81}\right)u^2 + \cdots , \end{aligned} \tag{2.91}$$

□

2.5 The relation between A-model and B-model

We now propose the relation between the A-model and the B-model as the conjecture [12, 15]. Here we provide the most general expression. In the non-compact toric Calabi–Yau manifold with genus-g mirror curve, the non-perturbative free energy including the perturbative part,

$$\begin{aligned} &J(\boldsymbol{\mu}, \boldsymbol{m}, \hbar) = \frac{C_{ijk}}{12\pi\hbar} t_i t_j t_k + \left(\frac{2\pi B_i}{\hbar} + \frac{\hbar \tilde{B}_i}{2\pi}\right) t_i + A(\boldsymbol{m}, \hbar) + F^{(\mathrm{n.p.})}(\boldsymbol{t}, g_s), \\ &g_s = \frac{4\pi^2}{\hbar}, \;\; -\log z_i = \sum_{j=1}^{g} D_{ij}\mu_j, \;\; \mathrm{e}^{\mu_i} = u_i \end{aligned} \tag{2.92}$$

is related to the B-model grand partition function as following equation,

$$\Xi_{\text{B-model}}(\boldsymbol{u}, \hbar) = \sum_{\boldsymbol{n} \in \mathbb{Z}^g} \mathrm{e}^{J_{\text{A-model}}(\boldsymbol{\mu} + 2\pi \mathrm{i} \boldsymbol{n}, \boldsymbol{m}, \hbar)}, \tag{2.93}$$

[11] Actually we can show that this expression is consistent with the previous definition of the grand partition function in the case of local $\mathbb{P}^1 \times \mathbb{P}^1$. In general, when we define the trace class operator, we can define the partition function given in terms of the Fermi gas, and the grand partition function is given by the determinant [56, 57].

where $\boldsymbol{z}$ and $\boldsymbol{m}$ are the complex moduli and the tunable parameters, B_i, $\tilde{B}_i$, C_{ijk}, D_{ij} are the constants, and $A(\boldsymbol{m},\hbar)$ is the function. The complex moduli $\boldsymbol{z}$ and the tunable parameters $\boldsymbol{m}$ are related to the Kähler parameters $\boldsymbol{t}$ by the quantum mirror map $\boldsymbol{t}=\boldsymbol{t}(\boldsymbol{z},\boldsymbol{m};\hbar)$.

Notice that we take the summation over $2\pi\mathrm{i}\boldsymbol{n}$. The grand partition function is the periodic function of $\boldsymbol{\mu}$. However, the grand potential is lost the periodicity. In order to restore the periodicity, we take the summation.[12]

e.g.) Local $\mathbb{P}^2$

Let us check the conjecture in the case of local $\mathbb{P}^2$. Before checking, we rewrite (2.93). By multiplying $\mathrm{e}^{N\mu}$ and integrating μ along the complex axis, we find

$$Z_{\text{B-model}}(N,\hbar)=\sum_{n\in\mathbb{Z}}\int_{-\pi\mathrm{i}}^{\pi\mathrm{i}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\mathrm{e}^{J_{\text{A-model}}(\mu+2\pi\mathrm{i}n,\hbar)-N\mu}. \tag{2.94}$$

In the right hand side, we can extend the integration contour to go along the full imaginary axis,

$$\begin{aligned}(\text{r.h.s.})&=\sum_{n\in\mathbb{Z}}\int_{-\pi\mathrm{i}+2\pi\mathrm{i}n}^{\pi\mathrm{i}+2\pi\mathrm{i}n}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\mathrm{e}^{J_{\text{A-model}}(\mu,\hbar)-N\mu}\\&=\left(\cdots+\int_{-3\pi\mathrm{i}}^{-\pi\mathrm{i}}+\int_{-\pi\mathrm{i}}^{\pi\mathrm{i}}+\int_{\pi\mathrm{i}}^{3\pi\mathrm{i}}+\cdots\right)\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\mathrm{e}^{J_{\text{A-model}}(\mu,\hbar)-N\mu}\\&=\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\mathrm{e}^{J_{\text{A-model}}(\mu,\hbar)-N\mu}.\end{aligned} \tag{2.95}$$

Furthermore, we deform the contour to $\mathcal{C}$ defined in figure 10. Then, by expanding $\mathrm{e}^{J_{\text{A-model}}(\mu,\hbar)}$

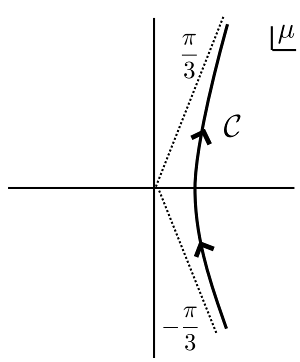

Figure 10: The contour.

as

$$\mathrm{e}^{J_{\text{A-model}}(\mu,\hbar)}=\mathrm{e}^{\frac{C(\hbar)}{3}\mu^3+B(\hbar)\mu+A(\hbar)}\sum_{l,n=0}^{\infty}a_{l,n}(\hbar)\mathrm{e}^{-rl\mu}\mu^n, \tag{2.96}$$

[12]This prescription also perform in the ABJM theory [45].

we can express the partition function as the Airy function $\mathrm{Ai}(z)$,

$$\int_{\mathcal{C}} \mathrm{e}^{\frac{C(\hbar)}{3}\mu^3+B(\hbar)\mu+A(\hbar)} \sum_{l,n=0}^{\infty} a_{l,n}(\hbar)\mathrm{e}^{-rl\mu-N\mu}\mu^n$$
$$= \frac{\mathrm{e}^{A(\hbar)}}{C^{\frac{1}{3}}(\hbar)} \sum_{l,n=0}^{\infty} a_{l,n}(\hbar)\left(-\frac{\partial}{\partial N}\right)^n \mathrm{Ai}\left(\frac{N+rl-B(\hbar)}{C^{\frac{1}{3}}(\hbar)}\right) =: Z_{\text{A-model}}(N,\hbar). \tag{2.97}$$

The leading term is $(l,n)=0$ term, and the correction terms are $(l,n)\neq(0,0)$ suppressed rapidly due to the property of the Airy function as in figure 11. Thus, even if we calculate the leading term and few corrections, we get the highly precise result as we will show below.

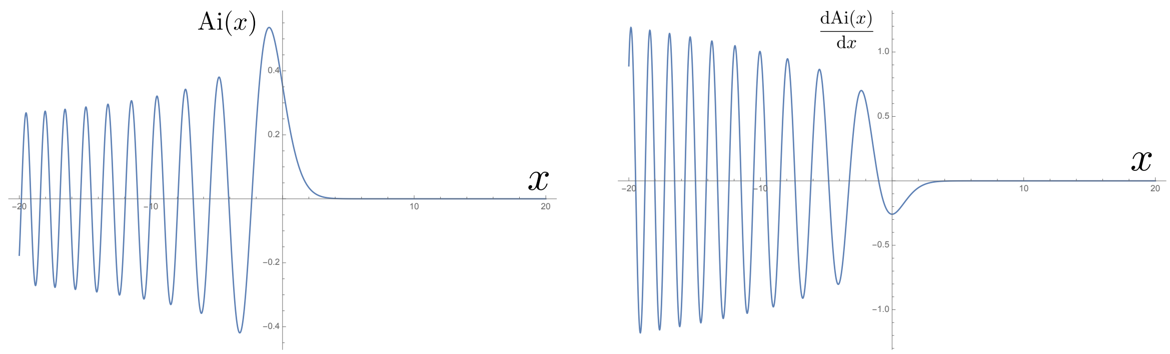

Figure 11: The Airy function and its derivative. These plots show that the Airy function and its derivative are suppressed rapidly as x grows.

The function $B(\hbar)$, $C(\hbar)$ is given in [58],

$$B(\hbar)=\frac{\pi}{2\hbar}-\frac{\hbar}{16\pi},\ C(\hbar)=\frac{9}{4\pi\hbar}. \tag{2.98}$$

In order to normalize the partition function, we use the following normalized partition function,

$$\hat{Z}_{\text{A-model}}(N,\hbar)=\frac{Z_{\text{A-model}}(N,\hbar)}{Z_{\text{A-model}}(0,\hbar)}, \tag{2.99}$$

so that we need not know the explicit form of $A(\hbar)$.

In the case of $\hbar=2\pi$, the instanton part of the grand potential up to $\mathcal{O}(\mathrm{e}^{-6\mu})$ is given by

$$J_{\text{inst.}}(\mu,2\pi)=\left(-\frac{45}{8\pi^2}\mu^2-\frac{9}{4\pi^2}\mu-\frac{3}{4\pi^2}+\frac{3}{8}\right)\mathrm{e}^{-3\mu}+\left(-\frac{999}{16\pi^2}\mu^2-\frac{63}{16\pi^2}\mu-\frac{45}{32\pi^2}+\frac{153}{16}\right)\mathrm{e}^{-6\mu}+\cdots. \tag{2.100}$$

Some coefficients $a_{l,n}$ are

$$
\begin{aligned}
&a_{0,0} = 1, \\
&a_{1,0} = \frac{3}{8} - \frac{3}{4\pi^2}, \quad a_{1,1} = -\frac{9}{4\pi^2}, \quad a_{1,2} = -\frac{45}{8\pi^2}, \quad a_{1,n\geq 3} = 0, \\
&a_{2,0} = \frac{1233}{128} + \frac{9}{32\pi^4} - \frac{27}{16\pi^2}, \quad a_{2,1} = \frac{27}{16\pi^4} - \frac{153}{32\pi^2}, \quad a_{2,2} = \frac{27}{4\pi^4} - \frac{4131}{64\pi^2}, \quad a_{2,3} = \frac{405}{32\pi^4}, \quad a_{2,4} = \frac{2025}{128\pi^4}.
\end{aligned}
\tag{2.101}
$$

By combining these results, we calculate the partition function numerically in some order,

$$
\begin{aligned}
\hat{Z}_{\text{A-model}}(1, 2\pi)_{\text{up to } l=0} &= 0.11106574082622089606963048862 0 \\
\hat{Z}_{\text{A-model}}(1, 2\pi)_{\text{up to } l=1} &= 0.11111111111428022765625149149 2. \\
\hat{Z}_{\text{A-model}}(1, 2\pi)_{\text{up to } l=2} &= 0.11111111111111111111104269367 87
\end{aligned}
\tag{2.102}
$$

By including the higher corrections, this value approaches to the B-model partition function,

$$
Z_{\text{B-model}}(1, 2\pi) = \frac{1}{9} = 0.1111111111111111111111111111111. \tag{2.103}
$$

□

2.6 Exact quantization condition

As a remarkable property in the B-model topological string theory, we would like to discuss the S-duality that we will use to propose the TS-CMP correspondence. The quantum B-period can be interpreted as the quantum corrected prepotential. These corrections are expressed as the Nekrasov–Shatashvili free energy [22],

$$
\begin{aligned}
\Pi_{B_i}(\boldsymbol{z}, \hbar) &= \frac{\partial \mathcal{F}_{\text{NS}}(t, \hbar)}{\partial t}, \\
\mathcal{F}_{\text{NS}}(t, \hbar) &= F_{\text{NS}}^{\text{pert.}}(t, \hbar) + F_{\text{NS}}(t, \hbar), \\
t &= t(\boldsymbol{z}, \boldsymbol{m}; \hbar)
\end{aligned}
\tag{2.104}
$$

On the other hand, we can impose the quantization condition in the quantum mechanics,

$$
\oint \mathrm{d}x\, p(x) = 2\pi\hbar\left(n + \frac{1}{2}\right), \tag{2.105}
$$

so that we obtain the following condition,

$$
\frac{\partial \mathcal{F}_{\text{NS}}(t, \hbar)}{\partial t} = 2\pi\hbar\left(n + \frac{1}{2}\right). \tag{2.106}
$$

By using the condition, we can solve the eigenvalue problem of the quantum mirror curve.

However, there is a crucial problem for this condition. As is the case in the ABJM theory, the left hand side of (2.106) has the poles for $\hbar = 2\pi a/b$, $a, b \in \mathbb{Z}_{>0}$. This is due to the resummation of the quantum correction to express as the Nekrasov–Shatashvili free energy [59].

To resolve this problem, we add the non-perturbative contribution [12, 21] to cancel the poles,

$$\begin{aligned} &\frac{\partial \mathcal{F}_{\rm NS}(t,\hbar)}{\partial t}+\frac{\partial \mathcal{F}_{\rm NS}(\tilde{t},\tilde{\hbar})}{\partial \tilde{t}}=2\pi\hbar\left(n+\frac{1}{2}\right),\\ &\tilde{t}=\frac{2\pi}{\hbar}t,\ \tilde{\hbar}=\frac{4\pi^2}{\hbar}. \end{aligned} \tag{2.107}$$

This condition is called as the exact quantization condition. Since we add the non-perturbative part by hand, (2.107) is conjecture. Let us check the conjecture.

e.g.)Local $\mathcal{B}_3$

In our paper, we discuss the spectral problem for local $\mathcal{B}_3$. To calculate the quantum mirror map, first we write the difference equation for local $\mathcal{B}_3$,

$$\begin{aligned} &(\mathrm{e}^{\mathsf{x}}+\mathrm{e}^{\mathsf{y}}+\mathrm{e}^{-\mathsf{x}-\mathsf{y}}+m_1\mathrm{e}^{-\mathsf{x}}+m_2\mathrm{e}^{-\mathsf{y}}+m_3\mathrm{e}^{\mathsf{x}+\mathsf{y}})\Psi(x)=\mathcal{E}\Psi(x)\\ &\Rightarrow X+\frac{m_1}{X}+V(X)+\frac{m_2}{V(qX)}+\frac{q^{-\frac{1}{2}}}{XV(qX)}+m_3q^{-\frac{1}{2}}XV(X)=\frac{1}{z}, \end{aligned} \tag{2.108}$$

$$V(X)=\frac{\Psi(x-\mathrm{i}\hbar)}{\Psi(x)},\ X=\mathrm{e}^{x},\ q=\mathrm{e}^{\mathrm{i}\hbar},\ z=\frac{1}{\mathcal{E}} \tag{2.109}$$

By expanding $V(X)$ as a series of z,

$$V(X)=\sum_{n=-1}^{\infty}v_n z^n, \tag{2.110}$$

we find the coefficients v_n,

$$v_{-1}(X)=\frac{1}{1+m_3q^{-\frac{1}{2}}X},\ v_0(X)=-\frac{m_1+q^{-\frac{1}{2}}X^2}{X(1+m_3q^{-\frac{1}{2}}X)},\ \cdots. \tag{2.111}$$

Finally, by taking the residues of $\log[V(X)]$, we obtain

$$\begin{aligned} -t(\mathcal{E},\boldsymbol{m},\hbar)=&\log z+(m_1+m_2+m_3)z^2+(q^{1/2}+q^{-1/2})(1+m_1m_2m_3)z^3\\ &+\left[\frac{3}{2}(m_1^2+m_2^2+m_3^2)+(4+q+q^{-1})(m_1m_2+m_2m_3+m_3m_1)\right]z^4+\mathcal{O}(z^5), \end{aligned} \tag{2.112}$$

$$z=\frac{1}{\mathcal{E}}. \tag{2.113}$$

In the case of $\hbar=2\pi$ and $m_1=m_2=m_3=1$, the problem is drastically simplified. The exact quantization condition and the quantum mirror map become

$$\begin{aligned} &3t^2-2\pi^2+t\frac{\partial}{\partial t}F_0^{\rm inst}(t+\pi\mathrm{i})-\frac{\partial}{\partial t}F_0^{\rm inst}(t+\pi\mathrm{i})=4\pi^2\left(n+\frac{1}{2}\right),\\ &-t(\mathcal{E},1,2\pi)=\log z+3z^2+4z^3+\frac{45}{2}z^4+\cdots, \end{aligned} \tag{2.114}$$

where $F_0^{\rm inst}(t+\pi{\rm i})$ is the instanton part of the genus-zero free energy of the usual topological string theory. This can be calculated from the topological vertex formalism,

$$F_0^{\rm inst}(t+\pi{\rm i}) = -6{\rm e}^{-t} - \frac{21}{4}{\rm e}^{-2t} - \frac{56}{9}{\rm e}^{-3t} - \frac{405}{32}{\rm e}^{-4t} - \frac{3756}{125}{\rm e}^{-5t} - \frac{751}{9}{\rm e}^{-6t} + \cdots . \tag{2.115}$$

By combining these results, we can calculate the eigenvalues $\mathcal{E}$ up to some order of $Q = {\rm e}^{-t}$. For example, in $n = 0$ corresponding the ground energy, we obtain

$$\begin{aligned} \text{up to } \mathcal{O}(Q^2) :& E_0 = \log \mathcal{E}_0 = 5.12332648396024387890839077424, \\ \text{up to } \mathcal{O}(Q^6) :& E_0 = \log \mathcal{E}_0 = 5.12332441505781350040103701489, \\ \text{Numerical value} :& E_0 = \log \mathcal{E}_0 = 5.12332441505673431832082360188, \end{aligned} \tag{2.116}$$

where the numerical value is obtained by diagonalizing the Hamiltonian in the harmonic oscillator basis [54]. This result shows the validity of the exact quantization condition. □

From this expression, we can see that the condition is invariant under the S-transformation,

$$(t, \hbar) \mapsto (\tilde{t}, \tilde{\hbar}). \tag{2.117}$$

We call this invariance as the S-duality. We will discuss what we can say from this duality in the section 4.

3 Hofstadter model

In this section, we introduce the Hofstadter model [20]. Its spectrum shows the fractal structure called as the Hofstadter butterfly.

In [19], they discuss the correspondence between the topological string theory on $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hofstadter model on the square lattice. In our paper [60], we discuss the correspondence between the topological string theory on local $\mathcal{B}_3$ and the Hofstadter model on the triangular lattice. Since the triangular lattice is the generalization of the square lattice mathematically, we discuss the case of the triangular lattice shown in figure 12.

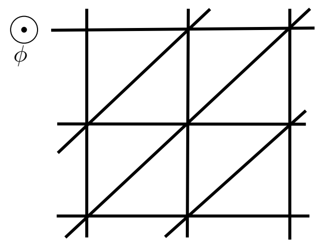

Figure 12: The triangular lattice. We put the magnetic flux ϕ which is perpendicular to the lattice.

3.1 Hamiltonian and Harper's equation

The Hamiltonian in the tight binding approximation is defined by

$$H_{\text{tri.}} = \sum_{\langle i,j \rangle} t_{i,j} c_j^\dagger c_i \mathrm{e}^{\mathrm{i} \int_i^j \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}}, \tag{3.1}$$

where $\boldsymbol{A}$ is the vector potential, $t_{i,j}$ are the hopping parameters between the nearest neighbors, and $c_i^\dagger$, c_i are the creation and the annihilation operators of the electron in the cite i satisfying the following commutation relations,

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = 0, \quad \{c_i^\dagger, c_j^\dagger\} = 0. \tag{3.2}$$

We assume that the hopping parameters along the vertical, horizontal, and diagonal direction are the same,

$$\begin{aligned} & t_{i+(m+1)1_x, i+m1_x} = t_{i+1_x, i}, \\ & t_{i+(m+1)1_y, i+m1_y} = t_{i+1_y, i}, \\ & t_{i+(m+1)1_{xy}, i+m1_xy} = t_{i+1_{xy}, i}, \\ & (m \in \mathbb{Z}) \end{aligned} \tag{3.3}$$

where $1_{x,y,xy}$ are the unit vectors along the vertical, horizontal, and diagonal direction. Then, by dividing the both side of the Hamiltonian by the hopping parameter $t_{i,i+1_x}$, and redefining the Hamiltonian to cancel the factor $1/t_{i,i+1_x}$, we obtain

$$H_{\text{tri.}} = \sum_j c^\dagger_{j+1_x} c_j \mathrm{e}^{\mathrm{i}\int_j^{j+1_x} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}} + \lambda_1 \sum_j c^\dagger_{j+1_y} c_j \mathrm{e}^{\mathrm{i}\int_j^{j+1_y} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}} + \lambda_2 \sum_j c^\dagger_{j+1_x+1_y} c_j \mathrm{e}^{\mathrm{i}\int_j^{j+1_x+1_y} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{l}} \tag{3.4}$$

where the parameters $\lambda_{1,2}$ are defined as follows,

$$\lambda_1 = \frac{t_{i,i+1_y}}{t_{i,i+x}},\ \lambda_2 = \frac{t_{i,i+1_y+1_y}}{t_{i,i+1_x}}. \tag{3.5}$$

In the case of $\lambda_2 = 0$, the Hamiltonian reduces to the one on the square lattice.

The vector potential is related to the magnetic flux ϕ,

$$\oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \phi, \tag{3.6}$$

where ∂S is the boundary of the square, so that the magnetic flux which is perpendicular to the unit cell (triangle) is $\phi/2$ (see figure 13). We fix the gauge as $\boldsymbol{A} = (0, \phi x, 0)$, so called

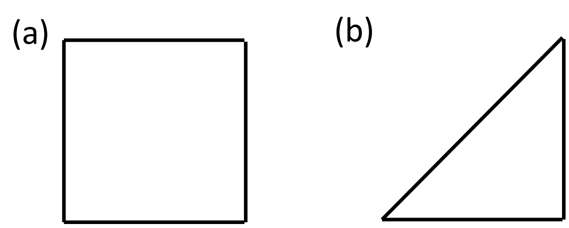

Figure 13: The unit cell of the square lattice (the left panel) and the triangular lattice (the right panel). The magnetic flux in the unit cell of the square lattice is ϕ, so that the magnetic flux in the unit cell of the triangular lattice is $\phi/2$.

Landau gauge. Then, the Hamiltonian becomes

$$\begin{aligned} H_{\text{tri.}} &= \sum_j c^\dagger_{j+1_x} c_j + \lambda_1 \sum_j c^\dagger_{j+1_y} c_j \mathrm{e}^{\mathrm{i}\phi j_x} + \lambda_2 \sum_j c^\dagger_{j+1_x+1_y} c_j \mathrm{e}^{\mathrm{i}\phi j_x + \mathrm{i}\frac{\phi}{2}}, \\ j &= (j_x, j_y) \end{aligned} \tag{3.7}$$

Let us consider the Schrödinger equation,

$$H_{\text{tri.}}|\psi\rangle = E|\psi\rangle, \tag{3.8}$$

under the following magnetic flux,

$$\phi = 2\pi\frac{a}{b}, \tag{3.9}$$

where a and b are the coprime integers. In this situation, the Hamiltonian is invariant under the translation along the horizontal direction, $j_x \to j_x + b$. Then, according to the Bloch's theorem, the wave function can be written as

$$\begin{aligned}&\psi(j) = \mathrm{e}^{\mathrm{i}(k_x j_x + k_y j_y)}\psi_{n_x}(k_x, k_y), \quad \psi_{j_x+b}(k_x, k_y) = \psi_{j_x}(k_x, k_y),\\&|\psi\rangle = \sum_l \psi_l(k_x, k_y)c_l^\dagger|0\rangle,\end{aligned} \tag{3.10}$$

where k_x and k_y are the momentum in the Brillouin zone $0 \le k_x \le 2\pi/b$, and $0 \le k_y \le 2\pi$, and the Schrödinger equation becomes as following difference equations,

$$\begin{aligned}&\tilde{B}^*_{j-1}\psi_{j-1} + A_j\psi_j + \tilde{B}_j\psi_{j+1} = E\psi_j,\\&A_j = -2\cos(k_x + \phi j), \quad \tilde{B}_j = -\mathrm{e}^{\mathrm{i}k_y}B_j, \quad B_j = \lambda_1 + \lambda_2\mathrm{e}^{\mathrm{i}(k_x+\phi j)}.\\&(j = 1, 2, ..., b)\end{aligned} \tag{3.11}$$

called as the Harper's equations. The spectrum of the Harper's equation is determined by the determinant,

$$\begin{aligned}&\det[D_{a/b}(E, \boldsymbol{\lambda})] = 0,\\&D_{a/b}(E, \boldsymbol{\lambda}) = \begin{pmatrix} A_1 - E & B_1 & 0 & \cdots & 0 & 0 & B_b^*\mathrm{e}^{-\mathrm{i}bk_y}\\ B_1^* & A_2 - E & B_2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & B^*_{b-2} & A_{b-1} - E & B_{b-1}\\ B_b\mathrm{e}^{\mathrm{i}bk_y} & 0 & 0 & \cdots & 0 & B^*_{b-1} & A_b - E\end{pmatrix}.\end{aligned} \tag{3.12}$$

We have two comments about the relation between the Hofstader model and the topological string theory. The Hamiltonian on the square lattice $H_{\mathrm{sq.}}$ is given by the magnetic translation operators $T_{x,y}$ [61, 24, 25],

$$\begin{aligned}&H_{\mathrm{sq.}} = H_{\mathrm{tri.}}|_{\lambda_2=0} = T_x + T_x^\dagger + \lambda_1(T_y + T_y^\dagger),\\&T_x = \sum_j c^\dagger_{j+1_x}c_j\mathrm{e}^{\mathrm{i}\int_j^{j+1_x}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}}, \ T_y = \sum_j c^\dagger_{j+1_y}c_j\mathrm{e}^{\mathrm{i}\int_j^{j+1_y}\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}}.\end{aligned} \tag{3.13}$$

Similarly, the Hamiltonian on the triangular lattice can be also expressed by the simple one,

$$H_{\mathrm{tri.}} = T_x + T_x^\dagger + \lambda_1(T_y + T_y^\dagger) + \lambda_2(\mathrm{e}^{-\frac{\mathrm{i}\phi}{2}}T_xT_y + \mathrm{e}^{\frac{\mathrm{i}\phi}{2}}T_y^\dagger T_x^\dagger). \tag{3.14}$$

The commutation relations of the magnetic translation operators are given by

$$T_xT_y = \mathrm{e}^{\mathrm{i}\phi}T_yT_x, \quad T_xT_x^\dagger = T_yT_y^\dagger = 1. \tag{3.15}$$

This expression is the same structure as the quantum mirror curve of local $\mathcal{B}_3$, as we explained in the section 1.5.

When we consider the Hofstadter model under the strong magnetic field, the tight binding approximation might be broken. However, in [62], they pointed out that even if we consider the case of the strong magnetic field, we obtain almost the same Harper's equation but the flux is given by $\frac{4\pi^2}{\phi}$. Therefore, we can study the Hofstadter model in the tight binding Hamiltonian for any values of the magnetic flux. This property corresponds to the S-dual structure in the topological string theory.

3.2 Spectrum and Density of state

In this subsection, we show the band spectrum, and calculate the density of state. First, we solve the determinant (3.12) numerically in the region of the Brillouin zone, and the result is in figure 14. The Hofstadter butterfly for $\lambda_2 \neq 0$ is not invariant under the shift $\phi \to \phi + 2\pi$

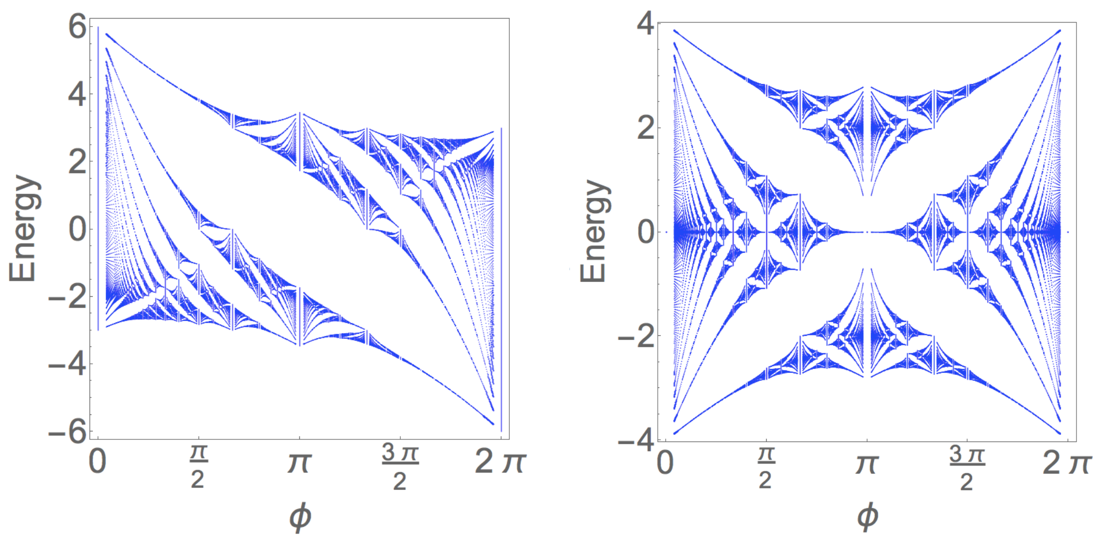

Figure 14: The Hofstadter butterfly. The horizontal axis and the vertical axis correspond to the magnetic flux and the energy of the electron. In the left panel we set $\lambda_1 = \lambda_2 = 1$. In the right panel we set $\lambda_1 = 1$, $\lambda_2 = 0$ which corresponds to the Hofstadter butterfly on the square lattice.

since there is the term having $\phi/2$ factor in the Hamiltonian.

Next, we calculate the density of state in this system. For simplicity, let us consider the case of $\lambda_1 = \lambda_2 = 1$. When the magnetic flux is turned off, by using the dispersion relation,

$$E(k_x, k_y) = 2\cos[k_x] + 2\cos[k_y] + 2\cos[k_x + k_y], \tag{3.16}$$

the density of state is given by

$$\rho_0(E) = \int_0^{2\pi} \mathrm{d}k_x \int_0^{2\pi} \mathrm{d}k_y \frac{1}{4\pi^2} \delta(2\cos[k_x] + 2\cos[k_y] + 2\cos[k_x + k_y] - E),$$

$$= \begin{cases} \dfrac{1}{2\pi^2(3+E)^{1/4}}\mathbb{K}\left(\dfrac{12-E^2+8\sqrt{3+E}}{16\sqrt{3+E}}\right), & -2<E\leq 6, \\ \dfrac{2}{\pi^2\sqrt{12-E^2+8\sqrt{3+E}}}\mathbb{K}\left(\dfrac{16\sqrt{3+E}}{12-E^2+8\sqrt{3+E}}\right), & -3\leq E<-2, \end{cases} \tag{3.17}$$

where $\mathbb{K}(z)$ is the complete elliptic integral of the first kind,

$$\mathbb{K}(z) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\alpha}{\sqrt{1-z\sin^2[\alpha]}}. \tag{3.18}$$

When we turn on the magnetic flux, the dispersion relation is given by

$$\begin{aligned} &F_{a/b}(E) = 2\cos[bk_x] + 2\cos[bk_y] + 2(-1)^{ab}\cos[bk_x+bk_y], \\ &F_{a/b}(E) = \widetilde{D}_{a/b}(E) + 2\{(1+(-1)^b+(-1)^{(a-1)b}\}, \\ &\widetilde{D}_{a/b}(E) = \det\begin{pmatrix} A_1+E & B_1 & 0 & \cdots & 0 & 0 & B_b^* \\ B_1^* & A_2+E & B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{b-2}^* & A_{b-1}+E & B_{b-1} \\ B_b & 0 & 0 & \cdots & 0 & B_{b-1}^* & A_b+E \end{pmatrix}, \\ &A_j = 2\cos(\phi j),\ B_j = -1+\mathrm{e}^{\mathrm{i}\phi j}. \end{aligned} \tag{3.19}$$

The density of state is almost the same as (3.17),

$$\begin{aligned} &\rho(E,\phi) = \\ &\begin{cases} \dfrac{|F'(E)|}{2\pi^2 b(3+F(E))^{1/4}}\mathbb{K}\left(\dfrac{12-F^2(E)+8\sqrt{3+F(E)}}{16\sqrt{3+F(E)}}\right), & -2<F(E)\leq 6, \\ \dfrac{2|F'(E)|}{\pi^2 b\sqrt{12-F^2(E)+8\sqrt{3+F(E)}}}\mathbb{K}\left(\dfrac{16\sqrt{3+F(E)}}{12-F^2(E)+8\sqrt{3+F(E)}}\right), & -3\leq F(E)<-2, \end{cases} \\ &F(E) = (-1)^{ab}F_{a/b}(E). \end{aligned} \tag{3.20}$$

We plot some examples of the density of state as in figure 15. Then, we find the poles called as the Van Hove singularity [63]. The reason that the Van Hove singularity arises is as following: we plot the dispersion relation (3.16) for $\phi = 0$ in figure 16. Then, we find that there is a flat direction. If the region where the integrand diverges is a point, the integral might be finite. However, the region is not a point due to the existence of the flat direction, so that the integral diverges.

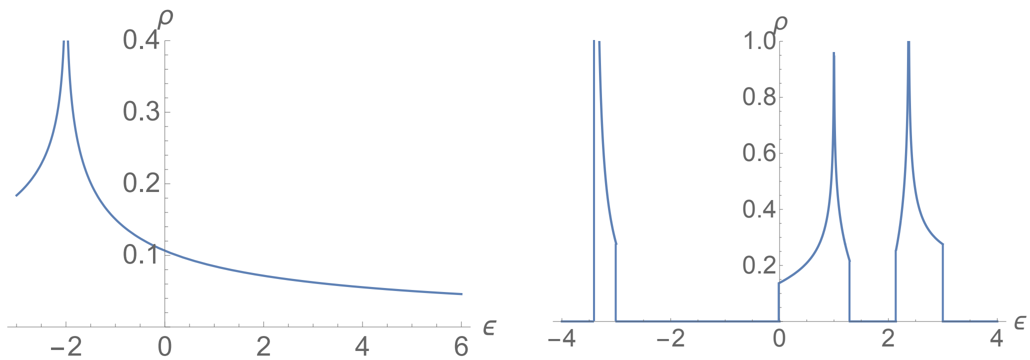

Figure 15: The density of state for $\lambda_1 = \lambda_2 = 1$. The horizontal axis and the vertical axis correspond to the energy and the density of state. In the left panel we set $\phi = 0$, and in the right panel we set $\phi = 4\pi/3$.

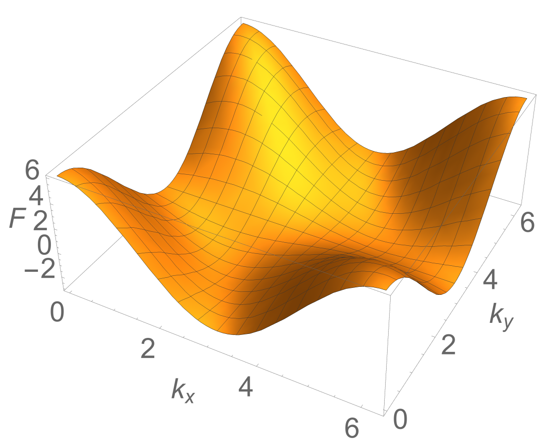

Figure 16: The dispersion relation.

4 The TS-CMP correspondence

We now propose the TS-CMP correspondence. This claims that the branch cuts of the quantum mirror map give the Hofstadter butterfly, and the imaginary part of the quantum mirror map on the cuts gives the density of state.

4.1 S-duality

In this subsection, we explain how to calculate the quantum mirror map in the closed form. In general, as we explained in the section 2.4, we need to calculate the quantum mirror map order by order. Again we write the result in the section 2.6,

$$\begin{aligned} -t(\mathcal{E},\boldsymbol{m},\hbar) &= \log z + (m_1+m_2+m_3)z^2 + (q^{1/2}+q^{-1/2})(1+m_1m_2m_3)z^3 \\ &\quad + \left[\frac{3}{2}(m_1^2+m_2^2+m_3^2) + (4+q+q^{-1})(m_1m_2+m_2m_3+m_3m_1)\right]z^4 + \mathcal{O}(z^5), \\ z &= \frac{1}{\mathcal{E}},\ q = \mathrm{e}^{\mathrm{i}\hbar}. \end{aligned} \tag{4.1}$$

From this expression, we observe that the quantum mirror map is invariant under the shift $\hbar \to \hbar \pm 4\pi$ and the flip $\hbar \to -\hbar$,

$$t(\mathcal{E},\boldsymbol{m},\hbar \pm 4\pi) = t(\mathcal{E},\boldsymbol{m},\hbar),\ t(\mathcal{E},\boldsymbol{m},-\hbar) = t(\mathcal{E},\boldsymbol{m},\hbar). \tag{4.2}$$

We call this transformation as T-transformation. By using the S- and the T-transformation, and setting $\hbar = 2\pi a/b$, we can calculate the quantum mirror map exactly.

The S-duality claims that the exact quantization condition is invariant under the S-transformation (2.117). This means that there is a $(\tilde{\hbar},\tilde{\mathcal{E}})$-system,

$$\begin{aligned} &\mathrm{e}^{\tilde{\mathsf{x}}} + \mathrm{e}^{\tilde{\mathsf{y}}} + \mathrm{e}^{-\tilde{\mathsf{x}}-\tilde{\mathsf{y}}} + \tilde{m}_1\mathrm{e}^{-\tilde{\mathsf{x}}} + \tilde{m}_2\mathrm{e}^{-\tilde{\mathsf{y}}} + \tilde{m}_3\mathrm{e}^{\tilde{\mathsf{x}}+\tilde{\mathsf{y}}} = \tilde{\mathcal{E}}, \\ &(\tilde{\mathsf{x}},\ \tilde{\mathsf{y}}) = \left(\frac{2\pi}{\hbar}\mathsf{x},\ \frac{2\pi}{\hbar}\mathsf{y}\right), \quad \tilde{m}_i = m_i^{\frac{2\pi}{\hbar}}, \\ &[\tilde{\mathsf{x}},\ \tilde{\mathsf{y}}] = \tilde{\hbar} = \frac{4\pi^2}{\hbar}, \end{aligned} \tag{4.3}$$

and the quantum mirror map in the dual system $\tilde{t}$ can be calculated from the WKB approximation,

$$t(\tilde{\mathcal{E}},\tilde{\boldsymbol{m}},\tilde{\hbar}) = \oint_A \mathrm{d}x\,\frac{\partial S(x)}{\partial x},\ \Psi(\tilde{x}) = \exp\left[\frac{1}{\tilde{\hbar}}S(\tilde{x})\right], \tag{4.4}$$

where we omit the tilde of the dual quantum mirror map since the form of the function is the same as the quantum mirror map. Furthermore, according to the exact quantization condition, t and $\tilde{t}$ are satisfied with the following relation,

$$\tilde{t} = \frac{2\pi}{\hbar}t. \tag{4.5}$$

By combining these facts, the dual complex modulus $\tilde{\mathcal{E}}$ is related to the complex modulus $\mathcal{E}$ through the quantum mirror map,

$$t(\tilde{\mathcal{E}},\tilde{\boldsymbol{m}},\tilde{\hbar})=\frac{2\pi}{\hbar}t(\mathcal{E},\boldsymbol{m},\hbar). \tag{4.6}$$

Let us demonstrate what this relation is.

e.g.) $\hbar=\pi,\ \tilde{\hbar}=4\pi$

We investigate the relation (4.6) in the case of $\hbar=\pi,\ \tilde{\hbar}=4\pi$, and $\boldsymbol{m}=\mathbf{1}$. From two difference equations, we obtain the quantum mirror map and the dual one,

$$-t(\mathcal{E},\boldsymbol{m}=\mathbf{1},\hbar=\pi)=\log z+3z^2+\frac{21}{2}z^4+56z^6+\frac{1485}{4}z^8+\frac{14058}{5}z^{10}+\mathcal{O}(z^{12}), \tag{4.7}$$

$$\begin{aligned}-t(\tilde{\mathcal{E}},\boldsymbol{m}=\mathbf{1},\tilde{\hbar}=4\pi)=&\log\tilde{z}+3\tilde{z}^2+4\tilde{z}^3+\frac{45}{2}\tilde{z}^4+72\tilde{z}^5+340\tilde{z}^6+1440\tilde{z}^7\\&+\frac{27405}{4}\tilde{z}^8+\frac{96880}{3}\tilde{z}^9++\frac{794178}{5}\tilde{z}^{10}+\mathcal{O}(\tilde{z}^{11}),\end{aligned} \tag{4.8}$$

$$\tilde{\mathcal{E}}=\frac{1}{\tilde{z}}. \tag{4.9}$$

Let us find the relation between the complex modulus and its dual. Since the quantum mirror curve and its dual curve are commute, we can diagonalize two operators simultaneously. After shifting the variable $y\to y-x/2$, we write down the difference equations for the quantum mirror curve and the dual one,

$$\begin{aligned}&2\cosh\left(\frac{x}{2}+\frac{\mathrm{i}\hbar}{4}\right)\Psi(x+\mathrm{i}\hbar)+2\cosh\left(\frac{x}{2}-\frac{\mathrm{i}\hbar}{4}\right)\Psi(x-\mathrm{i}\hbar)=(\mathcal{E}-2\cosh x)\Psi(x),\\&2\cosh\left(\frac{\tilde{x}}{2}+\frac{\mathrm{i}\tilde{\hbar}}{4}\right)\Psi(x+2\pi\mathrm{i})+2\cosh\left(\frac{\tilde{x}}{2}-\frac{\mathrm{i}\tilde{\hbar}}{4}\right)\Psi(x-2\pi\mathrm{i})=(\tilde{\mathcal{E}}-2\cosh\tilde{x})\Psi(x).\end{aligned} \tag{4.10}$$

By substituting $\hbar=\pi$, we find

$$2\cosh\left(\frac{x}{2}+\frac{\mathrm{i}\pi}{4}\right)\Psi(x+\mathrm{i}\pi)+2\cosh\left(\frac{x}{2}-\frac{\mathrm{i}\pi}{4}\right)\Psi(x-\mathrm{i}\pi)=(\mathcal{E}-2\cosh x)\Psi(x), \tag{4.11}$$

$$-2\cosh(x)\Psi(x+2\mathrm{i}\pi)-2\cosh(x)\Psi(x-2\mathrm{i}\pi)=(\tilde{\mathcal{E}}-2\cosh(2x))\Psi(x). \tag{4.12}$$

By shifting $x\to x\pm\mathrm{i}\pi$ in (4.11), we obtain two difference equations,

$$2\cosh\left(\frac{x}{2}+\frac{3\pi\mathrm{i}}{4}\right)\Psi(x+2\pi\mathrm{i})+2\cosh\left(\frac{x}{2}+\frac{\pi\mathrm{i}}{4}\right)\Psi(x)=(\mathcal{E}+2\cosh x)\Psi(x+\pi\mathrm{i}), \tag{4.13}$$

$$2\cosh\left(\frac{x}{2}-\frac{\pi\mathrm{i}}{4}\right)\Psi(x)+2\cosh\left(\frac{x}{2}-\frac{3\pi\mathrm{i}}{4}\right)\Psi(x-2\pi\mathrm{i})=(\mathcal{E}+2\cosh x)\Psi(x-\pi\mathrm{i}). \tag{4.14}$$

From (4.13) $\times\ 2\cosh(x/2+\pi\mathrm{i}/4)$ + (4.14) $\times\ 2\cosh(x/2-\pi\mathrm{i}/4)$, and using (4.11), we find

$$2\cosh\left(\frac{x}{2}+\frac{3\pi\mathrm{i}}{4}\right)\cosh\left(\frac{x}{2}+\frac{\pi\mathrm{i}}{4}\right)\Psi(x+2\pi\mathrm{i})+2\cosh\left(\frac{x}{2}-\frac{3\pi\mathrm{i}}{4}\right)\cosh\left(\frac{x}{2}-\frac{\pi\mathrm{i}}{4}\right)\Psi(x-2\pi\mathrm{i})$$

$$+2\left\{\cosh^2\left(\frac{x}{2}+\frac{\pi\mathrm{i}}{4}\right)+\cosh^2\left(\frac{x}{2}-\frac{\pi\mathrm{i}}{4}\right)\right\}\Psi(x)$$

$$=(\mathcal{E}+2\cosh x)\cosh\left(\frac{x}{2}-\frac{\pi\mathrm{i}}{4}\right)\Psi(x+\pi\mathrm{i})-(\mathcal{E}+2\cosh x)\cosh\left(\frac{x}{2}+\frac{\pi\mathrm{i}}{4}\right)\Psi(x-\pi\mathrm{i})$$

$$\Leftrightarrow -2\cosh(x)\Psi(x+2\mathrm{i}\pi)-2\cosh(x)\Psi(x-2\mathrm{i}\pi)=(\mathcal{E}^2-6+2\cosh x)\Psi(x). \tag{4.15}$$

Therefore, when the wave function is non-trivial, we find

$$\tilde{\mathcal{E}}=\mathcal{E}^2-6 \tag{4.16}$$

which satisfies (4.6). □

In general, the relation is given by

$$\mathcal{F}_{b/a}(\tilde{\mathcal{E}},\tilde{\boldsymbol{m}})=\mathcal{F}_{a/b}(\mathcal{E},\boldsymbol{m}), \tag{4.17}$$

where $\mathcal{F}_{a/b}(\mathcal{E},\boldsymbol{m})$ is the degree-b polynomial. In the case of local $\mathbb{P}^1\times\mathbb{P}^1$, we can calculate $\mathcal{F}_{a/b}(\mathcal{E},\boldsymbol{m})$ from the mirror curve and its dual,

$$\mathrm{e}^x+\mathrm{e}^{-x}+m(\mathrm{e}^y+\mathrm{e}^{-y})=\mathcal{E}_{\mathbb{P}^1\times\mathbb{P}^1}, \tag{4.18}$$

$$\mathrm{e}^{\tilde{x}}+\mathrm{e}^{-\tilde{x}}+\tilde{m}(\mathrm{e}^{\tilde{y}}+\mathrm{e}^{-\tilde{y}})=\tilde{\mathcal{E}}_{\mathbb{P}^1\times\mathbb{P}^1}. \tag{4.19}$$

The calculation method in [19] is as following: by shifting $x\to x+\mathrm{i}j\hbar$ in (4.18) and $x\to 2\pi\mathrm{i}j$ in (4.19) for the integers $j\in\mathbb{Z}$, these difference equations are written as the matrix forms,

$$\begin{pmatrix}\Psi_{j+1}\\ \Psi_j\end{pmatrix}=\begin{pmatrix}T_j & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_j\\ \Psi_{j-1}\end{pmatrix}, \tag{4.20}$$

$$\begin{pmatrix}\tilde{\Psi}_{j+1}\\ \tilde{\Psi}_j\end{pmatrix}=\begin{pmatrix}\tilde{T}_j & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\tilde{\Psi}_j\\ \tilde{\Psi}_{j-1}\end{pmatrix}, \tag{4.21}$$

where we define

$$\Psi_j=\Psi(x+\mathrm{i}j\hbar),\quad T_j=\frac{\mathcal{E}_{\mathbb{P}^1\times\mathbb{P}^1}-2\cosh[x+\mathrm{i}j\hbar]}{R^2}, \tag{4.22}$$

$$\tilde{\Psi}_j=\Psi(x+2\pi\mathrm{i}j),\quad \tilde{T}_j=\frac{\tilde{\mathcal{E}}_{\mathbb{P}^1\times\mathbb{P}^1}-2\cosh[\tilde{x}+4\pi^2\mathrm{i}j/\hbar]}{\tilde{R}^2}. \tag{4.23}$$

When we set $\hbar=2\pi a/b$, by using the periodicity,

$$T_{j+b}=T_j,\ \tilde{T}_{j+a}=\tilde{T}_j, \tag{4.24}$$

we can rewrite the difference equations as

$$\Psi_b+\Psi_{-b}=\mathrm{Tr}\left[\begin{pmatrix}T_{b-1} & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}T_{b-2} & -1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}T_0 & -1\\ 1 & 0\end{pmatrix}\right]\Psi_0, \tag{4.25}$$

$$\tilde{\Psi}_a + \tilde{\Psi}_{-a} = \mathrm{Tr}\left[\begin{pmatrix} \tilde{T}_{a-1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{T}_{a-2} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \tilde{T}_{0} & -1 \\ 1 & 0 \end{pmatrix}\right]\tilde{\Psi}_0. \tag{4.26}$$

Therefore,

$$\begin{aligned}&\mathrm{Tr}\left[\begin{pmatrix} \tilde{T}_{a-1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{T}_{a-2} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} \tilde{T}_{0} & -1 \\ 1 & 0 \end{pmatrix}\right] \\ &\quad = \mathrm{Tr}\left[\begin{pmatrix} T_{b-1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} T_{b-2} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} T_{0} & -1 \\ 1 & 0 \end{pmatrix}\right].\end{aligned} \tag{4.27}$$

The essence of the calculation is that e^x and e^y are separated in the (dual) mirror curve. However, local $\mathcal{B}_3$ is not the case, and the derivation of $\mathcal{F}_{a/b}(\mathcal{E},\boldsymbol{m})$ from the difference equations is difficult. Alternatively, we use the knowledge of the condensed matter physics to guess $\mathcal{F}_{a/b}(\mathcal{E},\boldsymbol{m})$. From the result in the previous section, we can guess the explicit form of $\mathcal{F}_{a/b}$ except for $b=2$,

$$\begin{aligned}&\mathcal{F}_{a/b}(\mathcal{E}',\boldsymbol{m}) = \mathcal{D}_{a/b}(\mathcal{E}',\boldsymbol{\lambda}) + 2\left\{\lambda_1^b + (-1)^{(a-1)b}\lambda_2^b + (-1)^b\right\}, \\ &\mathcal{D}_{a/b}(\mathcal{E}',\boldsymbol{\lambda}) = \det\begin{pmatrix} \mathcal{A}_1+\mathcal{E}' & \mathcal{B}_1 & 0 & \cdots & 0 & 0 & \mathcal{B}_b^* \\ \mathcal{B}_1^* & \mathcal{A}_2+\mathcal{E}' & \mathcal{B}_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{B}_{b-2}^* & \mathcal{A}_{b-1}+\mathcal{E}' & \mathcal{B}_{b-1} \\ \mathcal{B}_b & 0 & 0 & \cdots & 0 & \mathcal{B}_{b-1}^* & \mathcal{A}_b+\mathcal{E}' \end{pmatrix}, \\ &\mathcal{A}_j = 2\cos\left(\frac{2\pi a j}{b}\right),\ \mathcal{B}_j = -\lambda_1 + \lambda_2 \mathrm{e}^{\frac{2\pi \mathrm{i} a}{b}}, \\ &\mathcal{E}' = \frac{\mathcal{E}}{\sqrt{m_1}},\end{aligned} \tag{4.28}$$

where we identify the tunable parameters $\boldsymbol{m}$ with the hopping parameters $\boldsymbol{\lambda}$,

$$\lambda_1 = \sqrt{\frac{m_2}{m_1}}, \quad \lambda_2 = \sqrt{\frac{m_3}{m_1}}, \tag{4.29}$$

and impose the following condition to be Hermitian,

$$m_1 m_2 m_3 = 1. \tag{4.30}$$

Since the complex modulus $\mathcal{E}$ is replaced by $\mathcal{E}'$, we consider the $(\mathcal{E}',\boldsymbol{\lambda})$-system. Then, (4.6) is slightly modified as follows,

$$\begin{aligned}&\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},\tilde{\hbar}) = \frac{2\pi}{\hbar} t'(\mathcal{E}',\boldsymbol{\lambda},\hbar), \\ &t'(\mathcal{E}',\boldsymbol{\lambda},\hbar) = t(\mathcal{E},\boldsymbol{m},\hbar) - \frac{1}{2}\log m_1 \\ &\qquad = \log z' + (1+\lambda_1^2+\lambda_2^2) z'^2 + 2(q^{1/2}+q^{-1/2})\lambda_1\lambda_2 z'^3\end{aligned} \tag{4.31}$$

$$+\left[\frac{3}{2}(1+\lambda_1^4+\lambda_2^4)+(4+q+q^{-1})(\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2)\right]z'^4+\mathcal{O}(z'^5). \tag{4.32}$$

$(z'=\sqrt{m_1}z)$

4.2 Explicit calculation of Quantum mirror map

We now ready to calculate the quantum mirror map from the classical one. For example, let us consider the case of $a=2,\ b=5$. By using the S- and the T-transformation, the quantum mirror map reduces to the classical one,

$$\begin{aligned} t'(\mathcal{E}',\boldsymbol{\lambda},2\pi\cdot 2/5) &= \frac{2}{5}t'^d(\mathcal{E}'^d,\boldsymbol{\lambda}^d,2\pi\cdot 5/2)=\frac{2}{5}t'^d(\mathcal{E}'^d,\boldsymbol{\lambda}^d,2\pi\cdot 1/2)\\ &= \frac{2}{5}\cdot\frac{1}{2}\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},2\pi\cdot 2)=\frac{2}{5}\cdot\frac{1}{2}\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}};0), \end{aligned} \tag{4.33}$$

$$\mathcal{F}_{2/5}(\mathcal{E}',\boldsymbol{\lambda})=\mathcal{F}_{5/2}(\mathcal{E}'^d,\boldsymbol{\lambda}^d)=\mathcal{F}_{1/2}(\mathcal{E}'^d,\boldsymbol{\lambda}^d)=\mathcal{F}_{2/1}(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}})=\tilde{\mathcal{E}}', \tag{4.34}$$

where we denote “ d ” by S-transformation. In the case of $a=3,b=5$, we find

$$\begin{aligned} t'(\mathcal{E}',\boldsymbol{\lambda},2\pi\cdot 3/5) &= \frac{3}{5}t'^d(\mathcal{E}'^d,\boldsymbol{\lambda}^d,2\pi\cdot 5/3)=\frac{3}{5}t'^d(\mathcal{E}^d,\boldsymbol{\lambda},-2\pi\cdot 1/3)\\ &= \frac{3}{5}t'^d(\mathcal{E}'^d,\boldsymbol{\lambda}^d,2\pi\cdot 1/3)=\frac{3}{5}\cdot\frac{1}{3}\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},2\pi\cdot 3)=\frac{3}{5}\cdot\frac{1}{3}\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},2\pi), \end{aligned} \tag{4.35}$$

$$\mathcal{F}_{3/5}(\mathcal{E}',\boldsymbol{\lambda})=\mathcal{F}_{5/3}(\mathcal{E}'^d,\boldsymbol{\lambda}^d)=\mathcal{F}_{-1/3}(\mathcal{E}'^d,\boldsymbol{\lambda}^d)=\mathcal{F}_{-3}(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}})=\mathcal{F}_1(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}})=\tilde{\mathcal{E}}'. \tag{4.36}$$

Based on these results, we find the quantum mirror map in the general values of a and b,

$$\begin{aligned} &t'(\mathcal{E}',\boldsymbol{\lambda},\hbar=2\pi a/b)=\begin{cases}\frac{1}{b}\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},0)=\frac{1}{b}(\log[\tilde{\mathcal{E}}']-\Pi_A(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},0)), & \text{for}\quad ab:\ \text{even}\\ \\ \frac{1}{b}\tilde{t}'(\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},2\pi)=\frac{1}{b}(\log[\tilde{\mathcal{E}}']-\Pi_A(-\tilde{\mathcal{E}}',\tilde{\boldsymbol{\lambda}},0)), & \text{for}\quad ab:\ \text{odd}\end{cases}\\ &\tilde{\mathcal{E}}'=\mathcal{F}_{a/b}(\mathcal{E}',\tilde{\boldsymbol{\lambda}}),\ \ \tilde{\lambda}_i=\lambda_i^b. \end{aligned} \tag{4.37}$$

Therefore, by using the classical mirror map, we can obtain the quantum mirror map. Let us demonstrate the evaluation.

e.g.) $\boldsymbol{m}=\boldsymbol{1}$, $\mathcal{E}'=\mathcal{E}$

In this case, we can easily calculate the derivative of the classical mirror map from the definition,

$$\begin{aligned} t(\mathcal{E},\boldsymbol{1},\hbar=0)=\oint_A \mathrm{d}x\,y(x)=\frac{1}{\pi\mathrm{i}}\int_{x_-}^{x_+}\mathrm{d}x\,\mathrm{arccosh}\left(\frac{\mathcal{E}-2\cosh x}{4\cosh\frac{x}{2}}\right)\ \text{for}\ \mathcal{E}>6,\\ \frac{\mathcal{E}-2\cosh x_\pm}{4\cosh\frac{x_\pm}{2}}=\mp 1,\ \ x_\pm>0. \end{aligned} \tag{4.38}$$

By differentiating the classical mirror map, we find

$$\begin{aligned}\frac{\partial t(\mathcal{E},\mathbf{1},\hbar=0)}{\partial \mathcal{E}} &= \frac{1}{\pi\mathrm{i}}\int_{x_-}^{x_+}\mathrm{d}x\,\frac{1}{4\cosh\frac{x}{2}\sqrt{\left(\frac{\mathcal{E}-2\cosh x}{4\cosh\frac{x}{2}}\right)^2-1}}\\ &= \frac{1}{\pi\mathrm{i}}\int_{w_-}^{w_+}\mathrm{d}w\,\frac{1}{\sqrt{(w^2-1)\big((\mathcal{E}-2w)^2-8(w+1)\big)}}\\ &= \frac{1}{\pi}\int_{w_-}^{w_+}\mathrm{d}w\,\frac{1}{2\sqrt{(w^2-1)(w_+-w)(w-w_-)}}\\ &= \frac{2}{\pi\sqrt{\mathcal{E}^2-12+8\sqrt{3+\mathcal{E}}}}\mathbb{K}\left(\frac{16\sqrt{3+\mathcal{E}}}{\mathcal{E}^2-12+8\sqrt{3+\mathcal{E}}}\right), \end{aligned}\tag{4.39}$$

where we define the variable w as following,

$$w=\cosh x,\quad w_\pm=\cosh x_\pm=\frac{\mathcal{E}+2\pm 2\sqrt{\mathcal{E}+3}}{2},$$

and use the formula,

$$\int_a^b\frac{\mathrm{d}t}{\sqrt{(t-a)(b-t)(t-c)(t-d)}}=\frac{2}{\sqrt{(a-c)(b-d)}}\mathbb{K}\left(\frac{(b-a)(c-d)}{(a-c)(b-d)}\right),\qquad d<c<b<a.\tag{4.40}$$

By combining the above result with (4.37), we find the following formula,

$$\begin{aligned}&\frac{\partial t(\mathcal{E},\mathbf{1},\hbar=2\pi a/b)}{\partial\mathcal{E}}=\\ &\quad\frac{2\mathcal{F}'(\mathcal{E},\mathbf{1})}{\pi b\sqrt{\mathcal{F}^2(\mathcal{E},\mathbf{1})-12+8\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}}\mathbb{K}\left(\frac{16\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}{\mathcal{F}^2(\mathcal{E},\mathbf{1})-12+8\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}\right),\\ &\mathcal{F}(\mathcal{E},\mathbf{1})=(-1)^{ab}\mathcal{F}_{a/b}(\mathcal{E},\mathbf{1}).\end{aligned}\tag{4.41}$$

The branch cuts of the quantum mirror map are

$$\begin{aligned}&\mathcal{F}^2(\mathcal{E},\mathbf{1})-12+8\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}\le 0,\quad \left(\frac{16\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}{\mathcal{F}^2(\mathcal{E},\mathbf{1})-12+8\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}\right)^{-1}\ge 1\\ &\Rightarrow -3\le\mathcal{F}(\mathcal{E},\mathbf{1})\le 6.\end{aligned}\tag{4.42}$$

In the case of $a=1,b=4$, the above condition is

$$\begin{aligned}&-3\le\mathcal{E}^4-12\mathcal{E}^2-8\sqrt{2}\mathcal{E}+6\le 6\\ &\Rightarrow -2.8284\le\mathcal{E}\le-2.3751,\quad -1.9318\le\mathcal{E}\le-1.0352,\\ &\qquad 0\le\mathcal{E}\le 0.51763,\quad 3.7893\le\mathcal{E}\le 3.8637.\end{aligned}\tag{4.43}$$

When we consider the case of $a=1,b=3$, the branch cuts are given by

$$-6\le\mathcal{E}^3-9\mathcal{E}-6\le 3,$$

$$\Rightarrow -3.0000 \le \mathcal{E} \le -2.2266, \quad -1.1847 \le \mathcal{E} \le 0.0000, \quad 3.0000 \le \mathcal{E} \le 3.4114. \tag{4.44}$$

In both cases, the results give the band structure of the Hofstadter model.

Finally, we calculate the density of state. In the region $-3 \le \mathcal{F}(\mathcal{E}, 1) \le 6$, by using the formula,

$$\mathbb{K}\left(\frac{1}{z}\right) = \sqrt{z}\{\mathbb{K}(z) + \mathrm{i}\mathbb{K}(1-z)\}, \tag{4.45}$$

the imaginary part of the quantum mirror map is

$$\begin{aligned}
&\mathrm{Im}\left[\frac{\partial t(\mathcal{E}, \mathbf{1}, \hbar = 2\pi a/b)}{\partial \mathcal{E}}\right] \\
&= \begin{cases}
\frac{|\mathcal{F}'(\mathcal{E},\mathbf{1})|}{2\pi b(3+\mathcal{F}(\mathcal{E},\mathbf{1}))^{1/4}} \mathbb{K}\left(\frac{12-\mathcal{F}^2(\mathcal{E},\mathbf{1})+8\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}{16\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}\right), & -3 \le \mathcal{F}(\mathcal{E}, \mathbf{1}) \le -2, \\
\frac{2|\mathcal{F}'(\mathcal{E},\mathbf{1})|}{\pi b\sqrt{12-\mathcal{F}^2(\mathcal{E},\mathbf{1})+8\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}} \mathbb{K}\left(\frac{16\sqrt{3+\mathcal{F}(\mathcal{E},\mathbf{1})}}{12-\mathcal{F}^2(\mathcal{E},\mathbf{1})+8\sqrt{3+\mathcal{F}(\mathcal{E},1)}}\right), & -2 \le \mathcal{F}(\mathcal{E}, \mathbf{1}) \le 6.
\end{cases}
\end{aligned} \tag{4.46}$$

By multiplying $\frac{1}{\pi}$, this result completely agrees with the density of state in the Hofstadter model. □

In the general values of $\boldsymbol{m}$, it is useful to rewrite the mirror curve as the Weierstrass form by a coordinate transformation [14],

$$Y^2 = 4X^3 - g_2 X - g_3. \tag{4.47}$$

where g_2 and g_3 are given by

$$\begin{aligned}
g_2 &= \frac{1}{12z^4}[1 - 8(m_1+m_2+m_3)z^2 - 24(1+m_1m_2m_3)z^3 \\
&\quad + 16(m_1^2+m_2^2+m_3^2 - m_1m_2 - m_2m_3 - m_3m_1)z^4], \\
g_3 &= \frac{1}{216z^6}[1 - 12(m_1+m_2+m_3)z^2 - 36(1+m_1m_2m_3)z^3 \\
&\quad + 24(2m_1^2+2m_2^2+2m_3^2+m_1m_2+m_2m_3+m_3m_1)z^4 \\
&\quad + 144(m_1+m_2+m_3)(1+m_1m_2m_3)z^5 \\
&\quad + 8(-8m_1^3 - 8m_2^3 - 8m_3^3 + 12m_1^2m_2 + 12m_2^2m_3 + 12m_3^2m_1 \\
&\quad + 12m_1m_2^2 + 12m_2m_3^2 + 12m_3m_1^2 + 27 + 6m_1m_2m_3 + 27m_1^2m_2^2m_3^2)z^6].
\end{aligned} \tag{4.48}$$

Then, the derivative of the classical mirror map is given by

$$\frac{\partial t}{\partial z} = -\frac{1}{2\pi z^2}\frac{2}{\sqrt{e_1-e_3}}\mathbb{K}\left(\frac{e_2-e_3}{e_1-e_3}\right), \tag{4.49}$$

where e_1, e_2, and e_3 are three roots of the curve (4.47). From this expression, we can determine the branch cuts and the imaginary part of the quantum mirror map in various protocols of (λ_1, λ_2) as in figure 17. The result in the right panel of the figure is consistent with the result in [19] since this corresponds to the small value of m_3.

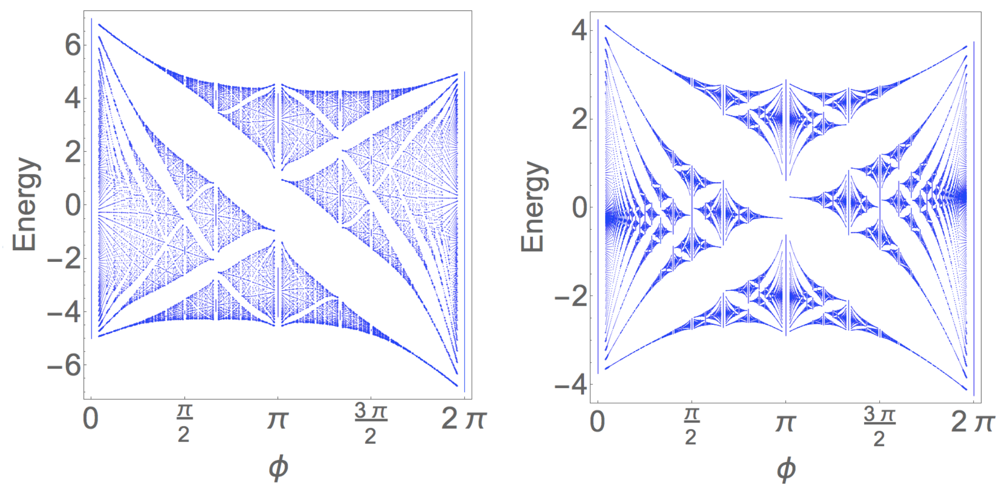

Figure 17: The Hofstadter butterfly. In the left panel we plot in the protocol $(\lambda_1, \lambda_2) = (2, 1/2)$. In the right panel we plot in the protocol $(\lambda_1, \lambda_2) = (1, 1/8)$.

5 Summary and Discussion

In this thesis, we have expressed how to define the non-perturbative topological string theory. In the A-model side, based on the duality between the ABJM theory and the topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$, we proposed the non-perturbative free energy on the general non-compact toric Calabi–Yau manifold. The non-perturbative part is given by the Nekrasov–Shatashvili free energy.

In the B-model side, based on the Fermi gas formalism in the ABJM theory, we have considered the quantization of the mirror curve in the mirror Calabi–Yau manifold. Then, we define the trace class operator and the grand partition function. As is the case in the perturbative topological string theory, the non-perturbative definitions in the A- and the B-model are related through the quantum mirror map.

In the point of view of the quantized mirror curve, this is quantum mechanics, and is the same as the Hofstadter model in the condensed matter physics. Here we have called this relation as the TS-CMP correspondence, and the correspondence has worked in two cases (table 2). In this thesis, we have discussed the second correspondence. To show the

Topological String side		Condensed Matter Theory side
Local $\mathbb{P}^1 \times \mathbb{P}^1$	$\leftrightarrow$	Square lattice
Local $\mathcal{B}_3$	$\leftrightarrow$	Triangular lattice

Table 2: The TS-CMP correspondence.

TS-CMP correspondence, we need to know the explicit form of the quantum mirror map. Compared with the first one, the direct derivation of the quantum mirror map from the quantum mirror curve is difficult. Alternatively, we use the knowledge of the condensed matter physics. Then, we find that the branch cut and the imaginary part of the quantum mirror map give the Hofstadter butterfly and the density of state. This shows that we can study the non-perturbative topological string theory from the knowledge of well-known condensed matter physics.

There are many interesting issues that we need to study. As a naive extension, the higher genus case would be interesting. In this thesis, we discuss the genus-one mirror curve. However, the S-duality structure exists in the higher-genus mirror curve. Therefore we would obtain the branch cuts drawing a fractal structure. Since the non-perturbative effects in the higher-genus case is less known, studying the quantum mirror curve itself is worthily.

Second, we expect the quantization of the mirror curve in the presence of the branes to have the branch cuts drawing a fractal structure. In [64], the author showed that the geometric transition can be applied in the A-model non-perturbative topological string theory. Based on this result, we would be able to see the fractal structure in the presence of the branes from the closed topological string theory by setting the complex moduli appropriately.

The third one is about the non-Hermitian Hamiltonian. In both correspondence in table 2, the Hofstadter model is Hermitian. Correspondingly, the mirror curve is invariant under the transformation $(x, y) \to (-x, -y)$. However, as is the case of local $\mathbb{P}^2$, there are some mirror curves which are not invariant under the above transformation. We plot the branch cuts of the local $\mathbb{P}^2$ as a example, and the result is in figure. 18. We do not know so far

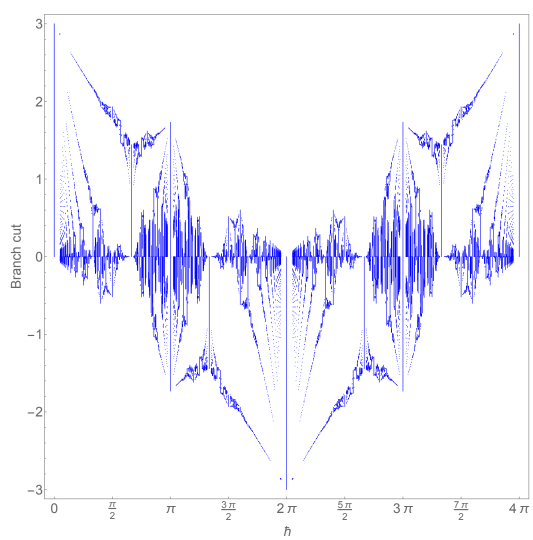

Figure 18: The branch cut of local $\mathbb{P}^2$ in each values of $\hbar$.

which there is a corresponding condensed matter physics or not. If this problem is solved, we can easily generalize the wide class of Calabi–Yau manifolds.

Finally, we expect the non-Hermitian Hofstadter model studied in [65] to solve the crucial problem about the quantization of the conifold. In the genus $g \geq 1$ mirror curve, we can define the trace class operator, so that we can obtain the grand partition function. However, the genus-zero mirror curve is not the case; we cannot define the trace class operator, and cannot compare the B-model with the A-model. It might correspond to the fact that the quantum mirror map is trivial,

$$t = \log z, \tag{5.1}$$

where z and t are the complex modulus and the Kähler parameter in the resolved conifold, respectively. Thus, the branch cuts have no fractal structure. However, the authors in [65] have obtained the non-trivial band structure, even if the Hamiltonian corresponds to the genus-zero mirror curve. If this conflict is solved, we will be able to define the quantization of the genus-zero mirror curve.

Acknowledgement

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A Notation, Definition and Formula

Young diagram

We define the Young diagram as in figure. 19. The variables μ_i, μ_j^t, $l(\mu)$ denote the number

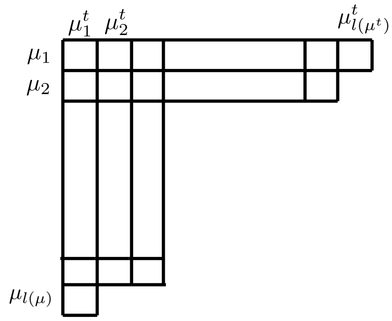

Figure 19: The Young diagram and its parameters.

of boxes in the i-th row, the number of boxes in the j-th column, and the number of the columns. We also define the following quantities,

$$|\mu| = \sum_{i}^{l(\mu)} \mu_i, \ \ ||\mu^2|| = \sum_{i}^{l(\mu)} \mu_i^2. \tag{A.1}$$

Schur function and skew Schur function

The Schur function $s_\mu(x)$ is defined by a ratio of the determinant,

$$s_\mu(x_1, \cdots, x_N) = \frac{\det x_j^{\mu_i+N-i}}{\det x_j^{N-i}}, \quad i = 1, \cdots, N. \tag{A.2}$$

For example, when we consider the case of $N = 3$ and $\mu_1 = 3,\ \mu_2 = 2,\ \mu_3 = 1$, the Schur function is

$$\begin{aligned} s_\mu(x) &= \det \begin{bmatrix} x_1^5 & x_2^5 & x_3^5 \\ x_1^3 & x_2^3 & x_3^3 \\ x_1^1 & x_2^1 & x_3^1 \end{bmatrix} \Bigg/ \det \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1^1 & x_2^1 & x_3^1 \\ x_1^0 & x_2^0 & x_3^0 \end{bmatrix} \\ &= x_1 x_2 x_3 (x_1 + x_2)(x_2 + x_3)(x_3 + x_1). \end{aligned} \tag{A.3}$$

The skew Schur function $s_{\mu/\eta}(x)$ is defined by

$$s_{\mu/\eta}(x) = \sum_\sigma N_{\nu\eta}^\mu s_\nu(x), \tag{A.4}$$

where the summation is taken over all shapes of the Young diagrams. The coefficients $N_{\nu\eta}^\mu$ are called as the Littlewood–Richardson coefficients, and have following properties:

- The product of two Schur functions can be expressed by the linear combination of single Schur functions,

$$s_\mu(x)s_\nu(x) = \sum_\sigma N^\sigma_{\mu\nu} s_\sigma(x). \tag{A.5}$$

- Unless three Young diagrams μ, ν and η satisfy the relation, $|\mu| = |\nu| + |\eta|$, the Littlewood–Richardson coefficients vanish, $N^\mu_{\nu\eta} = 0$.

- The Littlewood–Richardson coefficients are invariant under exchanging of μ and ν, $N^\sigma_{\mu\nu} = N^\sigma_{\nu\mu}$, and the transposing all Young diagrams, $N^{\sigma^t}_{\mu^t\nu^t} = N^\sigma_{\nu\mu}$.

To calculate the partition function of the (refined) topological string theory, we can use the formulae about the skew Schur function,

$$\begin{aligned}
&s_{\lambda/\mu}(\alpha x) = \alpha^{|\lambda|-|\mu|} s_{\lambda/\mu}(x), \\
&\sum_\eta s_{\eta/\lambda}(x) s_{\eta/\mu}(y) = \prod_{i,j=1}^\infty (1 - x_i y_j)^{-1} \sum_\eta s_{\mu/\eta}(x) s_{\lambda/\eta}(y), \\
&\sum_\eta s_{\eta^t/\lambda}(x) s_{\eta/\mu}(y) = \prod_{i,j=1}^\infty (1 + x_i y_j) \sum_\eta s_{\mu^t/\eta^t}(x) s_{\lambda^t/\eta}(y).
\end{aligned} \tag{A.6}$$

Quantum dilogarithm

The quantum dilogarithm $\Phi_b(x)$ is defined by [66]

$$\Phi_b(x) = \frac{(\mathrm{e}^{2\pi b(x+c_b)}; q)_\infty}{(\mathrm{e}^{2\pi b^{-1}(x-c_b)}; \tilde{q})_\infty}, \tag{A.7}$$

where,

$$\begin{aligned}
&(x, q)_\infty = \prod_{n=0}^\infty (1 - x q^n), \\
&q = \mathrm{e}^{2\pi \mathrm{i} b^2}, \;\; \tilde{q} = \mathrm{e}^{-2\pi \mathrm{i} b^{-2}}, c_b = \mathrm{i}\frac{b + b^{-1}}{2}, \;\; \mathrm{Im}(b^2) > 0.
\end{aligned} \tag{A.8}$$

The quantum dilogarithm satisfies the following relation,

$$\begin{aligned}
&\Phi_b(x)\Phi_b(-x) = \mathrm{e}^{\pi \mathrm{i} x^2} \Phi_b(0)^2, \quad \Phi_b(0) = \mathrm{e}^{\frac{\pi \mathrm{i}(b^2 + b^{-2})}{24}}, \\
&\Phi_b(x + c_b + \mathrm{i}b) = \frac{\Phi_b(x + c_b)}{1 - q\mathrm{e}^{2\pi b x}}, \\
&\Phi_b(x + c_b + \mathrm{i}b^{-1}) = \frac{\Phi_b(x + c_b)}{1 - \tilde{q}^{-1}\mathrm{e}^{2\pi b^{-1} x}}.
\end{aligned} \tag{A.9}$$

In addition, if b is real, we can use the following formula,

$$\overline{\Phi_b(z)} = \frac{1}{\Phi_b(z)}. \tag{A.10}$$

By combining them, one can show that [55]

$$\Phi_b(\mathsf{p})\mathrm{e}^{2\pi b\mathsf{q}}\overline{\Phi_b(\mathsf{p})} = \mathrm{e}^{2\pi b\mathsf{q}} + \mathrm{e}^{2\pi b(\mathsf{p}+\mathsf{q})}, \quad [\mathsf{q}, \mathsf{p}] = \frac{1}{2\pi\mathrm{i}}. \tag{A.11}$$

This formula is useful to calculate the inverse of the quantum mirror curve.

B Calabi–Yau manifold

B.1 General definition

In this section, we explain what the Calabi–Yau manifolds are. In general, the Calabi–Yau manifold is defined by the complex n-dimensional Kähler manifolds with a Riemannian metric g,

$$g_{ij} = g_{\bar{i}\bar{j}} = 0, \quad g_{i\bar{j}} = \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_{\bar{j}}} \log K(z, \bar{z}), \tag{B.1}$$

where $K(z, \bar{z})$ is a real function. In addition, the Ricci tensor has to vanish,

$$R_{i\bar{j}} = 0. \tag{B.2}$$

As a important property, the Calabi–Yau manifold has a nowhere vanishing holomorphic n-form,

$$\Omega = f(z)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n, \tag{B.3}$$

which is uniquely determined up to an overall complex constant.

B.2 Non-compact case

Here we focus on the non-compact toric Calabi–Yau manifold. The definition of this manifold is divided into two types: one is the manifold where the A-model topological string theory is defined, and the other is the mirror Calabi–Yau manifold where the B-model topological string theory is defined. Let us explain them.

B.2.1 A-model side

The non-compact toric Calabi–Yau manifold CY_A in the A-model topological string theory is expressed by a $\mathbb{T}^2 \times \mathbb{R}$ fibration on a three-dimensional space as in figure 20, called as the toric geometry. One of the cycle in the torus, (0,1)-cycle or (1,0)-cycle, shrinks along some

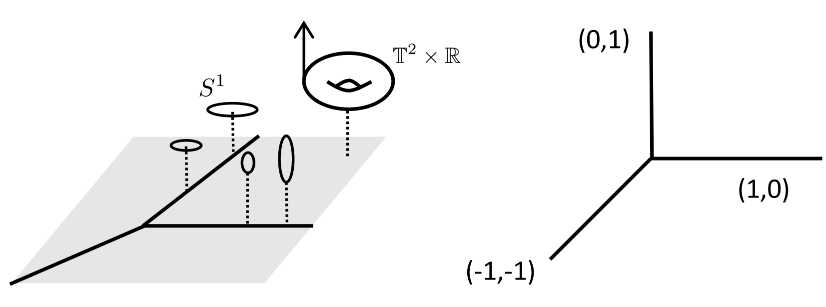

Figure 20: The Calabi–Yau manifold and the web diagram.

lines, and the shrinking cycles are the line specific. The web diagram which we use to express the topological vertex represents the degeneration lines of the torus. Let us explain how to define the non-compact toric Calabi–Yau manifold and how to obtain the web diagram.

Any CY_A can be expressed by the combination of the simplest CY_A, $\mathbb{C}^3$. Then, we first consider the $\mathbb{C}^3$ case whose coordinates denote z_1, z_2 and z_3. To express the $\mathbb{C}^3$ geometry in term of the $\mathbb{T}^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$, we define the following coordinates,

$$\begin{aligned} r_\alpha(z) &= |z_1|^2 - |z_3|^2, \\ r_\beta(z) &= |z_2|^2 - |z_3|^2, \\ r_\gamma(z) &= \mathrm{Im}(z_1 z_2 z_3). \end{aligned} \tag{B.4}$$

These variables define $\mathbb{R}^3$. We also define the Poisson bracket,

$$\begin{aligned} \partial_v z_i &= \{r_v, z_i\}_{\mathrm{PB}}, \quad v = \alpha,\ \beta,\ \gamma, \\ \{r_a, z_i\}_{\mathrm{PB}} &= \mathrm{i} \sum_{j=1}^{3} \Big(\frac{\partial r_a}{\partial \bar{z}_j} \frac{\partial z_i}{\partial z_j} - \frac{\partial r_a}{\partial z_j} \frac{\partial z_i}{\partial \bar{z}_j} \Big). \end{aligned} \tag{B.5}$$

They make the circle actions,

$$\mathrm{e}^{\alpha r_\alpha + \beta r_\beta} : (z_1, z_2, z_3) \to (\mathrm{e}^{\mathrm{i}\alpha} z_1, \mathrm{e}^{\mathrm{i}\beta} z_2, \mathrm{e}^{-\mathrm{i}\alpha - \mathrm{i}\beta} z_3), \tag{B.6}$$

and (α, β, γ) define $\mathbb{T}^2 \times \mathbb{R}$. We call S^1's made by α and β as the (0,1)-cycle and the (1,0)-cycle, respectively.

When $z_1 = z_3 = 0$ which corresponds to $r_\alpha = 0$, the (0,1)-cycle shrinks. Similarly, (1,0)-cycle shrinks when $z_2 = z_3 = 0$ corresponding to $r_\beta = 0$. Finally, a one-cycle defined by $\alpha + \beta$ shrinks when $z_1 = z_2 = 0$ corresponding to $r_\alpha = r_\beta$. The condition $r_\alpha = 0,\ r_\beta = 0$, and $r_\alpha = r_\beta$ correspond to the vertical, horizontal, and diagonal line in the right panel of the figure 20.

Next we define general non-compact toric Calabi–Yau manifold. Let us consider a $\mathbb{C}^{N+3}$ space defined by $N + 3$ coordinates $z_1, \cdots, z_{N+3}$, and define the space μ_A as following,

$$\mu_A = \Big\{ z_1, \cdots, z_{N+3} \Big| \sum_{j=1}^{N+1} Q_A^j |z_j|^2 = t_A,\ t_A > 0,\ A = 1 \cdots, N, \quad \sum_{j}^{N+3} Q_A^j = 0 \Big\}, \tag{B.7}$$

where Q_A are the charge vectors. The conditions $\sum_j^{N+3} Q_A^j = 0$ correspond to the Calabi–Yau condition $R_{i\bar{j}} = 0$. The equations $\sum_{j=1}^{N+1} Q_A^j |z_j|^2 = t_A$ are invariant under a $\mathrm{U}(1)^N$ transformation,

$$z_j \mapsto \mathrm{e}^{\mathrm{i} Q_A^j \alpha_A} z_j. \tag{B.8}$$

By dividing μ_A by the symmetry, we define the non-compact toric Calabi–Yau manifold CY_A,

$$CY_A = \bigcap_A \mu_A / \mathrm{U}(1)^N. \tag{B.9}$$

From this definition, we can express CY_A with the web diagram.

e.g.) Resolved Conifold

The charge vector of the resolved conifold is given by

$$Q_A = (1, -1, -1, 1), \tag{B.10}$$

so that (B.7) is

$$|z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2 = t. \tag{B.11}$$

Since the Kähler parameter t is positive, either $|z_1|^2$ or $|z_4|^2$ has to be non-zero value. Therefore, the conifold is defined by two patches: one is $U_1 = (z_1 = 0, z_2, z_3, z_4)$, and the other is $U_4 = (z_1, z_2, z_3, z_4 = 0)$.

In the U_4-patch, by taking the coordinates r_α, r_β as follows,

$$\begin{aligned} r_\alpha(z) &= |z_2|^2 - |z_1|^2 \\ r_\beta(z) &= |z_3|^2 - |z_1|^2 \end{aligned} \tag{B.12}$$

we obtain the web diagram which is the same as the right panel of the figure 20.

In the U_1-patch, the coordinates are

$$\begin{aligned} r_\alpha(z) &= |z_4|^2 - |z_3|^2 - t, \\ r_\beta(z) &= |z_4|^2 - |z_2|^2 - t. \end{aligned} \tag{B.13}$$

The web diagram consists of three degeneration loci which are defined by three vectors $(-1, 0)$, $(0, -1)$, and $(1, 1)$. By combining them, we find the web diagram description of the conifold as in figure 21. Since there is the S^1-fibration over the connected line, the geometry is topologically S^2. □

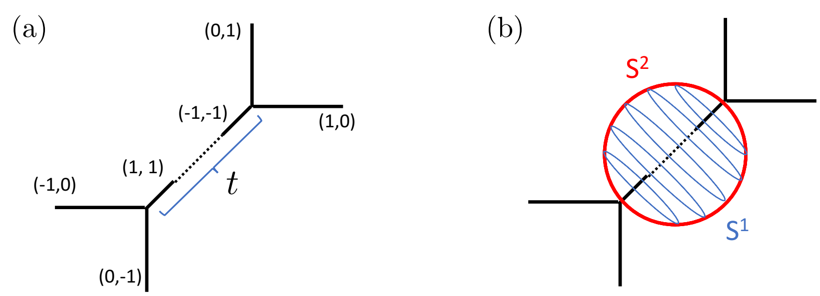

Figure 21: (a)The web diagram expression of the conifold. (b)The sphere structure in the conifold.

B.2.2 B-model side

Next, we explain the mirror Calabi–Yau manifold. As we mentioned in the section 2.2.2, the mirror Calabi–Yau manifold is defined by

$$vw - H(\mathrm{e}^x, \mathrm{e}^y; \boldsymbol{z}) = 0, \quad v, w \in \mathbb{C}, \tag{B.14}$$

and the information of the (non-perturbative) topological string theory is encoded in the equation,

$$H(\mathrm{e}^x, \mathrm{e}^y; \boldsymbol{z}) = 0. \tag{B.15}$$

For example, the mirror Calabi–Yau manifold of the conifold is defined by

$$\mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^{x+y} - u = 0, \tag{B.16}$$

where u is the complex modulus.

In general, $H(\mathrm{e}^x, \mathrm{e}^y; \boldsymbol{z})$ is expressed as the summation of $a_{m,n}\mathrm{e}^{mx+ny}$, where $m, n \in \mathbb{Z}$ and $a_{m,n}$ are the coefficients including the complex moduli. Again we express the mirror Calabi–Yau manifold pictorially: we regard e^{mx+ny} as the point (m, n) in a two-dimensional space. Then, the conifold is given by the three points as in the left panel of figure 22, called as the toric diagram.

The toric diagram is closely related to the web diagram; by connecting the dots to divide into triangles, and drawing the lines which are perpendicular each lines of the toric diagram, we can obtain the web diagram of the conifold in the A-model side. Conversely, we can

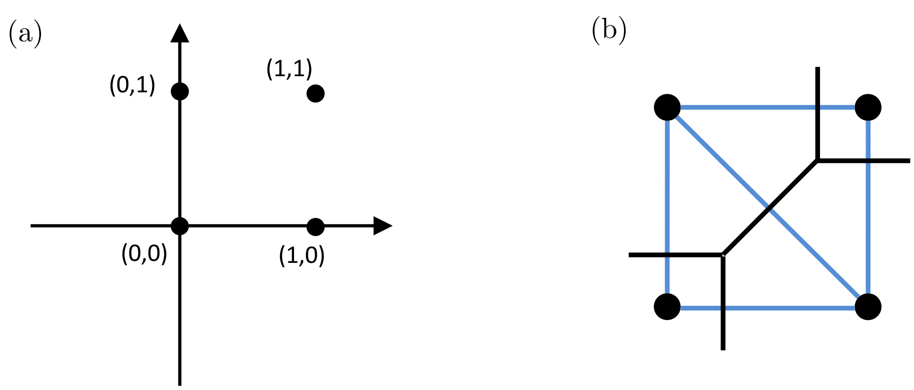

Figure 22: (a)The toric diagram. (b)The relation between the toric diagram and the web diagram.

obtain the toric diagram and the mirror curve from the web diagram.

The left panel of the figure 22 can be regarded as the lattice sites. In this sense, the topological string theory on the mirror curve could have the information of the Hofstadter model on a certain lattice. This is the pictorial explanation of the TS-CMT correspondence. Indeed, the toric diagrams of local $\mathbb{P}^1 \times \mathbb{P}^1$ and local $\mathcal{B}_3$ are the same as the square lattice and the triangular lattice, respectively.

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