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## COMPACT TRANSFORMATION GROUPS AND FIXED POINT SETS OF RESTRICTED ACTION TO MAXIMAL TORUS

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### 0. Introduction

Let  $G$  be a compact connected Lie group and let  $T$  be a maximal torus of  $G$ . Define

$$\begin{aligned} m(G) &= \max \{ \dim H \mid H \text{ is a proper closed subgroup of } G \} , \\ m_0(G) &= \max \{ \dim H \mid H \text{ is a proper closed subgroup of } G \\ &\quad \text{with } \text{rank } H = \text{rank } G \} . \end{aligned}$$

Let  $M$  be a connected manifold with a non-trivial smooth  $G$ -action and let  $H$  be a closed subgroup of  $G$ . Denote by  $F(H, M)$  the fixed point set of the restricted action of the given  $G$ -action to the subgroup  $H$ . Then each connected component  $F_a$  ( $a \in A$ ) of  $F(H, M)$  is a regular submanifold of  $M$ . Define

$$\dim F(H, M) = \max \{ \dim F_a \mid a \in A \}$$

if  $F(H, M)$  is non-empty and we put

$$\dim F(H, M) = -1$$

if  $F(H, M)$  is empty. Then we have the following results.

#### Theorem 1.

- (a) *In general,  $\dim M - \dim F(T, M) \geq \dim G - m(G)$ .*
- (b) *If  $G$  is semi-simple and*

$$\dim F(G, M) < \dim F(T, M) ,$$

*then*

$$\dim M - \dim F(T, M) \geq \dim G - m_0(G) .$$

#### Theorem 2. *If*

$$\dim M - \dim F(T, M) = \dim G - m(G) ,$$

then  $G$  is semi-simple,  $m(G)=m_0(G)$  and

$$\dim M - \dim F_a = \dim G - m(G)$$

for each connected component  $F_a$  of  $F(T, M)$ . Moreover

$$\dim H = m(G) \quad \text{and} \quad \text{rank } H = \text{rank } G$$

for a principal isotropy group  $H$ .

### 1. Preliminary lemmas

In this section we prepare several lemmas.

**Lemma 1.1.** *Let  $H$  be a closed subgroup of  $G$  and assume  $T \subset H$ . Then*

$$F(T, G/H) = N(T)H/H.$$

*In particular,  $F(T, G/H)$  is a non-empty finite set.*

Proof. It is clear that

$$F(T, G/H) = \{gH \mid g^{-1}Tg \subset H\}.$$

If  $g^{-1}Tg \subset H$ , then there is  $h \in H$  such that

$$g^{-1}Tg = hTh^{-1},$$

since  $T$  is a maximal torus of  $H^0$ , the identity component of  $H$ . Thus

$$gh \in N(T): \text{the normalizer of } T \text{ in } G.$$

Hence we obtain

$$F(T, G/H) = N(T)H/H.$$

Next, there is a natural surjection  $N(T)/T \rightarrow N(T)H/H$ , where  $N(T)/T$  is the Weyl group of  $G$  which is a finite group. Therefore  $F(T, G/H)$  is a non-empty finite set. q.e.d.

In the following, we assume that  $M$  is a connected manifold with a non-trivial smooth  $G$ -action. It is clear

$$(1.2) \quad \dim M \geq \dim G - m(G).$$

**Lemma 1.3.**  $\dim M - \dim F(G, M) > \dim G - m(G)$ .

Proof. If  $F(G, M)$  is empty, then the inequality is clear from (1.2). If  $F(G, M)$  is non-empty, let  $n = \dim F(G, M)$  and let  $F_a$  be an  $n$ -dimensional connected component of  $F(G, M)$ . For  $x \in F_a$ ,

$$T_x M = T_x(F_a) \oplus N_x$$

as  $G$ -vector spaces, where  $N_x$  is a normal space of  $F_a$  in  $M$ . Then there is a non-zero vector  $v \in N_x$  with  $G_v \neq G$ . Thus

$$\dim G - m(G) \leq \dim G/G_v < \dim N_x = \dim M - n.$$

q.e.d.

**Lemma 1.4.** *If*

$$\dim M - \dim F(T, M) \leq \dim G - m_0(G)$$

and

$$\dim F(G, M) < \dim F(T, M),$$

then

$$M = G \cdot F(H, M).$$

Here  $H$  is a compact connected subgroup of  $G$  such that

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

Proof. Let  $k = \dim F(T, M)$  and denote by  $F^k$  the union of  $k$ -dimensional connected components of  $F(T, M)$ . Then

$$F^k - F(G, M)$$

is non-empty by the assumption. For  $x \in F^k - F(G, M)$ ,

$$T_x M = T_x(G \cdot x) \oplus N_x$$

as  $G_x$ -vector spaces, where  $N_x$  is a normal space of the orbit  $G \cdot x$  in  $M$ . Since  $T \subset G_x$ ,  $F(T, G \cdot x)$  is a non-empty finite set by Lemma 1.1. Thus

$$\begin{aligned} k &= \dim F(T, T_x M) = \dim F(T, N_x) \\ &\leq \dim N_x = \dim M - \dim G/G_x \leq \dim M - \dim G + m_0(G). \end{aligned}$$

On the other hand,

$$k \geq \dim M - \dim G + m_0(G)$$

by the assumption. Therefore

$$(1) \quad \dim G_x = m_0(G),$$

$$(2) \quad F(T, N_x) = N_x.$$

Since the action of  $G_x$  on  $N_x$  is a slice representation at  $x$ , a principal isotropy group  $H'$  contains  $T$  by (2), and hence

$$\dim H' = m_0(G)$$

by (1). Let  $H$  be the identity component of the principal isotropy group  $H'$ . Then we have

$$M = G \cdot F(H, M) = \{g \cdot x \mid g \in G, x \in F(H, M)\}.$$

q.e.d.

**Lemma 1.5.** *If*

$$\dim M - \dim F(T, M) \leq \dim G - m(G),$$

*then  $m(G) = m_0(G)$  and*

$$M = G \cdot F(H, M).$$

*Here  $H$  is a compact connected subgroup of  $G$  such that*

$$\dim H = m(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

*Proof.* Taking account of Lemma 1.3 and using similar arguments as in the proof of Lemma 1.4, we can prove this lemma.

**Lemma 1.6.** *Let  $G$  be a compact connected Lie group and let  $H$  be a closed subgroup of  $G$  such that*

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H^0 = \text{rank } G.$$

*Then  $N(H)^0 = H^0$ , where  $H^0$  is the identity component of  $H$  and  $N(H)$  is the normalizer of  $H$  in  $G$ .*

*Proof.* Assume  $N(H)^0 \neq H^0$ . Then the assumption on  $H$  implies  $N(H) = G$ . Thus  $H$  is a normal subgroup of  $G$ , and hence

$$\text{rank } G = \text{rank } H^0 + \text{rank } G/H.$$

Then the assumption on  $H$  implies  $\text{rank } G/H = 0$  and hence  $G = H$ . But this is a contradiction to

$$\dim H = m_0(G) < \dim G.$$

q.e.d.

**Lemma 1.7.** *Let  $G$  be a compact connected semi-simple Lie group and let  $H$  be a closed connected subgroup of  $G$  such that*

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

*Let  $V$  be a real  $G$ -vector space such that*

$$V = G \cdot F(H, V) \quad \text{and} \quad F(G, V) = \{0\}.$$

Then  $S(V) = G/H$  as  $G$ -manifolds and  $N(H)/H = Z_2$ . Here  $S(V)$  is a  $G$ -invariant unit sphere of  $V$ .

Proof. By the assumption on  $H$  and  $V$ , the identity component of an isotropy subgroup at each point of  $S(V)$  is conjugate to  $H$  in  $G$ . Hence there is an equivariant diffeomorphism

$$S(V) = G/H \times_{N(H)/H} F(H, S(V))$$

as  $G$ -manifolds. Here  $F(H, S(V))$  is a unit sphere of  $F(H, V)$ . Since  $N(H)/H$  is a finite group by Lemma 1.6, the natural projection

$$G/H \times F(H, S(V)) \rightarrow S(V)$$

is a finite covering as  $G$ -manifolds. On the other hand,  $S(V)$  is simply connected, because  $G$  is semi-simple. Therefore

$$S(V) = G/H$$

as  $G$ -manifolds and  $F(H, S(V))$  is a zero-sphere  $S^0$ . Finally,

$$N(H)/H = F(H, G/H) = F(H, S(V)) = S^0.$$

Thus  $N(H)/H = Z_2$ , the cyclic group of order 2. q.e.d.

## 2. Proof of theorems

Let  $G$  be a compact connected Lie group and let  $T$  be a maximal torus of  $G$ . Let  $M$  be a connected manifold with a non-trivial smooth  $G$ -action. It is easy to see that

$$F(T, M) = M \quad \text{implies} \quad F(G, M) = M.$$

Thus

$$\dim M - \dim F(T, M) \geq 2,$$

because

$$\dim M \equiv \dim F_a \pmod{2}$$

for each connected component  $F_a$  of  $F(T, M)$ .

If  $G$  is not semi-simple, then

$$\dim G - m(G) = 1$$

and hence there is nothing to prove. In particular, if

$$\dim M - \dim F(T, M) = \dim G - m(G),$$

then  $G$  is semi-simple, and  $m(G) = m_0(G)$  by Lemma 1.5.

Now we assume that  $G$  is semi-simple and there is a closed connected subgroup  $H$  of  $G$  such that

$$(*) \quad M = G \cdot F(H, M), \dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

Moreover, (i) first suppose that  $F(G, M)$  is empty. Then by the assumption  $(*)$ , the identity component of an isotropy subgroup at each point of  $M$  is conjugate to  $H$  in  $G$ . Hence there is an equivariant diffeomorphism

$$M = G/H \times_{N(H)/H} F(H, M)$$

as  $G$ -manifolds. Since  $N(H)/H$  is a finite group by Lemma 1.6, the natural projection

$$p: G/H \times F(H, M) \rightarrow M$$

is a finite covering as  $G$ -manifolds. Hence we obtain

$$F(T, M) = p(F(T, G/H) \times F(H, M)).$$

Here  $F(T, G/H)$  is a non-empty finite set by Lemma 1.1. Therefore

$$\begin{aligned} \dim M - \dim F_a &= \dim M - \dim F(H, M) \\ &= \dim G/H = \dim G - m_0(G), \end{aligned}$$

for each connected component  $F_a$  of  $F(T, M)$ .

(ii) Next suppose that  $F(G, M)$  is non-empty. Then each fibre  $N_x$  of the normal  $G$ -vector bundle of  $F(G, M)$  in  $M$  satisfies the hypothesis of Lemma 1.7, and hence

$$N(H)/H = Z_2 \quad \text{and} \quad S(N_x) = G/H.$$

Let  $U$  be a  $G$ -invariant closed tubular neighborhood of  $F(G, M)$  in  $M$ . Then there is an equivariant diffeomorphism

$$M = \partial(D(V) \times F(H, M - \text{int } U))/Z_2$$

as  $G$ -manifolds. Here  $V$  is a real  $G$ -vector space (unique up to  $G$ -isomorphism) with  $S(V) = G/H$ ,  $Z_2$  acts on the unit disk  $D(V)$  as antipodal involution, and  $G$  acts naturally on  $D(V)$  and trivially on  $F(H, M - \text{int } U)$ . Hence we obtain

$$\begin{aligned} F(T, M) &= \partial(F(T, D(V)) \times F(H, M - \text{int } U))/Z_2 \\ &= \partial([-1, 1] \times F(H, M - \text{int } U))/Z_2. \end{aligned}$$

Therefore

$$\begin{aligned} \dim M - \dim F_a &= \dim M - \dim F(H, M - \text{int } U) \\ &= \dim D(V) - 1 \\ &= \dim G/H \\ &= \dim G - m_0(G), \end{aligned}$$

for each connected component  $F_a$  of  $F(T, M)$ .

Now the proofs of Theorem 1 and Theorem 2 are completed by Lemma 1.4 and Lemma 1.5.

### 3. Integers $m(G)$ and $m_0(G)$

In this section we show certain properties of  $m(G)$  and  $m_0(G)$ . It is easy to see that

$$(3.1) \quad m(G_1 \times G_2) \geq \max(m(G_1) + \dim G_2, \dim G_1 + m(G_2)),$$

and

$$(3.2) \quad m(G) \geq 1, \quad \text{if } G \neq S^1.$$

**Lemma 3.3.** *Let  $G_1$  and  $G_2$  be compact connected Lie groups. Suppose that  $G_1$  is simple and  $G_1 \neq S^1$ . Let  $H$  be a closed connected subgroup of  $G_1 \times G_2$  with  $\dim H = m(G_1 \times G_2)$ . Then*

$$H = H_1 \times G_2 \quad \text{or} \quad H = G_1 \times H_2$$

where  $H_a$  is a closed subgroup of  $G_a$  ( $a = 1, 2$ ) with  $\dim H_a = m(G_a)$ .

Proof. Let  $p_a: G_1 \times G_2 \rightarrow G_a$  ( $a = 1, 2$ ) be natural projections, and let  $i_a: G_a \rightarrow G_1 \times G_2$  be natural injections defined by

$$\begin{aligned} i_1(g) &= (g, e_2), g \in G_1 \\ i_2(g) &= (e_1, g), g \in G_2 \end{aligned}$$

where  $e_a$  is the identity element of  $G_a$  ( $a = 1, 2$ ). Define

$$H_a = p_a(H) \quad \text{and} \quad H_a' = i_a^{-1}(H).$$

Then  $H_a'$  is a normal subgroup of  $H_a$  ( $a = 1, 2$ ) and  $H_1' \times H_2'$  is a normal subgroup of  $H$ , and  $H \subset H_1 \times H_2$ . Moreover the projection  $p_a$  induces an isomorphism

$$p_a': H/H_1' \times H_2' \rightarrow H_a/H_a' \quad (a = 1, 2).$$

(i) First suppose  $H_1 \neq G_1$ . Then

$$H \subset p_1^{-1}(H_1) = H_1 \times G_2 \neq G_1 \times G_2.$$

Hence we obtain

$$H = H_1 \times G_2 \quad \text{and} \quad \dim H_1 = m(G_1)$$

from the assumption  $\dim H = m(G_1 \times G_2)$ .

(ii) Next suppose  $H_1 = G_1$ . Then  $H_1'$  is a normal subgroup of the simple Lie group  $G_1$  and hence  $H_1' = G_1$  or  $H_1'$  is a finite group. Since  $m(G_1) \geq 1$  and

there is an isomorphism

$$H/i_1(H_1') = H_2,$$

we obtain

$$\begin{aligned} m(G_1 \times G_2) &= \dim H = \dim H_1' + \dim H_2 \\ &< \dim H_1' + m(G_1) + \dim G_2 \leq \dim H_1' + m(G_1 \times G_2). \end{aligned}$$

Thus  $\dim H_1' \neq 0$ , and hence

$$H_1' = H_1 = G_1.$$

Therefore

$$H = G_1 \times H_2 \quad \text{and} \quad \dim H_2 = m(G_2)$$

from the assumption  $\dim H = m(G_1 \times G_2)$ . q.e.d.

**Corollary 3.4.** *Let  $G_1$  and  $G_2$  be compact connected Lie groups. Suppose that  $G_1$  is simple. Then*

$$\dim(G_1 \times G_2) - m(G_1 \times G_2) = \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)).$$

**Proof.** If  $G_1 \neq S^1$ , Then the equation follows from Lemma 3.3. If  $G_1 = S^1$ , then  $m(G_1 \times G_2) = \dim G_2$  and hence the equation holds. q.e.d.

**Theorem 3.5.** *Let  $G_1$  and  $G_2$  be compact connected Lie groups. Then*

$$\dim(G_1 \times G_2) - m(G_1 \times G_2) = \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)).$$

**Proof.** Let  $G^*$  be a compact connected covering group of  $G$ . Then it is easy to see that

$$m(G^*) = m(G).$$

There are covering groups  $G_a^*$  of  $G_a$  ( $a=1, 2$ ) such that

$$\begin{aligned} G_1^* &= H_1 \times \cdots \times H_r \times T^m \\ G_2^* &= K_1 \times \cdots \times K_s \times T^n \end{aligned}$$

where  $H_i$ ,  $K_j$  are compact connected non-abelian simple Lie groups, and  $T^m$ ,  $T^n$  are tori. If  $m$  or  $n$  is non-zero, then

$$\begin{aligned} \dim(G_1 \times G_2) - m(G_1 \times G_2) &= 1 \\ \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)) &= 1. \end{aligned}$$

Next, if  $m=n=0$ , then

$$\begin{aligned} \dim(G_1 \times G_2) - m(G_1 \times G_2) &= \min_{i,j}(\dim H_i - m(H_i), \dim K_j - m(K_j)) \\ &= \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)) \end{aligned}$$

be Corollary 3.4. q.e.d.

REMARK 3.6. The integer  $m_0(G)$  can be defined only when  $G$  is non-abelian (i.e.  $G$  does not coincide with its maximal torus).

**Theorem 3.7.** *Let  $G_1$  and  $G_2$  be compact connected non-abelian Lie groups. Then*

$$\dim (G_1 \times G_2) - m_0(G_1 \times G_2) = \min (\dim G_1 - m_0(G_1), \dim G_2 - m_0(G_2)).$$

Proof. Let  $H$  be a closed connected subgroup of  $G_1 \times G_2$  such that

$$\dim H = m_0(G_1 \times G_2) \quad \text{and} \quad \text{rank } H = \text{rank } (G_1 \times G_2).$$

Then there are closed connected subgroups  $H_a$  of  $G_a$  ( $a=1, 2$ ) such that

$$H = H_1 \times H_2 \quad \text{and} \quad \text{rank } H_a = \text{rank } G_a \quad (a=1, 2)$$

from the assumption  $\text{rank } H = \text{rank } (G_1 \times G_2)$ . Moreover

$$\dim H = m_0(G_1 \times G_2)$$

implies that

$$H_1 = G_1 \quad \text{and} \quad \dim H_2 = m_0(G_2)$$

or

$$H_2 = G_2 \quad \text{and} \quad \dim H_1 = m_0(G_1).$$

q.e.d.

Table of  $m(G)$  and  $m_0(G)$  for simple Lie group  $G$  (cf. [1], [2])

$G$	$\dim G$	$m(G)$	$H$	$m_0(G)$	$U$
$SU(n), n \neq 4$	$n^2 - 1$	$(n-1)^2$	$S(U(n-1) \times U(1))$	$(n-1)^2$	$S(U(n-1) \times U(1))$
$SU(4)$	15	10	$Sp(2)$	9	$S(U(3) \times U(1))$
$SO(2n+1)$	$2n^2 + n$	$2n^2 - n$	$SO(2n)$	$2n^2 - n$	$SO(2n)$
$Sp(n)$	$2n^2 + n$	$2n^2 - 3n + 4$	$Sp(n-1) \times Sp(1)$	$2n^2 - 3n + 4$	$Sp(n-1) \times Sp(1)$
$SO(2n), n > 3$	$2n^2 - n$	$2n^2 - 3n + 1$	$SO(2n-1)$	$2n^2 - 5n + 4$	$SO(2n-2) \times SO(2)$
$G_2$	14	8	$SU(3)$	8	$SU(3)$
$F_4$	52	36	$Spin(9)$	36	$Spin(9)$
$E_6$	78	52	$F_4$	46	
$E_7$	133	79		79	
$E_8$	248	136		136	

Here  $H, U$  are closed connected subgroups of  $G$  with  $\dim H = m(G)$ ,  $\dim U = m_0(G)$  and  $\text{rank } U = \text{rank } G$

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