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Author(s)	Uchida, Fuichi
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COMPACT TRANSFORMATION GROUPS AND FIXED POINT SETS OF RESTRICTED ACTION TO MAXIMAL TORUS

FUICHI UCHIDA

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0. Introduction

Let G be a compact connected Lie group and let T be a maximal torus of G . Define

$$m(G) = \max \{ \dim H \mid H \text{ is a proper closed subgroup of } G \},$$

$$m_0(G) = \max \{ \dim H \mid H \text{ is a proper closed subgroup of } G \\ \text{with rank } H = \text{rank } G \}.$$

Let M be a connected manifold with a non-trivial smooth G -action and let H be a closed subgroup of G . Denote by $F(H, M)$ the fixed point set of the restricted action of the given G -action to the subgroup H . Then each connected component F_a ($a \in A$) of $F(H, M)$ is a regular submanifold of M . Define

$$\dim F(H, M) = \max \{ \dim F_a \mid a \in A \}$$

if $F(H, M)$ is non-empty and we put

$$\dim F(H, M) = -1$$

if $F(H, M)$ is empty. Then we have the following results.

Theorem 1.

- (a) *In general, $\dim M - \dim F(T, M) \geq \dim G - m(G)$.*
- (b) *If G is semi-simple and*

$$\dim F(G, M) < \dim F(T, M),$$

then

$$\dim M - \dim F(T, M) \geq \dim G - m_0(G).$$

Theorem 2. *If*

$$\dim M - \dim F(T, M) = \dim G - m(G),$$

then G is semi-simple, $m(G)=m_0(G)$ and

$$\dim M - \dim F_a = \dim G - m(G)$$

for each connected component F_a of $F(T, M)$. Moreover

$$\dim H = m(G) \quad \text{and} \quad \text{rank } H = \text{rank } G$$

for a principal isotropy group H .

1. Preliminary lemmas

In this section we prepare several lemmas.

Lemma 1.1. *Let H be a closed subgroup of G and assume $T \subset H$. Then*

$$F(T, G/H) = N(T)H/H.$$

In particular, $F(T, G/H)$ is a non-empty finite set.

Proof. It is clear that

$$F(T, G/H) = \{gH \mid g^{-1}Tg \subset H\}.$$

If $g^{-1}Tg \subset H$, then there is $h \in H$ such that

$$g^{-1}Tg = hTh^{-1},$$

since T is a maximal torus of H^0 , the identity component of H . Thus

$$gh \in N(T): \text{ the normalizer of } T \text{ in } G.$$

Hence we obtain

$$F(T, G/H) = N(T)H/H.$$

Next, there is a natural surjection $N(T)/T \rightarrow N(T)H/H$, where $N(T)/T$ is the Weyl group of G which is a finite group. Therefore $F(T, G/H)$ is a non-empty finite set. q.e.d.

In the following, we assume that M is a connected manifold with a non-trivial smooth G -action. It is clear

$$(1.2) \quad \dim M \geq \dim G - m(G).$$

Lemma 1.3. $\dim M - \dim F(G, M) > \dim G - m(G)$.

Proof. If $F(G, M)$ is empty, then the inequality is clear from (1.2). If $F(G, M)$ is non-empty, let $n = \dim F(G, M)$ and let F_a be an n -dimensional connected component of $F(G, M)$. For $x \in F_a$,

$$T_x M = T_x(F_a) \oplus N_x$$

as G -vector spaces, where N_x is a normal space of F_a in M . Then there is a non-zero vector $v \in N_x$ with $G_v \neq G$. Thus

$$\dim G - m(G) \leq \dim G/G_v < \dim N_x = \dim M - n.$$

q.e.d.

Lemma 1.4. *If*

$$\dim M - \dim F(T, M) \leq \dim G - m_0(G)$$

and

$$\dim F(G, M) < \dim F(T, M),$$

then

$$M = G \cdot F(H, M).$$

Here H is a compact connected subgroup of G such that

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

Proof. Let $k = \dim F(T, M)$ and denote by F^k the union of k -dimensional connected components of $F(T, M)$. Then

$$F^k - F(G, M)$$

is non-empty by the assumption. For $x \in F^k - F(G, M)$,

$$T_x M = T_x(G \cdot x) \oplus N_x$$

as G_x -vector spaces, where N_x is a normal space of the orbit $G \cdot x$ in M . Since $T \subset G_x$, $F(T, G \cdot x)$ is a non-empty finite set by Lemma 1.1. Thus

$$\begin{aligned} k &= \dim F(T, T_x M) = \dim F(T, N_x) \\ &\leq \dim N_x = \dim M - \dim G/G_x \leq \dim M - \dim G + m_0(G). \end{aligned}$$

On the other hand,

$$k \geq \dim M - \dim G + m_0(G)$$

by the assumption. Therefore

$$\begin{aligned} (1) \quad & \dim G_x = m_0(G), \\ (2) \quad & F(T, N_x) = N_x. \end{aligned}$$

Since the action of G_x on N_x is a slice representation at x , a principal isotropy group H' contains T by (2), and hence

$$\dim H' = m_0(G)$$

by (1). Let H be the identity component of the principal isotropy group H' . Then we have

$$M = G \cdot F(H, M) = \{g \cdot x \mid g \in G, x \in F(H, M)\}.$$

q.e.d.

Lemma 1.5. *If*

$$\dim M - \dim F(T, M) \leq \dim G - m(G),$$

then $m(G) = m_0(G)$ and

$$M = G \cdot F(H, M).$$

Here H is a compact connected subgroup of G such that

$$\dim H = m(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

Proof. Taking account of Lemma 1.3 and using similar arguments as in the proof of Lemma 1.4, we can prove this lemma.

Lemma 1.6. *Let G be a compact connected Lie group and let H be a closed subgroup of G such that*

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H^0 = \text{rank } G.$$

Then $N(H)^0 = H^0$, where H^0 is the identity component of H and $N(H)$ is the normalizer of H in G .

Proof. Assume $N(H)^0 \neq H^0$. Then the assumption on H implies $N(H) = G$. Thus H is a normal subgroup of G , and hence

$$\text{rank } G = \text{rank } H^0 + \text{rank } G/H.$$

Then the assumption on H implies $\text{rank } G/H = 0$ and hence $G = H$. But this is a contradiction to

$$\dim H = m_0(G) < \dim G.$$

q.e.d.

Lemma 1.7. *Let G be a compact connected semi-simple Lie group and let H be a closed connected subgroup of G such that*

$$\dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

Let V be a real G -vector space such that

$$V = G \cdot F(H, V) \quad \text{and} \quad F(G, V) = \{0\}.$$

Then $S(V)=G/H$ as G -manifolds and $N(H)/H=Z_2$. Here $S(V)$ is a G -invariant unit sphere of V .

Proof. By the assumption on H and V , the identity component of an isotropy subgroup at each point of $S(V)$ is conjugate to H in G . Hence there is an equivariant diffeomorphism

$$S(V) = G/H \times_{N(H)/H} F(H, S(V))$$

as G -manifolds. Here $F(H, S(V))$ is a unit sphere of $F(H, V)$. Since $N(H)/H$ is a finite group by Lemma 1.6, the natural projection

$$G/H \times F(H, S(V)) \rightarrow S(V)$$

is a finite covering as G -manifolds. On the other hand, $S(V)$ is simply connected, because G is semi-simple. Therefore

$$S(V) = G/H$$

as G -manifolds and $F(H, S(V))$ is a zero-sphere S^0 . Finally,

$$N(H)/H = F(H, G/H) = F(H, S(V)) = S^0.$$

Thus $N(H)/H=Z_2$, the cyclic group of order 2.

q.e.d.

2. Proof of theorems

Let G be a compact connected Lie group and let T be a maximal torus of G . Let M be a connected manifold with a non-trivial smooth G -action. It is easy to see that

$$F(T, M) = M \quad \text{implies} \quad F(G, M) = M.$$

Thus

$$\dim M - \dim F(T, M) \geq 2,$$

because

$$\dim M \equiv \dim F_a \pmod{2}$$

for each connected component F_a of $F(T, M)$.

If G is not semi-simple, then

$$\dim G - m(G) = 1$$

and hence there is nothing to prove. In particular, if

$$\dim M - \dim F(T, M) = \dim G - m(G),$$

then G is semi-simple, and $m(G)=m_0(G)$ by Lemma 1.5.

Now we assume that G is semi-simple and there is a closed connected subgroup H of G such that

$$(*) \quad M = G \cdot F(H, M), \dim H = m_0(G) \quad \text{and} \quad \text{rank } H = \text{rank } G.$$

Moreover, (i) first suppose that $F(G, M)$ is empty. Then by the assumption (*), the identity component of an isotropy subgroup at each point of M is conjugate to H in G . Hence there is an equivariant diffeomorphism

$$M = G/H \times_{N(H)/H} F(H, M)$$

as G -manifolds. Since $N(H)/H$ is a finite group by Lemma 1.6, the natural projection

$$p: G/H \times F(H, M) \rightarrow M$$

is a finite covering as G -manifolds. Hence we obtain

$$F(T, M) = p(F(T, G/H) \times F(H, M)).$$

Here $F(T, G/H)$ is a non-empty finite set by Lemma 1.1. Therefore

$$\begin{aligned} \dim M - \dim F_a &= \dim M - \dim F(H, M) \\ &= \dim G/H = \dim G - m_0(G), \end{aligned}$$

for each connected component F_a of $F(T, M)$.

(ii) Next suppose that $F(G, M)$ is non-empty. Then each fibre N_x of the normal G -vector bundle of $F(G, M)$ in M satisfies the hypothesis of Lemma 1.7, and hence

$$N(H)/H = Z_2 \quad \text{and} \quad S(N_x) = G/H.$$

Let U be a G -invariant closed tubular neighborhood of $F(G, M)$ in M . Then there is an equivariant diffeomorphism

$$M = \partial(D(V) \times F(H, M - \text{int } U))/Z_2$$

as G -manifolds. Here V is a real G -vector space (unique up to G -isomorphism) with $S(V) = G/H$, Z_2 acts on the unit disk $D(V)$ as antipodal involution, and G acts naturally on $D(V)$ and trivially on $F(H, M - \text{int } U)$. Hence we obtain

$$\begin{aligned} F(T, M) &= \partial(F(T, D(V)) \times F(H, M - \text{int } U))/Z_2 \\ &= \partial([-1, 1] \times F(H, M - \text{int } U))/Z_2. \end{aligned}$$

Therefore

$$\begin{aligned} \dim M - \dim F_a &= \dim M - \dim F(H, M - \text{int } U) \\ &= \dim D(V) - 1 \\ &= \dim G/H \\ &= \dim G - m_0(G), \end{aligned}$$

for each connected component F_a of $F(T, M)$.

Now the proofs of Theorem 1 and Theorem 2 are completed by Lemma 1.4 and Lemma 1.5.

3. Integers $m(G)$ and $m_0(G)$

In this section we show certain properties of $m(G)$ and $m_0(G)$. It is easy to see that

$$(3.1) \quad m(G_1 \times G_2) \geq \max(m(G_1) + \dim G_2, \dim G_1 + m(G_2)),$$

and

$$(3.2) \quad m(G) \geq 1, \quad \text{if } G \neq S^1.$$

Lemma 3.3. *Let G_1 and G_2 be compact connected Lie groups. Suppose that G_1 is simple and $G_1 \neq S^1$. Let H be a closed connected subgroup of $G_1 \times G_2$ with $\dim H = m(G_1 \times G_2)$. Then*

$$H = H_1 \times G_2 \quad \text{or} \quad H = G_1 \times H_2$$

where H_a is a closed subgroup of G_a ($a=1, 2$) with $\dim H_a = m(G_a)$.

Proof. Let $p_a: G_1 \times G_2 \rightarrow G_a$ ($a=1, 2$) be natural projections, and let $i_a: G_a \rightarrow G_1 \times G_2$ be natural injections defined by

$$i_1(g) = (g, e_2), g \in G_1$$

$$i_2(g) = (e_1, g), g \in G_2$$

where e_a is the identity element of G_a ($a=1, 2$). Define

$$H_a = p_a(H) \quad \text{and} \quad H'_a = i_a^{-1}(H).$$

Then H'_a is a normal subgroup of H_a ($a=1, 2$) and $H'_1 \times H'_2$ is a normal subgroup of H , and $H \subset H_1 \times H_2$. Moreover the projection p_a induces an isomorphism

$$p'_a: H/H'_1 \times H'_2 \rightarrow H_a/H'_a \quad (a=1, 2).$$

(i) First suppose $H_1 \neq G_1$. Then

$$H \subset p_1^{-1}(H_1) = H_1 \times G_2 \neq G_1 \times G_2.$$

Hence we obtain

$$H = H_1 \times G_2 \quad \text{and} \quad \dim H_1 = m(G_1)$$

from the assumption $\dim H = m(G_1 \times G_2)$.

(ii) Next suppose $H_1 = G_1$. Then H'_1 is a normal subgroup of the simple Lie group G_1 and hence $H'_1 = G_1$ or H'_1 is a finite group. Since $m(G_1) \geq 1$ and

there is an isomorphism

$$H/i_1(H_1') = H_2,$$

we obtain

$$\begin{aligned} m(G_1 \times G_2) &= \dim H = \dim H_1' + \dim H_2 \\ &< \dim H_1' + m(G_1) + \dim G_2 \leq \dim H_1' + m(G_1 \times G_2). \end{aligned}$$

Thus $\dim H_1' \neq 0$, and hence

$$H_1' = H_1 = G_1.$$

Therefore

$$H = G_1 \times H_2 \quad \text{and} \quad \dim H_2 = m(G_2)$$

from the assumption $\dim H = m(G_1 \times G_2)$. q.e.d.

Corollary 3.4. *Let G_1 and G_2 be compact connected Lie groups. Suppose that G_1 is simple. Then*

$$\dim(G_1 \times G_2) - m(G_1 \times G_2) = \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)).$$

Proof. If $G_1 \neq S^1$, Then the equation follows from Lemma 3.3. If $G_1 = S^1$, then $m(G_1 \times G_2) = \dim G_2$ and hence the equation holds. q.e.d.

Theorem 3.5. *Let G_1 and G_2 be compact connected Lie groups. Then*

$$\dim(G_1 \times G_2) - m(G_1 \times G_2) = \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)).$$

Proof. Let G^* be a compact connected covering group of G . Then it is easy to see that

$$m(G^*) = m(G).$$

There are covering groups G_a^* of G_a ($a=1, 2$) such that

$$\begin{aligned} G_1^* &= H_1 \times \cdots \times H_r \times T^m \\ G_2^* &= K_1 \times \cdots \times K_s \times T^n \end{aligned}$$

where H_i, K_j are compact connected non-abelian simple Lie groups, and T^m, T^n are tori. If m or n is non-zero, then

$$\begin{aligned} \dim(G_1 \times G_2) - m(G_1 \times G_2) &= 1 \\ \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)) &= 1. \end{aligned}$$

Next, if $m=n=0$, then

$$\begin{aligned} \dim(G_1 \times G_2) - m(G_1 \times G_2) &= \min_{i,j}(\dim H_i - m(H_i), \dim K_j - m(K_j)) \\ &= \min(\dim G_1 - m(G_1), \dim G_2 - m(G_2)) \end{aligned}$$

be Corollary 3.4.

q.e.d.

REMARK 3.6. The integer $m_0(G)$ can be defined only when G is non-abelian (i.e. G does not coincide with its maximal torus).

Theorem 3.7. *Let G_1 and G_2 be compact connected non-abelian Lie groups. Then*

$$\dim(G_1 \times G_2) - m_0(G_1 \times G_2) = \min(\dim G_1 - m_0(G_1), \dim G_2 - m_0(G_2)).$$

Proof. Let H be a closed connected subgroup of $G_1 \times G_2$ such that

$$\dim H = m_0(G_1 \times G_2) \quad \text{and} \quad \text{rank } H = \text{rank}(G_1 \times G_2).$$

Then there are closed connected subgroups H_a of G_a ($a=1, 2$) such that

$$H = H_1 \times H_2 \quad \text{and} \quad \text{rank } H_a = \text{rank } G_a \quad (a = 1, 2)$$

from the assumption $\text{rank } H = \text{rank}(G_1 \times G_2)$. Moreover

$$\dim H = m_0(G_1 \times G_2)$$

implies that

$$H_1 = G_1 \quad \text{and} \quad \dim H_2 = m_0(G_2)$$

or

$$H_2 = G_2 \quad \text{and} \quad \dim H_1 = m_0(G_1).$$

q.e.d.

Table of $m(G)$ and $m_0(G)$ for simple Lie group G (cf. [1], [2])

G	$\dim G$	$m(G)$	H	$m_0(G)$	U
$SU(n), n \neq 4$	$n^2 - 1$	$(n-1)^2$	$S(U(n-1) \times U(1))$	$(n-1)^2$	$S(U(n-1) \times U(1))$
$SU(4)$	15	10	$Sp(2)$	9	$S(U(3) \times U(1))$
$SO(2n+1)$	$2n^2 + n$	$2n^2 - n$	$SO(2n)$	$2n^2 - n$	$SO(2n)$
$Sp(n)$	$2n^2 + n$	$2n^2 - 3n + 4$	$Sp(n-1) \times Sp(1)$	$2n^2 - 3n + 4$	$Sp(n-1) \times Sp(1)$
$SO(2n), n > 3$	$2n^2 - n$	$2n^2 - 3n + 1$	$SO(2n-1)$	$2n^2 - 5n + 4$	$SO(2n-2) \times SO(2)$
G_2	14	8	$SU(3)$	8	$SU(3)$
F_4	52	36	$Spin(9)$	36	$Spin(9)$
E_6	78	52	F_4	46	
E_7	133	79		79	
E_8	248	136		136	

Here H, U are closed connected subgroups of G with $\dim H = m(G)$, $\dim U = m_0(G)$ and $\text{rank } U = \text{rank } G$

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