

Title	On standard L-functions attached to $\text{alt}^{n-1}(\mathbb{C}^n)$ -valued Siegel modular forms
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Citation	Osaka Journal of Mathematics. 1995, 32(3), p. 547-563
Version Type	VoR
URL	<a href="https://doi.org/10.18910/6943">https://doi.org/10.18910/6943</a>
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## ON STANDARD $L$ -FUNCTIONS ATTACHED TO $\text{ALT}^{n-1}(\mathbf{C}^n)$ -VALUED SIEGEL MODULAR FORMS

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(Received February 8, 1993)

### Introduction

In [23], we studied some properties of standard  $L$ -functions attached to  $\text{sym}^l(V)$ -valued Siegel modular forms of weight  $\det^k \otimes \text{sym}^l$ . More precisely, let  $\det^k \otimes \text{sym}^l$  be an irreducible rational representation of  $GL(n, \mathbf{C})$  with representation space  $\text{sym}^l(V)$ , where  $V$  is isomorphic to  $\mathbf{C}^n$  and  $\text{sym}^l(V)$  is the  $l$ -th symmetric tensor product of  $V$ . Let  $f$  be a  $\text{sym}^l(V)$ -valued holomorphic cusp form of weight  $\det^k \otimes \text{sym}^l$  for  $Sp(n, \mathbf{Z})$  (size  $2n$ ). Suppose  $f$  is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra. Then we define the standard  $L$ -function attached to  $f$  by

$$(0.1) \quad L(s, f, \underline{\text{St}}) := \prod_p \left\{ (1-p^{-s}) \prod_{j=1}^n (1-\alpha_j(p)^{-1}p^{-s})(1-\alpha_j(p)p^{-s}) \right\}^{-1},$$

where  $p$  runs over all prime numbers and  $\alpha_j(p)$  ( $1 \leq j \leq n$ ) are the Satake  $p$ -parameters of  $f$ . The right-hand side of (0.1) converges absolutely and locally uniformly for  $\text{Re}(s) > n+1$ . We put

$$A(s, f, \underline{\text{St}}) := \Gamma_{\mathbf{R}}(s+\varepsilon) \Gamma_c(s+k+l-1) \left\{ \prod_{j=2}^n \Gamma_c(s+k-j) \right\} L(s, f, \underline{\text{St}}),$$

with

$$\Gamma_{\mathbf{R}}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_c(s) := 2(2\pi)^{-s} \Gamma(s)$$

and

$$\varepsilon := \begin{cases} 0 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Then we have the following (cf. Andrianov and Kalinin [Z], Böcherer [5] and Mizumoto [19] for  $l=0$ ).

**Theorem.** ([23, Theorems 2 and 3]) For  $k, l \in 2\mathbf{Z}$ ,  $k, l > 0$ ,  $\Lambda(s, f, \underline{\text{St}})$  has a meromorphic continuation to the whole  $s$ -plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\text{St}}) = \Lambda(1-s, f, \underline{\text{St}}).$$

Suppose  $k > n$ . Then  $\Lambda(s, f, \underline{\text{St}})$  is holomorphic except for possible simple poles at  $s=0$  and  $s=1$ ; it has a pole at  $s=1$  (or equivalently,  $s=0$ ) if and only if  $f$  belongs to the  $\mathbf{C}$ -vector space spanned by certain theta series in [24] which is invariant under the action of the Hecke algebra.

If we note that the signature of  $\det^k \otimes \text{sym}^l$  is  $(k+l, k, \dots, k) \in \mathbf{Z}^n$ , we expect the following [23, §3.1 Remark]:

(C). Let  $\rho$  be an irreducible rational representation of  $GL(n, \mathbf{C})$  with representation space  $V$  whose signature is  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbf{Z}^n$  with  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ . Let  $f$  be a  $V$ -valued holomorphic cusp form of weight  $\rho$  for  $Sp(n, \mathbf{Z})$ . Suppose that  $f$  is an eigenform. Then, it is expected that the completed Dirichlet series

$$\Lambda(s, f, \underline{\text{St}}) := \Gamma_{\mathbf{R}}(s + \varepsilon) \prod_{j=1}^n \Gamma_{\mathbf{C}}(s + \lambda_j - j) L(s, f, \underline{\text{St}})$$

should satisfy a functional equation.

Unfortunately, within our knowledge it is not verified so far whether (C) holds or not except  $\det^k$  and  $\det^k \otimes \text{sym}^l$  cases. We will give another example satisfying (C).

For  $l \in \mathbf{Z}$ ,  $0 \leq l \leq n$ , let  $\det^k \otimes \text{alt}^l$  be an irreducible rational representation of  $GL(n, \mathbf{C})$  with representation space  $\text{alt}^l(V)$ , where  $V$  is isomorphic to  $\mathbf{C}^n$  and  $\text{alt}^l(V)$  is the  $l$ -th alternating tensor product of  $V$ . Let  $M_k^n(\text{alt}^l(V))$  (resp.  $S_k^n(\text{alt}^l(V))$ ) be the  $\mathbf{C}$ -vector space consisting of  $\text{alt}^l(V)$ -valued holomorphic modular (resp. cusp) forms of weight  $\det^k \otimes \text{alt}^l$  for  $Sp(n, \mathbf{Z})$ .

Suppose that  $f \in S_k^n(\text{alt}^{n-1}(V))$  is an eigenform. We note that the signature of  $\det^k \otimes \text{alt}^{n-1}$  is  $(k+1, \dots, k+1, k)$ . We put

$$\Lambda(s, f, \underline{\text{St}}) := \Gamma_{\mathbf{R}}(s+1) \left\{ \prod_{j=1}^{n-1} \Gamma_{\mathbf{C}}(s+k+1-j) \right\} \Gamma_{\mathbf{C}}(s+k-n) L(s, f, \underline{\text{St}}).$$

Then the main result of this paper is the following (cf. Piatetski-Shapiro and Rallis [21], Weissauer [24]).

**Theorem 1.** Let  $k$  be an even integer,  $n$  an odd integer and  $2k \geq n > 2$ . Then  $\Lambda(s, f, \underline{\text{St}})$  has a meromorphic continuation to the whole  $s$ -plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\text{St}}) = \Lambda(1-s, f, \underline{\text{St}}).$$

Moreover, suppose  $k > n$ . Then,  $\Lambda(s, f, \underline{\text{St}})$  is entire.

NOTATION.

1°. As usual,  $\mathbf{Z}$  is the ring of rational integers,  $\mathbf{Q}$  the field of rational numbers,  $\mathbf{R}$  the field of real numbers,  $\mathbf{C}$  the field of complex numbers.

2°. Let  $m, n \in \mathbf{Z}$ ,  $m, n > 0$ . If  $A$  is an  $m \times n$ -matrix, then we write it also as  $A^{(m,n)}$ , and as  $A^{(m)}$  if  $m=n$ . The identity matrix of size  $n$  is denoted by  $1_n$ .

3°. For  $m, n \in \mathbf{Z}$ ,  $m, n > 0$ , and a commutative ring  $R$  containing 1, let  $R^{(m,n)}$  (resp.  $R^{(n)}$ ) be the  $R$ -module of all  $m \times n$  (resp.  $n \times n$ ) matrices with entries in  $R$ .

4°. For a real symmetric positive definite matrix  $S$ ,  $S^{1/2}$  is the unique real symmetric positive definite matrix such that  $(S^{1/2})^2 = S$ .

5°. For matrix  $A^{(m)}$ ,  $B^{(m,n)}$ , we define  $A[B] := {}^t \bar{B} A B$ , where  ${}^t B$  is the transpose of  $B$  and  $\bar{B}$  is the complex conjugate of  $B$ .

6°. For a matrix  $A^{(m)} = (a_{jh})_{1 \leq j, h \leq m}$ ,  $\tilde{a}_{jh}$  is the cofactor of  $a_{jh}$  and  $\tilde{A} = (\tilde{a}_{jh})$ .

7°. For  $n \in \mathbf{Z}$ ,  $n > 0$ , we put

$$\mathbf{T}^{(n)} := \left\{ T = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{pmatrix} \in \mathbf{Z}^{(n)} \mid t_j > 0 (1 \leq j \leq n), t_1 | \dots | t_n \right\}.$$

8°. For  $n \in \mathbf{Z}$ ,  $n > 0$ , let  $\Gamma^n := \text{Sp}(n, \mathbf{Z})$  be the Siegel modular group of degree  $n$  and let  $\mathfrak{H}_n$  be the Siegel upper half space of degree  $n$ , that is,

$$\mathfrak{H}_n := \{Z = X + iY \in \mathbf{C}^{(n)} \mid {}^t Z = Z, Y > 0\}.$$

For each  $r \in \mathbf{Z}$  with  $0 \leq r \leq n$ , we put

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ \mathbf{C}^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n \mid C = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{C}_4^{(r)} \end{pmatrix}, D = \begin{pmatrix} * & 0 \\ * & D_4^{(r)} \end{pmatrix} \right\}.$$

All these are subgroups of  $\Gamma^n$ .

9°. For  $n \in \mathbf{Z}$ ,  $n \geq 0$ , we put

$$\Gamma_n(s) := \prod_{j=1}^n \Gamma\left(s - \frac{j-1}{2}\right),$$

and

$$\gamma(s) := \begin{cases} \frac{\Gamma_n\left(\frac{s+n}{2}\right)}{\Gamma_n\left(\frac{s}{2}\right)} & \text{for } n \text{ even,} \\ \frac{\Gamma_{n-1}\left(\frac{s+n}{2}\right)}{\Gamma_{n-1}\left(\frac{s-1}{2}\right)} & \text{for } n \text{ odd,} \end{cases}$$

where  $\Gamma(s)$  is the gamma function. We note that

$$\gamma(s) = \gamma(1-s)$$

Moreover, we put

$$\xi(s) := \Gamma_R(s)\zeta(s) = \xi(1-s),$$

where  $\zeta(s)$  is the Riemann zeta function.

Throughout the paper we understand that a product (resp. a sum) over an empty set is equal to 1 (resp. 0).

**1. Preliminaries**

Let  $\rho$  be a finite-dimensional representation of  $GL(n, \mathbf{C})$  with representation space  $V$ . By definition,  $V$ -valued  $C^\infty$ -Siegel modular forms of weight  $\rho$  are  $C^\infty$ -functions from  $\mathfrak{H}_n$  to  $V$  satisfying

$$(1.1) \quad (f|_\rho M)(Z) = f(Z)$$

for all  $Z \in \mathfrak{H}_n$  and  $M = \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \in \Gamma^n$ , where

$$(f|_\rho M)(Z) := \rho((CZ + D)^{-1})f(M\langle Z \rangle) \text{ and } M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}.$$

The space of all such functions is denoted by  $M_\rho^n(V)^\infty$ .

We write  $|_k$  for  $\rho = \det^k$  and we omit subscripts  $\rho, k$  when there is no fear of confusion.

A holomorphic function  $f$  from  $\mathfrak{H}_n$  to  $V$  is called a  $V$ -valued Siegel modular form of weight  $\rho$  if it satisfies (1.1) and if it is holomorphic at the cusps when  $n = 1$ . The space of  $V$ -valued Siegel modular forms of weight  $\rho$  is denoted by  $M_\rho^n(V)$ .

We define the Siegel operator on  $M_\rho^n(V)$  by

$$(\Phi f)(Z) := \lim_{t \rightarrow \infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix}\right)$$

for  $Z \in \mathfrak{H}_{n-1}$ . Let  $W$  be the subspace of  $V$  generated by the values of  $\Phi f$  for all  $f \in M_\rho^n(V)$ . Then  $W$  is invariant under the transformations

$$\rho\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right), g \in GL(n-1, \mathbf{C}).$$

If we assume  $W \neq \{0\}$ , we get the representation  $\sigma$  of  $GL(n-1, \mathbf{C})$  with representation space  $W$ . Thus the operator  $\Phi$  defines the map

$$\Phi : M_\rho^n(V) \rightarrow M_\sigma^{n-1}(W).$$

Suppose  $f \in M_\rho^n(V)$ . Then it is called a cusp form if it satisfies  $\Phi f = 0$ , and we put

$$S_\rho^n(V) := \{f \in M_\rho^n(V) \mid f \text{ is a cusp form}\}.$$

If  $\rho$  is an irreducible rational representation,  $\rho$  is equivalent to an irreducible rational representation  $\tilde{\rho}$  satisfying the following condition: Let  $\tilde{V}$  be the representation space of  $\tilde{\rho}$ . Then, there exists a unique one-dimensional vector subspace  $C\tilde{v}$  of  $\tilde{V}$  such that for any upper triangular matrix of  $GL(n, \mathbf{C})$ ,

$$\tilde{\rho}\left(\begin{pmatrix} g_{11} & & * \\ & \ddots & \\ 0 & & g_{nn} \end{pmatrix}\right)\tilde{v} = \left(\prod_{j=1}^n g_{jj}^{\lambda_j}\right)\tilde{v},$$

where  $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbf{Z}^n$  and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ .

Then we call  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  the signature of  $\rho$ .

REMARK. Suppose the signature of  $\rho$  is  $(\lambda_1, \lambda_2, \dots, \lambda_n)$ . We note that  $M_\rho^n(V) = \{0\}$  if  $\lambda_n < 0$  and that  $M_\rho^n(V)^\infty = \{0\}$  if  $\lambda_1 + \dots + \lambda_n \not\equiv 0 \pmod{2}$ .

Now, we put

$$G^+Sp(n, \mathbf{Q}) := \left\{M \in GL(2n, \mathbf{Q}) \mid {}^t M \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} M = \mu(M) \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}, \mu(M) > 0\right\}.$$

For  $g \in G^+Sp(n, \mathbf{Q})$ , let  $\Gamma^n g \Gamma^n = \cup_{j=1}^r \Gamma^n g_j$  be a decomposition of the double coset  $\Gamma^n g \Gamma^n$  into left cosets. For  $f \in M_\rho^n(V)$  (resp.  $S_\rho^n(V)$ ,  $M_\rho^n(V)^\infty$ ), we define the Hecke operator  $(\Gamma^n g \Gamma^n)$  by

$$f|(\Gamma^n g \Gamma^n) := \sum_{j=1}^r f|g_j.$$

Let  $f \in S_\rho^n(V)$  be an eigenform. We define the standard L-function attached to  $f$  by (0.1). We also define the following series:

$$(1.2) \quad D(s, f) := \sum_{T \in \mathbf{T}^{(n)}} \lambda(f, T) \det(T)^{-s},$$

where  $\lambda(f, T)$  is the eigenvalue on  $f$  of the Hecke operator  $\left(\Gamma^n \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \Gamma^n\right)$ ,  $T \in \mathbf{T}^{(n)}$ . By Böcherer [6], we have:

$$(1.3) \quad \zeta(s) \prod_{j=1}^n \zeta(2s-2j) D(s, f) = L(s-n, f, \underline{\text{St}}).$$

For  $k \in 2\mathbf{Z}$ ,  $k > 0$ ,  $s \in \mathbf{C}$  and  $Z = (z_{jh}) \in \mathfrak{S}_n$  with  $z_{jh} = x_{jh} + iy_{jh}$ , we define the Eisenstein series by

$$E_k^n(Z, s) := \sum_{M = \begin{pmatrix} * & \\ C^{(n)} & D^{(n)} \end{pmatrix} \in P_{n,0} \backslash \Gamma^n} \det(CZ + D)^{-k} \det(\text{Im}(M\langle Z \rangle))^s.$$

Then  $E_k^n(Z, s) \in M_k^{n\infty}$ , where  $M_k^{n\infty}$  is the space of  $C^\infty$ -Siegel modular forms of weight  $k$ . The function  $E_k^n(Z, s) \det(\text{Im}(Z))^{-s}$  converges absolutely and locally uniformly for  $k + 2\text{Re}(s) > n + 1$ . Moreover, we have the following :

**Theorem 2.** (Langlands [18], Kalinin [13] and Mizumoto [19, 20]) *Let  $n \in \mathbf{Z}$ ,  $k \in 2\mathbf{Z}$  and  $n, k > 0$ . Then for  $Z \in \mathfrak{S}_n$ ,*

$$E_k^n(Z, s) := \frac{\Gamma_n\left(s + \frac{k}{2}\right)}{\Gamma_n(s)} \xi(2s) \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \xi(4s-2j) E_k^n\left(Z, s - \frac{k}{2}\right)$$

*is invariant under  $s \rightarrow \frac{n+1}{2} - s$  and it is an entire function in  $s$ .*

It is also known that every partial derivative (in  $z_{jh}$ 's) of the Eisenstein series  $E_k^n(Z, s)$  is slowly increasing (locally uniformly in  $s$ ).

**Theorem 3.** (Mizumoto [20]) *Let  $n \in \mathbf{Z}$ ,  $k \in 2\mathbf{Z}$  and  $n, k > 0$ .*

(i) *For each  $s_0 \in \mathbf{C}$ , there exist constants  $\delta > 0$  and  $d \in \mathbf{Z} (d \geq 0)$ , depending only on  $n, k$  and  $s_0$ , such that*

$$(s - s_0)^d E_k^n(X + iY, s)$$

*is holomorphic in  $s$  for  $|s - s_0| < \delta$ , and is  $C^\infty$  in  $(X, Y)$ .*

(ii) *Furthermore, for given  $\varepsilon > 0$  and  $N \in \mathbf{Z} (N \geq 0)$ , there exist constants  $\alpha > 0$  and  $\beta > 0$  depending only on  $n, k, d, s_0, \varepsilon, \delta$  and  $N$  such that*

$$|(s - s_0)^d D_{X,Y} E_k^n(X + iY, s)| \leq \alpha \det(\text{Im}(Z))^\beta$$

*for  $Y \geq \varepsilon 1_n$  and  $|s - s_0| < \delta$ , where  $D_{X,Y}$  is an arbitrary monomial of degree  $N$  in  $\frac{\partial}{\partial x_{jh}}$  and  $\frac{\partial}{\partial y_{jh}}$  ( $1 \leq j, h \leq n$ ).*

The assertion above for the case  $N = 0$  has been proved by Langlands [18] and Kalinin [13].

**2. Differential operators**

In what follows, we put

$$V_1 = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n, \mathbf{e}_1 = (e_1, \dots, e_n),$$

$$V_2 = \mathbb{C}e_{n+1} \oplus \cdots \oplus \mathbb{C}e_{2n}, \mathbf{e}_2 = (e_{n+1}, \dots, e_{2n}).$$

Let  $\text{alt}^{n-1}(V_1)$  (resp.  $\text{alt}^{n-1}(V_2)$ ) be the  $(n-1)$ -th alternating tensor product of  $V_1$  (resp.  $V_2$ ). If we put

$$t_j := (-1)^{j-1} e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n,$$

$$t_{n+j} := (-1)^{j-1} e_{n+1} \wedge \cdots \wedge e_{n+j-1} \wedge e_{n+j+1} \wedge \cdots \wedge e_{2n} \quad (1 \leq j \leq n),$$

we can write

$$\text{alt}^{n-1}(V_1) = \mathbb{C}t_1 \oplus \cdots \oplus \mathbb{C}t_n \text{ and } \text{alt}^{n-1}(V_2) = \mathbb{C}t_{n+1} \oplus \cdots \oplus \mathbb{C}t_{2n}.$$

Moreover, we put

$$\mathbf{t}_1 := (t_1, \dots, t_n) \text{ and } \mathbf{t}_2 := (t_{n+1}, \dots, t_{2n}).$$

If for each  $g \in GL(n, \mathbb{C})$ ,  $g$  acts on  $e_j$  ( $j=1, 2$ ) by  $e_j g$ , then  $\det^k \otimes \text{alt}^{n-1}(g)$  acts on  $t_j$  ( $j=1, 2$ ) by

$$\det^k \otimes \text{alt}^{n-1}(g)t_j := \det(g)^k t_j \tilde{g} = \det(g)^{k+1} t_j^t g^{-1}.$$

If we put  $\alpha = (a_1, \dots, a_n) \in \mathbb{C}^n$ ,  $\det^k \otimes \text{alt}^{n-1}(g)$  acts on  $\sum_{j=1}^n a_j t_j = \mathbf{t}_1^t \alpha \in \text{alt}^{n-1}(V_1)$  and  $\mathbf{t}_2^t \alpha \in \text{alt}^{n-1}(V_2)$  by

$$\det^k \otimes \text{alt}^{n-1}(g)(\mathbf{t}_j^t \alpha) := \det(g)^k t_j \tilde{g}^t \alpha = \det(g)^{k+1} t_j^t g^{-1} \alpha \quad (j=1, 2).$$

Thus we get the action of  $\det^k \otimes \text{alt}^{n-1}$  on  $\text{alt}^{n-1}(V_j)$  ( $j=1, 2$ ).

Let  $\iota$  be the isomorphism from  $V_1$  to  $V_2$  defined by  $\iota(e_j) = e_{n+j}$  ( $1 \leq j \leq n$ ). It induces the isomorphism (also denoted by  $\iota$ ) from  $\text{alt}^{n-1}(V_1)$  to  $\text{alt}^{n-1}(V_2)$ . For a  $\text{alt}^{n-1}(V_1)$ -valued function  $f$  on  $\mathfrak{H}_n$  and for  $Z \in \mathfrak{H}_n$ , we define  $\iota(f)$  by

$$(\iota(f))(Z) := \iota(f(Z)).$$

For a function  $F$  on  $\mathfrak{H}_{2n}$ ,  $\begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}_tU^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$ , we define the pullback  $d^*$  by

$$(d^*F)\left(\begin{pmatrix} Z & U \\ {}_tU & W \end{pmatrix}\right) := F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right).$$

We consider  $\Gamma^n \times \Gamma^n$  imbedded in  $\Gamma^{2n}$  by

$$\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \times \begin{pmatrix} A'^{(n)} & B'^{(n)} \\ C'^{(n)} & D'^{(n)} \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{pmatrix},$$



and when convenient will identify  $\Gamma^n \times \Gamma^n$  with its image in  $\Gamma^{2n}$ .

We summarize some facts on differential operators obtained from invariant pluri-harmonic polynomials in Ibukiyama [12]. Let  $\rho_0$  (resp.  $\rho'_0$ ) be an irreducible rational representation of  $GL(n, \mathbb{C})$  with representation space  $V$  (resp.  $V'$ ), where  $\rho_0$  is equivalent to  $\rho'_0$ . For  $n, k \in \mathbb{Z}, n, k > 0$ , let  $X = (x_{jv})$  be a variable on  $\mathbb{C}^{(n, 2k)}$ . We put

$$\Delta_{jh} := \sum_{v=1}^{2k} \frac{\partial^2}{\partial x_{jv} \partial x_{hv}}$$

A polynomial  $P(X)$  on  $\mathbb{C}^{(n, 2k)}$  is called pluri-harmonic if  $\Delta_{jh}P = 0$  for each  $j, h$  with  $1 \leq j \leq h \leq n$ .

From now on, we assume that  $2k \geq n$ . Suppose that a polynomial map

$$P : \mathbb{C}^{(n, 2k)} \times \mathbb{C}^{(n, 2k)} \rightarrow V \otimes V'$$

satisfies the following three conditions :

- (2.1)  $P(X_1, X_2)$  is pluri-harmonic for each  $X_j \ (j=1, 2)$ ,
- (2.2)  $P(X_1g, X_2g) = P(X_1, X_2)$  for each  $g \in O(2k)$ ,
- (2.3)  $P(a_1X_1, a_2X_2) = (\rho_0(a_1) \otimes \rho'_0(a_2))P(X_1, X_2)$  for each  $a_j \in GL(n, \mathbb{C}) \ (j=1, 2)$ .

Then there exists a unique polynomial map  $Q$  on  $\mathbb{C}^{(2n)}$  such that

$$P(X_1, X_2) = Q \begin{pmatrix} X_1^t X_1 & X_1^t X_2 \\ X_2^t X_1 & X_2^t X_2 \end{pmatrix}.$$

Let  $\mathfrak{z} = (z_{jh})$  be a variable on  $\mathfrak{S}_{2n}$ . We put

$$\frac{\partial}{\partial \mathfrak{z}} := \left( \frac{1 + \delta_{jh}}{2} \frac{\partial}{\partial z_{jh}} \right)_{1 \leq j, h \leq 2n},$$

where, for  $z_{jh} = x_{jh} + iy_{jh}$ ,

$$\frac{\partial}{\partial z_{jh}} = \frac{1}{2} \left( \frac{\partial}{\partial x_{jh}} - i \frac{\partial}{\partial y_{jh}} \right), \quad \frac{\partial}{\partial \bar{z}_{jh}} = \frac{1}{2} \left( \frac{\partial}{\partial x_{jh}} + i \frac{\partial}{\partial y_{jh}} \right).$$

If we put

$$D := d^* Q \left( \frac{\partial}{\partial \mathfrak{z}} \right),$$

we have the following :

**Theorem 4.** (Ibukiyama [12]) *Let  $n, k \in \mathbb{Z}$  and  $2k \geq n > 0$ .*

- (i) *Let  $F$  be any  $\mathbb{C}$ -valued  $C^\infty$ -function on  $\mathfrak{S}_{2n}$ . If we put  $\rho = \det^k \otimes \rho_0$  and*

$\rho' = \det^k \otimes \rho'_0$ , then for each  $(g, g') \in \Gamma^n \times \Gamma^n$  and  $\mathfrak{z} = \begin{pmatrix} Z^{(n)} & U^{(n)} \\ {}_tU^{(n)} & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$ , we get the following commutation relation :

$$((DF)|_{\rho}(g)_z|_{\rho'}(g')_w)(\mathfrak{z}) = (D(F|_k(g, g')))(\mathfrak{z}),$$

where  $(\ )_z$  (resp.  $(\ )_w$ ) denotes the action on  $Z$  (resp.  $W$ ).

(ii) The operator  $D$  sends modular forms to modular forms :

$$D : M_k^{2n\infty} \rightarrow M_{\rho}^n(V)^{\infty} \otimes M_{\rho'}^n(V')^{\infty}.$$

Moreover,  $D$  is a holomorphic operator and it satisfies

$$D : M_k^{2n} \rightarrow M_{\rho}^n(V) \otimes M_{\rho'}^n(V').$$

Now we apply it to  $\det^k \otimes \text{alt}^{n-1}$  cases. Let  $\rho_0 = \text{alt}^{n-1}$  (resp.  $\rho'_0 = \text{alt}^{n-1}$ ) be the representation of  $GL(n, \mathbb{C})$  with representation space  $\text{alt}^{n-1}(V_1)$  (resp.  $\text{alt}^{n-1}(V_2)$ ). For a variable  $\mathfrak{z} = (z_{jh})$  on  $\mathfrak{H}_{2n}$ , we put

$$u_{jh} : = z_{j\ n+h} (1 \leq j, h \leq n), \quad U^{(n)} : = (u_{jh}) \quad \text{and} \quad \frac{\partial}{\partial U} : = \left( \frac{\partial}{\partial u_{jh}} \right)_{1 \leq j, h \leq n}.$$

For functions on  $\mathfrak{H}_{2n}$ , we define the differential operator  $\mathcal{D}$  by

$$\mathcal{D} : = d^* \left( t_1 \widetilde{\frac{\partial}{\partial U}} t_2 \right).$$

Then we have :

**Proposition 1.** Let  $n, k \in \mathbb{Z}$  and  $2k \geq n > 2$ .

(i) Let  $F$  be any  $\mathbb{C}$ -valued  $C^{\infty}$ -function on  $\mathfrak{H}_{2n}$ . Then for each  $(g, g') \in \Gamma^n \times \Gamma^n$  and  $\mathfrak{z} = \begin{pmatrix} Z & U \\ {}_tU & W \end{pmatrix} \in \mathfrak{H}_{2n}$ , we get the following commutation relation :

$$((\mathcal{D}F)|_{\rho}(g)_z|_{\rho'}(g')_w)(\mathfrak{z}) = (\mathcal{D}(F|_k(g, g')))(\mathfrak{z}).$$

(ii) The operator  $\mathcal{D}$  sends modular forms to modular forms :

$$\mathcal{D} : M_k^{2n\infty} \rightarrow M_k^n(\text{alt}^{n-1}(V_1))^{\infty} \otimes M_k^n(\text{alt}^{n-1}(V_2))^{\infty}.$$

Moreover,  $\mathcal{D}$  is a holomorphic operator and it satisfies

$$\mathcal{D} : M_k^{2n} \rightarrow M_k^n(\text{alt}^{n-1}(V_1)) \otimes M_k^n(\text{alt}^{n-1}(V_2)).$$

Proof. Let  $X_j$  ( $j=1, 2$ ) be variables on  $\mathbb{C}^{(n, 2k)}$ . If we put

$$\frac{\partial}{\partial U} = X_1 {}^t X_2,$$

the polynomial  $t_1 \widetilde{X_1} {}^t X_2 t_2$  satisfies the three conditions (2. 1), (2. 2), (2. 3).

Therefore we get Proposition 1 by Theorem 4.  $\square$

**3. Proof of Theorem 1**

We prove Theorem 1 according to Böcherer’s method in [5]. We first apply the differential operator  $\mathcal{D}$  to the Eisenstein series  $E_k^{2n}(\mathfrak{z}, s)$ . For this, we use the coset decomposition by Garrett :

**Lemma 1.** (Garrett [9] and Mizumoto [19, Appendix B])

(i) *The double coset  $P_{2n,0} \backslash \Gamma^{2n} / \Gamma^n \times \Gamma^n$  has an irredundant set of coset representatives*

$$g_{\hat{T}} = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & \hat{T}^{(n)} & 1_n & 0 \\ \hat{T}^{(n)} & 0 & 0 & 1_n \end{pmatrix},$$

where  $\hat{T} = \begin{pmatrix} 0 & 0 \\ 0 & T^{(r)} \end{pmatrix}$ ,  $T \in \mathbf{T}^{(r)}$  ( $0 \leq r \leq n$ ).

(ii) *The left coset  $P_{2n,0} \backslash P_{2n,0} g_{\hat{T}} (\Gamma^n \times \Gamma^n)$  has an irredundant set of coset representatives  $g_{\hat{T}} \hat{g}_1 g_2 \hat{g}'_1 g'_2$ ,*

$$\hat{g}_1 \in G_{n,r}, g_2 \in P_{n,r} \backslash \Gamma^n, \hat{g}'_1 \in \Gamma^r(T) \backslash G_{n,r}, g'_2 \in P_{n,r} \backslash \Gamma^n,$$

where

$$G_{n,r} := \left\{ \begin{pmatrix} \hat{A}^{(n)} & \hat{B}^{(n)} \\ \hat{C}^{(n)} & \hat{D}^{(n)} \end{pmatrix} = \begin{pmatrix} 1_{n-r} & 0 & 0 & 0 \\ 0 & A^{(r)} & 0 & B^{(r)} \\ 0 & 0 & 1_{n-r} & 0 \\ 0 & C^{(r)} & 0 & D^{(r)} \end{pmatrix} \in \Gamma^n \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^r \right. \right\}$$

and for  $T \in \mathbf{T}^{(r)}$ ,

$$\Gamma^r(T) := \left\{ g \in \Gamma^r \left| \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^r \right. \right\}.$$

Now we prove the following (cf. Böcherer [4, Satz 9], [5, Satz 3]) :

**Proposition 2.** *Let  $k$  be an even integer,  $n$  an odd integer and  $s$  a complex number such that  $k + 2\text{Re}(s) > 2n + 1$ . Suppose that  $2k \geq n > 2$ . For  $\mathfrak{z} = \begin{pmatrix} Z^{(n)} \\ t U^{(n)} \end{pmatrix}$*

*$U^{(n)} \in \mathfrak{H}_{2n}$ ,  $\mathfrak{z}_0 = \begin{pmatrix} Z^{(n)} & 0 \\ 0 & W^{(n)} \end{pmatrix} \in \mathfrak{H}_{2n}$ , we get*

$$\begin{aligned} & (\mathcal{D} E_k^{2n})(\mathfrak{z}, s) \\ &= \frac{\Gamma(2k + 2s + 1)}{\Gamma(2k + 2s - n + 2)} \sum_{T \in \mathbf{T}^{(n)}} \left( \mathcal{P}(Z, W, s) \left| \begin{pmatrix} \Gamma^n(T & 0 \\ 0 & T^{-1}) \Gamma^n \end{pmatrix}_w \right. \right) \det(T)^{-k-2s} \end{aligned}$$

$$+ \frac{\Gamma(2k+2s+1)}{\Gamma(2k+2s-n+2)} \mathcal{R}(Z, W, s),$$

where

$$\begin{aligned} & \mathcal{P}(Z, W, s) \\ & := \sum_{g \in \Gamma^n} \{ \det(\text{Im}(Z))^s \det(\text{Im}(W))^s |\det(W+Z)|^{-2s} \rho((W+Z)^{-1})(\mathbf{t}_1 \ ^t \mathbf{t}_2) \} |(g)_Z, \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}(Z, W, s) & := \sum_{T \in \mathbb{T}^{(n-1)}} \sum_{g_2 \in P_{n,n-1} \backslash \Gamma^n} \sum_{g'_2 \in P_{n,n-1} \backslash \Gamma^n} \sum_{\hat{g}_1 \in G_{n,n-1}} \sum_{\hat{g}'_1 \in \Gamma^{n-1} \backslash G_{n,n-1}} \\ & \cdot \{ \det(\text{Im}(Z))^s \det(\text{Im}(W))^s |\det(1_n - \hat{T}W\hat{T}Z)|^{-2s} \\ & \cdot \rho((1_n - \hat{T}W\hat{T}Z)^{-1})(\mathbf{t}_1 \ \hat{T} \ ^t \mathbf{t}_2) \} |(\hat{g}'_1)_W |(\hat{g}_1)_Z |(g'_2)_W |(g_2)_Z. \end{aligned}$$

Proof. It follows from Proposition 1 and Lemma 1 that

$$\begin{aligned} (\mathcal{D}E_k^{2n})(\mathfrak{B}, s) & = \sum_{r=0}^n \sum_{T \in \mathbb{T}^{(r)}} \sum_{g_2 \in P_{n,r} \backslash \Gamma^n} \sum_{g'_2 \in P_{n,r} \backslash \Gamma^n} \sum_{\hat{g}_1 \in G_{n,r}} \sum_{\hat{g}'_1 \in \Gamma^r \backslash G_{n,r}} \\ & \{ \mathcal{D}(\det(\text{Im}\mathfrak{B})^s |_{\mathfrak{k}g_{\hat{T}}}) \} |(\hat{g}'_1)_W |(\hat{g}_1)_Z |(g'_2)_W |(g_2)_Z. \end{aligned}$$

If for each  $\hat{T}$  we put  $g_{\hat{T}} = \begin{pmatrix} * & * \\ \mathbb{C}^{(2n)} & \mathbb{D}^{(2n)} \end{pmatrix}$ , we get

$$\mathcal{D}(\det(\text{Im}\mathfrak{B})^s |_{\mathfrak{k}g_{\hat{T}}}) = \det(\mathbb{C}\overline{\mathfrak{B}}_0 + \mathbb{D})^{-s} \mathcal{D}(\det(\mathbb{C}\mathfrak{B} + \mathbb{D})^{-k-s} \det(\text{Im}(\mathfrak{B}))^s),$$

by the form of  $\mathcal{D}$  and that of  $\det(\text{Im}(\mathfrak{B}))$ ,

$$= \det(\mathbb{C}\overline{\mathfrak{B}}_0 + \mathbb{D})^{-s} \det(\text{Im}(\mathfrak{B}_0))^s \mathcal{D}(\det(\mathbb{C}\mathfrak{B} + \mathbb{D})^{-k-s}).$$

As an example, we compute

$$d^* \widetilde{\frac{\partial}{\partial u_{nn}}} (\det(\mathbb{C}\mathfrak{B} + \mathbb{D})^{-k-s}).$$

Let  $\mathfrak{S}_m$  be the symmetric group of degree  $m$ . We put

$$\delta := \det(\mathbb{C}\mathfrak{B} + \mathbb{D}), \delta_0 := \det(\mathbb{C}\overline{\mathfrak{B}}_0 + \mathbb{D}), \partial_{jh} := \frac{\partial}{\partial u_{jh}} \quad (1 \leq j, h \leq n)$$

and, for  $m, q \in \mathbb{Z}, 0 < m$  and  $0 \leq q < m$ ,

$$L_m^q := \left\{ (l_1, \dots, l_m) \in \mathbb{Z}^m \mid l_\nu \geq 0 \ (1 \leq \nu \leq m), \sum_{\nu=1}^m l_\nu = m - q, \sum_{\nu=1}^m \nu l_\nu = m \right\}.$$

For  $(l_1, \dots, l_m) \in L_m^q$ , let  $\Lambda(l_1, \dots, l_m)$  be the set consisting of  $J \in \mathfrak{S}_m$  such that, if  $l_\gamma \neq 0$  ( $1 \leq \gamma \leq m$ ),

$$1 \leq J \left( \sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma \lambda + 1 \right) < \dots < J \left( \sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma \lambda + \gamma \right) \leq m \quad (0 \leq \lambda < l_\gamma)$$

and

$$1 \leq J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + 1\right) < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma + 1\right) < \cdots < J\left(\sum_{\nu=0}^{\gamma-1} \nu l_\nu + \gamma(l_\gamma - 1) + 1\right) \leq m.$$

Then we get

$$\begin{aligned} d^* \widetilde{\partial}_{nn}(\delta^{-k-s}) &= d^* \left( \sum_{\tau \in \mathfrak{S}_{n-1}} \text{sgn}(\tau) \partial_{1\tau(1)} \cdots \partial_{n-1\tau(n-1)} \right) (\delta^{-k-s}) \\ &= \sum_{q=0}^{n-2} \left\{ \left( \prod_{\mu=0}^{n-2-q} (-k-s-\mu) \right) \delta_0^{-k-s-(n-1-q)} \right. \\ &\quad \left. \times d^* \sum_{\tau \in \mathfrak{S}_{n-1}} \sum_{(l_1, \dots, l_{n-1}) \in L_{n-1}^q} \sum_{J \in \Lambda} \text{sgn}(\tau) \partial_{\tau}^q(q; (l_1, \dots, l_{n-1}))(\delta) \right\}, \end{aligned}$$

where  $\Lambda = \Lambda(l_1, \dots, l_{n-1})$  and

$$\begin{aligned} \partial_{\tau}^q(q; (l_1, \dots, l_{n-1}))(\delta) &= \prod_{\gamma=1}^{n-1} \left\{ (\partial_{\tau(J(a^{\gamma+1}))} \cdots \partial_{\tau(J(a^{\gamma+r}))})(\delta) \right. \\ &\quad \times \cdots \\ &\quad \left. \times (\partial_{\tau(J(a^{\gamma+\gamma(l_\gamma-1)+1})} \cdots \partial_{\tau(J(a^{\gamma+1}))})(\delta) \right\} \end{aligned}$$

with  $a^\gamma := \sum_{\nu=0}^{\gamma-1} \nu l_\nu$ ,  $\partial_{\tau(J(\cdot))} := \partial_{J(\cdot)\tau(J(\cdot))}$ .

For each  $q$  ( $0 \leq q \leq n-2$ ),  $(l_1, \dots, l_{n-1}) \in L_{n-1}^q$ ,  $\tau \in \mathfrak{S}_{n-1}$  and  $J \in \Lambda$ , we define

$$(\partial_{J(J)\tau(J(h))}) := \begin{pmatrix} ((A_\tau^J)_{\xi\eta})^{(n-1-q)} & * \\ * & \partial_{nn} \end{pmatrix},$$

where, for  $\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \leq \xi \leq \sum_{\nu=1}^{\gamma} l_\nu$  and  $\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \leq \eta \leq \sum_{\nu=1}^{\gamma} l_\nu$ ,  $(A_\tau^J)_{\xi\eta}$  is a  $\gamma \times \gamma'$  matrix. In the same way, we define

$$(b_{J(J)\tau(J(h))}) := \begin{pmatrix} ((B_\tau^J)_{\xi\eta})^{(n-1-q)} & * \\ * & b_{nn} \end{pmatrix},$$

where  $(\mathfrak{C}\mathfrak{B}_0 + \mathfrak{D})^{-1}\mathfrak{C} = \begin{pmatrix} * & \mathfrak{B}^{(n)} \\ * & * \end{pmatrix}$  and  $\mathfrak{B} = (b_{jh})$ .

Then we have

$$\begin{aligned} & d^* \sum_{\tau \in \mathfrak{S}_{n-1}} \text{sgn}(\tau) \partial_{\tau}^q(q; (l_1, \dots, l_{n-1}))(\delta) \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}/\prod_{\nu=1}^{\gamma-1} \mathfrak{S}_{\nu}^{\nu}} \left\{ \text{sgn}(\sigma) \prod_{\xi=1}^{n-1-q} d^* \det((A_\sigma^J)_{\xi\xi})(\delta) \right\}, \end{aligned}$$

by  $d^* \det((A_\sigma^J)_{\xi\xi})(\delta) = (\gamma+1)! \delta_0 \det((B_\sigma^J)_{\xi\xi})$  ( $\sum_{\nu=1}^{\gamma-1} l_\nu + 1 \leq \xi \leq \sum_{\nu=1}^{\gamma} l_\nu$ ),

$$\begin{aligned} &= \delta_0^{n-1-q} \prod_{\gamma=1}^{n-1} \{(\gamma+1)!\}^{l_\gamma} \sum_{\sigma \in \mathfrak{S}_{n-1}/\prod_{\nu=1}^{\gamma-1} \mathfrak{S}_{\nu}^{\nu}} \left\{ \text{sgn}(\sigma) \prod_{\xi=1}^{n-1-q} \det((B_\sigma^J)_{\xi\xi}) \right\} \\ &= \delta_0^{n-1-q} \widetilde{b}_{nn} \prod_{\gamma=1}^{n-1} \{(\gamma+1)!\}^{l_\gamma}. \end{aligned}$$

Since the number of elements of  $\Lambda$  is  $\left(\prod_{\gamma=1}^{n-1} \frac{1}{l_\gamma!}\right) \frac{(n-1)!}{(1!)^{l_1} \cdots ((n-1)!)^{l_{n-1}}}$ , we obtain

$$(3.1) \quad d^* \widetilde{\partial}_{nm}(\delta^{-k-s}) = (-1)^{n-1} \sum_{q=0}^{n-2} \left\{ a_{n-1}(q) \prod_{\mu=0}^{n-2-q} (2s+2k+2\mu) \right\} \delta_0^{-k-s} \widetilde{b}_{nm},$$

where

$$a_m(q) = (-1)^q 2^{-(m-q)} m! \sum_{(l_1, \dots, l_m) \in L_m^*} \left( \prod_{\gamma=1}^m \frac{(\gamma+1)^{l_\gamma}}{l_\gamma!} \right) \quad (0 < m, 0 \leq q < m).$$

In the same way, we have

$$\begin{aligned} & \mathcal{D}(\det(\mathfrak{C}\mathfrak{B} + \mathfrak{D})^{-k-s}) \\ &= (-1)^{n-1} \sum_{q=0}^{n-2} \left\{ a_{n-1}(q) \prod_{\mu=0}^{n-2-q} (2s+2k+2\mu) \right\} \det(\mathfrak{C}\mathfrak{B}_0 + \mathfrak{D})^{-k-s} (\mathbf{t}_1 \widetilde{\mathfrak{B}}^t \mathbf{t}_2) \end{aligned}$$

and

$$\begin{aligned} \mathcal{D}(\det(\text{Im}\mathfrak{B})^s |_{kg_T}) &= (-1)^{n-1} \sum_{q=0}^{n-2} \left\{ a_{n-1}(q) \prod_{\mu=0}^{n-2-q} (2s+2k+2\mu) \right\} \\ &\quad \times \det(\mathfrak{C}\mathfrak{B}_0 + \mathfrak{D})^{-k} \det(\text{Im}(g_T \langle \mathfrak{B}_0 \rangle))^s (\mathbf{t}_1 \widetilde{\mathfrak{B}}^t \mathbf{t}_2). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} & \det(\mathfrak{C}\mathfrak{B}_0 + \mathfrak{D})^{-k} \det(\text{Im}(g_T \langle \mathfrak{B}_0 \rangle))^s (\mathbf{t}_1 \widetilde{\mathfrak{B}}^t \mathbf{t}_2) \\ &= \det(\text{Im}(Z))^s \det(\text{Im}(W))^s |\det(1_n - \widehat{T}W\widehat{T}Z)|^{-2s} \rho((1_n - \widehat{T}W\widehat{T}Z)^{-1}) (\mathbf{t}_1 \widetilde{\widehat{T}}^t \mathbf{t}_2). \end{aligned}$$

Therefore we have only to prove

$$\sum_{q=0}^{n-2} \left\{ a_{n-1}(q) \prod_{\mu=0}^{n-2-q} (2s+2k+2\mu) \right\} = \prod_{\mu=0}^{n-2} (2s+2k-\mu).$$

To prove the formula above, we put  $x=2s+2k$  and  $m=n-1$ . Then we have to prove

$$(3.2) \quad \sum_{q=0}^{m-1} \left\{ a_m(q) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\} = \prod_{\mu=0}^{m-1} (x-\mu).$$

We put  $a_m(q)=0$  if  $q \geq m$ ,  $0 > q$  or  $0 \geq m$ . We use induction on  $m$ . If  $m=1$ , the assertion is trivial. We suppose

$$\sum_{q=0}^{m'-1} \left\{ a_{m'}(q) \prod_{\mu=0}^{m'-1-q} (x+2\mu) \right\} = \prod_{\mu=0}^{m'-1} (x-\mu).$$

for any  $m' < m$ . Then we have

$$\begin{aligned} \prod_{\mu=0}^{m-1} (x-\mu) &= \left\{ \prod_{\mu=0}^{m-2} (x-\mu) \right\} (x-m+1) \\ &= \left\{ \sum_{q=0}^{m-2} \left\{ a_{m-1}(q) \prod_{\mu=0}^{m-2-q} (x+2\mu) \right\} \right\} (x+(2m-2-2q)-(3m-3-2q)) \\ &= \sum_{q=0}^{m-2} \left\{ a_{m-1}(q) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\} \\ &\quad - \sum_{q=1}^{m-1} \left\{ (3m-2q-1) a_{m-1}(q-1) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\} \end{aligned}$$

$$= \sum_{q=0}^{m-1} \left\{ (a_{m-1}(q) - (3m-2q-1)a_{m-1}(q-1)) \prod_{\mu=0}^{m-1-q} (x+2\mu) \right\}.$$

If we note  $3l_1 + \dots + (m+1)l_{m-1} = 3m - 2q - 1$  in  $L_{m-1}^{q-1}$ , we have

$$\begin{aligned} & a_{m-1}(q) - (3m-2q-1)a_{m-1}(q-1) \\ &= \frac{1}{m} (-1)^q 2^{-(m-q)} m! \sum_{L_{m-1}^q} \left\{ (l_1+1) \frac{2^{l_1+1}}{(l_1+1)!} \left( \prod_{\gamma=2}^{m-1} \frac{(\gamma+1)^{l_\gamma}}{l_\gamma!} \right) \right\} \\ & \quad + \frac{1}{m} (-1)^q 2^{-(m-q)} m! \sum_{L_{m-1}^q} \left\{ \left( \prod_{\gamma=1}^{m-1} \frac{(\gamma+1)^{l_\gamma}}{l_\gamma!} \right) \left( \sum_{\gamma=1}^{m-1} (\gamma+1)(l_{\gamma+1}+1) \frac{\gamma+2}{l_{\gamma+1}+1} \frac{l_\gamma}{\gamma+1} \right) \right\} \\ &= (-1)^q 2^{-(m-q)} m! \sum_{L_{m-1}^q} \left\{ \left( \prod_{\gamma=1}^m \frac{(\gamma+1)^{l_\gamma}}{l_\gamma!} \right) \frac{1}{m} \sum_{\gamma=1}^m \gamma l_\gamma \right\} \\ &= a_m(q). \end{aligned}$$

Thus we get (3.2). □

REMARK. Under the notation above, we note that the formula

$$d^* \partial_{jh} \widetilde{\delta}^{-k-s} = (-1)^{n-1} \prod_{\mu=0}^{n-2} (2s+2k-\mu) \delta_0^{-k-s} \widetilde{b}_{jh}$$

which is obtained from (3.1) and (3.2), and the formula

$$d^* \left( \det \left( \frac{\partial}{\partial U} \right) \right) (\delta^{-k-s}) = (-1)^n \prod_{\mu=0}^{n-1} (2s+2k-\mu) \delta_0^{-k-s-1} \det(\widehat{T})$$

in [4, Satz 9], [5, Satz 3] have the same meaning.

For  $\sum_{j=1}^n a_j t_{n+j}$ ,  $\sum_{j=1}^n b_j t_{n+j} \in \text{alt}^{n-1}(V_2)$ , we define the inner product of them by

$$\left\langle \sum_{j=1}^n a_j t_{n+j}, \sum_{j=1}^n b_j t_{n+j} \right\rangle := \sum_{j=1}^n a_j \bar{b}_j.$$

Suppose  $f, g \in M_k^n(\text{alt}^{n-1}(V_2))^\infty$ . The Petersson inner product of  $f$  and  $g$  is defined by

$$(f, g) := \int_{\Gamma^n \backslash \mathfrak{H}_n} \langle \rho'(\sqrt{\text{Im}(W)}) f(W), \rho'(\sqrt{\text{Im}(W)}) g(W) \rangle \det(\text{Im}(W))^{-n-1} dXdY$$

if the right-hand side is convergent. Here  $W = X + iY$  with real matrices  $X = (x_{jh})$  and  $Y = (y_{jh})$ ;

$$dX := \prod_{j \leq h} dx_{jh}, \quad dY := \prod_{j \leq h} dy_{jh};$$

the integral is taken over a fundamental domain of  $\Gamma^n \backslash \mathfrak{H}_n$ . We write  $dW = dXdY$  when there is no fear of confusion.

**Theorem 5.** *Let  $k$  be an even integer,  $n$  an odd integer and  $2k \geq n > 2$ . If*

$f \in S_k^n(\text{alt}^{n-1}(V_2))$  is an eigenform,

$$\begin{aligned} & \left( f, (\mathcal{D} E_k^{2n}) \left( \begin{pmatrix} -\bar{Z}^{(n)} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s} + n}{2} \right) \right) \\ &= 2\pi^{nk - \frac{1}{2}(n-1)^2} i^{nk+n-1} \gamma(s) \Lambda(s, f, \text{St})(\iota^{-1}(f))(Z). \end{aligned}$$

If Theorem 5 is proved, the functional equation of  $\Lambda(s, f, \text{St})$  is obtained from that of  $E_k^{2n}(\mathfrak{B}, s)$ . Since it follows from Theorem 3 that the location of poles of  $E_k^{2n}(\mathfrak{B}, s)$  is invariant under the operation of  $\mathcal{D}$ , its holomorphy is proved in the same way as that by Mizumoto [19, Theorem 1] (cf. Weissauer [24]). Thus we get Theorem 1.

Proof of Theorem 5. It follows from Theorem 3 that  $\left( f, (\mathcal{D} E_k^{2n}) \left( \begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right)$  converges absolutely and locally uniformly for  $k + 2\text{Re}(s) > 2n + 1$ . We note that  $\mathcal{R}(Z, W, s)$  is orthogonal to  $S_k^n(\text{alt}^{n-1}(V_2))$  in the variable  $W$  by the same reason as that in Klingen [15, Satz 2]. Since the Hecke operators are Hermitian operators and  $f$  is an eigenform, we have

$$\begin{aligned} & \left( f, (\mathcal{D} E_k^{2n}) \left( \begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \bar{s} \right) \right) \\ &= \frac{\Gamma(2k + 2s + 1)}{\Gamma(2k + 2s - n + 2)} D(k + 2s, f)(f, \mathcal{P}(-\bar{Z}, *, \bar{s})) \end{aligned}$$

by the definition (1.2). If we compute the integral  $(f, \mathcal{P}(-\bar{Z}, *, \bar{s}))$  according to Klingen [14, § 1] (see also [5], [7], [23]), we obtain

$$(f, \mathcal{P}(-\bar{Z}, *, \bar{s})) = 2^{n(n-2s-k)+2} i^{nk+n-1} \psi(\iota^{-1}(f))(Z)$$

and

$$\psi = \int_{S^n} \det(1_n - S\bar{S})^{k+s-n-1} \left( (1_n - \widetilde{S\bar{S}}) [{}^t \mathbf{p}_n] \right) dS,$$

where  $\mathbf{p}_n^{(1,n)} := (0, \dots, 0, 1)$  and  $S^n := \{S \in C^{(n)} \mid S = {}^t S, 1_n - \bar{S}S > 0\}$ . Moreover, by Hua [10, § 2.3] (see also [5], [7], [14], [23]), we get

$$\psi = \pi^{\frac{n(n+1)}{2}} \left( \frac{2k + 2s - n + 1}{2} \right) \frac{\Gamma(k + s - n)}{\Gamma(k + s + 1)} \prod_{j=1}^{n-1} \frac{\Gamma(2k + 2s - 2n + 1 + 2j)}{\Gamma(2k + 2s - n + 1 + j)}.$$

Thus, by (1.3), we obtain

$$\begin{aligned} & \left( f, (\mathcal{D} E_k^{2n}) \left( \begin{pmatrix} -\bar{Z} & 0 \\ 0 & * \end{pmatrix}, \frac{\bar{s} + n - k}{2} \right) \right) \\ &= 2^{n(1-s)+2} i^{nk+n-1} \pi^{\frac{n(n+1)}{2}} \zeta(s+n)^{-1} \prod_{j=1}^n \zeta(2s + 2n - 2j)^{-1} \end{aligned}$$



$$\times \frac{\Gamma(s+k)}{\Gamma(s+k-n+1)} \prod_{j=1}^n \frac{\Gamma(s+k-n-2+2j)}{\Gamma(s+k-1+j)} L(s, f, \underline{\text{St}})(\iota^{-1}(f))(Z)$$

and Theorem 5 is proved.  $\square$

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