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ON PATHWISE UNIQUENESS AND COMPARISON OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

SHINTARO NAKAO

(Received March 28, 1981)

1. Introduction

In this paper we shall discuss the pathwise uniqueness and comparison problems for solutions of one-dimensional stochastic differential equations. Let $a(t, x)$ and $b(t, x)$ be bounded Borel functions defined on $[0, \infty) \times R$ with values in $R$. Consider the following one-dimensional stochastic differential equation;

$$
\begin{cases}
    dx(t) = a(t, x(t))dB(t) + b(t, x(t))dt, \\
    x(0) = x_0,
\end{cases}
$$

where $B(t)$ is a one-dimensional Brownian motion with $B(0)=0$ and $x_0 \in R$ is a non-random initial value. In [3], I showed that $a(t, x)=a(x)$ is uniformly positive and of bounded variation on any compact interval and $b(t, x)$ is time independent, then the pathwise uniqueness holds for the equation (1). A. Yu. Veretennikov [5] extended the above result to the case that the coefficients are time dependent. The purpose of this paper is to obtain another extension of the result of [3] different from that of A. Yu. Veretennikov.

$VI([0, \infty) \times R)$ denotes the space of all functions defined on $[0, \infty) \times R$ such that for $t \geq 0$ $f(t, x)$ is nondecreasing in $x$ and for $x \in R$ $f(t, x)$ is of bounded variation in $t$ on any compact interval. Throughout this paper we shall assume that $a(t, x)$ satisfies the following condition.

CONDITION A. $a(t, x)$ satisfies the following conditions;

(i) $a(t, x)$ is Borel measurable and there exist positive constants $a_1$ and $a_2$ such that $0 < a_1 \leq a(t, x) \leq a_2$ for $(t, x) \in [0, \infty) \times R$,

(ii) there exist $\alpha_1(t, x) \in VI([0, \infty) \times R)$ and $\alpha_2(t, x) \in VI([0, \infty) \times R)$ such that $\frac{1}{a(t, x)} = \alpha_1(t, x) - \alpha_2(t, x)$ for a.e. $(t, x) \in [0, \infty) \times R$,

(iii) for $t > 0$ and $N > 0$ there exists a positive constant $L(t, N)$ such that
In this paper we adopt the definitions in [1] about the solution of (1) and the pathwise uniqueness of (1). We obtain the following theorem.

**Theorem 1.** Suppose that \( a(t, x) \) satisfies Condition A and \( b(t, x) \) is bounded Borel measurable. Then the pathwise uniqueness holds for the stochastic differential equation (1).

We now consider the following stochastic differential equations;

\[
\begin{align*}
\text{(2)} & \quad \left\{ \begin{array}{l}
x(t) = a(t, x(t))dt + b_1(t, x(t))dB(t), \\
x(0) = x_0 \in \mathbb{R}
\end{array} \right., \\
\text{(3)} & \quad \left\{ \begin{array}{l}
y(t) = a(t, y(t))dt + b_2(t, y(t))dB(t), \\
y(0) = x_0.
\end{array} \right.
\]

The following comparison theorem is a generalization of a result of [4].

**Theorem 2.** Suppose that \( a(t, x) \), \( b_1(t, x) \) and \( b_2(t, x) \) satisfy the following conditions:

(i) \( a(t, x) \) satisfies Condition A,

(ii) \( b_1(t, x) \) and \( b_2(t, x) \) are bounded Borel functions such that \( b_1(t, x) \leq b_2(t, x) \) for \((t, x) \in [0, \infty) \times \mathbb{R} \) a.e.

Let \((x(t), B(t))\) and \((y(t), B(t))\) be solutions of the stochastic differential equations (2) and (3) respectively defined on a same probability space \((\Omega, \mathcal{F}, P)\) with a reference family \((\mathcal{F}_t)_{t \geq 0}\) such that \(x(0) = y(0) = x_0 \in \mathbb{R}\). Then it holds that \(x(t) \leq y(t)\) a.s. for \(t \geq 0\).

In section 2 we prove Theorem 1 and give an example of \(a(t, x)\) which satisfies Condition A. In section 3 we prove Theorem 2 by a new method.

2. Proof of pathwise uniqueness theorem

First we shall prepare two lemmas for the proof of Theorem 1. Let \((\Omega, \mathcal{F}, P)\) be a probability space with a reference family \((\mathcal{F}_t)_{t \geq 0}\) and let \(B(t)\) be a one-dimensional \((\mathcal{F}_t)\)-Brownian motion defined on \((\Omega, \mathcal{F}, P)\) with \(B(0) = 0\). Consider the stochastic process defined by

\[
x(t) = x_0 + \int_0^t \sigma(s)dB(s) + \int_0^t \gamma(s)ds,
\]

where \(\sigma(s)\) and \(\gamma(s)\) are bounded measurable stochastic processes on \((\Omega, \mathcal{F}, P)\).

1) Let \(f(s)\) be a real function defined on \([0, \infty)\). \(\|f\|\) denotes the total variation of \(f(s)\) on \([0, t]\).
adapted to $(\mathcal{F}_t)$ and $x_0$ is a real number. Set $\sigma = \sup_{(t, \omega)} |\sigma(t, \omega)|$ and $\gamma = \sup_{(t, \omega)} |\gamma(t, \omega)|$. For $N > 0$, $\tau_N = \inf \{ t; |x(t)| \geq N \}$. Let $g(t, x)$ be a Lebesgue measurable function defined on $[0, \infty) \times \mathbb{R}$. Setting

$$G(t, x) = \int_0^t g(t, y) dy \quad \text{for} \quad (t, x) \in [0, \infty) \times \mathbb{R}$$

and

$$V(t) = G(t, x(t)) - G(0, x_0) - \int_0^t g(s, x(s)) \sigma(s) dB(s),$$

we shall estimate the expectation of $|||V|||_{t \wedge \tau_N}^2$.

**Lemma 1.** Suppose that $g(t, x)$ belongs to $VI([0, \infty) \times \mathbb{R})$ and is continuously differentiable in $(t, x)$. Then it holds that for $t > 0$ and $N > 0$

$$E[|||V|||_{t \wedge \tau_N}^2] \leq 2(N + t\gamma)M(t, N) + 4NK(t, N),$$

where $E$ denotes the expectation with respect to $P$, $M(t, N) = \sup \{ \|g(s, y)\|; (s, y) \in [0, t] \times [-N, N] \}$ and $K(t, N) = \sup \{ \|g(\cdot, y)\|_{t \wedge \tau_N}; y \in [-N, N] \}$.

**Proof.** Itô’s formula implies that

$$V(t) = \int_0^t g(s, x(s)) \gamma(s) ds + \int_0^t \frac{\partial}{\partial s} G(s, x(s)) ds + \frac{1}{2} \int_0^t \frac{\partial}{\partial x} g(s, x(s)) \sigma(s)^2 ds = I_1(t) + I_2(t) + I_3(t).$$

It is easy to see that $E[|||I_1|||_{t \wedge \tau_N}] \leq t\gamma M(t, N)$ and $E[|||I_2|||_{t \wedge \tau_N}] \leq 2NK(t, N)$. Since $|||I_3|||_{t \wedge \tau_N} = V(t \wedge \tau_N) - I_1(t \wedge \tau_N) - I_2(t \wedge \tau_N)$, we have $E[|||I_3|||_{t \wedge \tau_N}] \leq E[V(t \wedge \tau_N)] + t\gamma M(t, N) + 2NK(t, N)$. On the other hand it holds that $E[V(t \wedge \tau_N)] = E[G(t \wedge \tau_N, x(t \wedge \tau_N)) - G(0, x_0)] \leq 2NM(t, N)$. Combining the above estimates, we have $E[|||V|||_{t \wedge \tau_N}] \leq 2(N + t\gamma)M(t, N) + 4NK(t, N)$, which completes the proof.

Let $\rho(s, y)$ be a non-negative $C^\infty$-function defined on $\mathbb{R}^2$ such that its support is contained in the closed unit ball and $\int_{\mathbb{R}^2} \rho(s, y) ds dy = 1$. For $\delta > 0$ set

$$\rho_{\delta}(s, y) = \frac{1}{\delta^2} \rho \left( \frac{s}{\delta}, \frac{y}{\delta} \right).$$

We now consider

$$V_{\delta}(t) = G_{\delta}(t, x(t)) - G_{\delta}(0, x_0) - \int_0^t g_{\delta}(s, x(s)) \sigma(s) dB(s),$$

2) Let $a$ and $b$ be real numbers. $a \wedge b = \min\{a, b\}$. 


where
\[ g_s = g*p_\delta \quad \text{and} \quad G_s(t, x) = \int_0^x g_s(t, y) dy. \]

**Lemma 2.** Suppose that \( g(t, x) \in \mathcal{V}(\mathbb{R}^+) \) satisfies that for \( t > 0 \) and \( N > 0 \) there exists a positive constant \( K(t, N) \) such that \( |||g(\cdot, x)|||_t \leq K(t, N) \) for \( x \in [-N, N] \). Then it holds that for \( 0 < \delta \leq 1, \ t > 0 \) and \( N > 0 \)
\[
E[||V||^2_{t \wedge x}] \leq 2(N + t\gamma)M(t+1, N+1) + 4NK(t+1, N+1),
\]
where
\[
M(t, N) = \sup \{ |g(s, y)| ; (s, y) \in [0, t] \times [-N, N] \}.
\]

**Proof.** It is easy to see that \( |||g_s(\cdot, x)|||_t \leq K(t+\delta, N+\delta) \) for \( x \in [-N, N] \) and \( \sup \{ |g(s, y)| ; (s, y) \in [0, t] \times [-N, N] \} \leq M(t+\delta, N+\delta) \). Hence Lemma 2 is an easy consequence of Lemma 1.

**Proof of Theorem 1.** Let \( a_0 = 1 > a_1 > a_2 > \cdots > a_k > \cdots \to 0 \) be a sequence such that \( \frac{1}{a_k} u = ku = k \) for \( k = 1, 2, \ldots \). Then there exists a twice continuously differentiable and odd function \( \psi_k(u) \) on \( \mathbb{R} \) such that \( 0 < \psi_k(u) \leq 1 \) for \( u \in [0, \infty) \),
\[
\psi_k(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq a_k, \\ 1 & \text{for } a_k \leq u. \end{cases}
\]
and
\[
0 \leq \psi^{(2)}_k(u) \leq \frac{1}{ku} \quad \text{for } a_k < u < a_{k-1}.
\]

Set \( \alpha(t, x) = \alpha_1(t, x) - \alpha_2(t, x) \), \( \alpha_2 = \alpha*\rho_\delta \), and \( h_0(t, x) = \int_0^x \alpha_2(t, y) dy \), where \( \rho_\delta \) is the function defined by (4).

Let \( (x(t), B(t)) \) and \( (y(t), B(t)) \) be solutions of (1) defined on a same quadruplet \( (\Omega, \mathcal{F}, P, (\mathcal{F}_t)) \). Set \( \mathcal{N} = \{ t ; |x(t)| \geq N \text{ or } |y(t)| \geq N \} \). Theorem 2 of N. V. Krylov [2] assures that for \( k = 1, 2, \ldots \) there exists a positive constant \( \delta_k = \delta_k(t, N) \leq \frac{1}{k} \) such that
\[
\max_{s \in \mathcal{N}} |\psi_k^{(i)}(u)| E \left[ \int_0^{t \wedge \mathcal{N}} |a \cdot \alpha_\delta(s, x(s)) - 1|^i ds \right] \leq \frac{1}{k} \quad \text{for } i = 1, 2.
\]

Obviously the same estimates as (6) hold for \( (y(t)) \). For simplicity we set
\[ h_k = h_\delta, \quad \alpha_k = \alpha_\delta, \quad z_k(t) = h_k(t, x(t)) - h_k(t, y(t)) \quad \text{and} \quad J(k, t) = (x(t) - y(t))\psi_k(z_k(t)). \]

3) For a function \( g(t, x) \) defined on \([0, \infty) \times \mathbb{R} \), \( g(t, x) \) denotes the function on \( \mathbb{R} \times \mathbb{R} \) such that
\[
g(t, x) = \begin{cases} g(t, x) & t \geq 0, \\ g(0, x) & t < 0. \end{cases}
\]
\( g*\rho_\delta \) denotes the convolution of \( g \) and \( \rho_\delta \).

4) \( f^{(i)}(u) \) denotes the i-th derivative of \( f(u) \).
The martingale part $m_k(t)$ of $z_k(t)$ is $\int_0^t (a \cdot \partial_k(s, x(s)) - a \cdot \partial_k(s, y(s))) dB(s)$. Setting $v_k(t) = z_k(t) - m_k(t)$, we have by Itô’s formula

$$J(k, t) = \int_0^t \psi_k(x(s))d(x-y)(s) + \int_0^t (x(s) - y(s))\psi_k^{(1)}(z_k(s)) d\nu_k(s) + \int_0^t \psi_k^{(1)}(z_k(s))(a(s, x(s)) - a(s, y(s)))d\nu_k(s) + \frac{1}{2} \int_0^t (x(s) - y(s))\psi_k^{(2)}(z_k(s))(a \cdot \partial_k(s, x(s)) - a \cdot \partial_k(s, y(s)))^2 ds$$

$$= J_1(k, t) + J_2(k, t) + J_3(k, t) + J_4(k, t) + J_5(k, t).$$

Using that

$$0 < \frac{x-y}{h_k(t, x) - h_k(t, y)} \leq a_2 \text{ for } t \geq 0, x \neq y \text{ and } \delta > 0$$

and

$$\lim_{k \to \infty} \psi_k(u) = \chi(u) = \begin{cases} 
-1 & \text{for } u < 0 \\
0 & \text{for } u = 0 \\
1 & \text{for } u > 0, 
\end{cases}$$

it is easy to see that

$$J(k, t \wedge \eta_N) \xrightarrow{k \to \infty} \|x(t \wedge \eta_N) - y(t \wedge \eta_N)\| \text{ in } L^1(P)$$

and

$$J_1(k, t \wedge \eta_N) \xrightarrow{k \to \infty} \int_0^{t \wedge \eta_N} \chi(x(s) - y(s))d(x-y)(s) \text{ in } L^1(P).$$

By (5) and (7) we obtain

$$E[|J_2(k, t \wedge \eta_N)|] \leq (\frac{2a_2^2}{k} E\left[\int_0^{t \wedge \eta_N} (a \cdot \partial_k(s, x(s)) - a \cdot \partial_k(s, y(s)))^2 ds \right] \leq \frac{(a_2^2}{a_4 k}$$

and

$$E[|J_4(k, t \wedge \eta_N)|] \leq \frac{2a_2^2}{k} E[\|v_k\|_{t \wedge \eta_N}].$$

Since supremum $E[\|v_k\|_{t \wedge \eta_N}]$ is finite by Lemma 2, we have $\lim_{k \to \infty} E[|J_2(k, t \wedge \eta_N) + J_3(k, t \wedge \eta_N)|] = 0$. (6) implies that $\lim_{k \to \infty} E[|J_4(k, t \wedge \eta_N) + J_5(k, t \wedge \eta_N)|] = 0$. Consequently we have

$$|x(t \wedge \eta_N) - y(t \wedge \eta_N)| = \int_0^{t \wedge \eta_N} \chi(x(s) - y(s))d(x-y)(s).$$
Letting $N \to \infty$ it holds that

\begin{equation}
|x(t) - y(t)| = \int_0^t \chi(x(s) - y(s))d(x - y)(s).
\end{equation}

(9) implies that

\begin{align*}
(x(t) \wedge y(t)) & = \frac{1}{2} \{x(t) + y(t) - |x(t) - y(t)|\} \\
& = x_0 + \int_0^t \frac{1}{2} \{a(s, x(s)) + a(s, y(s)) - \chi(x(s) - y(s))(a(s, x(s)) - a(s, y(s)))\} dB(s) \\
& \quad + \int_0^t \frac{1}{2} \{b(s, x(s)) + b(s, y(s)) - \chi(x(s) - y(s))(b(s, x(s)) - b(s, y(s)))\} ds \\
& = x_0 + \int_0^t a(s, x(s) \wedge y(s))dB(s) + \int_0^t b(s, x(s) \wedge y(s))ds.
\end{align*}

In the same way max $\{x(t), y(t)\}$ is a solution of (1). Since the uniqueness in law holds for (1), we conclude $x(t) = y(t)$ a.s. The proof is completed.

**Remark.** Let $a(t, x)$ be a uniformly positive and bounded Borel function and let $b(t, x)$ be a bounded Borel function. Set $h(t, x) = \int_0^t \frac{1}{a(t, y)}dy$. Suppose that there exists a solution $(\bar{x}(t), \bar{B}(t))$ with $\bar{x}(t) = x_0 + \int_0^t a(s, \bar{x}(s))d\bar{B}(s)$ such that $h(t, \bar{x}(t)) - h(0, x_0)$ is a continuous quasimartingale and the martingale part of $h(t, \bar{x}(t)) - h(0, x_0)$ is the one-dimensional Brownian motion $\bar{B}(t)$. Let $(x_1(t), B(t))$ and $(x_2(t), B(t))$ be solutions defined on a same quadruplet $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that

\begin{align*}
x_i(t) = x_0 + \int_0^t a(s, x_i(s))dB(s) + \int_0^t b(s, x_i(s))ds \quad i = 1, 2.
\end{align*}

Then it holds that $x_1(t) = x_2(t)$ a.s. for $t \geq 0$.

**Proof.** By the assumption the sample paths of $h(t, x_1(t)) - h(t, x_2(t))$ are continuous and of bounded variation on any compact interval with probability one. Let $\psi_k(u) (k=1, 2, \ldots)$ be the function defined in the proof of Theorem 1. Ito’s formula implies

\begin{align*}
(x_1(t) - x_2(t))\psi_4(h(t, x_1(t)) - h(t, x_2(t))) \\
= \int_0^t \psi_4(h(s, x_1(s)) - h(s, x_2(s)))d(x_1(s) - x_2(s)) \\
\quad + \int_0^t (x_1(s) - x_2(s))\psi_4^{(1)}(h(s, x_1(s)) - h(s, x_2(s)))d(h(s, x_1(s)) - h(s, x_2(s))).
\end{align*}
Letting $k \to \infty$ we have
\[ |x_1(t) - x_2(t)| = \int_0^t \chi(x_1(s) - x_2(s))d(x_1 - x_2)(s), \]
which implies the conclusion of Remark.

Finally we state an example of $a(t, x)$ which satisfies Condition A.

**Example.** Let $f(t)$ be a continuous function defined on $[0, \infty)$. For $t > 0$ and $c \in \mathbb{R}$, $n(t, c)$ denotes the number of the connected components of \{s \in (0, t); f(s) < c\}. Define
\[
a(t, x) = \begin{cases} 
2 & \text{for } x \leq f(t) \\
1 & \text{for } x > f(t).
\end{cases}
\]
If $\sup_{t \in \mathbb{R}} n(t, c)$ is finite for $t > 0$ and $N > 0$, then $a(t, x)$ satisfies Condition A. But this example does not satisfy those sufficient conditions in the preceding papers [1], [3], [5].

### 3. Proof of comparison theorem

Let $W_x$ be the space of all continuous functions $w$ defined on $[0, \infty)$ with values in $\mathbb{R}$ such that $w(0) = x \in \mathbb{R}$. $\mathcal{B}(W_x)$ denotes the $\sigma$-field generated by $w(s)$ $0 \leq s \leq t$ and $P^w$ denotes the Wiener measure on $W_x$. Let $\overline{\mathcal{B}}(W_x)$ be the completion of $\mathcal{B}(W_x)$ with respect to $P^w$.

**Proof of Theorem 2.** Fix an initial value $x_0 \in \mathbb{R}$. If the pathwise uniqueness holds for the stochastic differential equation (1), then there exists a unique function $F(w)$ defined on $W_x$ with values in $W_{x_0}$ such that

(i) $F(w)$ is $\mathcal{B}(W_x)/\mathcal{B}(W_{x_0})$-measurable for each $t \geq 0$,

(ii) any solution $(x(t), B(t))$ of (1) with $x(0) = x_0$ can be represented in the form $x(\cdot) = F(B(\cdot))$ a.s. (cf. [1]).

Let $F_1(w)$ and $F_2(w)$ be the above functions for the stochastic differential equations (2) and (3) respectively. It is sufficient to prove that $F_1(w)*F_2(w) \leq F_2(w)$ a.s. ($P^w$).

Set $a^i = a^i \ast \rho_{1/k}$ and $b^i = b^i \ast \rho_{1/k}$ ($i = 1, 2$), where $\rho_\delta$ is the mollifier defined by (4). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a reference family $(\mathcal{F}_t)$ such that there exists a one-dimensional $(\mathcal{F}_t)$-Brownian motion $B(t)$ with $B(0) = 0$. Obviously there exist solutions $(x_k(t), B(t))$ and $(y_k(t), B(t))$ defined on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that for $k = 1, 2, \ldots$

\[ x_k(t) = x_0 + \int_0^t a^s(s, x_k(s))dB(s) + \int_0^t b^s(s, x_k(s))ds \]

and

5) For $w_1, w_2 \in W, w_1 \leq w_2$ means that $w_1(t) \leq w_2(t)$ for each $t \geq 0$. 

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\[ \text{Proof of Theorem 2.} \quad \text{Fix a initial value } x_0 \in \mathbb{R}. \quad \text{If the pathwise uniqueness holds for the stochastic differential equation (1), then there exists a unique function } F(w) \text{ defined on } W_x \text{ with values in } W_{x_0}, \text{such that}\]

(i) $F(w)$ is $\mathcal{B}(W_x)/\mathcal{B}(W_{x_0})$-measurable for each $t \geq 0$,

(ii) any solution $(x(t), B(t))$ of (1) with $x(0) = x_0$ can be represented in the form $x(\cdot) = F(B(\cdot))$ a.s. (cf. [1]).

Let $F_1(w)$ and $F_2(w)$ be the above functions for the stochastic differential equations (2) and (3) respectively. It is sufficient to prove that $F_1(w)*F_2(w) \leq F_2(w)$ a.s. ($P^w$).

Set $a^i = a^i \ast \rho_{1/k}$ and $b^i = b^i \ast \rho_{1/k}$ ($i = 1, 2$), where $\rho_\delta$ is the mollifier defined by (4). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a reference family $(\mathcal{F}_t)$ such that there exists a one-dimensional $(\mathcal{F}_t)$-Brownian motion $B(t)$ with $B(0) = 0$. Obviously there exist solutions $(x_k(t), B(t))$ and $(y_k(t), B(t))$ defined on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that for $k = 1, 2, \ldots$

\[ x_k(t) = x_0 + \int_0^t a^s(s, x_k(s))dB(s) + \int_0^t b^s(s, x_k(s))ds \]

and

5) For $w_1, w_2 \in W, w_1 \leq w_2$ means that $w_1(t) \leq w_2(t)$ for each $t \geq 0$. 

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\[
y_{k}(t) = x_{0} + \int_{0}^{t} a^{k}(s, y_{k}(s))dB(s) + \int_{0}^{t} b_{2}^{k}(s, y_{k}(s))ds.
\]

Since the family of the laws \( P^{2k} \) of \( Z_{k}(t) = (x_{k}(t), y_{k}(t), B(t)) \) (\( k=1, 2, \ldots \)) is tight, there exist a subsequence \((k_{n})\) and a sequence of stochastic process \((x_{k_{n}}(t), y_{k_{n}}(t), B_{k_{n}}(t))\) defined on a probability space \((\Omega, \mathcal{F}, P)\) satisfying the following conditions;

(i) for each \( k_{n} \) the law of \((x_{k_{n}}(t), y_{k_{n}}(t), B_{k_{n}}(t))\) is \( P^{2k_{n}} \),

(ii) there exists a stochastic process \((x(t), y(t), B(t))\) defined on \((\Omega, \mathcal{F}, P)\) such that \((x_{k_{n}}(t), y_{k_{n}}(t), B_{k_{n}}(t))\) converges to \((x(t), y(t), B(t))\) uniformly on each compact interval a.s.

Since \( b_{1}^{t}(t, x) \leq b_{2}^{t}(t, x) \), it holds that \( x_{k}(t) \leq y_{k}(t) \) a.s. for \( t \geq 0 \) and \( k=k_{1}, k_{2}, \ldots \) (cf. [1]). Noting that \((x(t), B(t))\) and \((y(t), B(t))\) are solutions of (2) and (3) respectively, we have \( F_{1}(B(\cdot)) = x(\cdot) \leq y(\cdot) = F_{2}(B(\cdot)) \) a.s. \((P^{w})\). Therefore we conclude \( F_{1}(w) \leq F_{2}(w) \) a.s. \((P^{w})\). The proof is completed.

The above method can be applicable for the following general case.

**Remark.** Let \( a(t, x) \) be a uniformly positive bounded Borel function on \([0, \infty) \times \mathbb{R}\). Let \( b_{1}(t, x) \) and \( b_{2}(t, x) \) be bounded Borel functions such that \( b_{1}(t, x) \leq b_{2}(t, x) \) for \((t, x) \in [0, \infty) \times \mathbb{R}\) a.e. If the pathwise uniqueness holds for the equations (2) and (3), then the conclusion of Theorem 2 holds.

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**References**


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