<table>
<thead>
<tr>
<th>Title</th>
<th>On pathwise uniqueness and comparison of solutions of one-dimensional stochastic differential equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Nakao, Shintaro</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 20(1) P.197–P.204</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1983</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6944">https://doi.org/10.18910/6944</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/6944</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
<tr>
<td>Note</td>
<td></td>
</tr>
</tbody>
</table>
ON PATHWISE UNIQUENESS AND COMPARISON OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

SHINTARO NAKAO

(Received March 28, 1981)

1. Introduction

In this paper we shall discuss the pathwise uniqueness and comparison problems for solutions of one-dimensional stochastic differential equations. Let \( a(t, x) \) and \( b(t, x) \) be bounded Borel functions defined on \([0, \infty) \times \mathbb{R}\) with values in \(\mathbb{R}\). Consider the following one-dimensional stochastic differential equation:

\[
\begin{cases}
    dx(t) = a(t, x(t))dB(t) + b(t, x(t))dt, \\
    x(0) = x_0,
\end{cases}
\]

where \(B(t)\) is a one-dimensional Brownian motion with \(B(0)=0\) and \(x_0 \in \mathbb{R}\) is a non-random initial value. In [3], I showed that \(a(t, x) = a(x)\) is uniformly positive and of bounded variation on any compact interval and \(b(t, x)\) is time independent, then the pathwise uniqueness holds for the equation (1). A. Yu. Veretennikov [5] extended the above result to the case that the coefficients are time dependent. The purpose of this paper is to obtain another extension of the result of [3] different from that of A. Yu. Veretennikov.

\(VI([0, \infty) \times \mathbb{R})\) denotes the space of all functions defined on \([0, \infty) \times \mathbb{R}\) such that for \(t \geq 0\) \(f(t, x)\) is nondecreasing in \(x\) and for \(x \in \mathbb{R}\) \(f(t, x)\) is of bounded variation in \(t\) on any compact interval. Throughout this paper we shall assume that \(a(t, x)\) satisfies the following condition.

CONDITION A. \(a(t, x)\) satisfies the following conditions:

(i) \(a(t, x)\) is Borel measurable and there exist positive constants \(a_1\) and \(a_2\) such that \(0 < a_1 \leq a(t, x) \leq a_2\) for \((t, x) \in [0, \infty) \times \mathbb{R}\),

(ii) there exist \(\alpha_1(t, x) \in VI([0, \infty) \times \mathbb{R})\) and \(\alpha_2(t, x) \in VI([0, \infty) \times \mathbb{R})\) such that

\[
\frac{1}{a(t, x)} = \alpha_1(t, x) - \alpha_2(t, x)
\]

for a.e. \((t, x) \in [0, \infty) \times \mathbb{R}\),

(iii) for \(t > 0\) and \(N > 0\) there exists a positive constant \(L(t, N)\) such that
Theorem 1. Suppose that $a(t, x)$ satisfies Condition A and $b(t, x)$ is bounded Borel measurable. Then the pathwise uniqueness holds for the stochastic differential equation (1).

We now consider the following stochastic differential equations:

\begin{align}
\text{(2)} & \quad \left\{ \begin{array}{l}
\, \, dx(t) = a(t, x(t))dB(t) + b_1(t, x(t))dt, \\
\, \, x(0) = x_0 \in \mathbb{R}
\end{array} \right. \\
\text{(3)} & \quad \left\{ \begin{array}{l}
\, \, dy(t) = a(t, y(t))dB(t) + b_2(t, y(t))dt, \\
\, \, y(0) = x_0.
\end{array} \right.
\end{align}

The following comparison theorem is a generalization of a result of [4].

Theorem 2. Suppose that $a(t, x)$, $b_1(t, x)$ and $b_2(t, x)$ satisfy the following conditions:

(i) $a(t, x)$ satisfies Condition A,

(ii) $b_1(t, x)$ and $b_2(t, x)$ are bounded Borel functions such that $b_1(t, x) \leq b_2(t, x)$ for $(t, x) \in [0, \infty) \times \mathbb{R}$ a.e.

Let $(x(t), B(t))$ and $(y(t), B(t))$ be solutions of the stochastic differential equations (2) and (3) respectively defined on a same probability space $(\Omega, \mathcal{F}, P)$ with a reference family $(\mathcal{F}_t)_{t \geq 0}$ such that $x(0) = y(0) = x_0 \in \mathbb{R}$. Then it holds that $x(t) \leq y(t)$ a.s. for $t \geq 0$.

In section 2 we prove Theorem 1 and give an example of $a(t, x)$ which satisfies Condition A. In section 3 we prove Theorem 2 by a new method.

2. Proof of pathwise uniqueness theorem

First we shall prepare two lemmas for the proof of Theorem 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a reference family $(\mathcal{F}_t)_{t \geq 0}$ and let $B(t)$ be a one-dimensional $(\mathcal{F}_t)$-Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with $B(0) = 0$. Consider the stochastic process defined by

$$x(t) = x_0 + \int_0^t \sigma(s)dB(s) + \int_0^t \gamma(s)ds,$$

where $\sigma(s)$ and $\gamma(s)$ are bounded measurable stochastic processes on $(\Omega, \mathcal{F}, P)$.

1) Let $f(s)$ be a real function defined on $[0, \infty)$. $\|f\|_1$ denotes the total variation of $f(s)$ on $[0, t]$. 
adapted to \((\mathcal{F}_t)\) and \(x_0\) is a real number. Set \(\sigma = \sup_{t \geq 0} |\sigma(t, \omega)|\) and \(\gamma = \sup_{(t, \omega)} |\gamma(t, \omega)|\). For \(N > 0\), \(\tau_N = \inf \{t; |x(t)| \geq N\}\). Let \(g(t, x)\) be a Lebesgue measurable function defined on \([0, \infty) \times \mathbb{R}\). Setting

\[
G(t, x) = \int_0^t g(t, y) dy \quad \text{for} \quad (t, x) \in [0, \infty) \times \mathbb{R}
\]

and

\[
V(t) = G(t, x(t)) - G(0, x_0) - \int_0^t g(s, x(s)) \sigma(s) dB(s),
\]

we shall estimate the expectation of \(||V||_{t \wedge \tau_N}^2\).

**Lemma 1.** Suppose that \(g(t, x)\) belongs to \(VI([0, \infty) \times \mathbb{R})\) and is continuously differentiable in \((t, x)\). Then it holds that for \(t > 0\) and \(N > 0\)

\[
E[||V||_{t \wedge \tau_N}] \leq 2(N + t \gamma)M(t, N) + 4NK(t, N),
\]

where \(E\) denotes the expectation with respect to \(P\),

\[
M(t, N) = \sup \{||g(s, y)||; (s, y) \in [0, t] \times [-N, N]\}
\]

and

\[
K(t, N) = \sup \{||g(\cdot, y)||; y \in [-N, N]\}.
\]

**Proof.** Itō’s formula implies that

\[
V(t) = \int_0^t g(s, x(s)) \gamma(s) ds + \int_0^t \frac{\partial}{\partial s} G(s, x(s)) ds + \frac{1}{2} \int_0^t \frac{\partial}{\partial x} g(s, x(s)) \sigma(s)^2 ds
\]

\[
= I_1(t) + I_2(t) + I_3(t).
\]

It is easy to see that \(E[||I_1||_{t \wedge \tau_N}] \leq t \gamma M(t, N)\) and \(E[||I_2||_{t \wedge \tau_N}] \leq 2NK(t, N)\). Since \(||I_3||_{t \wedge \tau_N} = V(t \wedge \tau_N) - I_1(t \wedge \tau_N) - I_2(t \wedge \tau_N)\), we have \(E[||I_3||_{t \wedge \tau_N}] \leq E[V(t \wedge \tau_N)] + t \gamma M(t, N) + 2NK(t, N)\). On the other hand it holds that \(E[V(t \wedge \tau_N)] = E[G(t \wedge \tau_N, x(t \wedge \tau_N)) - G(0, x_0)] \leq 2NM(t, N)\). Combining the above estimates, we have \(E[||V||_{t \wedge \tau_N}] \leq 2(N + t \gamma)M(t, N) + 4NK(t, N)\), which completes the proof.

Let \(\rho(s, y)\) be a non-negative \(C^\infty\)-function defined on \(\mathbb{R}^2\) such that its support is contained in the closed unit ball and \(\int_{\mathbb{R}^2} \rho(s, y) ds dy = 1\). For \(\delta > 0\) set

\[
\rho_\delta(s, y) = \frac{1}{\delta^2} \rho \left( \frac{s}{\delta}, \frac{y}{\delta} \right).
\]

We now consider

\[
V_\delta(t) = G_\delta(t, x(t)) - G_\delta(0, x_0) - \int_0^t g_\delta(s, x(s)) \sigma(s) dB(s),
\]

2) Let \(a\) and \(b\) be real numbers. \(a \wedge b = \min\{a, b\}\).
Lemma 2. Suppose that \( g(t, x) \in V_I([0, \infty) \times \mathbb{R}) \) satisfies that for \( t > 0 \) and \( N > 0 \) there exists a positive constant \( K(t, N) \) such that \( \| g(\cdot, x) \|_t \leq K(t, N) \) for \( x \in [-N, N] \). Then it holds that for \( 0 < \delta \leq 1 \), \( t > 0 \) and \( N > 0 \)

\[
E[\| V \|_{t \wedge N}] \leq 2(N + t\gamma)M(t + 1, N + 1) + 4NK(t + 1, N + 1),
\]

where

\[
M(t, N) = \sup \{ |g(s, y)| ; (s, y) \in [0, t] \times [-N, N] \}.
\]

Proof. It is easy to see that \( \| g(s, \cdot) \|_{t \wedge N} \leq K(t + \delta, N + \delta) \) for \( x \in [-N, N] \) and \( \sup \{ |g(s, y)| ; (s, y) \in [0, t] \times [-N, N] \} \leq M(t + \delta, N + \delta) \). Hence Lemma 2 is an easy consequence of Lemma 1.

Proof of Theorem 1. Let \( a_0 = 1 > a_1 > a_2 > \cdots > a_k > \cdots \to 0 \) be a sequence such that \( \int_{a_k}^{a_{k-1}} \frac{1}{u} du = k \) for \( k = 1, 2, \cdots \). Then there exists a twice continuously differentiable and odd function \( \psi_k(u) \) on \( \mathbb{R} \) such that \( 0 \leq \psi_k(u) \leq 1 \) for \( u \in [0, \infty) \),

\[
\psi_k(u) = \begin{cases} 
0 & \text{for } 0 \leq u \leq a_k \\
1 & \text{for } a_{k-1} \leq u,
\end{cases}
\]

and

\[
0 \leq \psi^{(1)}(u) \leq \frac{2}{k} \quad \text{for } a_k < u < a_{k-1}.
\]

Set \( \alpha(t, x) = \alpha_1(t, x) - \alpha_2(t, x) + \cdots - \alpha_k(t, x) \), \( \alpha_\delta = \alpha_\delta P_\delta \) and \( h_\delta(t, x) = \int_0^t \alpha_\delta(t, y)dy \), where \( P_\delta \) is the function defined by (4).

Let \( (x(t), B(t)) \) and \( (y(t), B(t)) \) be solutions of (1) defined on a same quadruplet \( (\Omega, \mathcal{F}, P, (\mathcal{F}_t)) \). Set \( \tau_N = \{ t ; |x(t)| \geq N \text{ or } |y(t)| \geq N \} \). Theorem 2 of N. V. Krylov [2] assures that for \( k = 1, 2, \cdots \) there exists a positive constant \( \delta_k = \delta_k(t, N) \leq \frac{1}{k} \) such that

\[
\max_{u \in \mathbb{R}} |\psi_k^{(i)}(u)| E \left[ \int_0^{t \wedge N} |a \cdot \alpha_\delta(s, x(s)) - 1|^i ds \right] \leq \frac{1}{k} \quad \text{for } i = 1, 2.
\]

Obviously the same estimates as (6) hold for \( (y(t)) \). For simplicity we set

\[
h_k = h_k \delta, \quad \alpha_k = \alpha_k \delta, \quad z_k(t) = h_k(t, x(t)) - h_k(t, y(t)) \text{ and } J(k, t) = (x(t) - y(t)) \psi_k(z_k(t)).
\]

3) For a function \( g(t, x) \) defined on \([0, \infty) \times \mathbb{R}, g(t, x) \) denotes the function on \( \mathbb{R} \times \mathbb{R} \) such that \( g(t, x) = g(t, x) \), \( g*\rho_\delta \) denotes the convolution of \( g \) and \( \rho_\delta \).

4) \( f^{(i)}(u) \) denotes the i-th derivative of \( f(u) \).
The martingale part \( m_k(t) \) of \( z_k(t) \) is \( \int_0^t (a \cdot \bar{\alpha}_k(s, x(s)) - a \cdot \bar{\alpha}_k(s, y(s))) dB(s) \). Setting \( v_k(t) = z_k(t) - m_k(t) \), we have by Itô's formula

\[
J(k, t) = \int_0^t \psi_k(x_k(s))d(x - y)(s) + \int_0^t (x(s) - y(s))\psi_k^{(1)}(z_k(s))dm_k(s) \\
+ \frac{1}{2} \int_0^t (x(s) - y(s))\psi_k^{(2)}(z_k(s))(a \cdot \bar{\alpha}_k(s, x(s)) - a \cdot \bar{\alpha}_k(s, y(s)))ds
\]

Using that

\[
(7) \quad 0 < \frac{x - y}{h_k(t, x) - h_k(t, y)} \leq a_2 \quad \text{for} \quad t \geq 0, \quad x \neq y \quad \text{and} \quad \delta > 0
\]

and

\[
(8) \quad \lim_{k \to \infty} \psi_k(u) = \chi(u) = \begin{cases} 
-1 & \text{for} \quad u < 0 \\
0 & \text{for} \quad u = 0 \\
1 & \text{for} \quad u > 0
\end{cases}
\]

it is easy to see that

\[
J(k, t \wedge \eta_N) \xrightarrow{k \to \infty} |x(t \wedge \eta_N) - y(t \wedge \eta_N)| \quad \text{in} \quad L^1(P)
\]

and

\[
J_3(k, t \wedge \eta_N) \xrightarrow{k \to \infty} \int_0^{t \wedge \eta_N} \chi(x(s) - y(s))d(x - y)(s) \quad \text{in} \quad L^1(P).
\]

By (5) and (7) we obtain

\[
E[|J_3(k, t \wedge \eta_N)|] \leq \left( \frac{2a_2}{k} \right)^2 E \left[ \int_0^{t \wedge \eta_N} (a \cdot \bar{\alpha}_k(s, x(s)) - a \cdot \bar{\alpha}_k(s, y(s)))^2 ds \right] \leq 8 \left( \frac{a_2}{a_1 k} \right)^2
\]

and

\[
E[|J_4(k, t \wedge \eta_N)|] \leq \frac{2a_2}{k} E[||v_k||_{t \wedge \eta_N}].
\]

Since \( \sup_k E[||v_k||_{t \wedge \eta_N}] \) is finite by Lemma 2, we have \( \lim_{k \to \infty} E[|J_3(k, t \wedge \eta_N) + J_3(k, t \wedge \eta_N)|] = 0 \). (6) implies that \( \lim_{k \to \infty} E[|J_4(k, t \wedge \eta_N) + J_4(k, t \wedge \eta_N)|] = 0 \). Consequently we have

\[
|x(t \wedge \eta_N) - y(t \wedge \eta_N)| = \int_0^{t \wedge \eta_N} \chi(x(s) - y(s))d(x - y)(s).
\]
Letting $N \to \infty$ it holds that

\[(9) \quad |x(t) - y(t)| = \int_0^t \mathcal{X}(x(s) - y(s))d(x-y)(s) .\]

(9) implies that

\[
x(t) \wedge y(t) \quad = \quad \frac{1}{2} \{x(t) + y(t) - |x(t) - y(t)|\}
\]
\[
= \quad x_0 + \int_0^t \frac{1}{2} \{a(s, x(s)) + a(s, y(s)) - \mathcal{X}(x(s) - y(s))(a(s, x(s)) - a(s, y(s)))\} dB(s)
\]
\[
+ \int_0^t \frac{1}{2} \{b(s, x(s)) + b(s, y(s)) - \mathcal{X}(x(s) - y(s))(b(s, x(s)) - b(s, y(s)))\} ds
\]
\[
= \quad x_0 + \int_0^t a(s, x(s) \wedge y(s))dB(s) + \int_0^t b(s, x(s) \wedge y(s))ds .
\]

In the same way $\max \{x(t), y(t)\}$ is a solution of (1). Since the uniqueness in law holds for (1), we conclude $x(t) = y(t)$ a.s. The proof is completed.

REMARK. Let $a(t, x)$ be a uniformly positive and bounded Borel function and let $b(t, x)$ be a bounded Borel function. Set $h(t, x) = \int_0^t \frac{1}{a(t, y)}dy$. Suppose that there exists a solution $(\mathfrak{x}(t), \mathfrak{B}(t))$ with $\mathfrak{x}(t) = x_0 + \int_0^t a(s, \mathfrak{x}(s))dB(s)$ such that $h(t, \mathfrak{x}(t)) - h(0, x_0)$ is a continuous quasimartingale and the martingale part of $h(t, \mathfrak{x}(t)) - h(0, x_0)$ is the one-dimensional Brownian motion $\mathfrak{B}(t)$. Let $(x_1(t), \mathfrak{B}(t))$ and $(x_2(t), \mathfrak{B}(t))$ be solutions defined on a same quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ such that

\[
x_i(t) = x_0 + \int_0^t a(s, x_i(s))dB(s) + \int_0^t b(s, x_i(s))ds \quad i = 1, 2 .
\]

Then it holds that $x_1(t) = x_2(t)$ a.s. for $t \geq 0$.

Proof. By the assumption the sample paths of $h(t, x_1(t)) - h(t, x_2(t))$ are continuous and of bounded variation on any compact interval with probability one. Let $\psi_k(u) (k = 1, 2, \ldots)$ be the function defined in the proof of Theorem 1. Ito’s formula implies

\[
(x_1(t) - x_2(t))\psi_k(h(t, x_1(t)) - h(t, x_2(t)))
\]
\[
= \int_0^t \psi_k(h(s, x_1(s)) - h(s, x_2(s)))d(x_1 - x_2)(s)
\]
\[
+ \int_0^t (x_1(s) - x_2(s))\psi_k^{[1]}(h(s, x_1(s)) - h(s, x_2(s)))d(h(s, x_1(s)) - h(s, x_2(s))) .
\]
Letting $k \to \infty$ we have

$$|x_1(t) - x_2(t)| = \int_0^t \chi(x_1(s) - x_2(s))d(x_1 - x_2)(s),$$

which implies the conclusion of Remark.

Finally we state an example of $a(t, x)$ which satisfies Condition A.

**Example.** Let $f(t)$ be a continuous function defined on $[0, \infty)$. For $t > 0$ and $c \in \mathbb{R}$, $n(t, c)$ denotes the number of the connected components of \{s$\in(0, t); f(s) < c$\}. Define

$$a(t, x) = \begin{cases} 2 & \text{for } x \leq f(t) \\
1 & \text{for } x > f(t). \end{cases}$$

If $\sup_{c \in [-\infty, x]} n(t, c)$ is finite for $t > 0$ and $N > 0$, then $a(t, x)$ satisfies Condition A. But this example does not satisfy those sufficient conditions in the preceding papers [1], [3], [5].

3. **Proof of comparison theorem**

Let $W_x$ be the space of all continuous functions $w$ defined on $[0, \infty)$ with values in $\mathbb{R}$ such that $w(0) = x \in \mathbb{R}$. $\mathcal{B}_t(W_x)$ denotes the $\sigma$-field generated by $w(s)$ $0 \leq s \leq t$ and $P^w$ denotes the Wiener measure on $W_0$. Let $\overline{\mathcal{B}}_t(W_0)$ be the completion of $\mathcal{B}_t(W_0)$ with respect to $P^w$.

Proof of Theorem 2. Fix a initial value $x_0 \in \mathbb{R}$. If the pathwise uniqueness holds for the stochastic differential equation (1), then there exists a unique function $F(w)$ defined on $W_0$ with values in $W_{x_0}$ such that

(i) $F(w)$ is $\mathcal{B}_t(W_{x_0})$-measurable for each $t \geq 0$,

(ii) any solution $(x(t), B(t))$ of (1) with $x(0) = x_0$ can be represented in the form $x(\cdot) = F(B(\cdot))$ a.s. (cf. [1]).

Let $F_1(w)$ and $F_2(w)$ be the above functions for the stochastic differential equations (2) and (3) respectively. It is sufficient to prove that $F_1(w) \preceq F_2(w)$ a.s. ($P^w$).

Set $a^i = a^i \ast \rho_{1/k}$ and $b^i = b^i \ast \rho_{1/k}$ ($i = 1, 2$), where $\rho_8$ is the mollifier defined by (4). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a reference family $(\mathcal{F}_t)$ such that there exists a one-dimensional $(\mathcal{F}_t)$-Brownian motion $B(t)$ with $B(0) = 0$. Obviously there exist solutions $(x_i(t), B(t))$ and $(y_i(t), B(t))$ defined on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that for $k = 1, 2, \ldots$

$$x_k(t) = x_0 + \int_0^t a^i(s, x_k(s))dB(s) + \int_0^t b^i(s, x_k(s))ds$$

and

5) For $w_1, w_2 \in W$, $w_1 \leq w_2$ means that $w_1(t) \leq w_2(t)$ for each $t \geq 0$. 


\[ y_k(t) = x_0 + \int_0^t a^k(s, y_k(s)) dB(s) + \int_0^t b^k_2(s, y_k(s)) ds. \]

Since the family of the laws \( P_{x_k} \) of \( Z_k(t) = (x_k(t), y_k(t), B(t)) \) \( (k = 1, 2, \ldots) \) is tight, there exist a subsequence \( (k_n) \) and a sequence of stochastic process \( (x_{k_n}(t), y_{k_n}(t), B_{k_n}(t)) \) defined on a probability space \( (\Omega, \mathcal{F}, P) \) satisfying the following conditions;

(i) for each \( k_n \) the law of \( (x_{k_n}(t), y_{k_n}(t), B_{k_n}(t)) \) is \( P_{x_{k_n}} \),

(ii) there exists a stochastic process \( (x(t), y(t), B(t)) \) defined on \( (\Omega, \mathcal{F}, P) \) such that \( (x_{k_n}(t), y_{k_n}(t), B_{k_n}(t)) \) converges to \( (x(t), y(t), B(t)) \) uniformly on each compact interval a.s.

Since \( b^1_1(t, x) \leq b^2_1(t, x) \), it holds that \( x_k(t) \leq y_k(t) \) a.s. for \( t \geq 0 \) and \( k = k_1, k_2, \ldots \) (cf. [1]). Noting that \( (x(t), B(t)) \) and \( (y(t), B(t)) \) are solutions of (2) and (3) respectively, we have \( F_1(B(\cdot)) = x(\cdot) \leq y(\cdot) = F_2(B(\cdot)) \) a.s. \( (P) \). Therefore we conclude \( F_1(w) \leq F_2(w) \) a.s. \( (P^w) \). The proof is completed.

The above method can be applicable for the following general case.

**Remark.** Let \( a(t, x) \) be a uniformly positive bounded Borel function on \([0, \infty) \times \mathbb{R}\). Let \( b_1(t, x) \) and \( b_2(t, x) \) be bounded Borel functions such that \( b_1(t, x) \leq b_2(t, x) \) for \((t, x) \in [0, \infty) \times \mathbb{R}\) a.e. If the pathwise uniqueness holds for the equations (2) and (3), then the conclusion of Theorem 2 holds.

**References**


