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ON PATHWISE UNIQUENESS AND COMPARISON OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper we shall discuss the pathwise uniqueness and comparison problems for solutions of one-dimensional stochastic differential equations. Let $a(t, x)$ and $b(t, x)$ be bounded Borel functions defined on $[0, \infty) \times R$ with values in R . Consider the following one-dimensional stochastic differential equation;

$$(1) \quad \begin{cases} dx(t) = a(t, x(t))dB(t) + b(t, x(t))dt, \\ x(0) = x_0, \end{cases}$$

where $B(t)$ is a one-dimensional Brownian motion with $B(0)=0$ and $x_0 \in R$ is a non-random initial value. In [3], I showed that $a(t, x)=a(x)$ is uniformly positive and of bounded variation on any compact interval and $b(t, x)$ is time independent, then the pathwise uniqueness holds for the equation (1). A. Yu. Veretennikov [5] extended the above result to the case that the coefficients are time dependent. The purpose of this paper is to obtain another extension of the result of [3] different from that of A. Yu. Veretennikov.

$VI([0, \infty) \times R)$ denotes the space of all functions defined on $[0, \infty) \times R$ such that for $t \geq 0$ $f(t, x)$ is nondecreasing in x and for $x \in R$ $f(t, x)$ is of bounded variation in t on any compact interval. Throughout this paper we shall assume that $a(t, x)$ satisfies the following condition.

CONDITION A. $a(t, x)$ satisfies the following conditions;

- (i) $a(t, x)$ is Borel measurable and there exist positive constants a_1 and a_2 such that $0 < a_1 \leq a(t, x) \leq a_2$ for $(t, x) \in [0, \infty) \times R$,
- (ii) there exist $\alpha_1(t, x) \in VI([0, \infty) \times R)$ and $\alpha_2(t, x) \in VI([0, \infty) \times R)$ such that $\frac{1}{a(t, x)} = \alpha_1(t, x) - \alpha_2(t, x)$ for a.e. $(t, x) \in [0, \infty) \times R$,
- (iii) for $t > 0$ and $N > 0$ there exists a positive constant $L(t, N)$ such that

$|||\alpha_i(\cdot, x)|||_t^{*1} \leq L(t, N)$ for $x \in [-N, N]$ and $i=1, 2$.

In this paper we adopt the definitions in [1] about the solution of (1) and the pathwise uniqueness of (1). We obtain the following theorem.

Theorem 1. *Suppose that $a(t, x)$ satisfies Condition A and $b(t, x)$ is bounded Borel measurable. Then the pathwise uniqueness holds for the stochastic differential equation (1).*

We now consider the following stochastic differential equations;

$$(2) \quad \begin{cases} dx(t) = a(t, x(t))dB(t) + b_1(t, x(t))dt, \\ x(0) = x_0 \in R \end{cases}$$

and

$$(3) \quad \begin{cases} dy(t) = a(t, y(t))dB(t) + b_2(t, y(t))dt, \\ y(0) = x_0. \end{cases}$$

The following comparison theorem is a generalization of a result of [4].

Theorem 2. *Suppose that $a(t, x)$, $b_1(t, x)$ and $b_2(t, x)$ satisfy the following conditions ;*

(i) $a(t, x)$ satisfies Condition A,

(ii) $b_1(t, x)$ and $b_2(t, x)$ are bounded Borel functions such that $b_1(t, x) \leq b_2(t, x)$

for $(t, x) \in [0, \infty) \times R$ a.e.

Let $(x(t), B(t))$ and $(y(t), B(t))$ be solutions of the stochastic differential equations (2) and (3) respectively defined on a same probability space (Ω, \mathcal{F}, P) with a reference family $(\mathcal{F}_t)_{t \geq 0}$ such that $x(0) = y(0) = x_0 \in R$. Then it holds that $x(t) \leq y(t)$ a.s. for $t \geq 0$.

In section 2 we prove Theorem 1 and give an example of $a(t, x)$ which satisfies Condition A. In section 3 we prove Theorem 2 by a new method.

2. Proof of pathwise uniqueness theorem

First we shall prepare two lemmas for the proof of Theorem 1. Let (Ω, \mathcal{F}, P) be a probability space with a reference family $(\mathcal{F}_t)_{t \geq 0}$ and let $B(t)$ be a one-dimensional (\mathcal{F}_t) -Brownian motion defined on (Ω, \mathcal{F}, P) with $B(0) = 0$. Consider the stochastic process defined by

$$x(t) = x_0 + \int_0^t \sigma(s)dB(s) + \int_0^t \gamma(s)ds,$$

where $\sigma(s)$ and $\gamma(s)$ are bounded measurable stochastic processes on (Ω, \mathcal{F}, P)

1) Let $f(s)$ be a real function defined on $[0, \infty)$. $|||f|||_t$ denotes the total variation of $f(s)$ on $[0, t]$.

adapted to (\mathcal{F}_t) and x_0 is a real number. Set $\sigma = \sup_{(t,\omega)} |\sigma(t, \omega)|$ and $\gamma = \sup_{(t,\omega)} |\gamma(t, \omega)|$. For $N > 0$, $\tau_N = \inf \{t; |x(t)| \geq N\}$. Let $g(t, x)$ be a Lebesgue measurable function defined on $[0, \infty) \times R$. Setting

$$G(t, x) = \int_0^x g(t, y) dy \quad \text{for } (t, x) \in [0, \infty) \times R$$

and

$$V(t) = G(t, x(t)) - G(0, x_0) - \int_0^t g(s, x(s)) \sigma(s) dB(s),$$

we shall estimate the expectation of $|||V|||_{t \wedge \tau_N}^{*2})$.

Lemma 1. *Suppose that $g(t, x)$ belongs to $VI([0, \infty) \times R)$ and is continuously differentiable in (t, x) . Then it holds that for $t > 0$ and $N > 0$*

$$E[|||V|||_{t \wedge \tau_N}] \leq 2(N + t\gamma)M(t, N) + 4NK(t, N),$$

where E denotes the expectation with respect to P ,

$$M(t, N) = \sup \{|g(s, y)|; (s, y) \in [0, t] \times [-N, N]\}$$

and

$$K(t, N) = \sup \{|||g(\cdot, y)|||_t; y \in [-N, N]\}.$$

Proof. Itô's formula implies that

$$\begin{aligned} V(t) &= \int_0^t g(s, x(s)) \gamma(s) ds + \int_0^t \frac{\partial}{\partial s} G(s, x(s)) ds + \frac{1}{2} \int_0^t \frac{\partial}{\partial x} g(s, x(s)) \sigma(s)^2 ds \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

It is easy to see that $E[|||I_1|||_{t \wedge \tau_N}] \leq t\gamma M(t, N)$ and $E[|||I_2|||_{t \wedge \tau_N}] \leq 2NK(t, N)$. Since $|||I_3|||_{t \wedge \tau_N} = V(t \wedge \tau_N) - I_1(t \wedge \tau_N) - I_2(t \wedge \tau_N)$, we have $E[|||I_3|||_{t \wedge \tau_N}] \leq E[V(t \wedge \tau_N)] + t\gamma M(t, N) + 2NK(t, N)$. On the other hand it holds that $E[V(t \wedge \tau_N)] = E[G(t \wedge \tau_N, x(t \wedge \tau_N)) - G(0, x_0)] \leq 2NM(t, N)$. Combining the above estimates, we have $E[|||V|||_{t \wedge \tau_N}] \leq 2(N + t\gamma)M(t, N) + 4NK(t, N)$, which completes the proof.

Let $\rho(s, y)$ be a non-negative C^∞ -function defined on R^2 such that its support is contained in the closed unit ball and $\int_{R^2} \rho(s, y) ds dy = 1$. For $\delta > 0$ set

$$(4) \quad \rho_\delta(s, y) = \frac{1}{\delta^2} \rho\left(\frac{s}{\delta}, \frac{y}{\delta}\right).$$

We now consider

$$V_\delta(t) = G_\delta(t, x(t)) - G_\delta(0, x_0) - \int_0^t g_\delta(s, x(s)) \sigma(s) dB(s),$$

2) Let a and b be real numbers. $a \wedge b = \min\{a, b\}$.

where

$$g_\delta = \bar{g} * \rho_\delta^{*3}) \quad \text{and} \quad G_\delta(t, x) = \int_0^x g_\delta(t, y) dy.$$

Lemma 2. Suppose that $g(t, x) \in VI([0, \infty) \times R)$ satisfies that for $t > 0$ and $N > 0$ there exists a positive constant $K(t, N)$ such that $\|g(\cdot, x)\|_t \leq K(t, N)$ for $x \in [-N, N]$. Then it holds that for $0 < \delta \leq 1$, $t > 0$ and $N > 0$

$$E[\|V\|_{t \wedge \tau_N}] \leq 2(N + t\gamma)M(t+1, N+1) + 4NK(t+1, N+1),$$

where

$$M(t, N) = \sup \{ |g(s, y)|; (s, y) \in [0, t] \times [-N, N] \}.$$

Proof. It is easy to see that $\|g_\delta(\cdot, x)\|_t \leq K(t + \delta, N + \delta)$ for $x \in [-N, N]$ and $\sup \{ |g_\delta(s, y)|; (s, y) \in [0, t] \times [-N, N] \} \leq M(t + \delta, N + \delta)$. Hence Lemma 2 is an easy consequence of Lemma 1.

Proof of Theorem 1. Let $a_0 = 1 > a_1 > a_2 > \dots > a_k > \dots \rightarrow 0$ be a sequence such that $\int_{a_k}^{a_{k-1}} \frac{1}{u} du = k$ for $k = 1, 2, \dots$. Then there exists a twice continuously differentiable and odd function $\psi_k(u)$ on R such that $0 \leq \psi_k(u) \leq 1$ for $u \in [0, \infty)$,

$$\psi_k(u) = \begin{cases} 0 & \text{for } 0 \leq u \leq a_k \\ 1 & \text{for } a_{k-1} \leq u, \end{cases}$$

and

$$(5) \quad 0 \leq \psi_k^{(1)}(u)^{*4} \leq \frac{2}{ku} \quad \text{for } a_k < u < a_{k-1}.$$

Set $\alpha(t, x) = \alpha_1(t, x) - \alpha_2(t, x)$, $\alpha_\delta = \bar{\alpha} * \rho_\delta$ and $h_\delta(t, x) = \int_0^x \alpha_\delta(t, y) dy$, where ρ_δ is the function defined by (4).

Let $(x(t), B(t))$ and $(y(t), B(t))$ be solutions of (1) defined on a same quadruplet $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Set $\eta_N = \{t; |x(t)| \geq N \text{ or } |y(t)| \geq N\}$. Theorem 2 of N. V. Krylov [2] assures that for $k = 1, 2, \dots$ there exists a positive constant $\delta_k = \delta_k(t, N) \leq \frac{1}{k}$ such that

$$(6) \quad \max_{u \in R} |\psi_k^{(i)}(u)| E \left[\int_0^{t \wedge \eta_N} |a \cdot \alpha_{\delta_k}(s, x(s)) - 1|^i ds \right] \leq \frac{1}{k} \quad \text{for } i = 1, 2.$$

Obviously the same estimates as (6) hold for $(y(t))$. For simplicity we set $h_k = h_{\delta_k}$, $\bar{\alpha}_k = \alpha_{\delta_k}$, $z_k(t) = h_k(t, x(t)) - h_k(t, y(t))$ and $J(k, t) = (x(t) - y(t))\psi_k(z_k(t))$.

3) For a function $g(t, x)$ defined on $[0, \infty) \times R$, $\bar{g}(t, x)$ denotes the function on $R \times R$ such that

$$\bar{g}(t, x) = \begin{cases} g(t, x) & t \geq 0 \\ g(0, x) & t < 0. \end{cases} \quad \bar{g} * \rho_\delta \text{ denotes the convolution of } \bar{g} \text{ and } \rho_\delta.$$

4) $f^{(i)}(u)$ denotes the i -th derivative of $f(u)$.

The martingale part $m_k(t)$ of $z_k(t)$ is $\int_0^t (a \cdot \tilde{\alpha}_k(s, x(s)) - a \cdot \tilde{\alpha}_k(s, y(s))) dB(s)$. Setting $v_k(t) = z_k(t) - m_k(t)$, we have by Itô's formula

$$\begin{aligned} J(k, t) &= \int_0^t \psi_k(z_k(s)) d(x-y)(s) + \int_0^t (x(s) - y(s)) \psi_k^{(1)}(z_k(s)) dm_k(s) \\ &\quad + \int_0^t (x(s) - y(s)) \psi_k^{(1)}(z_k(s)) dv_k(s) \\ &\quad + \int_0^t \psi_k^{(1)}(z_k(s)) (a(s, x(s)) - a(s, y(s))) (a \cdot \tilde{\alpha}_k(s, x(s)) - a \cdot \tilde{\alpha}_k(s, y(s))) ds \\ &\quad + \frac{1}{2} \int_0^t (x(s) - y(s)) \psi_k^{(2)}(z_k(s)) (a \cdot \tilde{\alpha}_k(s, x(s)) - a \cdot \tilde{\alpha}_k(s, y(s)))^2 ds \\ &= J_1(k, t) + J_2(k, t) + J_3(k, t) + J_4(k, t) + J_5(k, t). \end{aligned}$$

Using that

$$(7) \quad 0 < \frac{x-y}{h_\delta(t, x) - h_\delta(t, y)} \leq a_2 \quad \text{for } t \geq 0, x \neq y \text{ and } \delta > 0$$

and

$$(8) \quad \lim_{k \rightarrow \infty} \psi_k(u) = \chi(u) = \begin{cases} -1 & \text{for } u < 0 \\ 0 & \text{for } u = 0 \\ 1 & \text{for } u > 0, \end{cases}$$

it is easy to see that

$$J(k, t \wedge \eta_N) \xrightarrow[k \rightarrow \infty]{} |x(t \wedge \eta_N) - y(t \wedge \eta_N)| \quad \text{in } L^1(P)$$

and

$$J_1(k, t \wedge \eta_N) \xrightarrow[k \rightarrow \infty]{} \int_0^{t \wedge \eta_N} \chi(x(s) - y(s)) d(x-y)(s) \quad \text{in } L^1(P).$$

By (5) and (7) we obtain

$$E[J_2(k, t \wedge \eta_N)^2] \leq \left(\frac{2a_2}{k}\right)^2 E\left[\int_0^{t \wedge \eta_N} (a \cdot \tilde{\alpha}_k(s, x(s)) - a \cdot \tilde{\alpha}_k(s, y(s)))^2 ds\right] \leq 8 \left(\frac{a_2}{a_1 k}\right)^2$$

and

$$E[|J_4(k, t \wedge \eta_N)|] \leq \frac{2a_2}{k} E[|||v_k|||_{t \wedge \eta_N}].$$

Since $\sup_k E[|||v_k|||_{t \wedge \eta_N}]$ is finite by Lemma 2, we have $\lim_{k \rightarrow \infty} E[|J_2(k, t \wedge \eta_N) + J_3(k, t \wedge \eta_N)|] = 0$. (6) implies that $\lim_{k \rightarrow \infty} E[|J_4(k, t \wedge \eta_N) + J_5(k, t \wedge \eta_N)|] = 0$. Consequently we have

$$|x(t \wedge \eta_N) - y(t \wedge \eta_N)| = \int_0^{t \wedge \eta_N} \chi(x(s) - y(s)) d(x-y)(s).$$

Letting $N \rightarrow \infty$ it holds that

$$(9) \quad |x(t) - y(t)| = \int_0^t \chi(x(s) - y(s)) d(x - y)(s).$$

(9) implies that

$$\begin{aligned} & x(t) \wedge y(t) \\ &= \frac{1}{2} \{x(t) + y(t) - |x(t) - y(t)|\} \\ &= x_0 + \int_0^t \frac{1}{2} \{a(s, x(s)) + a(s, y(s)) - \chi(x(s) - y(s))(a(s, x(s)) - a(s, y(s)))\} dB(s) \\ &\quad + \int_0^t \frac{1}{2} \{b(s, x(s)) + b(s, y(s)) - \chi(x(s) - y(s))(b(s, x(s)) - b(s, y(s)))\} ds \\ &= x_0 + \int_0^t a(s, x(s) \wedge y(s)) dB(s) + \int_0^t b(s, x(s) \wedge y(s)) ds. \end{aligned}$$

In the same way $\max\{x(t), y(t)\}$ is a solution of (1). Since the uniqueness in law holds for (1), we conclude $x(t) = y(t)$ a.s. The proof is completed.

REMARK. Let $a(t, x)$ be a uniformly positive and bounded Borel function and let $b(t, x)$ be a bounded Borel function. Set $h(t, x) = \int_0^x \frac{1}{a(t, y)} dy$. Suppose that there exists a solution $(\tilde{x}(t), \tilde{B}(t))$ with $\tilde{x}(t) = x_0 + \int_0^t a(s, \tilde{x}(s)) d\tilde{B}(s)$ such that $h(t, \tilde{x}(t)) - h(0, x_0)$ is a continuous quasimartingale and the martingale part of $h(t, \tilde{x}(t)) - h(0, x_0)$ is the one-dimensional Brownian motion $\tilde{B}(t)$. Let $(x_1(t), B(t))$ and $(x_2(t), B(t))$ be solutions defined on a same quadruplet $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that

$$x_i(t) = x_0 + \int_0^t a(s, x_i(s)) dB(s) + \int_0^t b(s, x_i(s)) ds \quad i = 1, 2.$$

Then it holds that $x_1(t) = x_2(t)$ a.s. for $t \geq 0$.

Proof. By the assumption the sample paths of $h(t, x_1(t)) - h(t, x_2(t))$ are continuous and of bounded variation on any compact interval with probability one. Let $\psi_k(u)$ ($k=1, 2, \dots$) be the function defined in the proof of Theorem 1. Itô's formula implies

$$\begin{aligned} & (x_1(t) - x_2(t)) \psi_k(h(t, x_1(t)) - h(t, x_2(t))) \\ &= \int_0^t \psi_k(h(s, x_1(s)) - h(s, x_2(s))) d(x_1 - x_2)(s) \\ &\quad + \int_0^t (x_1(s) - x_2(s)) \psi_k^{(1)}(h(s, x_1(s)) - h(s, x_2(s))) d(h(s, x_1(s)) - h(s, x_2(s))). \end{aligned}$$

Letting $k \rightarrow \infty$ we have

$$|x_1(t) - x_2(t)| = \int_0^t \chi(x_1(s) - x_2(s)) d(x_1 - x_2)(s),$$

which implies the conclusion of Remark.

Finally we state an example of $a(t, x)$ which satisfies Condition A.

EXAMPLE. Let $f(t)$ be a continuous function defined on $[0, \infty)$. For $t > 0$ and $c \in R$, $n(t, c)$ denotes the number of the connected components of $\{s \in (0, t); f(s) < c\}$. Define

$$a(t, x) = \begin{cases} 2 & \text{for } x \leq f(t) \\ 1 & \text{for } x > f(t). \end{cases}$$

If $\sup_{c \in [-N, N]} n(t, c)$ is finite for $t > 0$ and $N > 0$, then $a(t, x)$ satisfies Condition A. But this example does not satisfy those sufficient conditions in the preceding papers [1], [3], [5].

3. Proof of comparison theorem

Let W_x be the space of all continuous functions w defined on $[0, \infty)$ with values in R such that $w(0) = x \in R$. $\mathcal{B}_t(W_x)$ denotes the σ -field generated by $w(s)$ $0 \leq s \leq t$ and P^w denotes the Wiener measure on W_0 . Let $\overline{\mathcal{B}}_t(W_0)$ be the completion of $\mathcal{B}_t(W_0)$ with respect to P^w .

Proof of Theorem 2. Fix a initial value $x_0 \in R$. If the pathwise uniqueness holds for the stochastic differential equation (1), then there exists a unique function $F(w)$ defined on W_0 with values in W_{x_0} such that

- (i) $F(w)$ is $\overline{\mathcal{B}}_t(W_0)/\mathcal{B}_t(W_{x_0})$ -measurable for each $t \geq 0$,
- (ii) any solution $(x(t), B(t))$ of (1) with $x(0) = x_0$ can be represented in the form $x(\cdot) = F(B(\cdot))$ a.s. (cf. [1]).

Let $F_1(w)$ and $F_2(w)$ be the above functions for the stochastic differential equations (2) and (3) respectively. It is sufficient to prove that $F_1(w)^{*5} \leq F_2(w)$ a.s. (P^w).

Set $a^k = \bar{a} * \rho_{1/k}$ and $b_i^k = \bar{b}_i * \rho_{1/k}$ ($i = 1, 2$), where ρ_δ is the mollifier defined by (4). Let (Ω, \mathcal{F}, P) be a probability space with a reference family (\mathcal{F}_t) such that there exists a one-dimensional (\mathcal{F}_t) -Brownian motion $B(t)$ with $B(0) = 0$. Obviously there exist solutions $(x_k(t), B(t))$ and $(y_k(t), B(t))$ defined on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that for $k = 1, 2, \dots$

$$x_k(t) = x_0 + \int_0^t a^k(s, x_k(s)) dB(s) + \int_0^t b_1^k(s, x_k(s)) ds$$

and

5) For $w_1, w_2 \in W$, $w_1 \leq w_2$ means that $w_1(t) \leq w_2(t)$ for each $t \geq 0$.

$$y_k(t) = x_0 + \int_0^t a^k(s, y_k(s))dB(s) + \int_0^t b_2^k(s, y_k(s))ds.$$

Since the family of the laws P^{Z_k} of $Z_k(t) = (x_k(t), y_k(t), B(t))$ ($k=1, 2, \dots$) is tight, there exist a subsequence (k_n) and a sequence of stochastic process $(\bar{x}_{k_n}(t), \bar{y}_{k_n}(t), \bar{B}_{k_n}(t))$ defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ satisfying the following conditions;

- (i) for each k_n the law of $(\bar{x}_{k_n}(t), \bar{y}_{k_n}(t), \bar{B}_{k_n}(t))$ is $P^{Z_{k_n}}$,
- (ii) there exists a stochastic process $(\bar{x}(t), \bar{y}(t), \bar{B}(t))$ defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ such that $(\bar{x}_{k_n}(t), \bar{y}_{k_n}(t), \bar{B}_{k_n}(t))$ converges to $(\bar{x}(t), \bar{y}(t), \bar{B}(t))$ uniformly on each compact interval a.s.

Since $b_1^k(t, x) \leq b_2^k(t, x)$, it holds that $\bar{x}_k(t) \leq \bar{y}_k(t)$ a.s. for $t \geq 0$ and $k = k_1, k_2, \dots$ (cf. [1]). Noting that $(\bar{x}(t), \bar{B}(t))$ and $(\bar{y}(t), \bar{B}(t))$ are solutions of (2) and (3) respectively, we have $F_1(\bar{B}(\cdot)) = \bar{x}(\cdot) \leq \bar{y}(\cdot) = F_2(\bar{B}(\cdot))$ a.s. (\bar{P}). Therefore we conclude $F_1(w) \leq F_2(w)$ a.s. (P^w). The proof is completed.

The above method can be applicable for the following general case.

REMARK. Let $a(t, x)$ be a uniformly positive bounded Borel function on $[0, \infty) \times R$. Let $b_1(t, x)$ and $b_2(t, x)$ be bounded Borel functions such that $b_1(t, x) \leq b_2(t, x)$ for $(t, x) \in [0, \infty) \times R$ a.e. If the pathwise uniqueness holds for the equations (2) and (3), then the conclusion of Theorem 2 holds.

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