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ON THE SMOOTHING PROBLEM AND THE SIZE OF A TOPOLOGICAL MANIFOLD

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0. Introduction

In this paper which is a direct continuation of [S], we treat the smoothing problem of a compact topological manifold.

We define in §3 a size $|X/d|$ of a compact topological manifold X relative to a distance function d on X. As is seen in §4, it is rather easy to see that if X admits a smooth structure and if d is a Riemannian metric relative to the structure, then $|X/d|=0$.

Our main result is the converse of this fact, that is, if $|X/d|$ is sufficiently small, then X admits a smoothing and d is approximated by a Riemannian metric on the smoothing in the sense of Lipschitz ratio. (See Theorem 1)

We call a smoothing σ of X compatible in the strong sense with its distance d, if σ admits a Riemannian metric whose Lipschitz ratio to d is less than $|X/d|$ (see §3). Then, by Part II of [S], it is easy to see that if $|X/d|$ is sufficiently small, any two compatible smoothings are differentiably equivalent. Therefore we conclude as follows:

"If $|X/d|$ is sufficiently small, then X admits unique smoothing which is compatible with its distance d."

Finally we define the absolute size $|X|$ of X by

$$X=\inf\{|X/d| \ d\text{: } \textit{distance function on } X\},$$

to get a criterion for the existence of a smoothing on X:

$$\text{"}|X|=0 \Leftrightarrow X \text{ is smoothable."}$$

1. Modification of ϕ-average

We start with the following lemma which might be well known;

Lemma 1. *Given relatively compact open sets U, V, W in R^n such that $\bar{U}\subset V$, $\bar{V}\subset W$, then there is a smooth function $0\leqq t(p)\leqq 1$ on R^n which satisfies*

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the following:
(1) $t=1$ *on* U, $t=0$ *on* W'.
(2) *when* F *denote the closed set defined by*

$$F=\{x\in R^n/t(x)=0\}\supset W'$$

then, for some positive δ,

$$t(p)\leqq \text{dist}(p, F)\delta \quad \textit{and} \quad F\subset V.$$

(3) *for any* $\xi\in T_p(R^n)$,

$$|\partial_\xi t(p)|<\alpha_0|\xi|/\delta,$$

where α_0 *is the constant of* (1.2)' *of* [S].

Proof. Take an open set V_0 so that

$$\bar{U}\subset V_0, \quad \bar{V}_0\subset V,$$

and define $u(p)$ by

$$u(p)=\min(1, \text{dist}(p, V')/\text{dist}(V_0, V')),$$

then $u(p)$ is continuous and is such that

$$0\leqq u(p)\leqq 1, \quad u(p)=1 \quad \textit{on } V_0, u(p)=0 \quad \textit{only on } V'.$$

Let $t(p)$ be the ϕ-average of $u(p)$ (see §1 of [S]):

$$t(p)=\int\phi_\delta(x, p)u(x)dv,$$

where $\delta>0$ is given by

$$\delta=\min(\text{dist}(U, V_0'), \text{dist}(V_0, V'), \text{dist}(V, W')).$$

Then $0\leqq t(p)\leqq 1$ and hence ,if $p\in U$, then Car $\phi_\delta(x, p)\subset V_0$, therefore

$$u(x)=1 \qquad \text{on} \quad \text{Car}\ \phi_\delta(x, p).$$

Consequently $t(p)=1$ on U. In case of $p\in W'$, Car $\phi_\delta(x, p)\subset V'$, therefore

$$u(x)=0 \qquad \text{on} \quad \text{Car}\ \phi_\delta(x, p)$$

Hence $t(p)=0$ on W' and similarly $F\subset V$.

In order to prove assertion (2), we prove first

(4) $\text{dist}(p, F)\geqq \text{dist}(x, V')$ *for any* x *for which* $\text{dist}(x, p)\leqq\delta$.
In case of $x\in V$, let $f\in F$ be a point such that

$$\text{dist}(p, f)=\text{dist}(p, F),$$

and let the line $\overrightarrow{xf}$ cross ∂V at y. Then it is easy to see the following:

$$\begin{aligned}&\operatorname{dist}(x, f) = \operatorname{dist}(x, y)+\operatorname{dist}(y, f),\\ &\operatorname{dist}(x, f) \leqq \operatorname{dist}(x, p)+\operatorname{dist}(p, f),\\ &\operatorname{dist}(x, y) \geqq \operatorname{dist}(x, V'), \quad \operatorname{dist}(y, f) \geqq \operatorname{dist}(V, f) \geqq \delta .\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{dist}(x, V') &\leqq \operatorname{dist}(x, y) = \operatorname{dist}(x, f)-\operatorname{dist}(y, f)\\ &\leqq \operatorname{dist}(x, p) - \operatorname{dist}(y, f)+\operatorname{dist}(p, f)\\ &\leqq \operatorname{dist}(p, F) .\end{aligned}$$

And if $x \notin V$, (4) is obvious, since $\operatorname{dist}(x, V')=0$, finishing the proof of (4).

Now (4) yields that

$$u(x) \leqq \operatorname{dist}(x, V')/\delta \leqq \operatorname{dist}(p, F)/\delta \qquad \text{on} \quad \operatorname{Car} \phi_\delta(x, p),$$

therefore, taking ϕ-average,

$$t(p) \leqq \operatorname{dist}(p, F)/\delta .$$

Thus (2) is proved, and (3) is proved as follows:

$$\begin{aligned}|\partial_\xi t(p)| &= \left|\int \partial_\xi \phi_\delta(x, p) u(x) dv\right|\\ &\leqq 4\gamma(n)\delta^n \max|u(x)|\,|\xi|/\kappa(n)\delta^{n+1}\\ &\leqq \alpha_0|\xi|/\delta ,\end{aligned}$$

where $\alpha_0=4\gamma(n)/\kappa(n)$ (see (1.2)$'$ of [S]).

Let U, V, W, δ, $t(p)$, F be as in Lemma 1 and assume there given a Lipschitz homeomorphism h of $\bar{W}$ into R^N. Define (modified) ϕ-average ϕh of h by

$$\phi h(p) = \begin{cases} h(p) & (p \in F \cap \bar{W}),\\ \int \phi_{K\delta t(p)}(x, p) h(x) dv & (p \in F' \cap \bar{W}),\end{cases}$$

where K is a positive <1.

Obviously ϕh is smooth on F' and because of (1.6) of Part II of [S], ϕh satisfies

(5) $$|\phi h(p)-h(p)| \leqq \mu_0(1+\lambda)K\delta\, t(p),$$

provided h is of λ^2-Lipschitz condition.

This particularly implies the continuity of ϕh on $\bar{W}$ and yields

Lemma 2. *If* $\lambda<1$, *then for* $K \leqq \min(1, 1/4\mu_0)$,

$$\phi h(F') \cap \phi h(F) = \emptyset .$$

Proof. Suppose $\phi h(p)=\phi h(q)$ with $p\in F'$, $q\in F$, then

$$|h(p)-h(q)|=|h(p)-\phi h(p)|\leqq \operatorname{dist}(p, F)/2,$$

On the other hand

$$|h(p)-h(q)|\geqq \operatorname{dist}(p, q)/(1+\lambda)>\operatorname{dist}(p, q)/2.$$

Therefore we should have

$$\operatorname{dist}(p, F)/2\leqq \operatorname{dist}(p, q)/2<\operatorname{dist}(p, F)/2,$$

which is a contradiction.

Lemma 3. *For $0<K<K_0$ and $\lambda<\lambda_0$, ϕh is non degenerate in F'.*

Proof. Set $\varepsilon=K\delta$ and let $\partial'_\xi\phi_{t\varepsilon}$, $\partial''_\xi\phi_{t\varepsilon}$ denote the differential of $\phi_{t\varepsilon}$ keeping t fixed and the differential only in t, respectively. Then

$$\partial_\xi\phi h=\int\partial'_\xi\phi_{t\varepsilon}h dv+\int\partial''_\xi\phi_{t\varepsilon}h dv.$$

As for ∂'_ξ type differential we have ((1.7) of Part II of [S])

$$\left|\int\partial'_\xi\phi_{t\varepsilon}(x, p)h(x)dv-h_\sigma(\xi)\right|\leqq\mu_1\lambda|\xi|, \tag{6}$$

for a simplex σ at p of diameter $t\varepsilon$. And we get easily,

$$\partial''_\xi\phi_{t\varepsilon}=(-\phi'|x-p|/\kappa\varepsilon^{n+1}t^{n+2}-n\phi/\kappa\varepsilon^n t^{n+1})(\partial_\xi t).$$

Therefore

$$\begin{aligned}\left|\int\partial''_\xi\phi_{t\varepsilon}h dv\right| &\leqq \varepsilon\sqrt{N}(4+n)(1+\lambda)|\partial_\xi t|/\kappa\\ &\leqq \alpha_1 K|\xi|.\end{aligned}$$

Thus

$$|\partial_\xi\phi h-h_\sigma(\xi)|\leqq(\mu_1\lambda+\alpha_1 K)|\xi|,$$

and an arguement similar to that in §3 of [S] yields the conclusion.

A calculation similar to that in p. 68 [S] gives an evaluation of $|h_\sigma(\xi)|-|\xi|$ and therefore gives

Corollary 1. *With a constant $\mu=\mu(n)$, $\partial_\xi\phi h$ satisfies*

$$||\partial_\xi\phi h|-|\xi||\leqq(\mu\lambda+\alpha_1 K)|\xi|.$$

2. Smoothing of homeomorphism

Let M be a smooth manifold (not necessarily closed) isometrically imbedded in R^N with tubular neighbourhood $T(M)$ of sufficiently small diameter. Then for any $x \in T(M)$ and for any $y \in M$, we may assume

$$(1) \qquad |y-\pi(x)|/4 \leqq |x-y|.$$

where π denotes the projection of $T(M)$ onto M.

Lemma 4. *Let h be a homeomorphism of a relatively compact open set W_1 of R^n into M and let U, V, W be open sets such that $\bar{U} \subset V$, $\bar{V} \subset W$, $\bar{W} \subset W_1$. Then if h satisfies the λ_0^2 Lipschitz condition on W_1, there is a positive K_0 such that for any $K<K_0$ the modified ϕ_K-average $f_K = \pi\phi_K h$ of h relative to U, V, W followed by the projection π maps F' into $h(F')$.*

Proof. Suppose on the contrary $f_K(p) \notin h(F')$ for some $p \in F'$, then obviously

$$\rho(h(p), f_K(p)) \geqq \rho(h(p), h(\partial F')) = \rho(h(p), h(q)),$$

for some point $q \in \partial F'$. Since h is of λ_0^2-Lipschitz,

$$\rho(h(p), f_K(p)) \geqq |p-q|/1+\lambda_0 \geqq \operatorname{dist}(p, \partial F')/1+\lambda_0$$

On the other hand (see (1, 2) (1, 5)).

$$\begin{aligned} |\phi h(p)-h(p)| &\leqq \mu_0(1+\lambda_0)K\delta t(p) \\ &\leqq \mu_0(1+\lambda_0)K \min(\delta, \operatorname{dist}(p, F)). \end{aligned}$$

Therefore, if K is small, we may assume that $|h(p)-f_K(p)|$ is small and approximates $\rho(h(p), f_K(p))$, in particular,

$$\rho(h(p), f_K(p))/2 \leqq |h(p)-f_K(p)|.$$

Thus we should have

$$\operatorname{dist}(p, \partial F')/2(1+\lambda_0) \leqq 4\mu_0(1+\lambda_0)K \operatorname{dist}(p, F).$$

which yields a contradiction for $K<1/8\mu_0(1+\lambda_0)^2$.

Corollary 2. *Be the notations same as in Lemma 4, then if $K<K_0$, the map h_s defined by*

$$h_s(p) = \begin{cases} f_{sK}(p), & \text{if } 0<s \leqq 1 \\ h(p), & \text{if } s=0 \end{cases}$$

gives a homotopy between f_K and h as maps of F' into $h(F')$ and therefore as maps of W_1 into $h(W_1)$.

Lemma 5. *Using the same notation as in Lemma 4, we can find a positive $K_1 (\leqq K_0)$ such that if $K<K_1$, then $f_K = \pi\phi_K h$ is non degenerate on F'.*

Proof. The evaluation (1.7) and an arguement similar to that in the proof of Proposition 3 of [S] yield easily the conclusion.

Lemma 6. *Be the notation same as in Lemma 5, then if $K<K_1$, f_K maps F' onto $h(F')$.*

Proof. Suppose on the contrary, $h(p)\notin f_K(F')$ with $p\in F'$, then the arc $h_s(p)$ from $h(p)$ to $f_K(p)$ should cross $\partial f_K(F')$ at $h(q)\in h(F')$ (see Lemma 4). Since F' is compact and since f_K is an open map (see Lemma 5), we get

$$h(q)\in\partial f_K(F')\subset f_K(\partial F')=h(\partial F'),$$

which is a contradiction. Combining Lemmas 5, 6 with Corollary 1, we get

Proposition 1. *There exists a positive α such that if a map h of an open set W into a Riemannian manifold M has the Lipschitz size less than α, then for any open set U for which $\bar{U}\subset W$, a homeomorphism f of W into M approximates h in such a way that*
(1) *$f=h$ on $\dot{W}$,*
(2) *f is differentiable on U,*
(3) *the differential df on U satisfies*

$$\mathfrak{l}(df)\leqq((\mathfrak{l}h))^{\gamma(n)}$$

*with a positive $\gamma=\gamma(n)$ depending on $n=$*dim *W.*

3. Construction of a smooth manifold

Let $C=\{(U_i, h_i)\}_{i\in I}$ be a local coordinate system of a compact topological manifold X consisting of a open covering $\mathcal{U}=\{U_i\}_{i\in I}$ of X and of a set of homeomorphismus h_i of discs in R^n onto U_i. We refer simply by $\mathfrak{l}(h_i)$ the Lipschitz size of h_i relative to a (fixed) distance d on X and the usual metric $|\ |$ on R^n, (see p. 66 [S]). Let $\mathfrak{l}(C)$ denote the maximum of $\mathfrak{l}(h_i)$ and let $m(\mathcal{U})$ be the multiplicity of the covering:

$$m(\mathcal{U})=\max_{i\in I}\#\{j\in I\,|\,U_i\cap U_j=\phi\}.$$

Then we define the size $|C/d|$ of C relative to the distance d by

$$|C/d|=(8\gamma)^{m(\mathcal{U})}\log\mathfrak{l}(C).$$

where $\gamma=\gamma(n)$ is the positive depending on $n=\dim X$ of Proposition 1. The size $|X/d|$ of the manifold is defined to be the infimum of the numbers $|C/d|$ taken over the set of the coordinate systems of finite coverings. Then the condition $|X/d|<\varepsilon/2$ implies that there exists a finite covering $\mathcal{U}=\{U_i\}_{i\in I}$ of X and a system of homeomorphismus h_i of discs D_i onto U_i satisfying

$$(\mathfrak{l}(C))^{(8r)^{m(\mathcal{U})}} < \exp(\varepsilon)\,.$$

We now construct a smooth manifold under the condition above, provided ε is sufficiently small.

Lemma 7. *From a given finite open covering $\mathcal{U}$ of X, we can construct an open covering $\{X_1, \cdots, X_m\}$ of X such that*

(1) $m=m(\mathcal{U})+1$

(2) *each open set X_i is a disjoint union of some of open sets of $\mathcal{U}$.*

(3) *every open set U_i of $\mathcal{U}$ appears in only one open set X_i.*

Proof. Using a suitably defined order in the index set I, we classify I into subsets $I_1, \cdots, I_m$ in the following way;

(1) $1\in I_1$ and $i\in I$, $1<i$ belongs I_1 if and only if for all $j\in I_1$, $j<i$, it holds that $U_j\cap U_i=\phi$.

(2) min $(I-I_1\cup\cdots\cup I_k)\in I_{k+1}$ and $i\in I-(I_1\cup\cdots\cup I_k)$, belongs I_{k+1} if and only if for all $j\in I_{k+1}$, $j<i$, it holds that $U_j\cap U_i=\phi$.

This process continues at most $m=m(\mathcal{U})+1$ times, in fact, suppose on the contrary that there is $i\in I$ such that $i\notin I_1\cup\cdots\cup I_m$, then $U_i\cap U_{k_j}\neq\phi$ with some $k_j\in I_k$ for each $k=1, \cdots, m$, indicating that $\sharp\{j\in I_i\cap U_j\neq\phi\}\geqq m=m(\mathcal{U})+1$.

Hence letting $X_k=\bigcup_{j\in I_k} U_j$, we get the covering.

Let E_i denote the disjoint union of discs D_j, $j\in I_i$ and let H_i be the homeomorphism of E_i onto X_i which agrees with h_j on each component D_j of E_i.

Take concentric m discs $D_j^m\subset\cdots\subset D_j^2\subset D_j^1\subset D_j$ such that the images $X^k{}_i$ of $E^k{}_i=\bigcup_{j\in I_i} D^k{}_j$ under the homeomorphism H_i form a covering of X for each $k=1, \cdots, m$.

Each open set X_i is obviously smoothable as an homeomorphic image of a smooth manifolds E_i having a naturally defined Riemannian metric d_i. The homeomorphism $H_{12}=H_1^{-1}H_2$ is defined on $E_{12}=H_2^{-1}(X_1\cap X_2)$ and has the Lipschitz size less than $\mathfrak{l}(H_1)\mathfrak{l}(H_2)$ (see (2.3) p. 66 [S]).

Therefore, if $\mathfrak{l}(H_1)\mathfrak{l}(H_2)\leqq\alpha$ on E_{12} (α of Proposition 1), then an application of Proposition 1 to H_{12} and $E_{12}^1=H_2^{-1}(X_1^1\cap X_2^1)\subset E_{12}$ yields that there exists a homeomorphism h_{12} of E_{12} into E_1 which is diffeomorphic on E_{12}^1. By the identification through h_{12}, $E_1^1\cup E_2^1$ (disjoint union) turns out to be a smooth manifold C_2^1 and then $X_1^1\cup X_2^1=Y_2^1$ to be a smoothable manifold by a homeomorphism F_2 of C_2^1 onto Y_2^1 defined by the following:

$$F_2(x)=\begin{cases} H_1p_1^{-1}(x) & \text{if } x\in p_1(E_1^1) \\ H_1h_{12}p_2^{-1}(x) & \text{if } x\in p_2(E_{12}\cap E_2^1) \\ H_2p_2^{-1}(x) & \text{if } x\in p_2(E_2^1)-p_2(E_{12})\,. \end{cases}$$

In order to make $Y_3^2=X_1^2\cup X_2^2\cup X_3^2$ smoothable, consider the homeomorphism $H_{23}=F_2^{-1}H_3$ of $E_{23}=H_3^{-1}(Y_2^1\cap X_3^1)$ into C_2^1. We may proceed as in the same way above, if the Lipschitz size of H_{23} is sufficiently small relative to a certain Riemannian metric ρ_2 on C_2^1 and d_3 on E_3. Define ρ_2 by the following bilinear form $\langle\,\rangle_x$ on the tangent space $T_x(C_2^1)$ of C_2^1;

$$\langle\xi,\eta\rangle_x=a_1(x)\langle dp_1^{-1}(\xi),\ dp_1^{-1}(\eta)\rangle+a_2(x)\langle dp_2^{-1}(\xi)dp_2^{-1}(\eta)\rangle,$$

where a_i is a partition of unity associated to the covering $\{p_i(E_i)\}_{i=1,2}$. Then an inequality

$$||dh_{12}\xi|^2-|\xi|^2|\leqq\beta^2|\xi|^2\qquad(\xi\in T(E_i))\tag{4}$$

yields that, for $\xi\in T_x(C_2^1)$,

$$\begin{aligned}||\xi|_x^2-|dp_i^{-1}(\xi)|^2|&\leqq a_j(x)||dh_{12}dp_i^{-1}(\xi)|^2-|dp_i^{-1}(\xi)|^2|\\&\leqq\beta^2|dp_i^{-1}(\xi)|^2\qquad(i\neq j)\,.\end{aligned}$$

Therefore we easily see that under the inequality (4),

$$\sqrt{1-\beta^2}d_i(p,q)\leqq\rho_2(p_i(p),p_i(q))\leqq\sqrt{1+\beta^2}d_i(p,q)$$

and we get the following:

Lemma 8. *If* $\mathfrak{l}^2(dh_{12})\leqq4/3$, *then* $\mathfrak{l}(p_i)$ *(rel.* ρ_2, d_i*)* $\leqq\mathfrak{l}^2(dh_{12})$.

Thus combining Lemma 7 with Proposition 1, (3), we get;

$$\mathfrak{l}(H_{23})(\text{rel. }\rho_2,d_3)\leqq\mathfrak{l}^{2\gamma}(H_{12})\mathfrak{l}^{\gamma}(H_{12})\mathfrak{l}^2(C)\leqq\mathfrak{l}^{8\gamma}(C)\,.$$

Therefore if $\mathfrak{l}^{8\gamma}(C)\leqq\alpha$ then we approximate H_{23} by h_{23} on $E_3^1=H_3^{-1}F_2(C_2^1)$ so as to the identified manifold $C_3^2=E_3^2\bigcup_{h_{23}}C_2^2$ through h_{23} is a differentiable manifold which covers Y_3^2 by a suitably defined homeomorphism F_3. We continue the process and get manifolds $C_{k+1}{}^k$ and homeomorphisms F_k, covering $Y_{k+1}{}^k=Y_1{}^k\cup\cdots\cup Y_{k+1}{}^k$, as long as $H_{k\,k+1}=F_k^{-1}H_{k+1}$ satisfy

$$\mathfrak{l}(H_{k\,k+1})\ (\text{rel. }\rho_k,\ d_{k+1})\leqq\alpha\,.$$

Since inductively we easily verify

$$\mathfrak{l}(H_{k\,k+1})\ (\text{rel. }\rho_k,\ d_{k+1})\leqq\mathfrak{l}^{(8\gamma)^k}(C)\,,$$

provided $\mathfrak{l}^{(8\gamma)^{k-1}}(C)\leqq2/\sqrt{3}$, the assumption that

$$\mathfrak{l}^{(8\gamma)^M}(C)\leqq\min(\alpha,\ 2/\sqrt{3})\,,$$

where $M=m(\mathfrak{U})$, yields that we can complete our construction.

4. A remark in the differentiable case

We remark that if X is differentiable then $|X|=0$. In fact, take a Riemannian metric d on X, then the exponential map $\exp_p$ defined around $p \in X$, relative to d is such that if diam $(U(p)) \to 0$, then $I(\exp_p)$ (rel. d, $|\ |$)$\to 1$ on $U(p)$. Thus to prove $|X/d|=0$, it is sufficient to show that for any $\delta>0$, there exists an open covering $\mathcal{U}=\{U_i\}_{i\in I}$ of X such that diam $(U_i)<\delta$ and $m(\mathcal{U}) \leqq M$ (M is independent of δ). Such a covering is constructed as follows; Take the triangulation of X, described in [Wy. p. 124–135] or [S. p. 72], for $\varepsilon=1/4\ \delta$, and let $U(p)=\{x\in X/d(p, x)<\delta/2\}$ for each $p\in K^0$, the 0-skelton of K. Then since diam $\sigma<\varepsilon=1/4\delta$ for any $\sigma\in K$, $\{U(p)\}_{p\in K^0}$ forms an open covering of X and each open set of the covering has diameter less than δ. To evaluate the multiplicity, consider the volume of n-simplex σ in K which is estimated in [S] as in the following form with positive functions $\theta(n)$, $\beta(n)$ of n;

$$\text{vol}\,\sigma \geqq 1/4\,\theta(n)\,\text{diam}^n\,(\sigma) \geqq \beta(n)\theta(n)\delta^n/4^{n+1}$$

provided δ is sufficiently small. Therefore the maximal number of verteces in $U(p)$ is less than

$$\text{vol}\,(U(p))/\text{vol}\,\sigma \leqq 4^{n+2}\Gamma(n)/\beta(n, N)\theta(n, N) = M\,,$$

where $\Gamma(n)$ is the ratio to the volume of n-sphere to its diameter, thus we see the multiplicity is less than M.

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