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ON THE SMOOTHING PROBLEM AND THE SIZE OF A TOPOLOGICAL MANIFOLD

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0. Introduction

In this paper which is a direct continuation of [S], we treat the smoothing problem of a compact topological manifold.

We define in §3 a size $|X/d|$ of a compact topological manifold X relative to a distance function d on X . As is seen in §4, it is rather easy to see that if X admits a smooth structure and if d is a Riemannian metric relative to the structure, then $|X/d|=0$.

Our main result is the converse of this fact, that is, if $|X/d|$ is sufficiently small, then X admits a smoothing and d is approximated by a Riemannian metric on the smoothing in the sense of Lipschitz ratio. (See Theorem 1)

We call a smoothing σ of X compatible in the strong sense with its distance d , if σ admits a Riemannian metric whose Lipschitz ratio to d is less than $|X/d|$ (see §3). Then, by Part II of [S], it is easy to see that if $|X/d|$ is sufficiently small, any two compatible smoothings are differentiably equivalent. Therefore we conclude as follows:

“If $|X/d|$ is sufficiently small, then X admits unique smoothing which is compatible with its distance d .”

Finally we define the absolute size $|X|$ of X by

$$|X| = \inf \{ |X/d| \mid d: \text{distance function on } X \},$$

to get a criterion for the existence of a smoothing on X :

$$“|X| = 0 \Leftrightarrow X \text{ is smoothable.}”$$

1. Modification of ϕ -average

We start with the following lemma which might be well known;

Lemma 1. *Given relatively compact open sets U, V, W in R^n such that $\bar{U} \subset V, \bar{V} \subset W$, then there is a smooth function $0 \leq t(p) \leq 1$ on R^n which satisfies*

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the following:

- (1) $t=1$ on U , $t=0$ on W' .
 (2) when F denote the closed set defined by

$$F = \{x \in R^n / t(x) = 0\} \supset W'$$

then, for some positive δ ,

$$t(p) \leq \text{dist}(p, F)\delta \quad \text{and} \quad F \subset V.$$

- (3) for any $\xi \in T_p(R^n)$,

$$|\partial_\xi t(p)| < \alpha_0 |\xi| / \delta,$$

where α_0 is the constant of (1.2)' of [S].

Proof. Take an open set V_0 so that

$$\bar{U} \subset V_0, \quad \bar{V}_0 \subset V,$$

and define $u(p)$ by

$$u(p) = \min(1, \text{dist}(p, V') / \text{dist}(V_0, V')),$$

then $u(p)$ is continuous and is such that

$$0 \leq u(p) \leq 1, \quad u(p) = 1 \quad \text{on} \quad V_0, \quad u(p) = 0 \quad \text{only on} \quad V'.$$

Let $t(p)$ be the ϕ -average of $u(p)$ (see § 1 of [S]):

$$t(p) = \int \phi_\delta(x, p) u(x) dv,$$

where $\delta > 0$ is given by

$$\delta = \min(\text{dist}(U, V_0'), \text{dist}(V_0, V'), \text{dist}(V, W')).$$

Then $0 \leq t(p) \leq 1$ and hence, if $p \in U$, then $\text{Car } \phi_\delta(x, p) \subset V_0$, therefore

$$u(x) = 1 \quad \text{on} \quad \text{Car } \phi_\delta(x, p).$$

Consequently $t(p) = 1$ on U . In case of $p \in W'$, $\text{Car } \phi_\delta(x, p) \subset V'$, therefore

$$u(x) = 0 \quad \text{on} \quad \text{Car } \phi_\delta(x, p)$$

Hence $t(p) = 0$ on W' and similarly $F \subset V$.

In order to prove assertion (2), we prove first

- (4) $\text{dist}(p, F) \geq \text{dist}(x, V')$ for any x for which $\text{dist}(x, p) \leq \delta$.

In case of $x \in V$, let $f \in F$ be a point such that

$$\text{dist}(p, f) = \text{dist}(p, F),$$

and let the line \overrightarrow{xf} cross ∂V at y . Then it is easy to see the following:

$$\begin{aligned} \text{dist}(x, f) &= \text{dist}(x, y) + \text{dist}(y, f), \\ \text{dist}(x, f) &\leq \text{dist}(x, p) + \text{dist}(p, f), \\ \text{dist}(x, y) &\geq \text{dist}(x, V'), \quad \text{dist}(y, f) \geq \text{dist}(V, f) \geq \delta. \end{aligned}$$

Therefore

$$\begin{aligned} \text{dist}(x, V') &\leq \text{dist}(x, y) = \text{dist}(x, f) - \text{dist}(y, f) \\ &\leq \text{dist}(x, p) - \text{dist}(y, f) + \text{dist}(p, f) \\ &\leq \text{dist}(p, F). \end{aligned}$$

And if $x \in V$, (4) is obvious, since $\text{dist}(x, V') = 0$, finishing the proof of (4).

Now (4) yields that

$$u(x) \leq \text{dist}(x, V')/\delta \leq \text{dist}(p, F)/\delta \quad \text{on } \text{Car } \phi_\delta(x, p),$$

therefore, taking ϕ -average,

$$t(p) \leq \text{dist}(p, F)/\delta.$$

Thus (2) is proved, and (3) is proved as follows:

$$\begin{aligned} |\partial_\xi t(p)| &= \left| \int \partial_\xi \phi_\delta(x, p) u(x) dv \right| \\ &\leq 4\gamma(n)\delta^n \max |u(x)| |\xi|/\kappa(n)\delta^{n+1} \\ &\leq \alpha_0 |\xi|/\delta, \end{aligned}$$

where $\alpha_0 = 4\gamma(n)/\kappa(n)$ (see (1.2)' of [S]).

Let $U, V, W, \delta, t(p), F$ be as in Lemma 1 and assume there given a Lipschitz homeomorphism h of \bar{W} into R^N . Define (modified) ϕ -average ϕh of h by

$$\phi h(p) = \begin{cases} h(p) & (p \in F \cap \bar{W}), \\ \int \phi_{K\delta t(p)}(x, p) h(x) dv & (p \in F' \cap \bar{W}), \end{cases}$$

where K is a positive < 1 .

Obviously ϕh is smooth on F' and because of (1.6) of Part II of [S], ϕh satisfies

$$(5) \quad |\phi h(p) - h(p)| \leq \mu_0(1 + \lambda)K\delta t(p),$$

provided h is of λ^2 -Lipschitz condition.

This particularly implies the continuity of ϕh on \bar{W} and yields

Lemma 2. *If $\lambda < 1$, then for $K \leq \min(1, 1/4\mu_0)$,*

$$\phi h(F') \cap \phi h(F) = \emptyset.$$

Proof. Suppose $\phi h(p) = \phi h(q)$ with $p \in F', q \in F$, then

$$|h(p) - h(q)| = |h(p) - \phi h(p)| \leq \text{dist}(p, F)/2,$$

On the other hand

$$|h(p) - h(q)| \geq \text{dist}(p, q)/(1 + \lambda) > \text{dist}(p, q)/2.$$

Therefore we should have

$$\text{dist}(p, F)/2 \leq \text{dist}(p, q)/2 < \text{dist}(p, F)/2,$$

which is a contradiction.

Lemma 3. For $0 < K < K_0$ and $\lambda < \lambda_0$, ϕh is non degenerate in F' .

Proof. Set $\varepsilon = K\delta$ and let $\partial'_{\xi} \phi_{t\varepsilon}, \partial''_{\xi} \phi_{t\varepsilon}$ denote the differential of $\phi_{t\varepsilon}$ keeping t fixed and the differential only in t , respectively. Then

$$\partial_{\xi} \phi h = \int \partial'_{\xi} \phi_{t\varepsilon} h dv + \int \partial''_{\xi} \phi_{t\varepsilon} h dv.$$

As for ∂'_{ξ} type differential we have ((1.7) of Part II of [S])

$$(6) \quad \left| \int \partial'_{\xi} \phi_{t\varepsilon}(x, p) h(x) dv - h_{\sigma}(\xi) \right| \leq \mu_1 \lambda |\xi|,$$

for a simplex σ at p of diameter $t\varepsilon$. And we get easily,

$$\partial''_{\xi} \phi_{t\varepsilon} = (-\phi' |x - p| / \kappa \varepsilon^{n+1} t^{n+2} - n\phi / \kappa \varepsilon^n t^{n+1})(\partial_{\xi} t).$$

Therefore

$$\begin{aligned} \left| \int \partial''_{\xi} \phi_{t\varepsilon} h dv \right| &\leq \varepsilon \sqrt{N} (4 + n)(1 + \lambda) |\partial_{\xi} t| / \kappa \\ &\leq \alpha_1 K |\xi|. \end{aligned}$$

Thus

$$|\partial_{\xi} \phi h - h_{\sigma}(\xi)| \leq (\mu_1 \lambda + \alpha_1 K) |\xi|,$$

and an argument similar to that in §3 of [S] yields the conclusion.

A calculation similar to that in p. 68 [S] gives an evaluation of $|h_{\sigma}(\xi)| - |\xi|$ and therefore gives

Corollary 1. With a constant $\mu = \mu(n)$, $\partial_{\xi} \phi h$ satisfies

$$||\partial_{\xi} \phi h| - |\xi|| \leq (\mu \lambda + \alpha_1 K) |\xi|.$$

2. Smoothing of homeomorphism

Let M be a smooth manifold (not necessarily closed) isometrically imbed-

ded in R^N with tubular neighbourhood $T(M)$ of sufficiently small diameter. Then for any $x \in T(M)$ and for any $y \in M$, we may assume

$$(1) \quad |y - \pi(x)|/4 \leq |x - y|.$$

where π denotes the projection of $T(M)$ onto M .

Lemma 4. *Let h be a homeomorphism of a relatively compact open set W_1 of R^n into M and let U, V, W be open sets such that $\bar{U} \subset V, \bar{V} \subset W, \bar{W} \subset W_1$. Then if h satisfies the λ_0^2 Lipschitz condition on W_1 , there is a positive K_0 such that for any $K < K_0$ the modified ϕ_K -average $f_K = \pi \phi_K h$ of h relative to U, V, W followed by the projection π maps F' into $h(F')$.*

Proof. Suppose on the contrary $f_K(p) \notin h(F')$ for some $p \in F'$, then obviously

$$\rho(h(p), f_K(p)) \geq \rho(h(p), h(\partial F')) = \rho(h(p), h(q)),$$

for some point $q \in \partial F'$. Since h is of λ_0^2 -Lipschitz,

$$\rho(h(p), f_K(p)) \geq |p - q|/1 + \lambda_0 \geq \text{dist}(p, \partial F')/1 + \lambda_0$$

On the other hand (see (1, 2) (1, 5)).

$$\begin{aligned} |\phi h(p) - h(p)| &\leq \mu_0(1 + \lambda_0)K\delta t(p) \\ &\leq \mu_0(1 + \lambda_0)K \min(\delta, \text{dist}(p, F)). \end{aligned}$$

Therefore, if K is small, we may assume that $|h(p) - f_K(p)|$ is small and approximates $\rho(h(p), f_K(p))$, in particular,

$$\rho(h(p), f_K(p))/2 \leq |h(p) - f_K(p)|.$$

Thus we should have

$$\text{dist}(p, \partial F')/2(1 + \lambda_0) \leq 4\mu_0(1 + \lambda_0)K \text{dist}(p, F).$$

which yields a contradiction for $K < 1/8\mu_0(1 + \lambda_0)^2$.

Corollary 2. *Be the notations same as in Lemma 4, then if $K < K_0$, the map h_s defined by*

$$h_s(p) = \begin{cases} f_{sK}(p), & \text{if } 0 < s \leq 1 \\ h(p), & \text{if } s = 0 \end{cases}$$

gives a homotopy between f_K and h as maps of F' into $h(F')$ and therefore as maps of W_1 into $h(W_1)$.

Lemma 5. *Using the same notation as in Lemma 4, we can find a positive $K_1 (\leq K_0)$ such that if $K < K_1$, then $f_K = \pi \phi_K h$ is non degenerate on F' .*

Proof. The evaluation (1.7) and an argument similar to that in the proof of Proposition 3 of [S] yield easily the conclusion.

Lemma 6. *Be the notation same as in Lemma 5, then if $K < K_1$, f_K maps F' onto $h(F')$.*

Proof. Suppose on the contrary, $h(p) \notin f_K(F')$ with $p \in F'$, then the arc $h_s(p)$ from $h(p)$ to $f_K(p)$ should cross $\partial f_K(F')$ at $h(q) \in h(F')$ (see Lemma 4). Since F' is compact and since f_K is an open map (see Lemma 5), we get

$$h(q) \in \partial f_K(F') \subset f_K(\partial F') = h(\partial F'),$$

which is a contradiction. Combining Lemmas 5, 6 with Corollary 1, we get

Proposition 1. *There exists a positive α such that if a map h of an open set W into a Riemannian manifold M has the Lipschitz size less than α , then for any open set U for which $\bar{U} \subset W$, a homeomorphism f of W into M approximates h in such a way that*

- (1) $f=h$ on \bar{W} ,
- (2) f is differentiable on U ,
- (3) the differential df on U satisfies

$$l(df) \leq ((lh))^{\gamma(n)}$$

with a positive $\gamma = \gamma(n)$ depending on $n = \dim W$.

3. Construction of a smooth manifold

Let $\mathcal{C} = \{(U_i, h_i)\}_{i \in I}$ be a local coordinate system of a compact topological manifold X consisting of a open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and of a set of homeomorphism h_i of discs in R^n onto U_i . We refer simply by $l(h_i)$ the Lipschitz size of h_i relative to a (fixed) distance d on X and the usual metric $||$ on R^n , (see p. 66 [S]). Let $l(\mathcal{C})$ denote the maximum of $l(h_i)$ and let $m(\mathcal{U})$ be the multiplicity of the covering:

$$m(\mathcal{U}) = \max_{i \in I} \#\{j \in I \mid U_i \cap U_j \neq \emptyset\}.$$

Then we define the size $|\mathcal{C}/d|$ of \mathcal{C} relative to the distance d by

$$|\mathcal{C}/d| = (8\gamma)^{m(\mathcal{U})} \log l(\mathcal{C}).$$

where $\gamma = \gamma(n)$ is the positive depending on $n = \dim X$ of Proposition 1. The size $|X/d|$ of the manifold is defined to be the infimum of the numbers $|\mathcal{C}/d|$ taken over the set of the coordinate systems of finite coverings. Then the condition $|X/d| < \varepsilon/2$ implies that there exists a finite covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and a system of homeomorphism h_i of discs D_i onto U_i satisfying

$$l(C)^{(8r)^{m^{(Q)}}} < \exp(\varepsilon).$$

We now construct a smooth manifold under the condition above, provided ε is sufficiently small.

Lemma 7. *From a given finite open covering \mathcal{Q} of X , we can construct an open covering $\{X_1, \dots, X_m\}$ of X such that*

- (1) $m = m(\mathcal{Q}) + 1$
- (2) each open set X_i is a disjoint union of some of open sets of \mathcal{Q} .
- (3) every open set U_i of \mathcal{Q} appears in only one open set X_i .

Proof. Using a suitably defined order in the index set I , we classify I into subsets I_1, \dots, I_m in the following way;

- (1) $1 \in I_1$ and $i \in I$, $1 < i$ belongs I_1 if and only if for all $j \in I$, $j < i$, it holds that $U_j \cap U_i = \phi$.
- (2) $\min(I - I_1 \cup \dots \cup I_k) \in I_{k+1}$ and $i \in I - (I_1 \cup \dots \cup I_k)$, belongs I_{k+1} if and only if for all $j \in I_{k+1}$, $j < i$, it holds that $U_j \cap U_i = \phi$.

This process continues at most $m = m(\mathcal{Q}) + 1$ times, in fact, suppose on the contrary that there is $i \in I$ such that $i \notin I_1 \cup \dots \cup I_m$, then $U_i \cap U_{k_j} \neq \phi$ with some $k_j \in I_k$ for each $k = 1, \dots, m$, indicating that $\#\{j \in I_i \cap U_j \neq \phi\} \geq m = m(\mathcal{Q}) + 1$.

Hence letting $X_k = \bigcup_{j \in I_k} U_j$, we get the covering.

Let E_i denote the disjoint union of discs D_j , $j \in I_i$ and let H_i be the homeomorphism of E_i onto X_i which agrees with h_j on each component D_j of E_i .

Take concentric m discs $D_j^m \subset \dots \subset D_j^2 \subset D_j^1 \subset D_j$ such that the images X^k_i of $E^k_i = \bigcup_{j \in I_i} D_j^k$ under the homeomorphism H_i form a covering of X for each $k = 1, \dots, m$.

Each open set X_i is obviously smoothable as an homeomorphic image of a smooth manifolds E_i having a naturally defined Riemannian metric d_i . The homeomorphism $H_{12} = H_1^{-1}H_2$ is defined on $E_{12} = H_2^{-1}(X_1 \cap X_2)$ and has the Lipschitz size less than $l(H_1)l(H_2)$ (see (2.3) p. 66 [S]).

Therefore, if $l(H_1)l(H_2) \leq \alpha$ on E_{12} (α of Proposition 1), then an application of Proposition 1 to H_{12} and $E_{12}^1 = H_2^{-1}(X_1^1 \cap X_2^1) \subset E_{12}$ yields that there exists a homeomorphism h_{12} of E_{12} into E_1 which is diffeomorphic on E_{12}^1 . By the identification through h_{12} , $E_1^1 \cup E_2^1$ (disjoint union) turns out to be a smooth manifold C_2^1 and then $X_1^1 \cup X_2^1 = Y_2^1$ to be a smoothable manifold by a homeomorphism F_2 of C_2^1 onto Y_2^1 defined by the following:

$$F_2(x) = \begin{cases} H_1 p_1^{-1}(x) & \text{if } x \in p_1(E_1^1) \\ H_1 h_{12} p_2^{-1}(x) & \text{if } x \in p_2(E_{12} \cap E_2^1) \\ H_2 p_2^{-1}(x) & \text{if } x \in p_2(E_2^1) - p_2(E_{12}). \end{cases}$$

In order to make $Y_3^2 = X_1^2 \cup X_2^2 \cup X_3^2$ smoothable, consider the homeomorphism $H_{23} = F_2^{-1}H_3$ of $E_{23} = H_3^{-1}(Y_2^1 \cap X_3^1)$ into C_2^1 . We may proceed as in the same way above, if the Lipschitz size of H_{23} is sufficiently small relative to a certain Riemannian metric ρ_2 on C_2^1 and d_3 on E_3 . Define ρ_2 by the following bilinear form $\langle \cdot \rangle_x$ on the tangent space $T_x(C_2^1)$ of C_2^1 ;

$$\langle \xi, \eta \rangle_x = a_1(x) \langle dp_1^{-1}(\xi), dp_1^{-1}(\eta) \rangle + a_2(x) \langle dp_2^{-1}(\xi) dp_2^{-1}(\eta) \rangle,$$

where a_i is a partition of unity associated to the covering $\{p_i(E_i)\}_{i=1,2}$. Then an inequality

$$(4) \quad ||dh_{12}\xi|^2 - |\xi|^2| \leq \beta^2 |\xi|^2 \quad (\xi \in T(E_i))$$

yields that, for $\xi \in T_x(C_2^1)$,

$$\begin{aligned} ||\xi|_x^2 - |dp_i^{-1}(\xi)|^2| &\leq a_j(x) ||dh_{12} dp_i^{-1}(\xi)|^2 - |dp_i^{-1}(\xi)|^2| \\ &\leq \beta^2 |dp_i^{-1}(\xi)|^2 \quad (i \neq j). \end{aligned}$$

Therefore we easily see that under the inequality (4),

$$\sqrt{1 - \beta^2} d_i(p, q) \leq \rho_2(p_i(p), p_i(q)) \leq \sqrt{1 + \beta^2} d_i(p, q)$$

and we get the following:

Lemma 8. *If $Y^2(dh_{12}) \leq 4/3$, then $I(p_i)$ (rel. ρ_2, d_i) $\leq Y^2(dh_{12})$.*

Thus combining Lemma 7 with Proposition 1, (3), we get;

$$I(H_{23}) \text{ (rel. } \rho_2, d_3) \leq Y^{2\gamma}(H_{12}) I^\gamma(H_{12}) I^2(C) \leq I^{8\gamma}(C).$$

Therefore if $I^{8\gamma}(C) \leq \alpha$ then we approximate H_{23} by h_{23} on $E_3^1 = H_3^{-1}F_2(C_2^1)$ so as to the identified manifold $C_3^2 = E_3^2 \cup_{h_{23}} C_2^2$ through h_{23} is a differentiable manifold which covers Y_3^2 by a suitably defined homeomorphism F_3 . We continue the process and get manifolds C_{k+1}^k and homeomorphisms F_k , covering $Y_{k+1}^k = Y_1^k \cup \dots \cup Y_{k+1}^k$, as long as $H_{kk+1} = F_k^{-1}H_{k+1}$ satisfy

$$I(H_{kk+1}) \text{ (rel. } \rho_k, d_{k+1}) \leq \alpha.$$

Since inductively we easily verify

$$I(H_{kk+1}) \text{ (rel. } \rho_k, d_{k+1}) \leq I^{(8\gamma)^k}(C),$$

provided $I^{(8\gamma)^{k-1}}(C) \leq 2/\sqrt{3}$, the assumption that

$$I^{(8\gamma)^M}(C) \leq \min(\alpha, 2/\sqrt{3}),$$

where $M = m(\mathcal{U})$, yields that we can complete our construction.

4. A remark in the differentiable case

We remark that if X is differentiable then $|X|=0$. In fact, take a Riemannian metric d on X , then the exponential map \exp_p defined around $p \in X$, relative to d is such that if $\text{diam}(U(p)) \rightarrow 0$, then $l(\exp_p)$ (rel. $d, || \cdot ||$) $\rightarrow 1$ on $U(p)$. Thus to prove $|X/d|=0$, it is sufficient to show that for any $\delta > 0$, there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X such that $\text{diam}(U_i) < \delta$ and $m(\mathcal{U}) \leq M$ (M is independent of δ). Such a covering is constructed as follows; Take the triangulation of X , described in [Wy. p. 124–135] or [S. p. 72], for $\varepsilon = 1/4 \delta$, and let $U(p) = \{x \in X / d(p, x) < \delta/2\}$ for each $p \in K^0$, the 0-skelton of K . Then since $\text{diam} \sigma < \varepsilon = 1/4 \delta$ for any $\sigma \in K$, $\{U(p)\}_{p \in K^0}$ forms an open covering of X and each open set of the covering has diameter less than δ . To evaluate the multiplicity, consider the volume of n -simplex σ in K which is estimated in [S] as in the following form with positive functions $\theta(n)$, $\beta(n)$ of n ;

$$\text{vol } \sigma \geq 1/4 \theta(n) \text{diam}^n(\sigma) \geq \beta(n) \theta(n) \delta^n / 4^{n+1}$$

provided δ is sufficiently small. Therefore the maximal number of vertices in $U(p)$ is less than

$$\text{vol}(U(p)) / \text{vol } \sigma \leq 4^{n+2} \Gamma(n) / \beta(n, N) \theta(n, N) = M,$$

where $\Gamma(n)$ is the ratio to the volume of n -sphere to its diameter, thus we see the multiplicity is less than M .

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