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ON REGULAR RINGS WHOSE MAXIMAL RIGHT QUOTIENT RINGS ARE TYPE $I_f$

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Introduction

This paper is written about the property (DF) on regular rings whose maximal right quotient rings are Type $I_f$. Hereafter regular rings whose maximal right quotient rings are Type $I_f$ are said to satisfy ($\dagger$). The property (DF) is very important property when we study on regular rings satisfying ($\dagger$), and it was treated in the paper [5] written by the first author, where (DF) for a ring $R$ is defined as that if the direct sum of any two directly finite projective $R$-modules is always directly finite. In the above paper, the equivalent condition that a regular ring $R$ of bounded index satisfies (DF) was discovered and called ($\ddagger$). Stillmore, we proved that the condition (DF) is equivalent to ($\ddagger$) for regular rings whose primitive factor rings are artinian in the paper [6]. Then we have the problem that (DF) is equivalent ($\ddagger$) for regular rings satisfying ($\dagger$) or not, where the condition ($\dagger$) is weaker than one that primitive factor rings are artinian.

In §2, we shall prove Theorem 2.4. This is important, and using this, Theorem 2.5 (i.e. if $R$ is a regular ring satisfying ($\dagger$) and $k$ is any positive integer, then $kP$ is directly finite for every directly finite projective $R$-module $P$) is proved. Moreover, we shall solve the above problem in Theorem 2.11.

In §3, we shall consider some applications of Theorem 2.11. We prove Theorem 3.3 that if $R$ is a regular ring satisfying ($\dagger$) whose maximal right quotient ring of $R$ satisfies (DF), then so does $R$. Though it is clear that a regular rings satisfying ($\dagger$) which has a nonzero essential socle satisfies (DF), we can prove that, for regular rings satisfying ($\dagger$), the condition having a nonzero essential socle is not equivalent to ($\ddagger$) in Example 3.4. Next, we shall consider that $(\Pi R)/(\bigoplus R)$ satisfies (DF) or not for a regular ring $R$ satisfying ($\dagger$). This problem is a generalization of Example 3.4, and we prove that, for a regular ring $R$ of bounded index, $(\Pi R)/(\bigoplus R)$ satisfies (DF) (Theorem 3.9).

Throughout this paper, $R$ is a ring with identity and $R$-modules are unitary right $R$-modules.
1. Definitions and notations

**Definition 1.** A ring $R$ is (Von Neumann) regular provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$.

**Note.** Every projective modules over regular rings have the exchange property.

**Definition 2.** A module $M$ is directly finite provided that $M$ is not isomorphic to a proper direct summand of itself. If $M$ is not directly finite, then $M$ is said to be directly infinite. A ring $R$ is said to be directly finite (resp. directly infinite) if so is $R$ as an $R$-module.

**Definition 3.** The index of a nilpotent element $x$ in a ring $R$ is the least positive integer such that $x^n = 0$ (In particular, $0$ is nilpotent of index $1$). The index of a two-sided ideal $J$ of $R$ is the supremum of the indices of all nilpotent elements of $J$.

If this supremum is finite, then $J$ is said to have bounded index. If $J$ does not have bounded index, $J$ is said to be index $\infty$.

**Note.** Let $R$ be a regular ring with index $\infty$. Then using [3, the proof of Lemma 2], there exists a family $\{A_n\}_{n=1}^\infty$ of independent right ideals of $R$ such that $A_n$ contains a direct sum of $n$ nonzero pairwise isomorphic right ideals. Therefore $R$ has a family $\{e_{ij}\}_{i,j=1,2,\ldots}$ of idempotents such that

$$
e_{21} R \simeq e_{22} R$$
$$
e_{31} R \simeq e_{32} R \simeq e_{33} R$$
$$
\ldots$$

, where $e_{ij} = 0 \ (i<j)$, and $\{e_{i1}, \ldots, e_{ij}\}$ are orthogonal for all $i$.

**Definition 4.** A ring $R$ has (DF) if the direct sum of two directly finite projective $R$-modules is directly finite.

**Definition 5.** A regular ring $R$ is abelian provided all idempotents in $R$ are central.

**Definition 6.** A ring $R$ satisfies (*) if every nonzero two-sided ideal of $R$ contains a nonzero two-sided ideal of bounded index.

**Definition 7.** A ring $R$ is unit-regular provided that for each $x \in R$ there is a unit $u \in R$ such that $xux = x$. 
NOTE. Every finitely generated projective module over a unit-regular ring has the cancellation property ([2, Theorem 4.14]).

**Definition 8.** Let \( e \) be an idempotent in a regular ring \( R \). Then \( e \) is called an **abelian idempotent** (of \( R \)) whenever the ring \( eRe \) is abelian.

**Definition 9.** Let \( e \) be an idempotent in a regular right self-injective ring \( R \). Then \( e \) is **faithful** (in \( R \)) if 0 is the only central idempotent of \( R \) which is orthogonal to \( e \). A regular right self-injective ring \( R \) is said to be **Type I** provided that it contains a faithful abelian idempotent, and \( R \) is **Type \( I_f \)** if \( R \) is Type I and directly finite.

**Note.** It is well-known from [4, Theorem 2] and [2, Lemma 7.17] that a regular ring \( R \) satisfies (*) if and only if the maximal right quotient ring of \( R \) is Type \( I_f \).

**Note.** Let \( R \) be a regular ring satisfying (*). If \( P \) is a finitely generated projective \( R \)-module, then \( \text{End}_R(P) \) is a regular ring satisfying (*).

**Proof.** Choose a positive integer \( n \) and an idempotent matrix \( e \in M_n(R) \) such that \( e(nR_R) \cong P \). Then \( \text{End}_R(P) \cong eM_n(R)e \). Using [2, Corollary 10.5], we see that \( eM_n(Q(R))e \cong Q(eM_n(R)e) \) is Type \( I_f \), where \( Q(R) \) is the maximal right quotient of \( R \). Since \( eM_n(R)e \leq eQ(eM_n(R)e) \) as an \( eM_n(R)e \)-module, we have that \( eM_n(R)e \) satisfies (*), and so has \( \text{End}_R(P) \).

**Notations.** Let \( A, B \) and \( A_i \) (\( i \in I \)) be \( R \)-modules, and \( k \) be a positive integer. Take \( x \in \Pi A_i \). Then we have some notations as following.

- \( A < B \); \( A \) is a submodule of \( B \).
- \( A \leq B \); \( B \) has a submodule isomorphic to \( A \).
- \( A < \oplus B \); \( A \) is a direct summand of \( B \).
- \( A \leq \oplus B \); \( B \) has a direct summand isomorphic to \( A \).
- \( A < _e B \); \( A \) is an essential submodule of \( B \).
- \( A \leq _e B \); \( B \) has an essential submodule isomorphic to \( A \).
- \( kA \); the \( k \)-copies of \( A \).
- \( x(i) \); the \( i \)-th component of \( x \).
- \( Q(R) \); the maximal right quotient ring of \( R \).

2. The property (DF) for regular rings satisfying (*)

**Lemma 2.1** ([2, Theorem 6.6]). Let \( R \) be a regular ring whose primitive factor rings are artinian. Then \( R \) satisfies (*).
Lemma 2.2. Let $R$ be a regular ring satisfying $(\ast)$. Then there exist abelian regular rings $\{S_t\}_{t \in T}$ and orthogonal central idempotents $\{e_t\}_{t \in T}$ of $R$ such that $R_R \cong \prod M_{n(t)}(S_t)$, $\oplus M_{n(t)}(S_t) \leq R$ and $e_t R = M_{n(t)}(S_t)$. Therefore $Q(R) \cong \prod M_{n(t)}(Q(S_t))$.

Proof. This theorem follows from [2, Lemma 7.17 and the proof of Theorem 7.18].

Lemma 2.3. Let $R$ be a regular ring of bounded index and $P$ be a finitely generated projective $R$-module. Then $P$ cannot contain a family $\{A_1, A_2, \ldots\}$ of nonzero finitely generated submodules such that $A_i \geq A_{i+1}$ and $iA_i \leq P$ for each $i = 1, 2, \ldots$.

Proof. By [2, Corollary 7.13], we see that $\text{End}_R(P)$ has bounded index. Note the claim in the proof of [2, Theorem 6.6], and applying [5, Lemma 5] to $\text{End}_R(P)$, we see that this lemma holds.

Theorem 2.4. Let $R$ be a regular ring satisfying $(\ast)$, and $P$ be a projective $R$-module with a cyclic decomposition $P = \bigoplus_{i \in I} P_i$. Then the following conditions (a)~(d) are equivalent:

(a) $P$ is directly infinite.
(b) There exists a nonzero cyclic projective $R$-module $X$ such that $X \leq \bigoplus_{i_1 \neq \cdots \neq i_n} P_i$ for any finite subset $\{i_1, \ldots, i_n\}$ of $I$.
(c) There exists a nonzero cyclic projective $R$-module $X$ such that $X \nleq \bigoplus_{i \in I} P_i$.
(d) There exists a nonzero cyclic projective $R$-module $X$ such that $X \nleq \bigoplus_{i \in I} P_i$.

Proof. It is clear that (a) $\rightarrow$ (b) and (c) $\rightarrow$ (d) $\rightarrow$ (a) hold, hence we shall prove that (b) $\rightarrow$ (c) holds. We may assume $\oplus M_{n(t)}(S_t) \leq R < \prod M_{n(t)}(S_t)$ for some set of abelian regular rings $\{S_t\}_{t \in T}$ by Lemma 2.2. Now we assume that (b) holds, hence there exists a nonzero principal right ideal $X$ of $R$ such that $X \nleq Q(X)$. Let $\{i_1, \ldots, i_n\}$ be a subset of $I$ and set $I' = I - \{i_1, \ldots, i_n\}$. Since $\oplus M_{n(t)}(S_t) \leq R < \prod M_{n(t)}(S_t)$, there exists $t' \in T$ such that $Y = [(\Pi_{i \neq t'} S_t) \times M_{n(t')} S_{t'}) \cap X \neq 0$. By the property of regular ring, it is clear that $Y$ is a principal right ideal of $R$. Then $\mathfrak{k}_0 Y \nleq P$, hence $Y \nleq \bigoplus P_i$. Thus for each $i \in I$, we have decompositions $P_i = P_i^1 \oplus P_i^{(1)}$ and $Y \nleq P_i^1 \oplus \cdots \oplus P_i^1 \oplus (\bigoplus_{t' \in T} P_i^{(1)}$. Set $(\Pi_{i \neq t'} S_t) \times M_{n(t')} S_{t'} = S$, and then there exists a central idempotent $e$ in $R$ such that $eR = S$.

Note that $S$ is a regular ring of bounded index. It is clear that

$$Y \otimes_R S \nleq (P_i^1 \otimes_R S) \oplus \cdots \oplus (P_i^1 \otimes_R S) \oplus (\bigoplus_{i \in I} (P_i^1 \otimes_R S))$$

and $2Y \otimes_R S \nleq P \otimes_R S$. Since $S$ is unit-regular, $Y \otimes_R S$ has the cancellation property. Hence
Thus for each $i$, we obtain that $P_i^{(1)} \otimes R S = \bar{P}_i^{(2)}$ for each $i \in I$ and

$$Y \otimes R S \simeq P_i^{(2)} \oplus \cdots \oplus P_{in}^{(2)} \oplus (\oplus_{t \in T} P_i^{(2)}).$$

Continuing this procedure, we have that $\bar{P}_i^{(m)} = \bar{P}_i^{m+1} \oplus \bar{P}_i^{(m+1)}$ and

$$Y \otimes R S \simeq \bar{P}_i^{m+1} \oplus \cdots \oplus \bar{P}_i^{m+1} \oplus (\oplus_{t \in T} P_i^{(2)}).$$

for each $i \in I$ and each positive integer $m$.

Now we set $A_m = \bar{P}_i^m \oplus \cdots \oplus \bar{P}_i^m$, where $A_1 = (P_i^{(1)} \otimes R S) \oplus \cdots \oplus (P_i^{(1)} \otimes R S)$. Then $A_1 \simeq \oplus A_2 \oplus (\oplus_{t \in T} P_i^{(2)})$, hence there exist a direct summand $B_2$ of $A_2$ and a direct summand $Q^2$ of $P_i^2$ such that $A_1 \simeq B_2 \oplus (\oplus_{t \in T} Q^2)$. Continuing this procedure, we obtain a family $\{B_1, B_2, \cdots\}$ ($A_1 = B_1$) of finitely generated projective $S$-submodules of $(P_i^{(1)} \oplus \cdots \oplus P_i^{(1)}) \otimes R S$ such that $B_m \simeq B_{m+1}$ and $mB_m \leq nS$ for all $m$. By Lemma 2.3, there exists a positive integer $k$ such that $B_m = 0$ for all $m \geq k$. Thus we have that $A_1 \simeq (\oplus_{t \in T} Q^2) \oplus \cdots \oplus (\oplus_{t \in T} Q^2)$ and $Y \otimes R S \simeq (\oplus_{t \in T} Q^2) \oplus \cdots \oplus (\oplus_{t \in T} Q^2)$.

Noting that $\mathbf{K}^0 \otimes R S < S$, we have that $Y \otimes R S \simeq (\oplus_{t \in T} P_i)$.

**Corollary 2.5.** Let $R$ be a regular ring satisfying (*). Then $R$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals. Hence $R$ is directly finite.

**Proof.** From Lemma 2.2, we may assume that, $R_R < e_\Pi M_{\alpha t}(S_i)$ for some abelian regular rings $\{S_i\}_{t \in T}$. Set $T = \Pi M_{\alpha t}(S_i)$. Now we assume that $R$ contains a direct sum of nonzero pairwise isomorphic right ideals, and so there exists a nonzero idempotent $e$ of $R$ such that $0 \neq R_0(eR) \leq R_R$. Then $R_0(eR) \otimes \mathcal{T} \leq R \otimes \mathcal{T}$, and so $R_0(eT) \leq T$, which contradicts to Theorem 2.4 because $T$ is a directly finite regular ring satisfying (*).

**Theorem 2.6.** Let $R$ be a regular ring satisfying (*) and $k$ be a positive integer. If $P$ is a directly finite projective $R$-module, then so is $kP$.

**Proof.** We may assume that $\oplus M_{\alpha t}(S_i) < R_R < [\Pi M_{\alpha t}(S_i)]_R$ for some abelian regular rings $\{S_i\}_{t \in T}$, and let $P = \oplus_{t \in T} P_i$ be a cyclic decomposition of $P$. It is sufficient to prove that this theorem holds in case $k=2$. Assume that $2P$ is directly infinite. Then Theorem 2.4 follows that there exists a nonzero principal right ideal $X$ of $R$ such that $X \leq \oplus_{t \in T} \otimes_{(i_1, \cdots, i_n) \neq 0} 2P_i$ for any finite subset $\{i_1, \cdots, i_n\}$ of $I$. By the proof of Theorem 2.4, we may assume that exists $t'$ of $T$ such that $X \subset (t' + t, 0) \times M_{\alpha t}(S_i)$. For any finite subset $\{i_1, \cdots, i_n\}$ of $I$, we have that $0 \neq X \otimes R S \leq \oplus_{t \in T} \otimes_{(i_1, \cdots, i_n) \neq 0}(2P_i \otimes R S)$. Since $S$ is a regular ring of bounded index, we see that $2(P \otimes R S)_S$ is directly infinite by Theorem 2.4 and so $(P \otimes R S)_S$ is directly
infinite by [5, Theorem 4]. Moreover, using Theorem 2.4 again, there exists a nonzero principal right ideal \( Y \) of \( S \) such that \( Y \leq \bigoplus_{i \in I - \{i_1, \ldots, i_n\}} (P_i \otimes_R S) \) for any finite subset \( \{i_1, \ldots, i_n\} \) of \( I \). Considering \( Y \) as an \( R \)-module, \( O \neq Y \leq \bigoplus_{i \in I - \{i_1, \ldots, i_n\}} P_i \). Therefore \( P \) is directly infinite, and so this theorem is complete.

**Corollary 2.7.** Let \( R \) be a regular ring satisfying (\(*\)). Then every finitely generated projective \( R \)-module is directly finite.

**Proof.** It is clear by Corollary 2.5 and Theorem 2.6.

**Corollary 2.8.** Let \( R \) be a regular ring satisfying (\(*\)).

(a) \( M_n(R) \) is directly finite for all positive integer \( n \), and so \( M_n(R) \) contains no infinite direct sums of nonzero pairwise isomorphic right ideals.

(b) If \( P \) and \( Q \) are finitely generated projective \( R \)-modules, then \( P \oplus Q \) is directly finite.

**Proof.** (a) \( R \) is a regular ring satisfying (\(*\)), and hence so is \( M_n(R) \). Therefore Corollary 2.5 shows that (a) holds. (b) follows from Corollary 2.7.

**Note.** In [1], Chuang and Lee have shown that there exists a regular ring satisfying (\(*\)) which is not unit-regular. Our Corollary 2.8 gives a partially solution for open problems 1 and 9 in Goodearl's book ([2]).

**Definition.** Let \( R \) be a regular ring and \( P \) be a projective \( R \)-module. We call that \( P \) satisfies (\( \# \)) provided that, for each nonzero finitely generated submodule \( I \) of \( P \) and any family \( \{A_1, B_1, \ldots\} \) of submodules of \( P \) with

\[
I = A_1 \oplus B_1,
A_i = A_{2i} \oplus B_{2i},
B_i = A_{2i+1} \oplus B_{2i+1}
\]

for each \( i = 1, 2, \ldots \), there exists a nonzero projective \( R \)-module \( X \) such that \( X \leq \bigoplus_{i=m}^{\infty} A_i \) or \( X \leq \bigoplus_{i=m}^{\infty} B \) for any positive integer \( m \).

**Lemma 2.9** ([5, Lemma 6]). Let \( P \) be a nonzero finitely generated projective module over a regular ring \( R \), and set \( T = \text{End}_R(P) \). Then the following conditions are equivalent:

(a) \( P \) satisfies (\( \# \)).

(b) \( T \) satisfies (\( \# \)) as a \( T \)-module.

**Lemma 2.10** ([5, Lemma 7]). Let \( P \) be a nonzero finitely generated projective module over a regular ring \( R \), and set \( T = \text{End}_R(P) \). Then the following conditions are equivalent:

(a) \( P \) satisfies (\( \# \)).

(b) \( T \) satisfies (\( \# \)) as a \( T \)-module.

Let \( P \) be a nonzero finitely generated projective
module over a regular ring \( R \), and set \( T = \text{End}_R(P) \). Then the following conditions are equivalent:

(a) \( R \) satisfies (\#) as an \( R \)-module.
(b) All nonzero finitely generated projective \( R \)-modules satisfy (\#).
(c) For any positive integer \( k \), \( kR \) satisfies (\#).
(d) There exists a positive integer \( k \) such that \( kR \) satisfies (\#).

**Theorem 2.11.** Let \( R \) be a regular ring satisfying (\#). Then the following conditions are equivalent:

(a) \( R \) has (DF).
(b) \( R \) satisfies (\#) as an \( R \)-module.
(c) For any nonzero finitely generated projective \( R \)-module \( P \), \( \text{End}_R(P) \) has (DF).
(d) For any positive integer \( k \), \( M_k(R) \) has (DF).
(e) There exists a positive integer \( k \) such that \( M_k(R) \) satisfies (DF).

Proof. Note that \( \text{End}_R(P) \) is a regular ring with (\#). [5, Theorem 8] was proved only using [5, Theorem 2]. Now [5, Theorem 2] holds on a regular ring satisfying (\#) by Theorem 2.4. Hence we see that this theorem holds under this condition using the similar proof of [5, Theorem 8] (Note that the unit-regularity is not needed).

3. Some applications

**Lemma 3.1.** Let \( R \) be a regular ring satisfying (\#), and let \( \{e_i\} \) be a set of nonzero orthogonal central idempotents of \( R \) such that \( \oplus e_iR \neq eR \). Then \( R \) has (DF) if and only if \( e_iR \) has (DF) for all \( i \).

Proof. Note that \( e_iR \) is a ring direct summand of \( R \). It is clear from Theorem 2.11 that "only if" part holds. We shall prove that "if" part holds. Let \( I \) be a nonzero direct summand of \( R \), and so \( e_iR \cap I \neq 0 \) for some \( i \). Setting \( J = e_iR \cap I \), \( J \) is a principal right ideal of both \( R \) and \( e_iR \). We consider decompositions

\[
I = A_1 \oplus B_1 \\
A_j = A_{2j} \oplus B_{2j} \\
B_j = A_{2j+1} \oplus B_{2j+1}
\]

for each \( j = 1, 2, \ldots \), and so there exist decompositions of \( J \) such that

\[
J = C_1 \oplus D_1 \\
C_j = C_{2j} \oplus D_{2j}
\]
\[ D_j = C_{2j+1} \oplus D_{2j+1} \]

\[ C_j \leq \oplus A_j \quad \text{and} \quad D_j \leq \oplus B_j \quad \text{for each } j = 1, 2, \ldots. \]

By the assumption, there exists a nonzero cyclic projective \( e_j R \)-module \( X \) such that \( X \leq \oplus_{j=m}^n C_j \) or \( X \leq \oplus_{j=m}^n D_j \) for each positive integer \( m \). Hence \( X \otimes e_j R \leq \oplus_{j=m}^n (C_j \otimes e_j R) \) or \( X \otimes e_j R \leq \oplus_{j=m}^n (D_j \otimes e_j R) \). Note that \( \oplus_{j=m}^n (C_j \otimes e_j R) \leq \oplus_{j=m}^n A_j \) and \( \oplus_{j=m}^n (D_j \otimes e_j R) \leq \oplus_{j=m}^n B_j \). Therefore \( X \otimes e_j R \leq \oplus_{j=m}^n A_j \) or \( X \otimes e_j R \leq \oplus_{j=m}^n B_j \). Since \( X \otimes e_j R \neq 0 \), this lemma has proved by Theorem 2.11.

**Lemma 3.2** ([6, Proposition 2.1]). Let \( R \) be an abelian regular ring. If \( Q(R) \) has (DF), then so has \( R \).

**Theorem 3.3.** Let \( R \) be a regular ring satisfying (*). If \( Q(R) \) has (DF), then so does \( R \).

**Proof.** By Lemma 2.2, we may assume that there exists a set \( \{S_t\} \) of abelian regular rings such that \( R \leq \prod M(S_t) \). Then \( Q(R) = \prod M(Q(S_t)) \). Assume that \( Q(R) \) has (DF), then so does \( M(Q(S_t)) \) for all \( t \) by Lemma 3.1. Moreover, Theorem 2.11 shows that \( Q(S_t) \) also has (DF), hence so has \( S_t \) by Lemma 3.2. Thus \( M(Q(S_t)) \) also has (DF) by Theorem 2.11. There exists the set \( \{e_t\} \) of orthogonal central idempotents of \( R \) such that \( e_t R = M(Q(S_t)) \times [\prod t \neq t] \) and \( \oplus e_t R \leq R \). Therefore \( R \) has (DF) by Lemma 3.1.

Now we shall give an example of a regular ring with a zero socle satisfying (*), which has (DF), as follows.

**Example 3.4.** Let \( F \) be a field, and set \( R = \prod_{i=1}^\infty F_i \) and \( \bar{R} = R / \text{soc}(R) \). Then \( \bar{R} \) is a regular ring satisfying (*), which has (DF).

**Proof.** Since it is clear that \( \bar{R} \) is a regular ring satisfying (*), we shall prove that \( \bar{R} \) has (DF) using Theorem 2.11. Let \( \Psi \) be the natural map from \( R \) to \( \bar{R} \), and let \( I \) be a nonzero direct summand of \( \bar{R} \) with following decompositions:

\[ I = A_1 \oplus B_1 \]

\[ A_i = A_{2i} \oplus B_{2i} \]

\[ B_i = A_{2i+1} \oplus B_{2i+1} \quad \text{for } i = 1, 2, \ldots. \]

Now assume that there does not exist \( \{C_j\} \) (\( C_j = A_j \) for some \( i \)) which is an infinite subset of \( \{A_i\}_{i=1}^\infty \) such that \( C_i > C_{i+1} \) and \( C_j \neq 0 \) for all \( j \). Let \( \{D_p\} \) (\( D_p = A_i \) for some \( i \)) be an infinite decreasing sequence of \( \{A_i\} \), and so there exists a positive integer \( p' \) such that \( D_p = 0 \) (\( p \leq p' \)). Hence \( 0 = D_{p'} = A_i \) for some
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Thus \( E_q \neq 0 \). Next, we take \( \{E_q\} \) (\( E_q = A_i \) for some \( i \)) which is an infinite decreasing sequence of \( \{A_i\} \), where \( E_q < B_{i_1} \) and \( B_{k_2} < B_{i_1} \) (\( A_{k_2} = E_q \)) for all positive integer \( q \). Similarly, there exists a positive integer \( q' \) such that \( E_q = 0 \) (\( q' \leq q \)). Hence there exists a positive integer \( i_2 (i_2 > i_1) \) such that \( E_q = A_{i_2} = 0 \). Therefore \( B_{i_2} \neq 0 \) and \( B_{i_1} > B_{i_2} \). Continuing this procedure, we can get an infinite set \( \{B_{i_k}\} \) such that \( \{B_{i_1} \} \supset \{B_{i_k}\} \) and \( B_{i_k} \neq 0 \) for all \( k \). From the above, we may assume that there exists an infinite decreasing sequence \( \{C_j\} \) such that \( \{A_j\} \supset \{C_j\} \), \( C_j > C_{j+1} \) and \( C_j \neq 0 \) for all \( j \).

We have a set \( \{e_j\} \) of idempotents of \( R \) such that \( \Psi(e_j R) = C_j \) and \( e_j R \supset e_{j+1} R \) for all \( j \). We take an idempotent \( f_1 (e_1 R) \) with \( \dim_f (f_1 R) = 1 \). Next we take an idempotent \( f_2 (e_2 R) \) such that \( \dim_f (f_2 R) = 1 \) and \( f_1 f_2 = 0 \). Continuing this procedure, we can take a set \( \{f_j\} \) of orthogonal idempotents of \( R \). Set \( e = \bigvee f_j \) and then \( \Psi(e) \neq 0 \). We have that \( e R = J \oplus (e R \cap e_j R) \) and \( J \subset \oplus F_i \) for some right ideal \( J \). Noting that \( J \otimes R = 0 \), we have that

\[
0 \neq \Psi(e) \bar{R} \cong e R \otimes_R \bar{R} \
\cong [J \oplus (e R \cap e_j R)] \otimes_R \bar{R} \
\cong e_j R \otimes_R \bar{R} \
\cong C_j \quad \text{for all } j.
\]

Therefore \( 0 \neq \Psi(e) \bar{R} \leq \oplus_{i=0}^\infty A_i \) for any positive integer \( m \). Hence \( \bar{R} \) has (DF) by Theorem 2.11.

By Example 3.4, we have a problem that, for any regular ring \( S, R = (\Pi_{i=0}^\infty S) / (\oplus S) \) satisfies (DF) or not. Example 3.5 shows that, even if \( S \) satisfies (\( * \)), \( R \) does not satisfy (\( * \)). Therefore we shall give the necessary and sufficient condition for that \( R \) satisfies (\( * \)), and we solve the above problem under this condition.

**Example 3.5.** Let \( F \) be a field and set \( S = \Pi_{i=0}^\infty M_n(F), \ \tilde{S} = S / (\oplus M_n(F)), \ T = \Pi_{i=1}^\infty S_1(S_1 = S), \) and \( R = T / (\oplus S_0) \). Then \( S \) satisfies (\( * \)), but \( R \) does not satisfy (\( * \)).

Proof. It is clear that \( S \) satisfies (\( * \)). Therefore we shall show that \( R \) does not satisfy (\( * \)). Set a central idempotent \( e \in T \) as following;

\[
e(n) = (0, \ldots, 0, \left[ \begin{array}{c} 1 \\ 1 \end{array} \right], 0, 0, \ldots)
\]

where \( e(n) \in S_n \).

Let \( \Phi \) be the natural map from \( S \) to \( \tilde{S} \), and \( \rho \) be the natural map from \( T \) to \( R \). Set \( \Psi = \rho|_T \).

Noting that \( e(n) S_n \cong M_n(F) \), we have \( e T \cong S \). Hence there exists a ring
isomorphism \( \kappa \) from \( eT \) to \( S \). Now, we define a ring homomorphism \( \alpha \) from \( \Psi(e)R \) to \( S \) as following; for each \( x \in \Psi(e)R \), we take any element \( y \) of \( \Psi^{-1}(x) \) and set \( \alpha(x) = \Phi \kappa(y) \).

\[
\Psi(e)R \xrightarrow{\alpha} S \\
\uparrow_{\Psi} \quad \uparrow_{\Phi} \\
eT \xrightarrow{\kappa} S
\]

Similarly we define a ring homomorphism \( \beta \) from \( S \) to \( \Psi(e)\Lambda \). Then we have that \( \beta \alpha = 1_{\Psi(e)R} \) and \( \alpha \beta = 1_{S} \). Hence \( \alpha \) and \( \beta \) are isomorphic. Therefore \( \Psi(e)R \cong S \). Let \( I \) be a nonzero two-sided ideal of \( S \), and so \( I = J/(\oplus S_i) \) for some nonzero two-sided ideal of \( S \) which contains \( \oplus S_i \). There exists \( 0 \neq x \in J - (\oplus S_i) \) with \( x(i) \neq 0 \) for almost all \( i \). Since \( S_i x(i) S_i = M_i(F) \) has index \( i \), there exists a nonzero central idempotent \( e(i) \) of \( M_i(F) \) which \( S_i x(i) S_i \) has index \( i \). Therefore \( SxS \) does not have bounded index, and so does not \( J/(\oplus S_i) \). Therefore \( S \) does not satisfy \( (*) \), and hence so does not \( \Psi(e)R \). Thus \( R \) does not satisfy \( (*) \).

**Lemma 3.6.** Let \( R \) be a ring, and \( e, f \) be idempotents of \( R \). Then \( eR \cong fR \) if and only if there exist \( u \) and \( v \) of \( R \) such that \( vu = e \) and \( vw = f \).

**Lemma 3.7.** Let \( S \) be a regular ring which has index \( \infty \), and set \( R = (\Pi_{i=1}^{\infty} S_i)/(\oplus S_i) \) (\( S_i = S \)). Then \( R \) has an infinite direct sum of nonzero pairwise isomorphic right ideals.

**Proof.** Let \( \Psi \) be the natural map from \( \Pi_{i=1}^{\infty} S_i \) to \( R \). Since \( S \) has index \( \infty \), there exists a set of idempotents \( \{ e_{ij} \}_{i,j=1,2,...} \) as following:

\[
e_{11}S \\
e_{21}S \approx e_{22}S \\
e_{31}S \approx e_{32}S \approx e_{33}S \\
.....
\]

where \( e_{ij} = 0 \) (\( i < j \)) and \( \{ e_{i1},...,e_{ii} \} \) are nonzero orthogonal for all \( i \). For all positive integer \( m \), we take idempotents \( \{ f_m \} \) such that \( f_m(k) = e_{km} \) for all positive integer \( k \). Since \( e_{k1}S \approx e_{k2}S \) for all \( k \), there exist \( u_k \) and \( v_k \) of \( S \) such that \( u_k v_k = e_{k2} \) and \( v_k u_k = e_{k1} \) by Lemma 3.6. Set \( u \) and \( v \) of \( \Pi_{i=1}^{\infty} S_i \) such that \( u(k) = u_k \) and \( v(k) = v_k \). Then \( uv = f_2 \) and \( vu = f_1 - e \), where \( e \) is an idempotent with \( e(1) = e_{11} \) and \( e(k) = 0 \) (\( k \neq 1 \)). Hence \( (f_1 - e)(\Pi S_i) \approx f_2(\Pi S_i) \) and \( (f_1 - e)(\Pi S_i) \cap f_2(\Pi S_i) = 0 \). Therefore we see from Lemma 3.6 that \( \Psi(f_1 - e)R \cong \Psi(f_2)R \) and \( \Psi(f_1 - e)R \cap \Psi(f_2)R = 0 \). Since \( \Psi(f_1 - e)R = \Psi(f_1)R \), we have that \( \Psi(f_1)R \cong \Psi(f_2)R \) and \( \Psi(f_1)R \cap \Psi(f_2)R = 0 \).
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Continuing this procedure, for all positive integers $i$ and $j$, $\Psi(f_i)R \simeq \Psi(f_j)R$ and $\Psi(f_i) \cap \Psi(f_j)R = 0$ ($i \neq j$). Thus $R$ has an infinite direct sum of nonzero pairwise isomorphic right ideals.

**Theorem 3.8.** Let $S$ be a regular ring, and set $R = (\Pi_{i=1}^\infty S_i)/(\oplus S_i)$ ($S_i = S$). Then the following conditions are equivalent:

(a) $R$ satisfies ($\ast$).

(b) $R$ is a regular ring whose primitive factor rings are artinian.

(c) $R$ has bounded index.

(d) $R$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals.

(e) $S$ has bounded index.

Proof. It is clear by Lemma 3.7 that (d) $\rightarrow$ (e) $\rightarrow$ (c) $\rightarrow$ (b) $\rightarrow$ (a) hold. (a) $\rightarrow$ (d) follows from Corollary 2.5. Therefore this theorem is complete.

**Theorem 3.9.** Let $S$ be a regular ring of bounded index. Set $R = (\Pi_{n=1}^\infty S_n)/(\oplus S_n)$ ($S_n = S$). Then $R$ has (DF).

Proof. Set $\Pi_{n=1}^\infty S_n = T$, and let $\Psi$ be the natural map from $T$ to $R$. Let $I$ be a nonzero direct summand of $R$ with following decompositions:

$I = A_1 \oplus B_1$

$A_i = A_{2i} \oplus B_{2i}$

$B_i = A_{2i+1} \oplus B_{2i+1}$ for $i = 1, 2, \ldots$.

Similarly to the proof of Example 3.4, we may assume that there exists an infinite subset $\{C_j\}$ of $\{A_i\}$ ($C_j = A_i$ for some $i$) such that $C_j > C_{j+1}$ and $C_j \neq 0$ for all positive integer $j$. We have the set of idempotents $\{e_j\}$ of $T$ such that $\Psi(e_j) = C_j$ and $e_j T > e_{j+1} T$. Set $J_n = S_n \times (\Pi_{i \neq n}^\infty S_i)$. Then, $J_{n_1} \cap e_1 T \neq 0$ for some positive integer $n_1$. There exists a nonzero idempotent $f_1 \in T$ such that $f_1 T = J_{n_1} \cap e_1 T$. Next we have a nonzero idempotent $f_2 \in T$ for some $n_2$ ($> n_1$) such that $f_2 R = J_{n_2} \cap e_2 R$. Continuing this procedure, we have the set $\{f_j\}$ of orthogonal idempotents of $T$. Now, we set an idempotent $g$ of $T$ as following:

$g(n_j) = f_j(n_j) = e_j(n_j)$

$g(k) = 0$ ($k \notin \{n_j\}$).

Put $K_j = f_1 T \oplus \cdots \oplus f_{j-1} T$ for all $j$. Then $g T = K_j \oplus (g T \cap e_j T)$. Noting $K_j \oplus T R = 0$, we have that
0 \neq \Psi(g)R \cong gT \otimes_T R
\cong [K_j \otimes (gT \cap e_j T)] \otimes_T R
\cong (gT \cap e_j T) \otimes_T R
\cong e_j T \otimes_T R
\cong C_j \quad \text{for all } j.

From the above, we have that \( \Psi(g)R \leq \bigoplus_{i=m}^\infty A_i \) for any positive integer \( m \). Therefore \( R \) has (DF) by Theorem 2.11.

References


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