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# ON REGULAR RINGS WHOSE MAXIMAL RIGHT QUOTIENT RINGS ARE TYPE $I_{f}$ 

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## Introduction

This paper is written about the property (DF) on regular rings whose maximal right quotient rings are Type $I_{f}$. Hereafter regular rings whose maximal right quotient rings are Type $I_{f}$ are said to satisfy (*). The property (DF) is very important property when we study on regular rings satisfying (*), and it was treated in the paper [5] written by the first author, where (DF) for a ring $R$ is defined as that if the direct sum of any two directly finite projective $R$-modules is always directly finite. In the above paper, the equivalent condition that a regular ring $R$ of bounded index satisfies (DF) was discovered and called (\#). Stillmore, we proved that the condition (DF) is equivalent to (\#) for regular rings whose primitive factor rings are artinian in the paper [6]. Then we have the problem that (DF) is equivalent (\#) for regular rings satisfying (*) or not, where the condition (*) is weaker than one that primitive factor rings are artinian.

In §2, we shall prove Theorem 2.4. This is important, and using this, Theorem 2.5 (i.e. if $R$ is a regular ring satisfying $(*)$ and k is any positive integer, then $k P$ is directly finite for every directly finite projective $R$-module $P$ ) is proved. Moreover, we shall solve the above problem in Theorem 2.11.

In § 3, we shall consider some applications of Theorem 2.11. We prove Theorem 3.3 that if $R$ is a regular ring satisfying (*) whose maximal right quotient ring of $R$ satisfies (DF), then so does $R$. Though it is clear that a regular rings satisfying (*) which has a nonzero essential socle satisfies (DF), we can prove that, for regular rings satisfying $(*)$, the condition having a nonzero essential socle is not equivalent to (\#) in Example 3.4. Next, we shall consider that $\left(\Pi_{1}^{\infty} R\right) /(\oplus R)$ satisfies (DF) or not for a regular ring $R$ satisfying (*). This problem is a generalization of Example 3.4, and we prove that, for a regular ring $R$ of bounded index, $\left(\Pi_{1}^{\infty} R\right) /(\oplus R)$ satisfies (DF) (Theorem 3.9).

Throughout this paper, $R$ is a ring with identity and $R$-modules are unitary right $R$-modules.

## 1. Definitions and notations

Definition 1. A ring $R$ is (Von Neumann) regular provided that for every $x \in R$ there exists $y \in R$ such that $x y x=x$.

Note. Every projective modules over regular rings have the exchange property.
Definition 2. A module $M$ is directly finite provided that $M$ is not isomorphic to a proper direct summand of itself. If $M$ is not directly finite, then $M$ is said to be directly infinite. A ring $R$ is said to be directly finite (resp. directly infinite) if so is $R$ as an $R$-module.

Definition 3. The index of a nilpotent element $x$ in a ring $R$ is the least positive integer such that $x^{n}=0$ (In particular, 0 is nilpotent of index 1). The index of a two-sided ideal $J$ of $R$ is the supremum of the indices of all nilpotent elements of $J$.
If this supremum is finite, then $J$ is said to have bounded index. If $J$ does not have bounded index, J is said to be index $\infty$.

Note. Let $R$ be a regular ring with index $\infty$. Then using [3, the proof of Lemma 2], there exists a family $\left\{A_{n}\right\}_{n=1}^{\infty}$ of independent right ideals of $R$ such that $A_{n}$ contains a direct sum of $n$ nonzero pairwise isomorphic right ideals. Therefore $R$ has a family $\left\{e_{i j}\right\}_{i, j=1,2, \ldots}$ of idempotents such that

$$
\begin{aligned}
& e_{21} R \simeq e_{22} R \\
& e_{31} R \simeq e_{32} R \simeq e_{33} R
\end{aligned}
$$

, where $e_{i j}=0(i<j)$, and $\left\{e_{i 1}, \cdots, e_{i i}\right\}$ are orthogonal for all $i$.
Defintion 4. A ring $R$ has (DF) if the direct sum of two directly finite projective $R$-modules is directly finite.

Defintition 5. A regular ring $R$ is abelian provided all idempotents in $R$ are central.

Definition 6. A ring $R$ satisfies (*) if every nonzero two-sided ideal of $R$ contains a nonzero two-sided ideal of bounded index.

Definition 7. A ring $R$ is unit-regular provided that for each $x \in R$ there is a unit $u \in R$ such that $x u x=x$.

Note. Every finitely generated projective module over a unit-regular ring has the cancellation property ([2, Theorem 4.14]).

Definition 8. Let $e$ be an idempotent in a regular ring $R$. Then $e$ is called an abelian idempotent (of $R$ ) whenever the ring $e R e$ is abelian.

Definition 9. Let $e$ be an idempotent in a regular right self-injective ring $R$. Then e is faithful (in $R$ ) if 0 is the only central idempotent of $R$ which is orthogonal to $e$. A regular right self-injective ring $R$ is said to be Type $I$ provided that it contains a faithful abelian idempotent, and $R$ is Type $I_{f}$ if $R$ is Type $I$ and directly finite.

Note. It is well-known from [4, Theorem 2] and [2, Lemma 7.17] that a regular ring $R$ satisfies (*) if and only if the maximal right quotient ring of $R$ is Type $I_{f}$.

Note. Let $R$ be a regular ring satisfying (*). If $P$ is a finitely generated projective $R$-module, then $\operatorname{End}_{R}(P)$ is a regular ring satisfying (*).

Proof. Choose a positive integer $n$ and an idempotent matrix $e \in M_{n}(R)$ such that $e\left(n R_{R}\right) \simeq P$. Then $\operatorname{End}_{R}(P) \simeq e M_{n}(R) e$. Using [2, Corollary 10.5], we see that $e M_{n}(Q(R)) e \simeq Q\left(e M_{n}(R) e\right)$ is Type $I_{f}$, where $Q(R)$ is the maximal right quotient of $R$. Since $e M_{n}(R) e \leqq \leqq_{e} Q\left(e M_{n}(R) e\right)$ as an $e M_{n}(R) e$-module, we have that $e M_{n}(R) e$ satisfies (*), and so has $\operatorname{End}_{R}(P)$.

Notations. Let $A, B$ and $A_{i}(i \in I)$ be $R$-modules, and $k$ be a positive integer. Take $x \in \Pi A_{i}$. Then we have some notations as following.
$A<B ; A$ is a submodule of $B$.
$A \lesssim B ; B$ has a submodule isomorphic to $A$.
$A<\oplus B ; A$ is a direct summand of $B$.
$A \lesssim \oplus B ; B$ has a direct summand isomorphic to $A$.
$A<{ }_{e} B ; A$ is an essential submodule of $B$.
$A \lesssim_{e} B ; B$ has an essential submodule isomorphic to $A$.
$k A$; the $k$-copies of $A$.
$x(i)$; the $i$-th component of $x$.
$Q(R)$; the maximal right quotient ring of $R$.

## 2. The property (DF) for regular rings satisfying (*)

Lemma 2.1 ([2, Theorem 6.6]). Let $R$ be a regular ring whose primitive factor rings are artinian. Then $R$ satisfies (*).

Lemma 2.2. Let $R$ be a regular ring satisfying (*). Then there exist abelian regular rings $\left\{S_{t}\right\}_{t \in T}$ and orthogonal central idempotents $\left\{e_{t}\right\}_{t \in T}$ of $R$ such that $R_{R} \lesssim_{e}\left[\Pi M_{n(t)}\left(S_{t}\right)\right]_{R}, \oplus M_{n(t)}\left(S_{t}\right) \lesssim R$ and $e_{t} R=M_{n(t)}\left(S_{t}\right)$. Therefore $Q(R) \simeq$ $\Pi M_{n(t)}\left(Q\left(S_{t}\right)\right)$.

Proof. This theorem follows from [2, Lemma 7.17 and the proof of Theorem 7.18].

Lemma 2.3. Let $R$ be a regular ring of bounded index and $P$ be a finitely generated projective $R$-module. Then $P$ can not contain a family $\left\{A_{1}, A_{2}, \cdots\right\}$ of nonzero finitely generated submodules such that $A_{i} \gtrsim A_{i+1}$ and $i A_{i} \lesssim P$ for each $i=1,2, \cdots$.

Proof. By [2, Corollary 7.13], we see that $\operatorname{End}_{R}(P)$ has bounded index. Note the claim in the proof of [2, Theorem 6.6], and applying [5, Lemma 5] to $\operatorname{End}_{R}(P)$, we see that this lemma holds.

Theorem 2.4. Let $R$ be a regular ring satisfying (*), and $P$ be a projective $R$-module with a cyclic decomposition $P=\oplus_{i \in I} P_{i}$. Then the following conditions (a) $\sim(d)$ are equivalent:
(a) $P$ is directly infinite.
(b) There exists a nonzero cyclic projective $R$-module $X$ such that $\aleph_{0} X \leqq P$.
(c) There exists a nonzero cyclic projective $R$-module $X$ such that $X \lesssim$ $\oplus_{i \in I-\left\{i_{1}, \cdots, i_{n}\right\}} P_{i}$ for any finite subset $\left\{i_{1}, \cdots, i_{n}\right\}$ of $I$.
(d) There exists a nonzero cyclic projective $R$-module $X$ such that $\aleph_{0} X \lesssim \oplus P$.

Proof. It is clear that $($ a $) \rightarrow$ (b) and $(c) \rightarrow(d) \rightarrow$ (a) hold, hence we shall prove that (b) $\rightarrow$ (c) holds. We may assume $\oplus M_{n(t)}\left(S_{t}\right)<R_{R}<{ }_{e}\left[\Pi M_{n(t)}\left(S_{t}\right)\right]_{R}$ for some set of abelian regular rings $\left\{S_{t}\right\}_{t \in T}$ by Lemma 2.2. Now we assume that (b) holds, hence there exists a nonzero principal right ideal $X$ of $R$ such that $\aleph_{0} X \leqq P$. Let $\left\{i_{1}, \cdots, i_{n}\right\}$ be a subset of $I$ and set $I^{\prime}=I-\left\{i_{1}, \cdots, i_{n}\right\}$. Since $\oplus M_{n(t)}\left(S_{t}\right)<R_{R}<_{e}\left[\Pi M_{n(t)}\left(S_{t}\right)\right]_{R}$, there exists $t^{\prime} \in T$ such that $Y=\left[\left(\Pi_{t^{\neq t^{\prime}}} 0\right) \times M_{n\left(t^{\prime}\right)}\left(S_{t}\right)\right] \cap X \neq 0$. By the property of regular ring, it is clear that $Y$ is a principal right ideal of $R$. Then $\aleph_{0} Y \lesssim$ $P$, hence $Y \lesssim \oplus P$. Thus for each $i \in I$, we have decompositions $P_{i}=P_{i}^{1} \oplus P_{i}^{(1)}$ and $Y \simeq P_{i_{1}}^{1} \oplus \cdots \oplus P_{i_{n}}^{1} \oplus\left(\oplus_{i \in I} P_{i}^{\prime}\right)$. Set $\left(\Pi_{t^{\neq t^{\prime}}} 0\right) \times M_{n\left(t^{\prime}\right)}\left(S_{t^{\prime}}\right)=S$, and then there exists a central idempotent $e$ in $R$ such that $e R=S$.
Note that $S$ is a regular ring of bounded index. It is clear that

$$
Y \otimes_{R} S_{S} \simeq\left(P_{i_{1}}^{1} \otimes_{R} S\right) \oplus \cdots \oplus\left(P_{i_{n}}^{1} \otimes_{R} S\right) \oplus\left[\oplus_{i \in I}\left(P_{i}^{1} \otimes_{R} S_{s}\right)\right]
$$

and $2 Y \otimes_{R} S_{S} \leqslant P \otimes_{R} S_{S}$. Since $S$ is unit-regular, $Y \otimes_{R} S_{S}$ has the cancellation property. Hence

$$
Y \otimes_{R} S_{S} \leqslant \oplus\left(P_{i_{1}}^{(1)} \otimes_{R} S\right) \oplus \cdots \oplus\left(P_{i_{n}}^{(1)} \otimes_{R} S\right) \oplus\left[\oplus_{i \in I^{\prime}}\left(P_{i}^{(1)} \otimes_{R} S_{s}\right)\right]
$$

Thus for each $i$, we obtain that $P_{i}^{(1)} \otimes_{R} S_{S}=\bar{P}_{i}^{2} \oplus \bar{P}_{i}^{(2)}$ for each $i \in I$ and

$$
Y \otimes_{R} S_{S} \simeq \bar{P}_{i_{i}}^{2} \oplus \cdots \oplus \bar{P}_{i_{n}}^{2} \oplus\left(\oplus_{i \in I} \bar{P}_{i}^{2}\right)
$$

Continuing this procedure, we have that $\bar{P}_{i}^{(m)}=\bar{P}_{i}^{m+1} \oplus \bar{P}_{i}^{(m+1)}$ and

$$
Y \otimes_{R} S_{S} \simeq \bar{P}_{i_{1}}^{m+1} \oplus \cdots \oplus \bar{P}_{i_{n}}^{m+1} \oplus\left(\oplus_{i \in I} \bar{P}_{i}^{m+1}\right)
$$

for each $i \in I$ and each positive integer $m$.
Now we set $A_{m}=\bar{P}_{i_{1}}^{m} \oplus \cdots \oplus \bar{P}_{i_{n}}^{m}$, where $A_{1}=\left(P_{i_{1}}^{1} \otimes_{R} S\right) \oplus \cdots \oplus\left(P_{i_{n}}^{1} \otimes_{R} S\right)$. Then $A_{1} \lesssim \oplus A_{2} \oplus\left(\oplus_{i \in I} \bar{P}_{i}^{2}\right)$, hence there exist a direct summand $B_{2}$ of $A_{2}$ and a direct summand $Q_{i}^{2}$ of $\bar{P}_{i}^{2}$ such that $A_{1} \simeq B_{2} \oplus\left(\oplus_{i \in I} Q_{i}^{2}\right)$. Continuing this procedure, we obtain a family $\left\{B_{1}, B_{2}, \cdots\right\}\left(A_{1}=B_{1}\right)$ of finitely generated projective $S$-submodules of $\left(P_{i_{1}} \oplus \cdots \oplus P_{i_{n}}\right) \otimes_{R} S$ such that $B_{m} \gtrsim B_{m+1}$ and $m B_{m} \lesssim n S$ for all $m$. By Lemma 2.3, there exists a positive integer $k$ such that $B_{m}=0$ for all $m(\geqq k)$. Thus we have that $A_{1} \simeq\left(\oplus_{i \in I^{\prime}} Q_{i}^{2}\right) \oplus \cdots \oplus\left(\oplus_{i \in I} Q_{i}^{k}\right)$ and $Y \otimes_{R} S_{S} \simeq\left(\oplus_{i \in I} Q_{i}^{1}\right) \oplus \cdots \oplus\left(\oplus_{i \in I} Q_{i}^{k}\right)$. Noting that $0 \neq Y<S$, we have that $Y_{R} \lesssim \oplus_{i \in I} P_{i}$.

Corollary 2.5. Let $R$ be a regular ring satisfying (*). Then $R$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals. Hence $R$ is directly finite.

Proof. From Lemma 2.2, we may assume that, $R_{R}<_{e} \Pi M_{n(t)}\left(S_{t}\right)$ for some abelian regular rings $\left\{S_{t}\right\}_{t \in T}$. Set $T=\Pi M_{n(t)}\left(S_{t}\right)$. Now we assume that $R$ contains a direct sum of nonzero pairwise isomophic right ideals, and so there exists a nonzero idempotent $e$ of $R$ such that $0 \neq \aleph_{0}(e R) \lesssim R_{R}$. Then $\aleph_{0}(e R) \otimes_{R} T \lesssim R \otimes_{R} T$, and so $\mathbb{K}_{0}(e T) \lesssim T$, which contradicts to Theorem 2.4 because $T$ is a directly finite regular ring satisfying (*).

Theorem 2.6. Let $R$ be a regular ring satisfying (*) and $k$ be a positive integer. If $P$ is a directly finite projective $R$-module, then so is $k P$.

Proof. We may assume that $\oplus M_{n(t)}\left(S_{t}\right)<R_{R}<{ }_{e}\left[\Pi M_{n(t)}\left(S_{t}\right)\right]_{R}$ for some abelian regular rings $\left\{S_{t}\right\}_{t \epsilon T}$, and let $P=\oplus_{i \in I} P_{i}$ be a cyclic decomposition of $P$. It is sufficient to prove that this theorem holds in case $k=2$. Assume that $2 P$ is directly infinite. Then Theorem 2.4 follows that there exists a nonzero principal right ideal $X$ of $R$ such that $X \leqq \oplus_{i \in I-\left\{i_{1}, \cdots, i_{n}\right\}} 2 P_{i}$ for any finite subset $\left\{i_{1}, \cdots, i_{n}\right\}$ of $I$. By the proof of Theorem 2.4, we may assume that exists $t^{\prime}$ of $T$ such that $X<\left(\Pi_{t \neq t^{\prime}} 0\right) \times M_{n\left(t^{\prime}\right)}\left(S_{t^{\prime}}\right)=S$. For any finite subset $\left\{i_{1}, \cdots, i_{n}\right\}$ of $I$, we have that $0 \neq X \otimes_{R} S_{S} \lesssim \oplus_{i \in I-\left\{i_{1}, \cdots, i_{n}\right\}}\left(2 P_{i} \otimes_{R} S\right)$. Since $S$ is a regular ring of bounded index, we see that $2\left(P \otimes_{R} S\right)_{S}$ is directly infinite by Theorem 2.4 and so $\left(P \otimes_{R} S\right)_{S}$ is directly
infinite by [5, Theorem 4]. Moreover, using Theorem 2.4 again, there exists a nonzero principal right ideal $Y$ of $S$ such that $Y \lesssim \oplus_{i \in I-\left\{i_{1}, \cdots, i_{n}\right\}}\left(P_{i} \otimes_{R} S_{S}\right)$ for any finite subset $\left\{i_{1}, \cdots, i_{n}\right\}$ of $I$. Considering $Y$ as an $R$-module, $O \neq Y_{R} \lesssim \oplus_{I-\left\{i_{1}, \cdots, i_{n}\right\}} P_{i}$. Therefore $P$ is directly infinite, and so this theorem is complete.

Corollary 2.7. Let $R$ be a regular ring satisfying (*). Then every finitely generated projective $R$-module is directly finite.

Proof. It is clear by Corollary 2.5 and Theorem 2.6.
Corollary 2.8. Let $R$ be a regular ring satisfying (*).
(a) $M_{n}(R)$ is directly finite for all positive integer $n$, and so $M_{n}(R)$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals.
(b) If $P$ and $Q$ are finitely generated projective $R$-modules, then $P \oplus Q$ is directly finite.

Proof. (a) $R$ is a regular ring satisfying (*), and hence so is $M_{n}(R)$. Therefore Corollary 2.5 shows that (a) holds. (b) follows from Corollary 2.7.

Note. In [1], Chuang and Lee have shown that there exists a regular ring satisfying (*) which is not unit-regular. Our Corollary 2.8 gives a partially solution for open problems 1 and 9 in Goodearl's book ([2]).

Definition. Let $R$ be a regular ring and $P$ be a projective $R$-module. We call that $P$ satisfies (\#) provided that, for each nozero finitely generated submodule $I$ of $P$ and any family $\left\{A_{1}, B_{1}, \cdots\right\}$ of submodules of $P$ with

$$
\begin{aligned}
& I=A_{1} \oplus B_{1} \\
& A_{i}=A_{2 i} \oplus B_{2 i}, \\
& B_{i}=A_{2 i+1} \oplus B_{2 i+1} \quad \text { for each } i=1,2, \cdots
\end{aligned}
$$

there exists a nonzero projective $R$-module $X$ such that $X \leqq \oplus_{i=m}^{\infty} A_{i}$ or $X \lesssim \oplus_{i=m}^{\infty} B$ for any positive integer $m$.

Lemma 2.9 ([5, Lemma 6]). Let $P$ be a nonzero finitely generated projective module over a regular ring $R$, and set $T=\operatorname{End}_{R}(P)$. Then the following conditions are equivalent:
(a) $P$ satisfies (\#).
(b) $T$ satisfies (\#) as a T-module.

Lemma 2.10 ([5, Lemma 7]). Let P be a nonzero finitely generated projective
module over a regular ring $R$, and set $T=\operatorname{End}_{R}(P)$. Then the following conditions are equivalent:
(a) $R$ satisfies (\#) as an $R$-module.
(b) All nonzero finitely generated projective $R$-modules satisfy (\#).
(c) For any positive integer $k, k R$ satisfies (\#).
(d) There exists a positive integer $k$ such that $k R$ satisfies (\#).

Theorem 2.11. Let $R$ be a regular ring satisfying (*). Then the following conditions are equivalent:
(a) $R$ has (DF).
(b) $R$ satisfies (\#) as an $R$-module.
(c) For any nonzero finitely generated projective $R$-module $P, \operatorname{End}_{R}(P)$ has (DF).
(d) For any positive integer $k, M_{k}(R)$ has (DF).
(e) There exists a positive integer $k$ such that $M_{k}(R)$ satisfies (DF).

Proof. Note that $\operatorname{End}_{\boldsymbol{R}}(P)$ is a regular ring with (*). [5, Theorem 8] was proved only using [5, Theorem 2]. Now [5, Theorem 2] holds on a regular ring satisfying (*) by Theorem 2.4. Hence we see that this theorem holds under this condition using the similar proof of [5, Theorem 8] (Note that the unit-regularlity is not needed).

## 3. Some applications

Lemma 3.1. Let $R$ be a regular ring satisfying (*), and let $\left\{e_{i}\right\}$ be a set of nonzero orthogonal central idempotents of $R$ such that $\oplus e_{i} R_{R}<_{e} R_{R}$. Then $R$ has (DF) if and only if $e_{i} R$ has (DF) for all $i$.

Proof. Note that $e_{i} R$ is a ring direct summand of $R$. It is clear from Theorem 2.11 that "only if" part holds. We shall prove that "if" part holds. Let $I$ be a nonzero direct summand of $R$, and so $e_{i} R \cap I \neq 0$ for some $i$. Setting $J=e_{i} R \cap I$, $J$ is a principal right ideal of both $R$ and $e_{i} R$. We consider decompositions

$$
\begin{aligned}
& I=A_{1} \oplus B_{1} \\
& A_{j}=A_{2 j} \oplus B_{2 j} \\
& B_{j}=A_{2 j+1} \oplus B_{2 j+1} \quad \text { for each } j=1,2, \cdots
\end{aligned}
$$

and so there exist decompositions of $J$ such that

$$
\begin{aligned}
& J=C_{1} \oplus D_{1} \\
& C_{j}=C_{2 j} \oplus D_{2 j}
\end{aligned}
$$

$$
\begin{aligned}
& D_{j}=C_{2 j+1} \oplus D_{2 j+1} \\
& C_{j} \lesssim \oplus A_{j} \quad \text { and } \quad D_{j} \lesssim \oplus B_{j} \quad \text { for each } j=1,2, \cdots
\end{aligned}
$$

By the assumption, there exists a nonzero cyclic projective $e_{i} R$-module $X$ such that $X \lesssim \oplus_{j=m}^{\infty} C_{j}$ or $X \lesssim \oplus_{j=m}^{\infty} D_{j}$ for each positive integer $m$. Hence $X \otimes_{R} e_{i} R \lesssim$ $\oplus_{j=m}^{\infty}\left(C_{j} \otimes_{R} e_{i} R\right) \quad$ or $\quad X \otimes_{R} e_{i} R \lesssim \oplus_{j=m}^{\infty}\left(D_{j} \otimes_{R} e_{i} R\right)$. Note that $\oplus_{j=m}^{\infty}\left(C_{j} \otimes_{R} e_{i} R\right) \lesssim$ $\oplus_{j=m}^{\infty} A_{j}$ and $\oplus_{j=m}^{\infty}\left(D_{j} \otimes_{R} e_{i} R\right) \lesssim \oplus_{j=m}^{\infty} B_{j}$. Therefore $X \otimes_{R} e_{i} R \lesssim \oplus_{j=m}^{\infty} A_{j}$ or $X \otimes_{R} e_{i} R$ $\lesssim \oplus_{j=m}^{\infty} B_{j}$. Since $X \otimes_{R} e_{i} R \neq 0$, this lemma has proved by Theorem 2.11.

Lemma 3.2 ([6, Proposition 2.1]). Let $R$ be an abelian regular ring. If $Q(R)$ has (DF), then so has $R$.

Theorem 3.3. Let $R$ be a regular ring satisfying (*). If $Q(R)$ has (DF), then so does $R$.

Proof. By Lemma 2.2, we may assume that there exists a set $\left\{S_{t}\right\}$ of abelian regular rings such that $R_{R}<_{e}\left[\Pi M_{n(t)}\left(S_{t}\right)\right]$. Then $Q(R)=\Pi M_{n(t)} Q\left(S_{t}\right)$. Assume that $Q(R)$ has (DF), then so does $M_{n(t)}\left(Q\left(S_{t}\right)\right)$ for all $t$ by Lemma 3.1. Moreover, Theorem 2.11 shows that $Q\left(S_{t}\right)$ also has (DF), hence so has $S_{t}$ by Lemma 3.2. Thus $M_{n(t)}\left(S_{t}\right)$ also has (DF) by Theorem 2.11. There exists the set $\left\{e_{t}\right\}$ of orthogonal central idempotents of $R$ such that $e_{t} R=M_{n(t)}\left(S_{t}\right) \times\left[\Pi_{t \neq t} 0\right]$ and $\oplus e_{t} R<{ }_{e} R_{R}$. Therefore $R$ has (DF) by Lemma 3.1.

Now we shall give an example of a regular ring with a zero socle satisfying (*) which has (DF), as following.

Example 3.4. Let $F$ be a field, and set $R=\prod_{i=1}^{\infty} F_{i}\left(F_{i}=F\right)$ and $\bar{R}=R / \operatorname{soc}(R)$. Then $\bar{R}$ is a regular ring satisfying (*) which has (DF).

Proof. Since it is clear that $\bar{R}$ is a regular ring satisfying (*), we shall prove that $\bar{R}$ has (DF) using Theorem 2.11. Let $\Psi$ be the natural map from $R$ to $\bar{R}$, and let $I$ be a nonzero direct summand of $\bar{R}$ with following decompositions:

$$
\begin{aligned}
& I=A_{1} \oplus B_{1} \\
& A_{i}=A_{2 i} \oplus B_{2 i} \\
& B_{i}=A_{2 i+1} \oplus B_{2 i+1} \quad \text { for } i=1,2, \cdots
\end{aligned}
$$

Now assume that there does not exist $\left\{C_{j}\right\}\left(C_{j}=A_{j}\right.$ for some $i$ ) which is an infinite subset of $\left\{A_{i}\right\}_{i=1}^{\infty}$ such that $C_{j}>C_{j+1}$ and $C_{j} \neq 0$ for all $j$. Let $\left\{D_{p}\right\}\left(D_{p}=A_{i}\right.$ for some $i$ ) be an infinite decreasing sequence of $\left\{A_{i}\right\}$, and so there exists a positive integer $p^{\prime}$ such that $D_{p}=0 \quad\left(p \leqq p^{\prime}\right)$. Hence $0=D_{p^{\prime}}=A_{i_{1}}$ for some
$i_{1}$. Thus $B_{i_{1}} \neq 0$. Next, we take $\left\{E_{q}\right\}\left(E_{q}=A_{i}\right.$ for some $\left.i\right)$ which is an infinite decreasing sequence of $\left\{A_{i}\right\}$, where $E_{q}<B_{i_{1}}$ and $B_{k_{q}}<B_{i_{1}}\left(A_{k_{q}}=E_{q}\right)$ for all positive integer $q$. Similarly, there exists a positive integer $q^{\prime}$ such that $E_{q}=0\left(q^{\prime} \leqq q\right)$. Hence there exists a positive integer $i_{2}\left(i_{2}>i_{1}\right)$ such that $E_{q^{\prime}}=A_{i_{2}}=0$. Therefore $B_{i_{2}} \neq 0$ and $B_{i_{1}}>B_{i_{2} .}$. Continuing this procedure, we can get an infinite set $\left\{B_{i_{k}}\right\}$ such that $\left\{B_{i}\right\} \supset\left\{B_{i_{k}}\right\}$ and $B_{i_{k}} \neq 0$ for all $k$. From the above, we may assume that there exists an infinite decreasing sequence $\left\{C_{j}\right\}$ such that $\left\{A_{i}\right\} \supset\left\{C_{j}\right\}, C_{j}>C_{j+1}$ and $C_{j} \neq 0$ for all $j$.
We have a set $\left\{e_{j}\right\}$ of idempotents of $R$ such that $\Psi\left(e_{j} R\right)=C_{j}$ and $e_{j} R \geqq e_{j+1} R$ for all $j$. We take an idempotent $f_{1}\left(\in e_{1} R\right)$ with $\operatorname{dim}_{F}\left(f_{1} R\right)=1$. Next we take an idempotent $f_{2}\left(\in e_{2} R\right)$ such that $\operatorname{dim}_{F}\left(f_{2} R\right)=1$ and $f_{1} f_{2}=0$. Continuing this procedure, we can take a set $\left\{f_{j}\right\}$ of orthogonal idempotents of $R$. Set $e=\vee f_{j}$, and then $\Psi(e) \neq 0$. We have that $e R=J \oplus\left(e R \cap e_{j} R\right)$ and $J<\oplus F_{i}$ for some right ideal $J$. Noting that $J \otimes_{R} \bar{R}=0$, we have that

$$
\begin{aligned}
0 \neq \Psi(e) \bar{R} & \simeq e R \otimes_{R} \bar{R} \\
& \simeq\left[J \oplus\left(e R \cap e_{j} R\right)\right] \otimes_{R} \bar{R} \\
& \lesssim e_{j} R \otimes_{R} \bar{R} \\
& \simeq C_{j} \quad \text { for all } j .
\end{aligned}
$$

Therefore $0 \neq \Psi(e) \bar{R} \lesssim \oplus_{i=m}^{\infty} A_{i}$ for any positive integer $m$. Hence $\bar{R}$ has (DF) by Theorem 2.11.

By Example 3.4, we have a problem that, for any regular ring $S, R$ $=\left(\Pi_{1}^{\infty} S\right) /(\oplus S)$ satisfies (DF) or not. Example 3.5 shows that, even if $S$ satisfies (*), $R$ does not satisfy (*). Therefore we shall give the necessary and sufficient condition for that $R$ satisfies (*), and we solve the above problem under this condition.

Example 3.5. Let $F$ be a field and set $S=\prod_{n=1}^{\infty} M_{n}(F), \bar{S}=S /\left(\oplus M_{n}(F)\right)$, $T=\prod_{i=1}^{\infty} S_{i}\left(S_{i}=S\right)$ and $R=T /\left(\oplus S_{i}\right)$. Then $S$ satisfies (*), but $R$ does not satisfy (*).

Proof. It is clear that $S$ satisfies (*). Therefore we shall show that $R$ does not satisfy (*). Set a central idempotent $e(\in T)$ as following;

$$
\begin{aligned}
e(n)= & \left(0, \cdots, 0,\left[\begin{array}{ll}
1 & \\
& 1
\end{array}\right], 0,0, \cdots\right), \\
& \lfloor n-1\rfloor\lfloor n\rfloor
\end{aligned}
$$

where $e(n) \in S_{n}$.
Let $\Phi$ be the natural map from $S$ to $\bar{S}$, and $\rho$ be the natural map from $T$ to $R$. Set $\Psi=\left.\rho\right|_{e r}$. Noting that $e(n) S_{n} \simeq M_{n}(F)$, we have $e T \simeq S$. Hence there exists a ring
isomorphism $\kappa$ from $e T$ to $S$. Now, we define a ring homomorphism $\alpha$ from $\Psi(e) R$ to $\bar{S}$ as following; for each $x(\in \Psi(e) R)$, we take any element $y$ of $\Psi^{-1}(x)$ and set $\alpha(x)=\Phi \kappa(y)$.

$$
\begin{array}{cc}
\Psi(e) R \xrightarrow{\alpha} & \bar{S} \\
\uparrow^{\psi} & \stackrel{\uparrow^{\Phi}}{ } \\
e T & \xrightarrow{\kappa} \\
\hline
\end{array}
$$

Similarly we define a ring homomorphiism $\beta$ from $\bar{S}$ to $\Psi(e) R$. Then we have that $\beta \alpha=1_{\psi(e) R}$ and $\alpha \beta=1_{\bar{S}} . \quad$ Hence $\alpha$ and $\beta$ are isomorphic. Therefore $\Psi(e) R \simeq \bar{S}$. Let $I$ be a nonzero two-sided ideal of $\bar{S}$, and so $I=J /\left(\oplus S_{i}\right)$ for some nonzero two-sided ideal of $S$ which contains $\oplus S_{i}$. There exists $0 \neq x \in J-\left(\oplus S_{i}\right)$ with $x(i) \neq 0$ for almost all $i$. Since $S_{i} x(i) S_{i}=M_{i}(F)$ has index $i$, there exists a nonzero central idempotent $e(i)$ of $M_{i}(F)$ which $S_{i} x(i) S_{i}$ has index $i$. Therefore $S x S$ does not have bounded index, and so does not $J /\left(\oplus S_{i}\right)$. Therefore $\bar{S}$ does not satisfy (*), and hence so does not $\Psi(e) R$. Thus $R$ does not satisfy (*).

Lemma 3.6. Let $R$ be a ring, and $e, f$ be idempotents of $R$. Then $e R \simeq f R$ if and only if there exist $u$ and $v$ of $R$ such that $v u=e$ and $u v=f$.

Lemma 3.7. Let $S$ be a regular ring which has index $\infty$, and set $R=\left(\Pi_{i=1}^{\infty} S_{i}\right) /\left(\oplus S_{i}\right)\left(S_{i}=S\right)$. Then $R$ has an infinite direct sum of nonzero pairwise isomorphic right ideals.

Proof. Let $\Psi$ be the natural map from $\prod_{i=1}^{\infty} S_{i}$ to $R$. Since $S$ has index $\infty$, there exists a set of idempotents $\left\{e_{i j}\right\}_{i, j=1,2, \ldots}$ as following:

$$
\begin{aligned}
& e_{11} S \\
& e_{21} S \simeq e_{22} S \\
& e_{31} S \simeq e_{32} S \simeq e_{33} S
\end{aligned}
$$

, where $e_{i j}=0(i<j)$ and $\left\{e_{i 1}, \cdots, e_{i i}\right\}$ are nonzero orthogonal for all $i$. For all positive integer $m$, we take idempotents $\left\{f_{m}\right\}$ such that $f_{m}(k)=e_{k m}$ for all positive integer $k$. Since $e_{k 1} S \simeq e_{k 2} S$ for all $k$, there exist $u_{k}$ and $v_{k}$ of $S$ such that $u_{k} v_{k}=e_{k 2}$ and $v_{k} u_{k}=e_{k 1}$ by Lemma 3.6. Set $u$ and $v$ of $\Pi_{i=1}^{\infty} S_{i}$ such that $u(k)=u_{k}$ and $v(k)=v_{k}$. Then $u v=f_{2}$ and $v u=f_{1}-e$, where $e$ is an idempotent with $e(1)=e_{11}$ and $e(k)=0 \quad(k \neq 1)$. Hence $\left(f_{1}-e\right)\left(\Pi S_{i}\right) \simeq f_{2}\left(\Pi S_{i}\right)$ and $\left(f_{1}-e\right)\left(\Pi S_{i}\right) \cap f_{2}\left(\Pi S_{i}\right)=0$. Therefore we see from Lemma 3.6 that $\Psi\left(f_{1}-e\right) R \simeq \Psi\left(f_{2}\right) R$ and $\Psi\left(f_{1}-e\right) R \cap \Psi\left(f_{2}\right) R$ $=0$. Since $\Psi\left(f_{1}-e\right) R=\Psi\left(f_{1}\right) R$, we have that $\Psi\left(f_{1}\right) R \simeq \Psi\left(f_{2}\right) R$ and $\Psi\left(f_{1}\right) R \cap \Psi\left(f_{2}\right) R$
$=0$. Continuing this produre, for all positive integers $i$ and $j, \Psi\left(f_{i}\right) R \simeq \Psi\left(f_{j}\right) R$ and $\Psi\left(f_{i}\right) \cap \Psi\left(f_{j}\right) R=0(i \neq j)$. Thus $R$ has an infinite direct sum of nonzero pairwise isomorphic right ideals.

Theorem 3.8. Let $S$ be a regular ring, and set $R=\left(\Pi_{i=1}^{\infty} S_{i}\right) /\left(\oplus S_{i}\right)$ $\left(S_{i}=S\right)$. Then the following conditions are equivalent:
(a) $R$ satisfies (*).
(b) $R$ is a regular ring whose primitive factor rings are artinian.
(c) $R$ has bounded index.
(d) $R$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals.
(e) $S$ has bounded index.

Proof. It is clear by Lemma 3.7 that $(\mathrm{d}) \rightarrow(\mathrm{e}) \rightarrow(\mathrm{c}) \rightarrow(\mathrm{b}) \rightarrow(\mathrm{a})$ hold. (a) $\rightarrow$ (d) follows from Corollary 2.5. Therefore this theorem is complete.

Theorem 3.9. Let $S$ be a regular ring of bounded index. Set $R=\left(\Pi_{n=1}^{\infty} S_{n}\right) /\left(\oplus S_{n}\right)$ $\left(S_{n}=S\right)$. Then $R$ has (DF).

Proof. Set $\Pi_{n=1}^{\infty} S_{n}=T$, and let $\Psi$ be the natural map from $T$ to $R$. Let $I$ be a nonzero direct summand of $R$ with following decompositions:

$$
\begin{aligned}
& I=A_{1} \oplus B_{1} \\
& A_{i}=A_{2 i} \oplus B_{2 i} \\
& B_{i}=A_{2 i+1} \oplus B_{2 i+1} \quad \text { for } i=1,2, \cdots
\end{aligned}
$$

Similarly to the proof of Example 3.4, we may assume that there exists an infinite subset $\left\{C_{j}\right\}$ of $\left\{A_{i}\right\}\left(C_{j}=A_{i}\right.$ for some $i$ ) such that $C_{j}>C_{j+1}$ and $C_{j} \neq 0$ for all positive integer $j$. We have the set of idempotents $\left\{e_{j}\right\}$ of $T$ such that $\Psi\left(e_{j} T\right)=C_{j}$ and $e_{j} T>e_{j+1} T$. Set $J_{n}=S_{n} \times\left(\Pi_{i \neq n} 0\right)$. Then, $J_{n_{1}} \cap e_{1} T \neq 0$ for some positive integer $n_{1}$. There exists a nonzero idempotent $f_{1} \in T$ such that $f_{1} T=J_{n_{1}} \cap e_{1} T$. Next we have a nonzero idempotent $f_{2} \in T$ for some $n_{2}\left(>n_{1}\right)$ such that $f_{2} R=J_{n_{2}} \cap e_{2} R$. Continuing this procedure, we have the set $\left\{f_{j}\right\}$ of orthogonal idempotents of $T$. Now, we set an idempotent $g$ of $T$ as following;

$$
\begin{aligned}
& g\left(n_{j}\right)=f_{j}\left(n_{j}\right)=e_{j}\left(n_{j}\right) \\
& g(k)=0\left(k \notin\left\{n_{j}\right\}\right) .
\end{aligned}
$$

Put $K_{j}=f_{1} T \oplus \cdots \oplus f_{j-1} T$ for all $j$. Then $g T=K_{j} \oplus\left(g T \cap e_{j} T\right)$. Noting $K_{j} \otimes_{T} R=0$, we have that

$$
\begin{aligned}
& 0 \neq \Psi(g) R \simeq g T \otimes_{T} R \\
& \simeq\left[K_{j} \oplus\left(g T \cap e_{j} T\right)\right] \otimes_{T} R \\
& \simeq\left(g T \cap e_{j} T\right) \otimes_{T} R \\
& \lesssim e_{j} T \otimes_{T} R \\
& \simeq C_{j} \quad \text { for all } j .
\end{aligned}
$$

From the above, we have that $\Psi(g) R \lesssim \oplus_{i=m}^{\infty} A_{i}$ for any positive integer $m$. Therefore $R$ has (DF) by Theorem 2.11.

## References

[1] C.-L. Chuang and P.-H. Lee: On regular subdirect products of simple artinian ring. Pacific J. Math. 142 (1990), 17-21.
[2] K.R. Goodearl: Von Neumann regular rings, Kreiger, Florida, 1991.
[3] K.R. Goodearl and J. Moncasi: Cancellation of finitely generated modules over regular rings, Osaka J. Math. 26 (1989), 679-685.
[4] H. Kambara and S. Kobayahi: On regular self-injective rings, Osaka J. Math. 22 (1985), 71-79.
[5] M. Kutami: Projective modules over regular rings of bounded index, Math. J. Okayama Univ. 30 (1988), 53-62.
[6] M. Kutami and I. Inoue: The property $(D F)$ for regular rings whose primitive factor rings are artinian, Math. J. Okayama Univ. 35 (1993), 169-179

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