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<td>Author(s)</td>
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<td>Citation</td>
<td>Osaka Journal of Mathematics. 32(2) P.501–P.512</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1995</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6952">https://doi.org/10.18910/6952</a></td>
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Osaka University
ON REGULAR RINGS WHOSE MAXIMAL RIGHT QUOTIENT RINGS ARE TYPE I_

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(Received July 26, 1993)

Introduction

This paper is written about the property (DF) on regular rings whose maximal right quotient rings are Type $I_f$. Hereafter regular rings whose maximal right quotient rings are Type $I_f$ are said to satisfy (*). The property (DF) is very important property when we study on regular rings satisfying (*), and it was treated in the paper [5] written by the first author, where (DF) for a ring $R$ is defined as that if the direct sum of any two directly finite projective $R$-modules is always directly finite. In the above paper, the equivalent condition that a regular ring $R$ of bounded index satisfies (DF) was discovered and called (#). Stillmore, we proved that the condition (DF) is equivalent to (#) for regular rings whose primitive factor rings are artinian in the paper [6]. Then we have the problem that (DF) is equivalent (#) for regular rings satisfying (*) or not, where the condition (*) is weaker than one that primitive factor rings are artinian.

In § 2, we shall prove Theorem 2.4. This is important, and using this, Theorem 2.5 (i.e. if $R$ is a regular ring satisfying (*) and $k$ is any positive integer, then $kP$ is directly finite for every directly finite projective $R$-module $P$) is proved. Moreover, we shall solve the above problem in Theorem 2.11.

In § 3, we shall consider some applications of Theorem 2.11. We prove Theorem 3.3 that if $R$ is a regular ring satisfying (*) whose maximal right quotient ring of $R$ satisfies (DF), then so does $R$. Though it is clear that a regular rings satisfying (*) which has a nonzero essential socle satisfies (DF), we can prove that, for regular rings satisfying (*), the condition having a nonzero essential socle is not equivalent to (#) in Example 3.4. Next, we shall consider that $(\Pi^\infty R)/(\oplus R)$ satisfies (DF) or not for a regular ring $R$ satisfying (*). This problem is a generalization of Example 3.4, and we prove that, for a regular ring $R$ of bounded index, $(\Pi^\infty R)/(\oplus R)$ satisfies (DF) (Theorem 3.9).

Throughout this paper, $R$ is a ring with identity and $R$-modules are unitary right $R$-modules.
1. Definitions and notations

Definition 1. A ring $R$ is (Von Neumann) regular provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$.

Note. Every projective modules over regular rings have the exchange property.

Definition 2. A module $M$ is directly finite provided that $M$ is not isomorphic to a proper direct summand of itself. If $M$ is not directly finite, then $M$ is said to be directly infinite. A ring $R$ is said to be directly finite (resp. directly infinite) if so is $R$ as an $R$-module.

Definition 3. The index of a nilpotent element $x$ in a ring $R$ is the least positive integer such that $x^n = 0$ (In particular, 0 is nilpotent of index 1). The index of a two-sided ideal $J$ of $R$ is the supremum of the indices of all nilpotent elements of $J$.

If this supremum is finite, then $J$ is said to have bounded index. If $J$ does not have bounded index, $J$ is said to be index $\infty$.

Note. Let $R$ be a regular ring with index $\infty$. Then using [3, the proof of Lemma 2], there exists a family $\{A_n\}_{n=1}^\infty$ of independent right ideals of $R$ such that $A_n$ contains a direct sum of $n$ nonzero pairwise isomorphic right ideals. Therefore $R$ has a family $\{e_i\}_{i=1,2,\ldots}$ of idempotents such that

\[ e_{21}R \simeq e_{22}R \]
\[ e_{31}R \simeq e_{32}R \simeq e_{33}R \]
\[ \ldots \]

, where $e_{ij} = 0$ ($i < j$), and $\{e_{i1}, \ldots, e_{ii}\}$ are orthogonal for all $i$.

Definition 4. A ring $R$ has (DF) if the direct sum of two directly finite projective $R$-modules is directly finite.

Definition 5. A regular ring $R$ is abelian provided all idempotents in $R$ are central.

Definition 6. A ring $R$ satisfies (*) if every nonzero two-sided ideal of $R$ contains a nonzero two-sided ideal of bounded index.

Definition 7. A ring $R$ is unit-regular provided that for each $x \in R$ there is a unit $u \in R$ such that $xux = x$. 

NOTE. Every finitely generated projective module over a unit-regular ring has the cancellation property ([2, Theorem 4.14]).

**Definition 8.** Let \( e \) be an idempotent in a regular ring \( R \). Then \( e \) is called an *abelian idempotent* (of \( R \)) whenever the ring \( eRe \) is abelian.

**Definition 9.** Let \( e \) be an idempotent in a regular right self-injective ring \( R \). Then \( e \) is *faithful* (in \( R \)) if \( 0 \) is the only central idempotent of \( R \) which is orthogonal to \( e \). A regular right self-injective ring \( R \) is said to be *Type I* provided that it contains a faithful abelian idempotent, and \( R \) is *Type I* if \( R \) is Type I and directly finite.

NOTE. It is well-known from [4, Theorem 2] and [2, Lemma 7.17] that a regular ring \( R \) satisfies (\( \ast \)) if and only if the maximal right quotient ring of \( R \) is Type \( I_f \).

NOTE. Let \( R \) be a regular ring satisfying (\( \ast \)). If \( P \) is a finitely generated projective \( R \)-module, then \( \text{End}_R(P) \) is a regular ring satisfying (\( \ast \)).

**Proof.** Choose a positive integer \( n \) and an idempotent matrix \( e \in M_n(R) \) such that \( e(nR_R) \cong P \). Then \( \text{End}_R(P) \cong eM_n(R)e \). Using [2, Corollary 10.5], we see that \( eM_n(Q(R))e \cong Q(eM_n(R)e) \) is Type \( I_f \), where \( Q(R) \) is the maximal right quotient of \( R \). Since \( eM_n(R)e \leq eQ(eM_n(R)e) \) as an \( eM_n(R)e \)-module, we have that \( eM_n(R)e \) satisfies (\( \ast \)), and so has \( \text{End}_R(P) \).

**Notations.** Let \( A, B \) and \( A_i (i \in I) \) be \( R \)-modules, and \( k \) be a positive integer. Take \( x \in \prod A_i \). Then we have some notations as following.

\[
\begin{align*}
A &\leq B ; A \text{ is a submodule of } B. \\
A &\triangleleft B ; B \text{ has a submodule isomorphic to } A. \\
A &\triangleleft \oplus B ; A \text{ is a direct summand of } B. \\
A &\triangleleft eB ; B \text{ has a direct summand isomorphic to } A. \\
A &\triangleleft e B ; A \text{ is an essential submodule of } B. \\
A &\triangleleft eB ; B \text{ has an essential submodule isomorphic to } A. \\
kA &; \text{ the } k\text{-copies of } A. \\
x(i) &; \text{ the } i\text{-th component of } x. \\
Q(R) &; \text{ the maximal right quotient ring of } R.
\end{align*}
\]

2. The property (DF) for regular rings satisfying (\( \ast \))

**Lemma 2.1** ([2, Theorem 6.6]). Let \( R \) be a regular ring whose primitive factor rings are artinian. Then \( R \) satisfies (\( \ast \)).
Lemma 2.2. Let $R$ be a regular ring satisfying $(\ast)$. Then there exist abelian regular rings $\{S_t\}_{t \in T}$ and orthogonal central idempotents $\{e_t\}_{t \in T}$ of $R$ such that $R_R \leq \bigoplus M_{n(t)}(S_t) \leq R$ and $e_t R = M_{n(t)}(S_t)$. Therefore $Q(R) \simeq \prod M_{n(t)}(Q(S_t))$.

Proof. This theorem follows from [2, Lemma 7.17 and the proof of Theorem 7.18].

Lemma 2.3. Let $R$ be a regular ring of bounded index and $P$ be a finitely generated projective $R$-module. Then $P$ can not contain a family $\{A^1, A^2, \ldots \}$ of nonzero finitely generated submodules such that $A^i > A^{i+1}$ and $iA^i \leq P$ for each $i = 1, 2, \ldots$.

Proof. By [2, Corollary 7.13], we see that $\text{End}_R(P)$ has bounded index. Note the claim in the proof of [2, Theorem 6.6], and applying [5, Lemma 5] to $\text{End}_R(P)$, we see that this lemma holds.

Theorem 2.4. Let $R$ be a regular ring satisfying $(\ast)$, and $P$ be a projective $R$-module with a cyclic decomposition $P = \bigoplus_{i \in I} P_i$. Then the following conditions (a)~(d) are equivalent:

(a) $P$ is directly infinite.
(b) There exists a nonzero cyclic projective $R$-module $X$ such that $X \leq \bigoplus_{i \neq t} P_i$ for any finite subset $\{i_1, \ldots, i_n\}$ of $I$.
(c) There exists a nonzero cyclic projective $R$-module $X$ such that $X \leq \bigoplus_{i \in I - \{i_1, \ldots, i_n\}} P_i$ for any finite subset $\{i_1, \ldots, i_n\}$ of $I$.
(d) There exists a nonzero cyclic projective $R$-module $X$ such that $X \leq \bigoplus_{i \in I} P_i$.

Proof. It is clear that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) $\Rightarrow$ (a) hold, hence we shall prove that (b) $\Rightarrow$ (c) holds. We may assume $\bigoplus M_{n(t)}(S_t) \leq R_R \leq \bigoplus \prod M_{n(t)}(S_t)$ for some set of abelian regular rings $\{S_t\}_{t \in T}$ by Lemma 2.2. Now we assume that (b) holds, hence there exists a nonzero principal right ideal $X$ of $R$ such that $X \leq \bigoplus_{i \in I} P_i$.

Note that $S$ is a regular ring of bounded index. It is clear that

$$Y \otimes_R S \simeq (P^1 \otimes_R S) \oplus \cdots \oplus (P^i \otimes_R S) \oplus \bigoplus_{i \neq t} (P^i \otimes_R S)$$

and $2Y \otimes_R S \leq P \otimes_R S$. Since $S$ is unit-regular, $Y \otimes_R S$ has the cancellation property. Hence
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\[ Y \otimes_{R} S \leq \oplus (P_{1}^{1} \otimes_{R} S) \oplus \cdots \oplus (P_{n}^{1} \otimes_{R} S) \oplus \left[ \oplus_{i \in I} (P_{1}^{1} \otimes_{R} S) \right] \]

Thus for each \( i \), we obtain that \( P_{1}^{1} \otimes_{R} S = P_{1}^{2} \oplus P_{1}^{2} \) for each \( i \in I \) and

\[ Y \otimes_{R} S \simeq P_{1}^{m} \oplus \cdots \oplus P_{m}^{m} \oplus \left[ \oplus_{i \in I} (P_{1}^{1} \otimes_{R} S) \right] \]

Continuing this procedure, we have that \( P_{1}^{m} = P_{1}^{m+1} \oplus P_{1}^{m+1} \) for each \( i \in I \) and each positive integer \( m \).

Now we set \( ^{w} = P_{1}^{0} \otimes_{R} S \), where \( ^{w} = (P_{1}^{1} \otimes_{R} S) \oplus \left[ \oplus_{i \in I} (P_{1}^{1} \otimes_{R} S) \right] \). Then \( A_{1} \leq \oplus A_{2} \oplus (\oplus_{i \in I} Q_{i}) \), hence there exist a direct summand \( B_{2} \leq A_{2} \) and a direct summand \( Q_{i} \) of \( P_{1}^{1} \) such that \( A_{1} \simeq B_{2} \oplus (\oplus_{i \in I} Q_{i}) \). Continuing this procedure, we obtain a family \( \{ B_{1}, B_{2}, \ldots \} \) of finitely generated projective \( S \)-submodules of \( (P_{1}^{1} \oplus \cdots \oplus P_{1}^{1}) \otimes_{R} S \) such that \( B_{m} \geq B_{m+1} \) and \( mB_{m} \leq nS \) for all \( m \). By Lemma 2, 3, there exists a positive integer \( k \) such that \( B_{m} = 0 \) for all \( m \geq k \). Thus we have that \( A_{1} \simeq (\oplus_{i \in I} Q_{i}) \oplus \cdots \oplus (\oplus_{i \in I} Q_{i}) \) and \( Y \otimes_{R} S \simeq (\oplus_{i \in I} Q_{i}) \oplus \cdots \oplus (\oplus_{i \in I} Q_{i}) \).

Noting that \( \oplus_{i \in I} Q_{i} \leq S \), we have that \( Y \otimes_{R} S \leq \oplus_{i \in I} P_{i} \).

**Corollary 2.5.** Let \( R \) be a regular ring satisfying (*) Then \( R \) contains no infinite direct sums of nonzero pairwise isomorphic right ideals. Hence \( R \) is directly finite.

**Proof.** From Lemma 2.2, we may assume that, \( R_{R} < e \cdot \Pi_{i \in T} \left( S_{i} \right) \) for some abelian regular rings \( \{ S_{i} \}_{i \in T} \). Set \( T = \Pi_{i \in T} \left( S_{i} \right) \). Now we assume that \( R \) contains a direct sum of nonzero pairwise isomorphic right ideals, and so there exists a nonzero idempotent \( e \) of \( R \) such that \( 0 \neq \chi_{0}(eR) \leq R_{R} \). Then \( R_{R} \otimes_{R} T \leq R \otimes_{R} T, \) and so \( \chi_{0}(eT) \leq T, \) which contradicts to Theorem 2.4 because \( T \) is a directly finite regular ring satisfying (*).

**Theorem 2.6.** Let \( R \) be a regular ring satisfying (*) and \( k \) be a positive integer. If \( P \) is a directly finite projective \( R \)-module, then so is \( kP \).

**Proof.** We may assume that \( \oplus_{i \in I} \left( S_{i} \right) \leq \Pi_{i \in T} \left( S_{i} \right) \) for some abelian regular rings \( \{ S_{i} \}_{i \in T} \), and let \( P = \oplus_{i \in I} P_{i} \) be a cyclic decomposition of \( P \). It is sufficient to prove that this theorem holds in case \( k = 2 \). Assume that \( 2P \) is directly infinite. Then Theorem 2.4 follows that there exists a nonzero principal right ideal \( X \) of \( R \) such that \( X \leq \oplus_{i \in I} \left( \oplus_{i_{1}, \ldots, i_{n}} 2P_{i} \right) \) for any finite subset \( \{ i_{1}, \ldots, i_{n} \} \) of \( I \). By the proof of Theorem 2.4, we may assume that exists \( t \) of \( T \) such that \( \chi_{t}(0) \cdot \Pi \left( S_{t} \right) = S. \) For any finite subset \( \{ i_{1}, \ldots, i_{n} \} \) of \( I \), we have that \( 0 \neq \chi_{t}(0) \leq \oplus_{i \in I \setminus \{ i_{1}, \ldots, i_{n} \} } (2P_{i} \otimes_{R} S) \) Since \( S \) is a regular ring of bounded index, we see that \( 2P \otimes_{R} S \) is directly infinite by Theorem 2.4 and so \( (P \otimes_{R} S)_{S} \) is directly
infinite by [5, Theorem 4]. Moreover, using Theorem 2.4 again, there exists a 
nonzero principal right ideal \( Y \) of \( S \) such that 
\[ Y \leq \bigoplus_{i \in I - \{i_1, \ldots, i_n\}} (P_i \otimes_R S) \] 
for any finite subset \( \{i_1, \ldots, i_n\} \) of \( I \). Considering \( Y \) as an \( R \)-module, 
\[ O \neq Y \leq \bigoplus_{i \in I - \{i_1, \ldots, i_n\}} P_i \] 
Therefore \( P \) is directly infinite, and so this theorem is complete.

**Corollary 2.7.** Let \( R \) be a regular ring satisfying \((*)\). Then every finitely 
generated projective \( R \)-module is directly finite.

**Proof.** It is clear by Corollary 2.5 and Theorem 2.6.

**Corollary 2.8.** Let \( R \) be a regular ring satisfying \((*)\).
(a) \( M_n(R) \) is directly finite for all positive integer \( n \), and so \( M_n(R) \) contains no 
infinite direct sums of nonzero pairwise isomorphic right ideals.
(b) If \( P \) and \( Q \) are finitely generated projective \( R \)-modules, then \( P \oplus Q \) is directly 
finite.

**Proof.** (a) \( R \) is a regular ring satisfying \((*)\), and hence so is \( M_n(R) \). Therefore 
Corollary 2.5 shows that (a) holds. (b) follows from Corollary 2.7.

**NOTE.** In [1], Chuang and Lee have shown that there exists a regular ring 
satisfying \((*)\) which is not unit-regular. Our Corollary 2.8 gives a partially solution 
for open problems 1 and 9 in Goodearl's book ([2]).

**DEFINITION.** Let \( R \) be a regular ring and \( P \) be a projective \( R \)-module. We 
call that \( P \) satisfies \((\#)\) provided that, for each nonzero finitely generated submodule 
\( I \) of \( P \) and any family \( \{A_1, B_1, \ldots\} \) of submodules of \( P \) with
\[ I = A_1 \oplus B_1, \]
\[ A_i = A_{2i} \oplus B_{2i}, \]
\[ B_i = A_{2i+1} \oplus B_{2i+1} \quad \text{for each } i = 1, 2, \ldots, \]
there exists a nonzero projective \( R \)-module \( X \) such that 
\[ X \leq \bigoplus_{i=m}^\infty A_i \text{ or } X \leq \bigoplus_{i=m}^\infty B_i \] 
for any positive integer \( m \).

**Lemma 2.9** ([5, Lemma 6]). Let \( P \) be a nonzero finitely generated projective 
module over a regular ring \( R \), and set \( T = \text{End}_R(P) \). Then the following conditions 
are equivalent:
(a) \( P \) satisfies \((\#)\).
(b) \( T \) satisfies \((\#)\) as a \( T \)-module.

**Lemma 2.10** ([5, Lemma 7]). Let \( P \) be a nonzero finitely generated projective
module over a regular ring $R$, and set $T = \text{End}_R(P)$. Then the following conditions are equivalent:

(a) $R$ satisfies (\#) as an $R$-module.
(b) All nonzero finitely generated projective $R$-modules satisfy (\#).
(c) For any positive integer $k$, $kR$ satisfies (\#).
(d) There exists a positive integer $k$ such that $kR$ satisfies (\#).

**Theorem 2.11.** Let $R$ be a regular ring satisfying (\*). Then the following conditions are equivalent:

(a) $R$ has (DF).
(b) $R$ satisfies (\#) as an $R$-module.
(c) For any nonzero finitely generated projective $R$-module $P$, $\text{End}_R(P)$ has (DF).
(d) For any positive integer $k$, $M_k(R)$ has (DF).
(e) There exists a positive integer $k$ such that $M_k(R)$ satisfies (DF).

Proof. Note that $\text{End}_R(P)$ is a regular ring with (\*). [5, Theorem 8] was proved only using [5, Theorem 2]. Now [5, Theorem 2] holds on a regular ring satisfying (\*) by Theorem 2.4. Hence we see that this theorem holds under this condition using the similar proof of [5, Theorem 8] (Note that the unit-regularity is not needed).

3. Some applications

**Lemma 3.1.** Let $R$ be a regular ring satisfying (\*), and let $\{e_i\}$ be a set of nonzero orthogonal central idempotents of $R$ such that $\bigoplus e_i R_e R < e_i R$. Then $R$ has (DF) if and only if $e_i R$ has (DF) for all $i$.

Proof. Note that $e_i R$ is a ring direct summand of $R$. It is clear from Theorem 2.11 that "only if" part holds. We shall prove that "if" part holds. Let $I$ be a nonzero direct summand of $R$, and so $e_i R \cap I \neq 0$ for some $i$. Setting $J = e_i R \cap I$, $J$ is a principal right ideal of both $R$ and $e_i R$. We consider decompositions

$$I = A_1 \oplus B_1$$
$$A_j = A_{2j} \oplus B_{2j}$$
$$B_j = A_{2j+1} \oplus B_{2j+1} \quad \text{for each } j = 1, 2, \ldots,$$

and so there exist decompositions of $J$ such that

$$J = C_1 \oplus D_1$$
$$C_j = C_{2j} \oplus D_{2j}$$
$D_j = C_{2j+1} \oplus D_{2j+1}$

$C_j \leq \oplus A_j$ and $D_j \leq \oplus B_j$ for each $j = 1, 2, \ldots$.

By the assumption, there exists a nonzero cyclic projective $e_i R$-module $X$ such that $X \leq \oplus_{j=m}^{\infty} C_j$ or $X \leq \oplus_{j=m}^{\infty} D_j$ for each positive integer $m$. Hence $X \otimes_{R} e_i R \leq \oplus_{j=m}^{\infty} (C_j \otimes_{R} e_i R)$ or $X \otimes_{R} e_i R \leq \oplus_{j=m}^{\infty} (D_j \otimes_{R} e_i R)$. Note that $\oplus_{j=m}^{\infty} (C_j \otimes_{R} e_i R) \leq \oplus_{j=m}^{\infty} A_j$ and $\oplus_{j=m}^{\infty} (D_j \otimes_{R} e_i R) \leq \oplus_{j=m}^{\infty} B_j$. Therefore $X \otimes_{R} e_i R \leq \oplus_{j=m}^{\infty} A_j$ or $X \otimes_{R} e_i R \leq \oplus_{j=m}^{\infty} B_j$. Since $X \otimes_{R} e_i R \neq 0$, this lemma has been proved by Theorem 2.11.

**Lemma 3.2** ([(6, Proposition 2.1)]). Let $R$ be an abelian regular ring. If $Q(R)$ has (DF), then so has $R$.

**Theorem 3.3.** Let $R$ be a regular ring satisfying $(\ast)$. If $Q(R)$ has (DF), then so does $R$.

**Proof.** By Lemma 2.2, we may assume that there exists a set $\{S_i\}$ of abelian regular rings such that $R_R \leq \bigoplus_{i \in I} M_{n_i} Q(S_i)$. Then $Q(R) = \prod_{i \in I} M_{n_i} Q(S_i)$. Assume that $Q(R)$ has (DF), then so does $M_{n_i} (Q(S_i))$ for all $i$ by Lemma 3.1. Moreover, Theorem 2.11 shows that $Q(S_i)$ also has (DF), hence so has $S_i$ by Lemma 3.2. Thus $M_{n_i} (S_i)$ also has (DF) by Theorem 2.11. There exists the set $\{e_i\}$ of orthogonal central idempotents of $R$ such that $e_i R = M_{n_i} (S_i) \times [\prod_{i \neq t} 0]$ and $e_i R \leq R_R$. Therefore $R$ has (DF) by Lemma 3.1.

Now we shall give an example of a regular ring with a zero socle satisfying (\ast) which has (DF), as follows.

**Example 3.4.** Let $F$ be a field, and set $R = \prod_{i=1}^{\infty} F_i (F_i = F)$ and $\bar{R} = R / \text{soc}(R)$. Then $\bar{R}$ is a regular ring satisfying (\ast) which has (DF).

**Proof.** Since it is clear that $\bar{R}$ is a regular ring satisfying (\ast), we shall prove that $\bar{R}$ has (DF) using Theorem 2.11. Let $\Psi$ be the natural map from $R$ to $\bar{R}$, and let $I$ be a nonzero direct summand of $\bar{R}$ with following decompositions:

$I = A_1 \oplus B_1$

$A_i = A_{2i} \oplus B_{2i}$

$B_i = A_{2i+1} \oplus B_{2i+1}$ for $i = 1, 2, \ldots$.

Now assume that there does not exist $\{C_j\}$ ($C_j = A_j$ for some $i$) which is an infinite subset of $\{A_i\}_{i=1}^{\infty}$ such that $C_{i} > C_{i+1}$ and $C_j \neq 0$ for all $j$. Let $\{D_p\}$ ($D_p = A_i$ for some $i$) be an infinite decreasing sequence of $\{A_i\}$, and so there exists a positive integer $p'$ such that $D_{p'} = 0$ ($p \leq p'$). Hence $0 = D_{p'} = A_i$ for some
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Thus $B_i \neq 0$. Next, we take $\{E_q\}$ ($E_q = A_i$ for some $i$) which is an infinite decreasing sequence of $\{A_i\}$, where $E_q < B_i$ and $B_{kq} < B_i$ ($A_{kq} = E_q$) for all positive integer $q$. Similarly, there exists a positive integer $q'$ such that $E_q = 0$ ($q' \leq q$). Hence there exists a positive integer $i_2$ ($i_2 > i_1$) such that $E_q = A_{i_2} = 0$. Therefore $B_{i_2} \neq 0$ and $B_{i_1} > B_{i_2}$. Continuing this procedure, we can get an infinite set $\{B_k\}$ such that $\{B_{i_2} \gg \{B_k\}\}$ and $B_k \neq 0$ for all $k$. From the above, we may assume that there exists an infinite decreasing sequence $\{C_j\}$ such that $\{A_j\} \gg \{C_j\}$, $C_j > C_{j+1}$ and $C_j \neq 0$ for all $j$

We have a set $\{e_j\}$ of idempotents of $R$ such that $\Psi(e_jR) = C_j$ and $e_jR \gg e_{j+1}R$ for all $j$. We take an idempotent $f_1(e_1R)$ with $\dim(f_1R) = 1$. Next we take an idempotent $f_2(e_2R)$ such that $\dim(f_2R) = 1$ and $f_1f_2 = 0$. Continuing this procedure, we can take a set $\{f_j\}$ of orthogonal idempotents of $R$. Set $e = \vee f_j$ and then $\Psi(e) \neq 0$. We have that $eR = J \oplus (eR \cap e_jR)$ and $J \subset \subset F_i$ for some right ideal $J$. Noting that $J \otimes_R R = 0$, we have that

$$0 \neq \Psi(e)R \simeq eR \otimes_R R$$

$$\simeq [J \oplus (eR \cap e_jR)] \otimes_R R$$

$$\simeq e_jR \otimes_R R$$

$$\simeq C_j \quad \text{for all } j.$$ 

Therefore $0 \neq \Psi(e)R \leq \otimes_{i=1}^\infty A_i$ for any positive integer $m$. Hence $R$ has (DF) by Theorem 2.11.

By Example 3.4, we have a problem that, for any regular ring $S$, $R = (\Pi_1^m S)/(\oplus S)$ satisfies (DF) or not. Example 3.5 shows that, even if $S$ satisfies (*), $R$ does not satisfy (*). Therefore we shall give the necessary and sufficient condition for that $R$ satisfies (*), and we solve the above problem under this condition.

**Example 3.5.** Let $F$ be a field and set $S = \Pi_{n=1}^\infty M_n(F)$, $\bar{S} = S/(\oplus M_n(F))$, $T = \Pi_{n=1}^\infty S_1(S_1 = S)$ and $R = T/(\oplus S_1)$. Then $S$ satisfies (*), but $R$ does not satisfy (*).

**Proof.** It is clear that $S$ satisfies (*). Therefore we shall show that $R$ does not satisfy (*). Set a central idempotent $e \in T$ as following;

$$e(n) = (0, \ldots, 0, 1_{n-1}, 0, 0, \ldots),$$

$$\left[ \begin{array}{c} n-1 \\ n \end{array} \right]$$

where $e(n) \in S_n$. Let $\Phi$ be the natural map from $S$ to $\bar{S}$, and $\rho$ be the natural map from $T$ to $R$. Set $\Psi = \rho|_T$. Noting that $e(n)S_n \simeq M_n(F)$, we have $eT \simeq S$. Hence there exists a ring
isomorphism $\kappa$ from $eT$ to $S$. Now, we define a ring homomorphism $\alpha$ from $\Psi(e)R$ to $S$ as following; for each $x \in \Psi(e)R$, we take any element $y$ of $\Psi^{-1}(x)$ and set $\alpha(x) = \Phi\kappa(y)$.

$$\Psi(e)R \xrightarrow{\Phi} S$$

Similarly we define a ring homomorphism $\beta$ from $S$ to $\Psi(e)R$. Then we have that $\beta\alpha = 1_{\Psi(e)R}$ and $\alpha\beta = 1_S$. Hence $\alpha$ and $\beta$ are isomorphic. Therefore $\Psi(e)R \cong S$. Let $I$ be a nonzero two-sided ideal of $S$, and so $I = J/(\oplus S_i)$ for some nonzero two-sided ideal of $S$ which contains $\oplus S_i$. There exists $0 \neq x \in J - (\oplus S_i)$ with $x(i) \neq 0$ for almost all $i$. Since $S_i x(i) S_i = M_i(F)$ has index $i$, there exists a nonzero central idempotent $e(i)$ of $M_i(F)$ which $S_i x(i) S_i$ has index $i$. Therefore $S x S$ does not have bounded index, and so does not $(\oplus S_i)$. Therefore $S$ does not satisfy $(\ast)$, and hence so does not $\Psi(e)R$. Thus $R$ does not satisfy $(\ast)$.

**Lemma 3.6.** Let $R$ be a ring, and $e$, $f$ be idempotents of $R$. Then $eR \cong fR$ if and only if there exist $u$ and $v$ of $R$ such that $uv = e$ and $vu = f$.

**Lemma 3.7.** Let $S$ be a regular ring which has index $\infty$, and set $R = (\Pi_{i=1}^\infty S_i)/(\oplus S_i)$ ($S_i = S$). Then $R$ has an infinite direct sum of nonzero pairwise isomorphic right ideals.

**Proof.** Let $\Psi$ be the natural map from $\Pi_{i=1}^\infty S_i$ to $R$. Since $S$ has index $\infty$, there exists a set of idempotents $\{e_{ij}\}_{i,j=1,2,\ldots}$ as following:

$$e_{11}S$$

$$e_{21}S \approx e_{22}S$$

$$e_{31}S \approx e_{32}S \approx e_{33}S$$

$$\ldots$$

where $e_{ij} = 0 (i < j)$ and $\{e_{i1}, \ldots, e_{ii}\}$ are nonzero orthogonal for all $i$. For all positive integer $m$, we take idempotents $\{f_{km}\}$ such that $f_{km}(k) = e_{km}$ for all positive integer $k$. Since $e_{k1}S \approx e_{k2}S$ for all $k$, there exist $u_k$ and $v_k$ of $S$ such that $u_kv_k = e_{k1}$ and $v_ku_k = e_{k1}$ by Lemma 3.6. Set $u$ and $v$ of $\Pi_{i=1}^\infty S_i$ such that $u(k) = u_k$ and $v(k) = v_k$. Then $uv = f_1$ and $vu = f_1 - e$, where $e$ is an idempotent with $e(1) = e_{11}$ and $e(k) = 0$ ($k \neq 1$). Hence $(f_1 - e)(\Pi S_i) \approx f_2(\Pi S_i)$ and $(f_1 - e)(\Pi S_i) \cap f_2(\Pi S_i) = 0$. Therefore we see from Lemma 3.6 that $\Psi(f_1 - e)R \cong \Psi(f_2)R$ and $\Psi(f_1 - e)R \cap \Psi(f_2)R = 0$. Since $\Psi(f_1 - e)R = \Psi(f_1)R$, we have that $\Psi(f_1)R \cong \Psi(f_2)R$ and $\Psi(f_1)R \cap \Psi(f_2)R = 0$.
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=0. Continuing this procedure, for all positive integers $i$ and $j$, $\Psi(f_i)R \simeq \Psi(f_j)R$ and $\Psi(f_i) \cap \Psi(f_j)R = 0 \ (i \neq j)$. Thus $R$ has an infinite direct sum of nonzero pairwise isomorphic right ideals.

**Theorem 3.8.** Let $S$ be a regular ring, and set $R = (\Pi_{i=1}^{\infty} S_i) / (\bigoplus S_i)$ ($S_i = S$). Then the following conditions are equivalent:

(a) $R$ satisfies (*).
(b) $R$ is a regular ring whose primitive factor rings are artinian.
(c) $R$ has bounded index.
(d) $R$ contains no infinite direct sums of nonzero pairwise isomorphic right ideals.
(e) $S$ has bounded index.

Proof. It is clear by Lemma 3.7 that (d) $\rightarrow$ (e) $\rightarrow$ (c) $\rightarrow$ (b) $\rightarrow$ (a) hold. (a) $\rightarrow$ (d) follows from Corollary 2.5. Therefore this theorem is complete.

**Theorem 3.9.** Let $S$ be a regular ring of bounded index. Set $R = (\Pi_{i=1}^{\infty} S_i) / (\bigoplus S_n)$ ($S_n = S$). Then $R$ has (DF).

Proof. Set $\Pi_{n=1}^{\infty} S_n = T$, and let $\Psi$ be the natural map from $T$ to $R$. Let $I$ be a nonzero direct summand of $R$ with following decompositions:

$I = A_1 \oplus B_1$
$A_i = A_{2i} \oplus B_{2i}$
$B_i = A_{2i+1} \oplus B_{2i+1}$ for $i = 1, 2, \ldots$.

Similarly to the proof of Example 3.4, we may assume that there exists an infinite subset $\{C_j\}$ of $\{A_i\}$ ($C_j = A_i$ for some $i$) such that $C_j > C_{j+1}$ and $C_j \neq 0$ for all positive integer $j$. We have the set of idempotents $\{e_j\}$ of $T$ such that $\Psi(e_j T) = C_j$ and $e_j T > e_{j+1} T$. Set $J_n = S_n \times (\Pi_{i=n}^{\infty} S)$, then, $J_{n_1} \cap e_1 T \neq 0$ for some positive integer $n_1$. There exists a nonzero idempotent $f_1 \in T$ such that $f_1 T = J_{n_1} \cap e_1 T$. Next we have a nonzero idempotent $f_2 \in T$ for some $n_2 (> n_1)$ such that $f_2 R = J_{n_2} \cap e_2 R$. Continuing this procedure, we have the set $\{f_j\}$ of orthogonal idempotents of $T$. Now, we set an idempotent $g$ of $T$ as following:

$g(n_j) = f_j(n_j) = e_j(n_j)$
$g(k) = 0 \ (k \notin \{n_j\})$.

Put $K_j = f_1 T \oplus \cdots \oplus f_{j-1} T$ for all $j$. Then $g T = K_j \oplus (g T \cap e_j T)$. Noting $K_j \oplus T R = 0$, we have that
\[ 0 \neq \Phi(g)R \simeq gT \otimes \gamma R \]
\[ \simeq [K_j \oplus (gT \cap e_jT)] \otimes \gamma R \]
\[ \simeq (gT \cap e_jT) \otimes \gamma R \]
\[ \leq e_jT \otimes \gamma R \]
\[ \simeq C_j \quad \text{for all } j. \]

From the above, we have that \( \Phi(g)R \leq \bigoplus_{i=1}^{m^m} A_i \) for any positive integer \( m \). Therefore \( R \) has (DF) by Theorem 2.11.

References


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