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SPECTRA AND EIGENFORMS OF THE LAPLACIAN ON S^n AND $P^n(C)$

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Introduction

Let M be a compact Riemannian manifold. We consider the Laplace operator Δ acting on the space of differential forms on M. It is a strongly elliptic self-adjoint differential operator, so it has discrete eigenvalues with finite multiplicities. For a given Riemannian manifold, it may be an interesting problem to determine explicitly the spectra and eigenforms of Δ on M. As for the spectra and eigenfunctions of Δ acting on the space of functions, they are known for the cases where M are the following manifolds; flat tori, Klein bottles [3], symmetric spaces [12] and the Hopf manifolds [1]. On the other hand, as for the spectra and eigenforms of Δ acting on the differential forms, we have known no results except for flat tori. But, E. Calabi (unpublished) and recently S. Gallot et D. Meyer [7] have computed the eigenvalues of differential forms on the standard sphere by using the harmonic polynomial forms on $\boldsymbol{R}^{n+1}$.

In this paper, applying the representation theory we compute the eigenvalues of Δ and determine the spaces of eigenforms as representation spaces, on the standard sphere S^n and the complex projective space $P^n(\boldsymbol{C})$ with Fubini-Study metric. Our method is as follows: Let $M=G/K$ be a Riemannian homogeneous space with G acting as transitive isometry group on M. Then Δ is a G-invariant differential operator, so its eigenspaces are G-modules. First, we decompose the space of differential forms on M into G-irreducible modules. In the case where M is S^n, $P^n(\boldsymbol{C})$, or more generally a symmetric space, roughly speaking $\Delta=$ $-$Casimir operator on G. So from Freudenthal's Formula, we can compute the eigenvalues. But the first step of decomposing the space of differential forms on M into G-irreducible modules is generally not easy. In virtue of Frobenius reciprocity law, the problem can be reduced to the following problem: For a given irreducible G-module, how does it decompose into irreducible K-modules? For this problem, a few results are known (cf. H. Boerner [3] and D. P. Zelobenko [15]), and in case $M=S^n$, we apply the known results.

As for the Laplacian Δ acting on the space of functions on S^n and $P^n(\boldsymbol{C})$, its eigenfunctions have been obtained by the restriction to S^n and $P^n(\boldsymbol{C})$ of the harmonic homogeneous polynomials on $\boldsymbol{R}^{n+1}$ and $\boldsymbol{C}^{n+1}$ (cf. [12]). In **6** (Theorem 6.8) and **7** (Theorem 7.13), we give the analogy for differential forms on S^n and $P^n(\boldsymbol{C})$ using harmonic polynomial forms.

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1. Preliminaries

Let G be a compact connected Lie group, K a closed subgroup and M the quotient space G/K. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. A K-invariant inner product on $\mathfrak{g}/\mathfrak{k}$ determines a G-invariant Riemannian metric on M. We fix a G-invariant Riemannian metric on M and extend it canonically to a hermitian metric, denoted by $\langle\ ,\ \rangle$, on $\Lambda^p M$, the p-th exterior power of the complexified cotangent bundle of M. For a smooth vector bundle E over M, we denote by $C^\infty(E)$ the vector space of smooth sections of E. When E is a homogeneous vector bundle, $C^\infty(E)$ is considered as a G-module by $(g\cdot s)(x)=g\cdot s(g^{-1}x)$ for $g\in G$, $s\in C^\infty(E)$, and $x\in M$, in particular, $C^\infty(\Lambda^p M)$ has a natural G-module structure over $\boldsymbol{C}$. Now, we define the inner product $(\ ,\ )_M$ on $C^\infty(\Lambda^p M)$ by

$$(1.1)\qquad (\phi, \psi)_M = \int_M \langle\phi, \psi\rangle\, dm \qquad (\phi, \psi\in C^\infty(\Lambda^p M)),$$

where dm is the smooth measure on M defined by the Riemannian metric. From the construction, the G-action preserves $(\ ,\ )_M$, i.e.,

$$(g\cdot\phi, g\cdot\psi)_M = (\phi, \psi)_M \qquad \text{for all } \phi, \psi\in C^\infty(\Lambda^p M) \text{ and } g\in G.$$

By means of this inner product, we define the codifferentiation δ as the operator formally adjoint to the exterior differentiation d. We set $\Delta=d\delta+\delta d$, and call it the Laplace operator or Laplacian. Δ is a self-adjoint, strongly elliptic differential operator on $C^\infty(\Lambda^p M)$ for each p, and commutes with the G-action on $C^\infty(\Lambda^p M)$. The set of eigenvalues of Δ on $C^\infty(\Lambda^p M)$ is a discrete set of non-negative real numbers; $0\leqq\lambda_1^p<\lambda_2^p<\cdots\to\infty$. Moreover, each eigenspace $E^p_{\lambda_i^p}$ is a finite dimensional G-submodule and the algebraic sum $\sum_{i=1}^\infty E^p_{\lambda_i^p}$ is a dense subspace of $C^\infty(\Lambda^p M)$, the topology being defined by the inner product $(\ ,\ )_M$.

First, we recall some effects of d, δ and $*$ on the spectra and the eigenspaces of Laplacian. Hodge decomposition theorem asserts that

$$C^\infty(\Lambda^p M) = E_0^p\oplus dC^\infty(\Lambda^{p-1}M)\oplus\delta C^\infty(\Lambda^{p+1}M),$$

where E_0^p is the space of harmonic p-forms. This is a direct sum as G-modules. For each eigenvalue λ of Laplacian, we set

$$'E_\lambda^p = \{\phi \in E_\lambda^p;\ d\phi = 0\}\ ,\quad ''E_\lambda^p = \{\phi \in E_\lambda^p;\ \delta\phi = 0\}\ ,$$

which are both G-submodules of E_λ^p. We see

$$\text{(1.2)}\qquad \begin{cases} E_\lambda^p = 'E_\lambda^p \oplus ''E_\lambda^p & \text{for } \lambda \neq 0\,, \\ E_\lambda^p = 'E_\lambda^p = ''E_\lambda^p & \text{for } \lambda = 0\,, \end{cases}$$

and

$$\text{(1.3)}\qquad d\colon ''E_\lambda^p \xrightarrow{\sim} 'E_\lambda^{p+1},\ \delta\colon 'E_\lambda^p \xrightarrow{\sim} ''E_\lambda^{p-1} \qquad \text{for } \lambda \neq 0\,.$$

Here and in the following, "$\xrightarrow{\sim}$" means a G-isomorphism, unless specially mentioned.

Suppose that M is orientable and is oriented. Then, the so-called star operator

$$*\colon \Lambda^p M \xrightarrow{\sim} \Lambda^{n-p} M \qquad (n = \dim M)$$

is defined, and the codifferentiation is expressed as

$$\text{(1.4)}\qquad \delta = (-1)^{np+n+1} * d * \qquad \text{on } C^\infty(\Lambda^p M)\,.$$

This together with $*\Delta = \Delta *$ implies

$$\text{(1.5)}\qquad \begin{cases} \lambda_i^p = \lambda_i^{n-p} & (i = 1, 2, \cdots) \text{ and} \\ *\colon 'E_\lambda^p \xrightarrow{\sim} ''E_\lambda^{n-p} & (p = 0, 1, \cdots, n)\,. \end{cases}$$

Now, for a finite dimensional vector space U, we denote by $C^\infty(G; U)$ the vector space of all smooth functions of G with values in U, and consider it as a G-module by $(g\cdot f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C^\infty(G; U)$, and $x \in G$. Further, when U is a K-module, we denote by $C^\infty(G, K; U)$ the G-submodule consisting of $\phi \in C^\infty(G; U)$ such that $\phi(gk) = k^{-1}\phi(g)$ for any $k \in K$ and $g \in G$. On the other hand, the K-module U defines a homogeneous vector bundle $G \times_K U$ over G/K. Then, the G-module $C^\infty(G, K; U)$ can be identified with $C^\infty(G \times_K U)$, in particular, $C^\infty(\Lambda^p M)$ is identified with $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})$, $\mathfrak{g}/\mathfrak{k}$ being a K-module by the adjoint action of K in $\mathfrak{g}$, since $\Lambda^p M$ is canonically isomorphic to $G \times_K \Lambda^p (\mathfrak{g}/\mathfrak{k})^{*C}$.

Let $\mathcal{J}_G$ be a complete set of inequivalent irreducible representations of G over $\boldsymbol{C}$. For an element $\rho \in \mathcal{J}_G$, we define a G-homomorphism

$$\iota_\rho\colon \mathrm{Hom}_G(V_\rho, C^\infty(\Lambda^p M)) \otimes_C V_\rho \to C^\infty(\Lambda^p M)$$

by $\phi \otimes u \mapsto \phi(u)$, where V_ρ is the representation space of ρ. ι_ρ is clearly injective. We set $\mu_\rho = \dim_C \mathrm{Hom}_G(V_\rho, C^\infty(\Lambda^p M))$ and Γ_ρ^p = the image of ι_ρ. Then, it is easy to see that μ_ρ and Γ_ρ^p depend only on the equivalence class of ρ, and that Γ_ρ^p is isomorphic to the direct sum of μ_ρ-copies of V_ρ. To compute μ_ρ, we apply the following Frobenius' reciprocity law.

Proposition 1.1. *Let F be* $\boldsymbol{R}$ *or* $\boldsymbol{C}$. *For a finite dimensional K-module U over F and a finite dimensional G-module V over F, we have a canonical isomorphism as vector spaces*

$$\mathrm{Hom}_G(V, C^\infty(G, K; U)) \cong \mathrm{Hom}_K(V, U).$$

For a proof, see R. Bott [4].

By this proposition, we get

$$\mu_\rho = \dim \mathrm{Hom}_K(V_\rho, \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C}) \qquad \text{for any } \rho \in \mathcal{G}_G, \tag{1.6}$$

hence, in particular, μ_ρ is finite.

Proposition 1.2. *Under the above notations, we have*

$$\sum_i^\infty E_{\lambda_i^p}^p = \sum_{\rho\in\mathcal{G}_G} \Gamma_\rho^p \tag{1.7}$$

Proof. The left hand side of (1.7) is clearly included in the right. The converse is proved as follows. Since the inner product $(\ ,\ )_M$ on $C^\infty(\Lambda^p M)$ is invariant under G, Γ_ρ^p's are orthogonal to each other. On the other hand, the left hand side of (1.7) is dense in $C^\infty(\Lambda^p M)$ with respect to the topology defined by $(\ ,\ )_M$, and included in the right. From these, we get easily the proposition. q.e.d.

In the case that (G, K) is a compact symmetric pair with a semisimple Lie group G, we shall see later that each Γ_ρ^p is contained in a certain eigenspace E_λ^p.

2. The Laplace operator and the Casimir operator

In this section, we shall give some relations between E_λ^p's and Γ_ρ^p's, using the Casimir operator. Let G be a compact semisimple Lie group and B the Killing form of the Lie algebra $\mathfrak{g}$ of G. Let $\{X_1, \cdots, X_N\}$ be a basis of $\mathfrak{g}$, and put

$$\mathcal{C} = \sum_{i,j} C^{ij} X_i \cdot X_j, \qquad \text{where } (C^{ij}) = (B(X_i, X_j))^{-1}, \tag{2.1}$$

which is an element in the universal enveloping algebra $U_{\mathfrak{g}}$ of $\mathfrak{g}$. $\mathcal{C}$ is in the center of $U_{\mathfrak{g}}$ and is called the Casimir element. Since each X_i may be regarded as a left invariant differential operator on G, $\mathcal{C}$ can be considered as a differential operator on G, and it is a two-sided invariant differential operator. Then, $\mathcal{C}$ is called the Casimir operator.

The following fact is easily proved.

Proposition 2.1. *Let G and* $\mathcal{C}$ *be as above, and U a finite dimensional vector space over* $\boldsymbol{C}$. *Then, for any finite dimensional G-submodule* (V, ρ) *of* $C^\infty(G; U)$, *we have*

$$(2.2) \qquad Cf = \rho(C)f \quad \text{for all } f \text{ in } V,$$

ρ being extended to the representation of $U_{\mathfrak{g}}$.

For a proof, see M. Takeuchi [12].

Corollary. *Each finite dimensional G-submodule of $C^\infty(G; U)$ is stable under the Casimir operator.*

In Proposition 2.1, if ρ is irreducible, then $\rho(C)$ is a scalar operator by Schur's Lemma, and the value of $\rho(C)$ is given by the following

Proposition 2.2. *Let G and C be as before. We fix a lexicographic order on the dual space of a Cartan subalgebra of $\mathfrak{g}$. Then, for any irreducible representation (V, ρ) of G over $\boldsymbol{C}$ with the highest weight λ_ρ, we have*

$$(2.3) \qquad \rho(C) = -4\pi^2\langle\lambda_\rho+2\delta_G, \lambda_\rho\rangle id_V ,$$

where δ_G denotes the half sum of all positive roots of $\mathfrak{g}$, and $\langle\ ,\ \rangle$ denotes the inner product on the dual space of the Cartan subalgebra of $\mathfrak{g}$, induced from the Killing form sign changed.

For a proof, see N. Jacobson [9].

Now, we assume that (G, K) is a compact symmetric pair with a compact connected semisimple Lie group G. Let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Then, we have the Cartan decomposition,

$$\mathfrak{g} = \mathfrak{k}+\mathfrak{m} .$$

Restricting the Killing form sign changed to $\mathfrak{m}$, we can define a G-invariant Riemannian metric on $M=G/K$. Then, we have the following formula.

Proposition 2.3. *Let G, K and M be as above. Let Δ be the Laplace operator on M defined by the metric given above. Then, under the identification $C^\infty(\Lambda^p M)=C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})$, we have*

$$(2.4) \qquad \Delta = -C .$$

Analogous formula is given by Y. Matsushima—S. Murakami [10], when G is of non-compact type. We shall prove the formula along the same way as in [10].

Proof. Let $\pi: G\to M$ be the projection. The identification

$$C^\infty(\Lambda^p M) \cong C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C}) ,$$

denoted by $\alpha\mapsto\tilde{\alpha}$, is given by

$$(\tilde{\alpha}(g))(Y_1, \cdots, Y_p) = (\pi^*\alpha)(Y_1, \cdots, Y_p)(g) ,$$

for $g\in G$ and $Y_1, \cdots, Y_p \in \mathfrak{g}$, where $(\mathfrak{g}/\mathfrak{k})^*$ is considered as the set of linear forms on $\mathfrak{g}$ vanishing on $\mathfrak{k}$. Now, we choose an orthonormal basis $\{X_1, \cdots, X_n, X_{n+1}, \cdots, X_N\}$ of $\mathfrak{g}$, with respect to the inner product induced from the Killing form, in such a way that $\{X_1, \cdots, X_n\}$ forms a basis of $\mathfrak{m}$ and $\{X_{n+1}, \cdots, X_N\}$ a basis of $\mathfrak{k}$. An element η in $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})$ is determined by the system of C^∞-functions $\eta(X_{i_1}, \cdots, X_{i_p})$ $(1\leqq i_1<\cdots<i_p\leqq n)$.

Then, we define a linear map D: $C^\infty(G,K;\Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})\to C^\infty(G,K;\Lambda^{p+1}(\mathfrak{g}/\mathfrak{k})^{*C})$ by

$$(1)\qquad (D\eta)(X_{i_1}, \cdots, X_{i_{p+1}}) = \sum_{u=1}^{p+1} (-1)^{u-1} X_{i_u} \eta(X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_{p+1}}) \quad (1\leqq i_1<\cdots<i_{p+1}\leqq n).$$

We have

$$\widetilde{d\alpha} = D\tilde{\alpha} \qquad (\alpha\in C^\infty(\Lambda^p M)).$$

This follows from the fact that $[X_{i_u}, X_{i_v}]\in\mathfrak{k}$ for $X_{i_u}, X_{i_v}\in\mathfrak{m}$.

Next, we define an inner product $(\ ,\ )^*$ on $C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})$, using a K-invariant inner product on $\Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C}$ by

$$(2)\qquad (\xi, \eta)^* = \int_G \langle \xi(g), \eta(g)\rangle dg$$

for $\xi, \eta\in C^\infty(G,K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})$, where dg denotes a G-invariant smooth measure on G. Then, it is easy to see that

$$(3)\qquad (\alpha, \beta)_M = c\cdot(\alpha, \beta)^* \qquad (\alpha, \beta\in C^\infty(\Lambda^p M))$$

for some constant c.

The adjoint operator D^* of D is given by

$$(4)\qquad (D^*\xi)(X_{i_1}, \cdots, X_{i_{p-1}}) = -\sum_{k=1}^{n} X_k \xi(X_k, X_{i_1}, \cdots, X_{i_{p-1}}), \quad (1\leqq i_1<\cdots<i_{p-1}\leqq n),$$

where $\xi\in C^\infty(G, K; \Lambda^p(\mathfrak{g}/\mathfrak{k})^{*C})$.

In fact, for $\eta\in C^\infty(G, K; \Lambda^{p-1}(\mathfrak{g}/\mathfrak{k})^{*C})$, we have

$$\begin{aligned}(\xi, D\eta)^* &= \int_G \frac{1}{p!} \sum_{i_1,\cdots,i_p=1}^{n} \xi(X_{i_1}, \cdots, X_{i_p}) \\ &\quad \cdot \sum_{u=1}^{p} (-1)^{u-1} X_{i_u} \eta(X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p}) dg \\ &= -\frac{1}{p!} \sum_{i_1,\cdots,i_p=1}^{n} \int_G \sum_{u=1}^{p} (-1)^{u-1} X_{i_u}\cdot\xi(X_{i_1}, \cdots, X_{i_p}) \\ &\quad \cdot\eta(X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p}) dg \qquad \text{(cf. [10])}\end{aligned}$$

$$= -\frac{1}{(p-1)!}\sum_{j_1,\cdots,j_{p-1}=1}^{n}\int_G \sum_{k=1}^{n} X_k\xi(X_k, X_{j_1}, \cdots, X_{j_{p-1}}) \cdot\eta(X_{j_1}, \cdots, X_{j_{p-1}})dg\,.$$

We set $\Delta^\circ = DD^* + D^*D$. Then, we see

$$(\Delta\alpha, \beta)_M = c\cdot(\Delta^\circ\tilde{\alpha}, \tilde{\beta})^* \qquad \text{for } \alpha, \beta \in C^\infty(\Lambda^p M)\,,$$

consequently, $\widetilde{\Delta\alpha} = \Delta^\circ\tilde{\alpha}$.

To prove the proposition, it suffices to show that

$$\widetilde{\Delta\alpha} = -\sum_{k=1}^{N} X_k^2\tilde{\alpha} \qquad \text{for } \alpha \in C^\infty(\Lambda^p M)\,.$$

From (1) and (4), for $X_{i_1}, \cdots, X_{i_p}$ with $1 \leqq i_1 < \cdots < i_p \leqq n$, we have

$$(DD^*\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} X_{i_u}X_k\tilde{\alpha}(X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})$$

and

$$(D^*D\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{k=1}^{n} X_k(D\tilde{\alpha})(X_k, X_{i_1}, \cdots, X_{i_p})$$

$$= -\sum_{k=1}^{n} X_k^2\tilde{\alpha}(X_{i_1}, \cdots, X_{i_p})$$

$$-\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^u X_k X_{i_u}\tilde{\alpha}(X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})$$

$$-\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^u X_k\tilde{\alpha}([X_k, X_{i_u}], X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})$$

$$-\sum_{k=1}^{n}\sum_{u<v}(-1)^{u+v} X_k\tilde{\alpha}([X_{i_u}, X_{i_v}], X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, \widehat{X_{i_v}}, \cdots, X_{i_p})\,.$$

Hence, we have

$$(\Delta^\circ\tilde{\alpha})(X_{i_1}, \cdots, X_{i_p}) = -\sum_{k=1}^{n} X_k^2\tilde{\alpha}(X_{i_1}, \cdots, X_{i_p})$$

$$-\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^{u-1}[X_{i_u}, X_{i_k}]\tilde{\alpha}(X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})\,.$$

The second term in the right hand side of the above equation will be denoted by II. Set $[X_i, X_k] = \sum_{j=1}^{N} b_i{}^j{}_k X_j$. Since G is compact, the structure constants $b_i{}^j{}_k$ are skew symmetric with respect to i, j, k. Thus, we have

$$\mathrm{II} = -\sum_{k=1}^{n}\sum_{u=1}^{p}(-1)^{u-1}\sum_{j=n+1}^{N} b_{i_u}{}^j{}_k X_j\tilde{\alpha}(X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})$$

by virtue of $[X_{i_u}, X_k] \in \mathfrak{k}$. On the other hand, for $j = n+1, \cdots, N$,

$$X_j\tilde{\alpha}(X_{i_1}, \cdots, X_{i_p})(g) = \frac{d}{dt}\Big|_{t=0} \tilde{\alpha}(g\cdot\exp tX_j)(X_{i_1}, \cdots, X_{i_p})$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{\alpha}(g)(\exp tX_j\cdot X_{i_1}, \cdots, \exp tX_j\cdot X_{i_p})$$

$$= \sum_{u=1}^{p}(-1)^{u-1}\tilde{\alpha}(g)([X_j, X_{i_u}], X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})$$

$$= \sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} b_j{}^k{}_{i_u}\tilde{\alpha}(X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p})$$

$$= \sum_{u=1}^{p}(-1)^{u-1}\sum_{k=1}^{n} b_{i_u k}^{\ j}\tilde{\alpha}(X_k, X_{i_1}, \cdots, \widehat{X_{i_u}}, \cdots, X_{i_p}).$$

Consequently, we have

$$\mathrm{II} = -\sum_{j=n+1}^{N} X_j^2\tilde{\alpha}(X_{i_1}, \cdots, X_{i_p}),$$

which proves the desired formula. q.e.d.

3. Some remarks in Kählerian cases

In this section, we assume that $M=G/K$ is a homogeneous Kählerian manifold acted on by a compact Lie group G. Denoting $\mathfrak{m}=\mathfrak{g}/\mathfrak{k}$, we identify $\mathfrak{m}$ with the tangent space of M at the origin $e\cdot K$ by the projection $G\to G/K$. We denote by J the complex structure on $\mathfrak{m}$ and by $\langle\ ,\ \rangle$ the J-invariant inner product on $\mathfrak{m}$ defined by the Kähler metric on M. Then, the K-action on $\mathfrak{m}$, which is identical to the isotropy representation at the origin, preserves J and $\langle\ ,\ \rangle$. Let $U(\mathfrak{m})$ be the group of linear automorphisms of $\mathfrak{m}$ leaving J and $\langle\ ,\ \rangle$ invariant. Complexifying $\mathfrak{m}$ and J, we set $\mathfrak{m}^C=\mathfrak{m}\otimes_R \boldsymbol{C}$ and $\mathfrak{m}^{\pm}=\{X\in\mathfrak{m}^C; JX=\pm\sqrt{-1}X\}$. We have the direct sum as $U(\mathfrak{m})$-modules and also as K-modules over $\boldsymbol{C}$

$$\mathfrak{m}^C = \mathfrak{m}^+\oplus\mathfrak{m}^-.$$

$\mathfrak{m}^+$ and $\mathfrak{m}^-$ are conjugate-linearly K-isomorphic to each other by the conjugation $X\to\bar{X}$ of $\mathfrak{m}^C$ with respect to $\mathfrak{m}$, and the complex bilinear extension $\langle\ ,\ \rangle^C$ of $\langle\ ,\ \rangle$ to $\mathfrak{m}^C$ gives rise to canonical K-isomorphisms over $\boldsymbol{C}$

$$(\mathfrak{m}^{\pm})^* \cong \mathfrak{m}^{\mp}. \tag{3.1}$$

Now, we define a 2-form ω of type (1.1) and a hermitian inner product $\langle\ ,\ \rangle^h$ on $\mathfrak{m}^C$ by

$$\omega(X, Y) = \langle X, JY\rangle^C \qquad (X, Y\in\mathfrak{m}^C),$$
$$\langle X, Y\rangle^h = \langle X, \bar{Y}\rangle^C \qquad (X, Y\in\mathfrak{m}^C).$$

The $U(\mathfrak{m})$-action and the hermitian inner product $\langle\ ,\ \rangle^h$ will be canonically extended to $\Lambda^p\mathfrak{m}^-\otimes\Lambda^q\mathfrak{m}^+$, and the later will be denoted simply by $\langle\ ,\ \rangle$ in the sequel. $\Lambda^p\mathfrak{m}^-\otimes\Lambda^q\mathfrak{m}^+$ will be written by $\Lambda^{p,q}\mathfrak{m}$. ω is regarded as an element of $\Lambda^{1,1}\mathfrak{m}$ by (3.1).

We set $\Lambda_0^{r,s}=\{u\in\Lambda^{r,s}\mathfrak{m};\ \langle u, \omega\wedge v\rangle=0 \text{ for all } v\in\Lambda^{r-1,s-1}\mathfrak{m}\}$.

Then, the following fact is known (See A. Weil [13]).

Proposition 3.1.

(i) *For p and q with $p+q\leqq n$, the $U(\mathfrak{m})$-irreducible decomposition of $\Lambda^{p,q}\mathfrak{m}$ is given by*

$$\Lambda^{p,q}\mathfrak{m} = \begin{cases} \Lambda_0^{p,q}\oplus\omega\wedge\Lambda_0^{p-1,q-1}\oplus\cdots\oplus\omega^q\wedge\Lambda_0^{p-q,0}\ (p\geqq q)\,, \\ \Lambda_0^{p,q}\oplus\omega\wedge\Lambda_0^{p-1,q-1}\oplus\cdots\oplus\omega^p\wedge\Lambda_0^{0,q-p}\ (p\leqq q)\,. \end{cases} \tag{3.2}$$

(ii) *The mapping $u\mapsto\omega^k\wedge u$ of $\Lambda_0^{p,q}$ into $\omega^k\wedge\Lambda_0^{p,q}$ is a K-isomorphism, when max. $\{p+k, q+k\}\leqq n$.*

The irreducibility of $\Lambda_0^{r,s}$ follows from Weyl's dimension formula.

Now, we denote by $\Lambda^{p,q}M$ the bundle of $(p+q)$-covectors on M of type (p,q). Since the G-action preserves the complex structure on M, $\Lambda^{p,q}M$ is a homogeneous vector bundle;

$$\Lambda^{p,q}M\cong G\times_K\Lambda^{p,q}\mathfrak{m}\,, \tag{3.3}$$

by virtue of (3.1). Under this isomorphism, the Kähler form of M corresponds to the section of the bundle $G\times_K\Lambda^{1,1}$ given by

$$\Omega_x = g\times_K\omega \qquad (x=gK,\ x\in M)\,,$$

where $g\times_K\omega$ denotes the equivalence class of (g,ω). Then, we get immediately the corresponding decomposition of $\Lambda^{p,q}M$ for p and q with $p+q\leqq n$,

$$\Lambda^{p,q}M = \sum_i \Omega^i\wedge G\times_K\Lambda_0^{p-i,q-i} \tag{3.4}$$

where $\wedge$ is defined in an obvious manner.

Further, we define a bundle homomorphism L of $\Lambda^{p,q}M$ into $\Lambda^{p+1,q+1}M$ by

$$L\phi = \Omega_x\wedge\phi \qquad \text{for}\ \ \phi\in\Lambda^{p,q}M_x$$

on each fiber of $\Lambda^{p,q}M$. The G-invariance of Ω implies that L commutes with the G-action and that the linear map induced from L of $C^\infty(\Lambda^{p,q}M)$ into $C^\infty(\Lambda^{p+1,q+1}M)$ is a G-homomorphism, which we denote by the same letter L. By Proposition 3.1, the linear map L^k of $C^\infty(\Lambda^{p,q}M)$ into $C^\infty(\Lambda^{p+k,q+k}M)$ is a G-isomorphism, when max. $\{p+k, q+k\}\leqq n$. Now, we denote by L^* the bundle map adjoint to L with respect to the fiber metric on $\Lambda^{p,q}M$

$$L^*\colon \Lambda^{p,q}M\to\Lambda^{p-1,q-1}M\,.$$

The operator L^* commutes with the G-action and the subbundle Ker L^* of $\Lambda^{p,q}M$ coincides with $G\times_K\Lambda_0^{p,q}$. A section of $G\times_K\Lambda_0^{p,q}$ is said to be a *primitive form* of type (p, q). It is known that the linear maps L and L^* commute with the Laplace operator. Laplacian preserves types of complex differential forms on M. (See A. Weil [13].)

From the above arguments, in order to decompose $C^\infty(\Lambda^p M)$ or $C^\infty(\Lambda^{p,q}M)$ into the eigenspaces of the Laplace operator, or into the irreducible G-submodules, we may work on the space of primitive forms of type (r, s), $C^\infty(G\times_K\Lambda_0^{r,s})$.

Next, we recall effects of several operators acting on M on the spectra of Δ. More precisely, the exterior differentiation d decomposes

$$d = \partial+\bar\partial\,,$$

where

$$\partial\colon C^\infty(\Lambda^{p,q}M) \to C^\infty(\Lambda^{p+1,q}M)$$
$$\bar\partial\colon C^\infty(\Lambda^{p,q}M) \to C^\infty(\Lambda^{p,q+1}M)\,.$$

We denote by ∂^* and $\bar\partial^*$ the operators formally adjoint to ∂ and $\bar\partial$ respectively. Then, the following formulas are known (A. Weil [13]);

$$\partial^* = -*\bar\partial*,\ \bar\partial^* = -*\partial*\,,$$

$$(3.5)\qquad \begin{aligned}&[L, \partial^*] = \sqrt{-1}\bar\partial,\ [L, \bar\partial^*] = -\sqrt{-1}\partial\,,\\ &[L^*, \partial] = \sqrt{-1}\bar\partial^*,\ [L^*, \bar\partial] = -\sqrt{-1}\partial^*\,,\end{aligned}$$

where [,] denotes the commutator of two operators, for example,

$$[L, \partial^*] = L\partial^*-\partial^*L\,.$$

We have

$$\Delta = 2(\partial\partial^*+\partial^*\partial) = 2(\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)\,.$$

With respect to the splitting $\Lambda^r M=\sum_{p+q=r}\Lambda^{p,q}M$ and the above operators, we have the following decompositions of each eigenspace.

For each eigenvalue λ of Δ on $C^\infty(\Lambda^{p+q}M)$, we set

$$\begin{aligned}&E_\lambda^{p,q} = E_\lambda^{p+q}\cap C^\infty(\Lambda^{p,q}M)\,,\\ &{}'E_\lambda^{p,q} = \{\phi\in E_\lambda^{p,q};\ \partial\phi = 0\}\,,\quad {}''E_\lambda^{p,q} = \{\phi\in E_\lambda^{p,q};\ \partial^*\phi = 0\}\,,\\ &{}_{\prime}E_\lambda^{p,r} = \{\phi\in E_\lambda^{p,q};\ \bar\partial\phi = 0\}\,,\quad {}_{\prime\prime}E_\lambda^{p,q} = \{\phi\in E_\lambda^{p,q};\ \bar\partial^*\phi = 0\}\,.\end{aligned}$$

These are all finite dimensional G-submodules of $C^\infty(\Lambda^{p,q}M)$.

In our case, Hodge's decomposition theorem is expressed as

$$C^\infty(\Lambda^{p,q}M) = E_0^{p,q}\oplus\partial C^\infty(\Lambda^{p-1,q}M)\oplus\partial^* C^\infty(\Lambda^{p+1,q}M)$$

$$= E_0^{p,q} \oplus \bar{\partial} C^\infty(\Lambda^{p,q-1}M) \oplus \bar{\partial}^* C^\infty(\Lambda^{p,q+1}M) .$$

Hence, for $\lambda \neq 0$, we have

$$E_\lambda^{p,q} = {}'E_\lambda^{p,q} \oplus {}''E_\lambda^{p,q} \qquad (3.6)$$
$$= {}_{\prime}E_\lambda^{p,q} \oplus {}_{\prime\prime}E_\lambda^{p,q} .$$

Further, we have the following G-isomorphisms,

$$\partial : {}''E_\lambda^{p,q} \overset{\sim}{\rightarrow} {}'E_\lambda^{p+1,q}, \ \partial^* : {}'E_\lambda^{p,q} \overset{\sim}{\rightarrow} {}''E_\lambda^{p-1,q} , \qquad (3.7)$$
$$\bar{\partial} : {}_{\prime\prime}E_\lambda^{p,q} \overset{\sim}{\rightarrow} {}_{\prime}E_\lambda^{p,q+1}, \ \bar{\partial}^* : {}_{\prime}E_\lambda^{p,q} \overset{\sim}{\rightarrow} {}_{\prime\prime}E_\lambda^{p,q-1} ,$$

where $\lambda \neq 0$, and

$$* : {}_{\prime}E_\lambda^{p,q} \overset{\sim}{\rightarrow} {}''E_\lambda^{n-q,n-p}, \ * : {}_{\prime\prime}E_\lambda^{p,q} \overset{\sim}{\rightarrow} {}'E_\lambda^{n-q,n-p} . \qquad (3.8)$$

4. The spectra of Laplacian on the spheres

We retain the notations used in the preceding sections. In this section, we employ the following notations;

$$G = SO(n+1),\ K = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in M_{n+1}(\boldsymbol{R});\ A \in SO(n) \right\} ,$$
$$\mathfrak{g} = \{X \in M_{n+1}(\boldsymbol{R});\ {}^tX + X = 0\},\ \mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} \in M_{n+1}(\boldsymbol{R});\ {}^tY + Y = 0 \right\} ,$$
$$\mathfrak{f} = \left\{ 2\pi \begin{pmatrix} (0) & & & & \\ & 0 & -\lambda_1 & & \\ & \lambda_1 & 0 & & \\ & & & \ddots & \\ & & & 0 & -\lambda_m \\ & & & \lambda_m & 0 \end{pmatrix} ;\ \lambda_1, \cdots, \lambda_m \in \boldsymbol{R} \right\} ,$$

where $n=2m-1$ for an odd n and $n=2m$ for an even n. $\mathfrak{f}$ is a Cartan subalgebra of $\mathfrak{g}$. And $\mathfrak{f}$ is also a Cartan subalgebra of $\mathfrak{k}$ when n is even. If n is odd, then, the subspace $\mathfrak{f}_1$ consisting of the elements of $\mathfrak{f}$ with $\lambda_1=0$ forms a Cartan subalgebra of $\mathfrak{k}$. We consider $\lambda_1, \cdots, \lambda_m$ as linear forms on $\mathfrak{f}$ and take a linear order on $\mathfrak{f}^*$ such that $\lambda_1 > \cdots > \lambda_m > 0$. The Killing form B of $\mathfrak{g}$ is given by

$$B(X, Y) = (n-1)\, tr.XY \qquad (X, Y \in \mathfrak{g})$$

and $\mathfrak{m}$ is naturally identified with

$$\left\{ \begin{pmatrix} 0 & -x_1, \cdots, -x_n \\ x_1 & \\ \vdots & \\ x_n & 0 \end{pmatrix} ;\ x_1, \cdots, x_n \in \boldsymbol{R} \right\} .$$

We adopt the usual metric on S^n imbedded in $\boldsymbol{R}^{n+1}$ with radius one. We may identify $S^n = G/K$ and $TS^n = G \times_K \mathfrak{m}$. Note that the metric induced from the Killing form B is $2(n-1)$-times the usual one and that the formulas recalled in **1** and **2** hold with certain constant multiples.

Any dominant integral form Λ of G with respect to $\mathfrak{t}$ is uniquely expressed as

$$\Lambda = k_1\lambda_1 + \cdots + k_m\lambda_m ,$$

where $k_1, \cdots, k_m$ are integers satisfying

$$\begin{cases} k_1 \geqq k_2 \geqq \cdots \geqq k_{m-1} \geqq |k_m| & (n = 2m-1), \\ k_1 \geqq k_2 \geqq \cdots \geqq k_{m-1} \geqq k_m \geqq 0 & (n = 2m). \end{cases} \tag{4.1}$$

When n is odd, we set

$$z_j = \lambda_{j+1} | \mathfrak{t}_1 \qquad (j = 1, \cdots, m-1),$$

where right hand side denotes the restriction of the linear form λ_{j+1} on $\mathfrak{t}$ to the subspace $\mathfrak{t}_1$, and $z_1, \cdots, z_{m-1}$ are ordered as $z_1 > \cdots > z_{m-1} > 0$.

Proposition 4.1. *Let (V, ρ) be an irreducible G-module over $\mathbf{C}$ with the highest weight $\lambda_\rho = k_1\lambda_1 + \cdots + k_m\lambda_m$, where $k_1, \cdots, k_m$ satisfy* (4.1). *Then, as a K-module, V decomposes into K-irreducible submodules as follows;*

(i) *In case $n = 2m$,*

$$V = \sum V_{k'_1\lambda_1 + \cdots + k'_m\lambda_m} ,$$

where the summation runs over all integers $k'_1, \cdots, k'_m$ such that

$$k_1 \geqq k'_1 \geqq k_2 \geqq k'_2 \geqq \cdots \geqq k'_{m-1} \geqq k_m \geqq |k'_m| ,$$

and $V_{k'_1\lambda_1 + \cdots + k'_m\lambda_m}$ denotes the irreducible K-submodule of V with the highest weight $k'_1\lambda_1 + \cdots + k'_m\lambda_m$.

(ii) *In case $n = 2m-1$,*

$$V = \sum V_{k'_1 z_1 + \cdots + k'_{m-1} z_{m-1}} ,$$

where the summation runs over all integers $k'_1, \cdots, k'_m$ such that

$$k_1 \geqq k'_1 \geqq k_2 \geqq k'_2 \geqq \cdots \geqq k'_{m-1} \geqq |k_m| ,$$

and the meaning of $V_{k'_1 z_1 + \cdots + k'_{m-1} z_{m-1}}$ is similar to the above.

For a proof, see H. Boerner [3].

Applying this proposition to our problem, we shall give explicitly the irreducible representations of $SO(n+1)$ intervening in $C^\infty(\Lambda^p S^n)$.

The multiplicity μ_ρ of an irreducible representation (V, ρ) of G in $C^\infty(\Lambda^p S^n)$ is equal to $\dim_{\mathbf{C}} \mathrm{Hom}_K(V, \Lambda^p \mathfrak{m}^*)$ by Proposition 1.1, and the latter can be computed, applying Schur's Lemma, from the K-irreducible decomposition of $\Lambda^p \mathfrak{m}^*$

and Proposition 4.1.

Now, identifying $K=SO(n)$, the K-module $\mathfrak{m}$ is isomorphic to $\boldsymbol{R}^n$ with the standard representation of $SO(n)$, and we know the following facts;

A. Suppose n is odd, $n=2m-1$. Then, $\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^p\boldsymbol{R}^n$ is an irreducible $SO(n)$-module with the highest weight $\lambda_1+\cdots+\lambda_p$ for each $p\leqq m-1$. For $0\leqq p\leqq n$, $\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^p\boldsymbol{R}^n\cong\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^{n-p}\boldsymbol{R}^n$ as $SO(n)$-modules.

B. Suppose n is even, $n=2m$. Then, $\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^p\boldsymbol{R}^n$ is an irreducible $SO(n)$-module with the highest weight $\lambda_1+\cdots+\lambda_p$ for each $p<m$, and $\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^m\boldsymbol{R}^n$ splits into two irreducible submodules with the highest weight $\lambda_1+\cdots+\lambda_{m-1}-\lambda_m$ and $\lambda_1+\cdots+\lambda_{m-1}+\lambda_m$ respectively. For $p>m$, $\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^p\boldsymbol{R}^n\cong\boldsymbol{C}\otimes_{\boldsymbol{R}}\Lambda^{n-p}\boldsymbol{R}^n$ as $SO(n)$-modules.

Now, we put

$$\Lambda_0 = 0\,,$$
$$\Lambda_j = \lambda_1+\cdots+\lambda_j \qquad (j=1,2,\cdots,m-2)\,,$$
$$\Lambda_{m-1} = \begin{cases} \lambda_1+\cdots+\lambda_{m-1} & (n=2m)\,,\\ \dfrac{1}{2}(\lambda_1+\cdots+\lambda_{m-1}-\lambda_m) & (n=2m-1)\,,\end{cases}$$
$$\Lambda_m = \frac{1}{2}(\lambda_1+\cdots+\lambda_m)\,.$$

$\Lambda_1,\cdots,\Lambda_m$ are the so-called fundamental weights of $\mathfrak{g}$ and every dominant integral form of G is uniquely expressed as a linear combination with non-negative integer coefficients of $\Lambda_1,\cdots,\Lambda_{m-1},2\Lambda_m$, when $n=2m$, and of $\Lambda_1,\cdots,\Lambda_{m-2},2\Lambda_{m-1},\Lambda_{m-1}+\Lambda_m,2\Lambda_m$, when $n=2m-1$. In our case, we remark $\mathfrak{m}^*\cong\mathfrak{m}$ (as K-modules). With these terminologies, by the above procedure we get easily the spectra of Laplacian on S^n as given in the following theorem. Since S^n is orientable, it would be sufficient to write down them for $p\leqq\dfrac{n}{2}$.

Theorem 4.2.

(a) *Suppose* $p\leqq\dfrac{n}{2}$. *The highest weights* λ_ρ *of the irreducible representations* ρ *intervening in* $C^\infty(\Lambda^pS^n)$, *that is,* ρ *in* $\mathcal{J}_G$ *with* $\mu_\rho\geqq1$, *are as follows;*

(i) *In case* $n=2m$,

$$\lambda_\rho = \begin{cases} k\Lambda_1+\Lambda_p,\ k\Lambda_1+\Lambda_{p+1} & (0\leqq p\leqq m-2)\,,\\ k\Lambda_1+\Lambda_{m-1},\ k\Lambda_1+2\Lambda_m & (p=m-1)\,,\\ k\Lambda_1+2\Lambda_m & (p=m)\,,\end{cases}$$

where k runs over all non-negative integers.

(ii) *In case* $n=2m-1$,

$$\lambda_\rho = \begin{cases} k\Lambda_1+\Lambda_p,\ k\Lambda_1+\Lambda_{p+1}\ (0\leqq p\leqq m-3)\,, \\ k\Lambda_1+\Lambda_{m-2},\ k\Lambda_1+\Lambda_{m-1}+\Lambda_m\ (p=m-2)\,, \\ k\Lambda_1+2\Lambda_{m-1},\ k\Lambda_1+\Lambda_{m-1}+\Lambda_m,\ k\Lambda_1+2\Lambda_m\ (p=m-1)\,, \end{cases}$$

where k runs over all non-negative integers.

Further, the multiplicity μ_ρ of the above ρ in $C^\infty(\Lambda^p S^n)$ is exactly one except for the case $n=2m$ and $p=m$, and μ_ρ is two in this exceptional case.

(b) *The Laplace operator has eigenvalue $4\pi^2\langle\lambda_\rho+2\delta_G, \lambda_\rho\rangle$ on an $SO(n+1)$-irreducible submodule of differential forms on S^n with the highest weight λ_ρ, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathfrak{t}^*$ induced from $\frac{-1}{2(n-1)}$-times the Killing form.*

The values $4\pi^2\langle\lambda_\rho+2\delta_G, \lambda_\rho\rangle$ are given in the following table.

	λ_ρ	$4\pi^2\langle\lambda_\rho+2\delta_G, \lambda_\rho\rangle$
case $n=2m$	$k\Lambda_1$	$k(k+n-1)$
	$k\Lambda_1+\Lambda_p$ $(1\leqq p\leqq m-1)$	$(k+p)(k+n+1-p)$
	$k\Lambda_1+2\Lambda_m$	$(k+m)(k+m+1)$
case $n=2m-1$	$k\Lambda_1$	$k(k+n-1$
	$k\Lambda_1+\Lambda_p$ $(1\leqq p\leqq m-2)$	$(k+p)(k+n+1-p)$
	$k\Lambda_1+2\Lambda_{m-1}$	$(k+m)^2$
	$k\Lambda_1+\Lambda_{m-1}+\Lambda_m$	$(k+m-1)(k+m+1)$
	$k\Lambda_1+2\Lambda_m$	$(k+m)^2$

REMARK. For the space of real differential forms on S^n, its irreducible decomposition can be obtained from Theorem 4.2 together with the results of N. Bourbaki [5] and N. Iwahori [8].

Except for the case $n=2m-1$ and m is even, every irreducible submodule of $C^\infty(\Lambda^p S^n)$ is closed under the complex conjugation, and the space of real forms contained in this submodule is an irreducible $SO(n+1)$-module over $\boldsymbol{R}$. When $n=2m-1$ and m is even, two irreducible submodules of differential forms with the highest weight $k\Lambda_1+2\Lambda_{m-1}$ and $k\Lambda_1+2\Lambda_m$ are transformed to each other under the complex conjugation, and the space of real forms in the sum of the two submodules is irreducible over $\boldsymbol{R}$. Conversely, any irreducible submodule of real forms on S^n is obtained in this way.

5. The spectra of Laplacian on the complex projective spaces

In this section, we employ the following notations;

$$G = SU(n+1)\,,$$

$$K=S(U(1)\times U(n))=\left\{\begin{pmatrix}\varepsilon & 0\\ 0 & A\end{pmatrix}\in SU(n+1);\ \varepsilon\in U(1),\ A\in U(n)\right\},$$

$$\mathfrak{g}=\{X\in M_{n+1}(\boldsymbol{C});\ {}^t\bar{X}+X=0,\ tr.X=0\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix}\sqrt{-1}a & 0\\ 0 & Y\end{pmatrix}\in M_{n+1}(\boldsymbol{C});\ {}^t\bar{Y}+Y=0,\ a\in\boldsymbol{R},\ \sqrt{-1}a+tr.Y=0\right\},$$

$$B(X, Y)=2(n+1)tr.XY \qquad (X, Y\in\mathfrak{g}),$$

$$\mathfrak{m}=\left\{\begin{pmatrix}0 & -\bar{\zeta}_1,\cdots,-\bar{\zeta}_n\\ \zeta_1 & \\ \vdots & O\\ \zeta_n & \end{pmatrix}\in M_{n+1}(\boldsymbol{C});\ \zeta_1,\cdots,\zeta_n\in\boldsymbol{C}\right\},$$

$$\mathfrak{t}=\left\{2\pi\sqrt{-1}\begin{pmatrix}x_1 & & 0\\ & \ddots & \\ 0 & & x_{n+1}\end{pmatrix};\ x_1,\cdots,x_{n+1}\in\boldsymbol{R},\ x_1+\cdots+x_{n+1}=0\right\}.$$

The natural complex structure J on $\mathfrak{m}$ is given by

$$J\begin{pmatrix}0 & -\bar{\zeta}_1,\cdots,-\bar{\zeta}_n\\ \zeta_1 & \\ \vdots & O\\ \zeta_n & \end{pmatrix}=\begin{pmatrix}0 & -\sqrt{-1}\bar{\zeta}_1,\cdots,-\sqrt{-1}\bar{\zeta}_n\\ \sqrt{-1}\zeta_1 & \\ \vdots & O\\ \sqrt{-1}\zeta_n & \end{pmatrix},$$

and hence $\mathfrak{m}$ may be identified with $\boldsymbol{C}^n$.
Then, we may identify $P^n(\boldsymbol{C})=G/K$, and as complex vector bundle, we have $TP^n(\boldsymbol{C})\cong G\times_K\mathfrak{m}$.

We adopt the usual Fubini-Study metric on $P^n(\boldsymbol{C})$. Note that the metric induced from the Killing form B is $(n+1)$-times the Fubini-Study metric and the results of **1** and **2** hold with certain constant multiples.

Now, we consider the above $x_1, \cdots, x_{n+1}$ as linear forms on $\mathfrak{t}$ and introduce a linear order on $\mathfrak{t}^*$ such that

$$x_1>x_2>\cdots>x_n>0>x_{n+1}.$$

We note that the image of K in $GL(\mathfrak{m})$ under the adjoint representation in $\mathfrak{m}$ is exactly $U(\mathfrak{m})$, the unitary group of $\mathfrak{m}=\boldsymbol{C}^n$. Hence, the K-modules $\Lambda_0^{p,q}$ are all irreducible. The weights of $\mathfrak{m}$ with respect to $\mathfrak{t}$ are $\{x_i-x_1;\ i=2,3,\cdots,n+1\}$ and the highest weight of $\Lambda_0^{p,q}$ is $(x_2-x_1)+\cdots+(x_{q+1}-x_1)+(x_1-x_{n+1})+\cdots+(x_1-x_{n+2-p})$.

Next, we shall give a general formula to decompose any irreducible G-module into irreducible K-modules. In our case, K is of maximal rank, and every dominant integral form Λ of G and K with respect to $\mathfrak{t}$ is uniquely expressed as

$$\Lambda=k_1x_1+k_2x_2+\cdots+k_nx_n,$$

where $k_1, k_2, \cdots, k_n$ are integers satisfying $k_1\geqq\cdots\geqq k_n\geqq 0$ or $k_2\geqq\cdots\geqq k_n\geqq 0$, according as the group is G or K. We denote the Weyl groups of G and K by W_G and W_K respectively. W_G is the permutation group of $x_1, \cdots, x_{n+1}$ and W_K is the permutation group of $x_2, \cdots, x_{n+1}$.

Proposition 5.1. *Let (V, ρ) be an irreducible G-module with the highest weight $\lambda_\rho = m_1x_1 + \cdots + m_nx_n$ $(m_1 \geqq \cdots \geqq m_n \geqq 0)$. Then, (V, ρ) decomposes, as a K-module, into irreducible K-modules as follows;*

$$V = \sum V_{k_1x_1+k_2x_2+\cdots+k_nx_n}$$

where the summation runs over all the integers $k_1, \cdots, k_n$ for which there exists an integer k satisfying

$$m_1 \geqq k_2+k \geqq m_2 \geqq k_3+k \geqq m_3 \geqq \cdots \geqq m_{n-1} \geqq k_n+k \geqq m_n \geqq k \geqq 0\,,$$
$$k_1 = \sum_{i=1}^{n} m_i - \sum_{j=2}^{n} k_j - (n+1)k\,.$$

Proof. For an element $x \in \mathfrak{t}^*$, we denote by ξ_x^G and ξ_x^K the principal alternating sum under the Weyl groups W_G and W_K respectively.

$$\xi_x^G = \sum_{\sigma \in W_G} (-1)^\sigma e^{\sigma x}\,,$$
$$\xi_x^K = \sum_{\tau \in W_K} (-1)^\tau e^{\tau x}\,,$$

where $e^x = \exp 2\pi\sqrt{-1}x$, and $(-1)^\sigma$ denotes the signature of permutation σ. For irreducible representations (V, ρ) and (V', ρ') of G and K respectively, we denote their characters by χ_ρ and $\chi_{\rho'}$. Then, by the character formula of Weyl, we have

(1)
$$\xi_{\delta_G}^G \chi_\rho = \xi_{\lambda_\rho+\delta_G}^G \quad \text{on } \mathfrak{t}\,,$$
$$\xi_{\delta_K}^K \chi_{\rho'} = \xi_{\lambda_{\rho'}+\delta_K}^K \quad \text{on } \mathfrak{t}\,.$$

Now, we have

$$\delta_G = \frac{1}{2}\sum_{i<j}(x_i - x_j) = \sum_{i=1}^{n+1}(n+1-i)x_i\,,$$
$$\delta_K = \frac{1}{2}\sum_{2 \leqq i<j}(x_i - x_j) = \sum_{i=2}^{n+1}(n+1-i)x_i + \frac{n-1}{2}x_1\,,$$

and put

$$\begin{cases} p_i = m_i + n + 1 - i & (i = 1, 2, \cdots, n) \\ p_{n+1} = 0\,, \\ \lambda_\rho + \delta_G = p_1x_1 + p_2x_2 + \cdots + p_nx_n\,. \end{cases}$$

Then, we have

(2)
$$\xi_{\lambda_\rho+\delta_G}^G = \det(e^{p_ix_j})_{i,j=1,2,\cdots,n+1}\,,$$

$$\xi^G_{\delta_G} = \prod_{i<j} \left(e^{(x_i-x_j)/2} - e^{(x_j-x_i)/2}\right),$$

$$\xi^K_{\delta_K} = \prod_{2\leqq i<j} \left(e^{(x_i-x_j)/2} - e^{(x_j-x_i)/2}\right).$$

Hence, putting $z_i = x_i - x_1$ $(i=2, \cdots, n+1)$, we have

$$\frac{\xi^K_{\delta_K}}{\xi^G_{\delta_G}} = \prod_{i=2}^{n+1} (-1) e^{(z_i)/2} (e^{z_i}-1)^{-1} \tag{3}$$

We compute the right hand side of (2) as follows;

$$\begin{vmatrix} e^{p_1x_1}, \cdots, e^{p_1x_{n+1}} \\ \vdots \qquad \vdots \\ e^{p_nx_1}, \cdots, e^{p_nx_{n+1}} \\ 1 \quad, \cdots, 1 \end{vmatrix} = e^{(p_1+\cdots+p_n)x_1} \begin{vmatrix} 1, e^{p_1(x_2-x_1)}, \cdots, e^{p_1(x_{n+1}-x_1)} \\ \vdots \quad \vdots \qquad \vdots \\ 1, e^{p_n(x_2-x_1)}, \cdots, e^{p_n(x_{n+1}-x_1)} \\ 1, 1, \quad \cdots\cdots\cdots, 1 \end{vmatrix}$$

$$= (-1)^n e^{Px_1} \begin{vmatrix} e^{p_1z_2}-1, \cdots\cdots\cdots, e^{p_1z_{n+1}}-1 \\ \vdots \qquad \vdots \\ e^{p_nz_2}-1, \cdots\cdots\cdots, e^{p_nz_{n+1}}-1 \end{vmatrix} \quad (P = p_1+\cdots+p_n)$$

$$= (-1)^n e^{Px_1} \prod_{j=2}^{n+1} (e^{z_j}-1) \begin{vmatrix} \sum_{q_1=0}^{p_1-1} e^{q_1z_2}, \cdots, \sum_{q_1=0}^{p_1-1} e^{q_1z_{n+1}} \\ \vdots \qquad \vdots \\ \sum_{q_n=0}^{p_n-1} e^{q_nz_2}, \cdots, \sum_{q_n=0}^{p_n-1} e^{q_nz_{n+1}} \end{vmatrix}$$

$$= (-1)^n e^{Px_1} \prod_{j=2}^{n+1} (e^{z_j}-1) \begin{vmatrix} \sum_{q_1=p_2}^{p_1-1} e^{q_1z_2}, \cdots, \sum_{q_1=p_2}^{p_1-1} e^{q_1z_{n+1}} \\ \sum_{q_2=p_3}^{p_2-1} e^{q_2z_2}, \cdots, \sum_{q_2=p_3}^{p_2-1} e^{q_2z_{n+1}} \\ \vdots \qquad \vdots \\ \sum_{q_n=0}^{p_n-1} e^{q_nz_2}, \cdots, \sum_{q_n=0}^{p_n-1} e^{q_nz_{n+1}} \end{vmatrix}$$

$$= (-1)^n e^{Px_1} \prod_{j=2}^{n+1} (e^{z_j}-1) \sum{}' \det (e^{q_iz_j})_{\substack{i=1,2,\cdots,n \\ j=2,3,\cdots,n+1}}$$

where the summation $\sum'$ runs over all integers $q_1, \cdots, q_n$ satisfying

$$p_1-1 \geqq q_1 \geqq p_2,\ p_2-1 \geqq q_2 \geqq p_3, \cdots, p_n-1 \geqq q_n \geqq 0\,.$$

Further, we have

$$\det (e^{q_iz_j})_{i,j} = e^{-(q_1+\cdots+q_n)x_1} \cdot \det (e^{q_ix_j})_{i,j}\,.$$

Hence, from (1) and (3), we get

$$\xi^K_{\delta_K} \chi_\rho = \sum{}' e^{[P-(q_1+\cdots+q_n)]x_1 + 1/2(z_2+\cdots+z_{n+1})} \cdot \det (e^{q_ix_j}),$$

Here we note that $z_2+\cdots+z_{n+1} = -(n+1)x_1$.
We put

$$\begin{cases} k_1 = P-(q_1+\cdots+q_n)-n-q_n\,, \\ k_{j+1} = q_j-q_n-n+j \qquad (j=1,2,\cdots,n-1)\,. \end{cases}$$

Then, we get easily

$$\xi^K_{\delta_K}\chi_\rho = \sum \xi^K{}_{k_1x_1+\cdots+k_nx_n+\delta_K}\,,$$

where the summation runs over all integers $k_1, \cdots, k_n$ satisfying

$$m_1 \geqq k_2+q_n \geqq m_2 \geqq k_3+q_n \geqq \cdots \geqq k_n+q_n \geqq m_n \geqq q_n \geqq 0\,.$$

Therefore, by the character formula of Weyl, we get immediately the proposition. q.e.d.

The highest weight of $\Lambda_0^{p,q}$ with $p+q\leqq n$, which we have seen before, is described according to the type of (p, q) as follows;

(i) $p=0, q=0$ $\quad 0,$
(ii) $p=0, n>q>0,$ $\quad -qx_1+x_2+\cdots+x_{q+1},$
(iii) $n>p>0, q=0$ $\quad (p+1)x_1+x_2+\cdots+x_{n+1-p},$
(iv) $p, q>0, n>p+q$ $\quad (p-q+1)x_1+2x_2+\cdots+2x_{q+1}+x_{q+2}+\cdots+x_{n+1-p},$
(v) $p, q>0, n=p+q$ $\quad (p-q+1)x_1+2x_2+\cdots+2x_{q+1},$
(vi) $p=0, q=n$ $\quad -(n+1)x_1,$
(vii) $p=n, q=0$ $\quad (n+1)x_1.$

By Proposition 5.1 and Frobenius' reciprocity, an irreducible G-module with the highest weight $\lambda_\rho=m_1x_1+\cdots+m_nx_n$ appears in the irreducible decomposition of $C^\infty(G\times_K\Lambda_0^{p,q})$, if and only if m_i's satisfy the condition given by Proposition 5.1. For instance, in case (iii), there exists an integer k such that

$$m_1\geqq 1+k\geqq m_2\geqq 1+k\geqq\cdots\geqq 1+k\geqq m_{n+1-p}\geqq k\geqq m_{n+2-p}\geqq k\geqq\cdots\geqq m_n\geqq k\geqq 0\,.$$

The conditions are similar to the case above in the other cases.

We set $\Lambda_0=0$, $\Lambda_j=x_1+x_2+\cdots+x_j$ $(j=1, \cdots, n)$ and $\Lambda_{n+1}=0$. Then, $\Lambda_1, \cdots, \Lambda_n$ are the so-called fundamental weights of G so that every dominant integral form of G is uniquely written as a non-negative integral linear combination of them. We set

$$\Lambda(k, r, s) = k(\Lambda_1+\Lambda_n)+(r-s)\Lambda_1+\Lambda_s+\Lambda_{n-r+1}\,,$$

where k satisfies the condition

$$(*) \qquad k+r-s\geqq 0 \text{ and } k\geqq 0\,.$$

Then, the highest weights of irreducible G-modules appearing in $C^\infty(G\times_K\Lambda_0^{p,q})$ are of the following forms

$(\sharp, p, q)$ in case (i), $\Lambda(k, 0, 0)$,
in case (ii), $\Lambda(k, 0, q)$, $\Lambda(k, 0, q+1)$,

in case (iii), $\Lambda(k, p, 0)$, $\Lambda(k, p+1, 0)$,
in case (iv), $\Lambda(k, p, q)$, $\Lambda(k, p, q+1)$, $\Lambda(k, p+1, q)$, $\Lambda(k, p+1, q+1)$,
in case (v), $\Lambda(k, p, q)$, $\Lambda(k, p, q+1)$, $\Lambda(k, p+1, q)$,
in case (vi), $\Lambda(k, 0, n)$,
in case (vii), $\Lambda(k, n, 0)$,
where k's are integers satisfying $(*)$.

Moreover, the multiplicity of the representation with the highest weight mentioned above in $C^\infty(G\times_K\Lambda_q^{p,0})$ is exactly one.

Now, we denote by $\langle\ ,\ \rangle$ the inner product on $\mathfrak{t}^*$ induced from $\frac{-1}{n+1}B$. For $\Lambda=k(\Lambda_1+\Lambda_n)+(r-s)\Lambda_1+\Lambda_s+\Lambda_{n-r+1}$, we have by simple computations,

$$4\pi^2\langle\Lambda+2\delta_G, \Lambda\rangle=\begin{cases}(k+r)(k+n+2-s) & (1\leqq r,\ s\leqq n),\\ (k+r)(k+n+1) & (1\leqq r\leqq n,\ s=0),\\ k(k+n+1-s) & (r=0,\ 1\leqq s\leqq n),\\ k(k+n) & (r=0,\ s=0),\\ (k+n)(k+n+1) & (r=n+1,\ s=0),\\ k(k+1) & (r=0,\ s=n+1).\end{cases}$$

Thus, we get

Theorem 5.2. *Let $P^n(\boldsymbol{C})=SU(n+1)/S(U(1)\times U(n))$.*

(a) *Let p and q be non-negative integers with $p+q\leqq n$. The highest weight λ_ρ of the irreducible representation ρ of $SU(n+1)$ intervening in $C^\infty(G\times_K\Lambda_0^{p,q})$, that is, ρ in $\mathcal{J}_G$ with $\mu_\rho\geqq 1$, are those of $(\sharp, r, s)$ with $r-s=p-q$, $r\leqq p$ and $s\leqq q$.*

For $P^n(\boldsymbol{C})$ with the Fubini-Study metric, the multiplicity of the above representation ρ in the space of primitive forms of type (r, s) is one.

(b) *On the irreducible $SU(n+1)$-submodule of differential forms with the highest weight $\Lambda(k, r, s)$, the Laplace operator defined by the Fubini-Study metric has the eigenvalue $(k+r)(k+n+2-s)$ for $1\leqq r$, $s\leqq n$, $(k+r)(k+n+1)$ for $1\leqq r\leqq n$ and $s=0$, $k(k+n+1-s)$ for $r=0$ and $1\leqq s\leqq n$, $k(k+n)$ for $r=s=0$, $(k+n)(k+n+1)$ for $r=n+1$ and $s=0$, and $k(k+1)$ for $r=0$ and $s=n+1$.*

REMARK. For the space of real differential forms on $P^n(\boldsymbol{C})$, its irreducible decomposition can be obtained from Theorem 5.2 together with the results of N. Bourbaki [5] and N. Iwahori [8]. Two irreducible representations of $SU(n+1)$ over $\boldsymbol{C}$ with the highest weight Λ_j and Λ_{n-j+1} are anti-isomorphic to each other $(j=1, \cdots, n)$. We denote by ${}^{p,q}V_{r,s}^k$ the $SU(n+1)$-irreducible submodule of primitive forms of type (p, q) with the highest weight $k(\Lambda_1+\Lambda_n)+(r-s)\Lambda_1+\Lambda_s+\Lambda_{n-r+1}$. ${}^{p,q}V_{r,s}^k$ is complex conjugate of ${}^{q,p}V_{s,r}^{k+r-s}$ and the space ${}^{p,q}U_{r,s}^k$ of real forms in ${}^{p,q}V_{r,s}^k+{}^{q,p}V_{s,r}^{k+r-s}$ is an irreducible $SU(n+1)$-module over $\boldsymbol{R}$. Conversely, any irreducible $SU(n+1)$-submodule over $\boldsymbol{R}$ in the space of real forms on $P^n(\boldsymbol{C})$ is of the form $\Omega^h \wedge {}^{p,q}U^k_{r,s}$ for some h, k, p, q, r and s.

6. Eigenforms on S^n and the harmonic polynomial forms on $\boldsymbol{R}^{n+1}$

In the following **6** and **7**, we designate simply by $\Lambda^p M$ for $C^\infty(\Lambda^p M)$, the space of all differential p-forms on M.

Let $\boldsymbol{R}^{n+1}$ be an $(n+1)$-dimensional real Euclidean space and $(x^0, x^1, \cdots, x^n)$ be the standard coordinate system on $\boldsymbol{R}^{n+1}$. We put $r^2 = \sum_{i=0}^{n}(x^i)^2$, $r\frac{d}{dr} = \sum_{i=0}^{n} x^i \frac{d}{dx^j}$, $rdr = \sum_{i=0}^{n} x^i dx^i$. Let d_0 be the differential on $\Lambda^*(\boldsymbol{R}^{n+1}) = \sum_{j=0}^{n+1} \Lambda^p(\boldsymbol{R}^{n+1})$, δ_0 the codifferential and $*_0$ the Poincaré duality on $\Lambda^*(\boldsymbol{R}^{n+1})$. We define a linear operator $e(rdr)$ of $\Lambda^*(\boldsymbol{R}^{n+1})$ into itself by $e(rdr)\alpha = rdr \wedge \alpha$, for $\alpha \in \Lambda^*(\boldsymbol{R}^{n+1})$, and denote by $i\left(r\frac{d}{dr}\right)$ the interior product by $r\frac{d}{dr}$ on $\Lambda^*(\boldsymbol{R}^{n+1})$. Then the following lemma is easily verified.

Lemma 6.1. *Let α be a differential p-form on $\boldsymbol{R}^{n+1}$. Then we have the following formulas;*

$$(1)\quad i\left(r\frac{d}{dr}\right)*_0\alpha = (-1)^p *_0 e(rdr)\alpha\,,$$

$$*_0 i\left(r\frac{d}{dr}\right)\alpha = (-1)^{p+1} e(rdr) *_0 \alpha\,.$$

$$(2)\quad e(rdr)d_0\alpha + d_0 e(rdr)\alpha = 0\,,$$

$$i\left(r\frac{d}{dr}\right)\delta_0\alpha + \delta_0 i\left(r\frac{d}{dr}\right)\alpha = 0\,.$$

$$(3)\quad d_0 i\left(r\frac{d}{dr}\right)\alpha + i\left(r\frac{d}{dr}\right)d_0\alpha = L_{r(d/dr)}\alpha\,.$$

$$(4)\quad \delta_0 e(rdr)\alpha + e(rdr)\delta_0\alpha = (-1)^{np+1} *_0 L_{r(d/dr)} *_0 \alpha\,,$$

where L_X denotes the Lie derivation by a vector field X on $\boldsymbol{R}^{n+1}$.

We denote by $\bar{\Delta} = d_0\delta_0 + \delta_0 d_0$ the Laplace operator on the space of differential forms on $\boldsymbol{R}^{n+1}$. Then the operators d_0, δ_0, $\bar{\Delta}$, $e(rdr)$ and $i\left(r\frac{d}{dr}\right)$ acting on $\Lambda^*(R^{n+1})$ commute with the natural action of $O(n+1)$, the orthogonal group of $\boldsymbol{R}^{n+1}$, on $\Lambda^*(\boldsymbol{R}^{n+1})$.

Now, let P^p_k be the set of $\alpha \in \Lambda^p(\boldsymbol{R}^{n+1})$ of the form

$$\alpha = \sum_{0 \leq i_1 < \cdots < i_p \leq n} \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,,$$

where $\alpha_{i_1 \cdots i_p}$ are homogeneous polynomials of degree k.

Proposition 6.2. *Let* $\alpha\in P_k^p$. *Then we have*

(1) $d_0 i\left(r\frac{d}{dr}\right)\alpha+i\left(r\frac{d}{dr}\right)d_0\alpha = (k+p)\alpha$,

(2) $\delta_0 e(rdr)\alpha+e(rdr)\delta_0\alpha = -(n+1-p+k)\alpha$

Proof. Let $\alpha\in P_k^p$. Then we can see easily $L_{r(d/dr)}\alpha=(k+p)\alpha$. Combining this with (3), (4) in Lemma 6.1, we have Proposition 6.2.

q.e.d.

Corollary 6.3. *We have the following direct sum decompositions*;

$$P_k^p = (\mathrm{Ker}\ d_0 \cap P_k^p)\oplus\left(\mathrm{Ker}\ i\left(r\frac{d}{dr}\right)\cap P_k^p\right) \qquad (k+p\neq 0)\,,$$

$$P_k^p = (\mathrm{Ker}\ \delta_0 \cap P_k^p)\oplus(\mathrm{Ker}\ e(rdr) \cap P_k^p) \qquad (n+1-p+k\neq 0)$$

REMARK. We have

$$P_0^0 = \{\text{constant functions on } \boldsymbol{R}^{n+1}\}\ ,$$
$$P_0^{n+1} = \{a\ dx^0\wedge\cdots\wedge dx^n\colon a\in\boldsymbol{C}\}\ .$$

DEFINITION. A *harmonic polynomial form* is an element $\alpha\in P_k^p$ such that $\bar{\Delta}\alpha=0$ and $\delta_0\alpha=0$. Let H_k^p be the set of these forms;

$$H_k^p = \mathrm{Ker}\ \bar{\Delta} \cap \mathrm{Ker}\ \delta_0 \cap P_k^p\,.$$

Lemma 6.4. *We have the direct sum decomposition*;

$$P_k^p = H_k^p\oplus(r^2P_{k-2}^p+e(rdr)P_{k-1}^{p-1})\,.$$

Proof. Let $S^k(\boldsymbol{R}^{n+1})$ and $\Lambda^p(\boldsymbol{R}^{n+1})$ be the real vector spaces of a symmetric tensors of degree k and antisymmetric tensors of degree p over $\boldsymbol{R}^{n+1}$. Then we have the natural isomorphism between P_k^p and $S^k(\boldsymbol{R}^{n+1})\otimes\Lambda^p(\boldsymbol{R}^{n+1})\otimes\boldsymbol{C}$ as $O(n+1)$-modules, where $O(n+1)$ acts identically on $\boldsymbol{C}$. We consider $S^k(\boldsymbol{R}^{n+1})\otimes\Lambda^p(\boldsymbol{R}^{n+1})$ as a subspace of $\boldsymbol{T}^{k+p}(\boldsymbol{R}^{n+1})$, the real vector space of tensors of degree $k+p$. We define linear operators s_{ij} of $\boldsymbol{T}^{k+p}(\boldsymbol{R}^{n+1})$ to $\boldsymbol{T}^{k+p-2}(\boldsymbol{R}^{n+1})$ by

$$s_{ij}(e_1\otimes\cdots\otimes e_{k+p}) = \langle e_i, e_j\rangle e_1\otimes\cdots\otimes\widehat{e_i}\otimes\cdots\otimes\widehat{e_j}\otimes\cdots\otimes e_{k+p}\,,$$

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\boldsymbol{R}^{n+1}$. Put ${}_{R}\bar{\boldsymbol{H}}^{k+p}=\bigcap_{1\leq i<j\leq k+p}$ Ker s_{ij} and ${}_{R}\bar{\boldsymbol{H}}_k^p={}_{R}\bar{\boldsymbol{H}}^{k+p}\cap S^k(\boldsymbol{R}^{n+1})\otimes\Lambda^p(\boldsymbol{R}^{n+1})$. We denote by ${}_{R}\bar{\boldsymbol{Q}}_k^p$ the subspace orthogonal complement to ${}_{R}\bar{\boldsymbol{H}}_k^p$ in $S^k(\boldsymbol{R}^{n+1})\otimes\Lambda^p(\boldsymbol{R}^{n+1})$. Put $\bar{\boldsymbol{Q}}_k^p={}_{R}\bar{\boldsymbol{Q}}_k^p\otimes_{R}\boldsymbol{C}$ and $\bar{\boldsymbol{H}}_k^p={}_{R}\bar{\boldsymbol{H}}_k^p\otimes\boldsymbol{C}$. Then the subspaces in P_k^p corresponding to $\bar{Q}_k^p$ and $\bar{H}_k^p$ by the natural isomorphism in P_k^p are $Q_k^p=e(rdr)P_{k-1}^{p-1}+r^2P_{k-2}^p$ and H_k^p.

q.e.d.

Proposition 6.5. *Let* S^n *be the unit sphere in* $\boldsymbol{R}^{n+1}$ *and* i *be the inclusion of* S^n *into* $\boldsymbol{R}^{n+1}$. *Then we have*

(1) i^*: Ker $\delta_0 \cap P_k^p \to \Lambda^p(S^n)$ *is injective,*

(2) $i^*(P_k^p) = \sum_{i \geq 0} i^*(H_{k-2i}^p)$ *(direct sum).*

Proof. (2) follows easily by induction on k using Lemma 6.4, since $i^*(r^2) = 1$ and $i^*(rdr)=0$. We shall show (1). Let $\alpha \in$ Ker $\delta_0 \cap P_k^p$. Suppose $i^*\alpha=0$. Then α is in Ker $e(rdr) \cap P_k^p$. On the other hand $\delta_0\alpha=0$, and so from Corollary 6.3, we have $\alpha=0$.

q.e.d.

Corollary 6.6. $i^*: \sum_{k \geq 0} H_k^p \to \Lambda^p(S^n)$ *is injective and its image is dense in* $\Lambda^p(S^n)$.

Proof. By the polynomial approximation, $i^*(\sum_{k \geq 0} P_k^p)$ is dense in $\Lambda^p(S^n)$. Proposition 6.5 implies that $i^*(\sum_{k \geq 0} P_k^p) = i^*(\sum_{k \geq 0} H_k^p)$. Thus the density is proved.

Now, let $\alpha_1 \in \sum_{k \geq 0} H_{2k}^p$ and $\alpha_2 \in \sum_{k \geq 0} H_{2k-1}^p$. Assume $i^*(\alpha_1+\alpha_2)=0$. Then we shall show that $\alpha_1=\alpha_2=0$. By Proposition 6.5, we may take $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ in $P_{2k_0}^p$ and $P_{2k_0-1}^p$ respectively for some k_0 satisfying $i^*(\tilde{\alpha}_1)=i^*(\alpha_1)$ and $i^*(\tilde{\alpha}_2)=i^*(\alpha_2)$. We have $(\tilde{\alpha}_1+\tilde{\alpha}_2) \wedge rdr=0$ on S^n. To prove the corollary, it is now sufficient to show that if f_1, f_2 are homogeneous polynomials of degree $2k_0$, $2k_0-1$ respectively, and if f_1+f_2 is zero on S^n, then $f_1=f_2=0$ on $\boldsymbol{R}^{n+1}$. By the assumption, we see that f_1+rf_2 is zero on S^n and homogeneous on $\boldsymbol{R}^{n+1}$. Thus $f_1+rf_2 \equiv 0$ on $\boldsymbol{R}^{n+1}$. Substituting $-x_i$ for $x_i (0 \leq i \leq n)$, we have $f_1-rf_2 \equiv 0$ on $\boldsymbol{R}^{n+1}$. Thus we have $f_1=f_2=0$.

q.e.d.

Lemma 6.7. *Let* $\alpha \in H_k^p$. *Then we have* $d_0\alpha \in H_{k-1}^{p+1}$ *and* $i\left(r\frac{d}{dr}\right)\alpha \in H_{k+1}^{p-1}$.

Proof. The first statement follows easily from the facts that $\bar{\Delta}=d_0\delta_0+\delta_0 d_0$ and $d_0\bar{\Delta}=\bar{\Delta}d_0$. We see $i\left(r\frac{d}{dr}\right)\alpha \in H_{k+1}^{p-1}$, since $\delta_0 i\left(r\frac{d}{dr}\right)\alpha = -i\left(r\frac{d}{dr}\right)\delta_0\alpha=0$ by Lemma 6.1 (2) and we have

$$\begin{aligned}
\bar{\Delta} i\left(r\frac{d}{dr}\right)\alpha &= (\delta_0 d_0+d_0\delta_0)\, i\left(r\frac{d}{dr}\right)\alpha \\
&= \delta_0 L_{r(d/dr)}\alpha - \delta_0 i\left(r\frac{d}{dr}\right)d_0\alpha \\
&= (k+p)\delta_0\alpha + i\left(r\frac{d}{dr}\right)\delta_0 d_0\alpha \\
&= 0\,.
\end{aligned}$$

q.e.d.

Put $'H_k^p=$Ker $d_0 \cap H_k^p$ and $''H_k^p=$Ker $i\left(r\frac{d}{dr}\right) \cap H_k^p$. By Proposition 6.2 (1) and Lemma 6.7, we get

(6.1) $H_k^p = {'H_k^p} \oplus {''H_k^p} \qquad (p+k \neq 0,\ n+1-p+k \neq 0)$.

We see $H_0^0 = {'H_0^0} = {''H_0^0}$,

$H_0^{n+1} = {'H_0^{n+1}} = {''H_0^{n+1}}$.

Theorem 6.8. (1) *The modules* $'H_k^0$ $(k>0)$ *and* $''H_0^p$ $(0<p<n+1)$ *are reduced to zero.*

(2) *The module* $'H_k^p$ $(p>0)$ *is decomposed into the irreducible* $SO(n+1)$-*modules with the following highest weights and each of these appears with multiplicity one.*

(i) *case* $n=2m$,

$k\Lambda_1+\Lambda_p \qquad (0<p<m)$

$k\Lambda_1+2\Lambda_m \qquad (p=m,\ m+1)$

$k\Lambda_1+\Lambda_{n+1-p} \qquad (m+1<p\leqq 2m)$.

(ii) *case* $n=2m-1$

$k\Lambda_1+\Lambda_p \qquad (0<p<m-1)$

$k\Lambda_1+\Lambda_{m-1}+\Lambda_m \qquad (p=m-1,\ m+1)$

$k\Lambda_1+2\Lambda_{m-1},\ k\Lambda_1+2\Lambda_m \qquad (p=m)$

$k\Lambda_1+\Lambda_{n+1-p} \qquad (m+1<p\leqq 2m-1)$.

(3) *The module* $''H_k^p$ $(k\geqq 1, p\leqq n)$ *is decomposed into the irreducible* $SO(n+1)$-*modules with the following highest weights and each of these appears with multiplicity one.*

(i) *case* $n=2m$,

$(k-1)\Lambda_1+\Lambda_{p+1} \qquad (0\leqq p<m-1)$

$(k-1)\Lambda_1+2\Lambda_m \qquad (p=m-1,\ m)$

$(k-1)\Lambda_1+\Lambda_{n-p} \qquad (m+1\leqq p<2m-1)$.

(ii) *case* $n=2m-1$,

$(k-1)\Lambda_1+\Lambda_{p+1} \qquad (0\leqq p<m-2)$

$(k-1)\Lambda_1+\Lambda_{m-1}+\Lambda_m \qquad (p=m-2,\ m)$

$(k-1)\Lambda_1+2\Lambda_{m-1},\ (k-1)\Lambda_1+2\Lambda_m \qquad (p=m-1)$

$(k-1)\Lambda_1+\Lambda_{n-p} \qquad (m<p\leqq 2m-2)$.

(4) *The module* $i^*('H_k^p)$ *is consist of d-closed forms and the module* $i^*(''H_k^p)$ *of* δ-*closed forms.*

Proof. We shall give the proof for the case $n=2m$, the case $n=2m-1$, being treated quite analogously. The statement (1) can be easily seen by Proposition 6.2. We shall show (2) and (3). The module P_k^p is isomorphic to $\boldsymbol{S}^k(\boldsymbol{R}^{n+1})\otimes\boldsymbol{\Lambda}^p(\boldsymbol{R}^{n+1})\otimes C$ as $SO(n+1)$-modules. Therefore, the module P_k^p contains the $SO(n+1)$-irreducible submodule with the highest weight $k\Lambda_1+\Lambda_p$.

We denote this submodule by E_k^p. By Lemma 6.4, we have $E_k^p \subset H_k^p$. The module $d_0 E_k^p$ is isomorphic to E_k^p or reduced to zero. But the highest weight of the module P_{k-1}^{p+1} is $(k-1)\Lambda_1+\Lambda_{p+1}$ which is strictly lower than $k\Lambda_1+\Lambda_p$. Thus the module $d_0 E_k^p$ is reduced to zero and this implies $E_k^p \subset 'H_k^p$. On the other hand the module $i\left(r\frac{d}{dr}\right)E_{k-1}^{p+1}$ is contained in $''H_k^p$. By Proposition 6.2, the module $d_0 i\left(r\frac{d}{dr}\right)E_{k-1}^{p+1}$ is equal to E_{k-1}^{p+1}. Therefore, by (6.1) the module $i\left(r\frac{d}{dr}\right)E_{k-1}^{p+1}$ is also isomorphic to the module E_{k-1}^{p+1}. Thus we have shown that the module $''H_k^p$ contains the irreducible submodules with the highest weights in (3). Then by (6.1) and Theorem 4.2, no other submodules appear in H_k^p. Thus we have proved (2) and (3). Next we shall show (4). By the well known formula $di^*=i^*d_0$, we see that the module $i^*('H_k^p)$ is d-closed. When we note that d, δ and $*$ are $SO(n+1)$-homomorphisms, we see that the module $i^*(''H_k^p)$ is δ-closed by comparing the irreducible submodules appearing in $''H_k^p$ and $\sum_{k\geq 0} H_k^{p-1}$.

q.e.d.

7. Eigenforms on $P^n(C)$ and the harmonic polynomial forms on C^{n+1}

Let $(z^0, z^1, \cdots, z^n)$ be a standard holomorphic coordinate on C^{n+1} and $g=\sum_{i=0}^n dz^i \cdot d\bar{z}^i$ be the flat Kähler metric on C^{n+1}. We denote by $\Lambda^{p,q}(C^{n+1})$ the space of differential forms of type (p, q). We designate by d_0 the differential and δ_0 the codifferential on $\Lambda^*(C^{n+1})$. The operators d_0 and δ_0 are decomposed as follows;

$$d_0 = \partial_0+\bar{\partial}_0$$

with

$$\partial_0\colon \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p+1,q}(C^{n+1})$$
$$\bar{\partial}_0\colon \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p,q+1}(C^{n+1}),$$

and

$$\delta_0 = \partial_0^*+\bar{\partial}_0^*$$

with

$$\partial_0^*\colon \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p-1,q}(C^{n+1})$$
$$\bar{\partial}_0^*\colon \Lambda^{p,q}(C^{n+1}) \to \Lambda^{p,q-1}(C^{n+1}).$$

Furthermore, we denote by $*_0$ the Poincaré duality and put

$$W_0 = \sum_{i=0}^n z^i\frac{\partial}{\partial z^i},\ \bar{W}_0 = \sum_{i=0}^n \bar{z}^i\frac{\partial}{\partial \bar{z}^i},\ W_0^* = \sum_{i=0}^n \bar{z}^i dz^i, \text{ and } \bar{W}_0^* = \sum_{i=0}^n z^i d\bar{z}^i.$$

For $\alpha\in\Lambda^*(C^{n+1})$, we define $e(W_0^*)\,\alpha$ and $e(\bar{W}_0^*)\,\alpha$ by

$$e(W_0^*)\alpha = W_0^*\wedge\alpha$$

and

$$e(\bar{W}_0^*)\alpha = \bar{W}_0^*\wedge\alpha.$$

We designate by $i(W_0^*)$ (resp. $i(\bar{W}_0^*)$) the interior product by W_0 (resp. $\bar{W}_0^*$). Then these operators d_0, δ_0, ∂_0, ∂_0^*, $\bar{\partial}_0$, $\bar{\partial}_0^*$, $e(W_0^*)$, $e(\bar{W}_0^*)$, $i(W_0)$ and $i(\bar{W}_0^*)$ on $\Lambda^*(\boldsymbol{C}^{n+1})$ commute with the action of $U(n+1)$ on $\Lambda^*(\boldsymbol{C}^{n+1})$. Now the following lemma is easily verified.

Lemma 7.1. *For any* $\alpha \in \Lambda^{p,q}(\boldsymbol{C}^{n+1})$, *we have*

(1) $i(W_0)*_0\alpha = (-1)^{p+q}*_0 e(\bar{W}_0^*)\alpha$

$i(\bar{W}_0)*_0\alpha = (-1)^{p+q}*_0 e(W_0^*)\alpha$.

(2) $i(W_0)\partial_0\alpha + \partial_0 i(W_0)\alpha = L_{W_0}\alpha$

$i(\bar{W}_0)\bar{\partial}_0\alpha + \bar{\partial}_0 i(\bar{W}_0)\alpha = L_{\bar{W}_0}\alpha$

$i(\bar{W}_0)\partial_0\alpha + \partial_0 i(\bar{W}_0)\alpha = 0$

$i(W_0)\bar{\partial}_0\alpha + \bar{\partial}_0 i(W_0)\alpha = 0$.

(3) $\partial_0^* e(W_0^*)\alpha + e(W_0^*)\partial_0^*\alpha = (-1)^{p+q+1}*_0 L_{\bar{W}_0}*_0\alpha$

$\bar{\partial}_0^* e(\bar{W}_0^*)\alpha + e(\bar{W}_0^*)\bar{\partial}_0^*\alpha = (-1)^{p+q+1}*_0 L_{W_0}*_0\alpha$

$\partial_0^* e(\bar{W}_0^*)\alpha + e(\bar{W}_0^*)\partial_0^*\alpha = 0$

$\bar{\partial}_0^* e(W_0^*)\alpha + e(W_0^*)\bar{\partial}_0^*\alpha = 0$.

Now, we put $\Omega_0 = -\sqrt{-1}\sum_{i=0}^{n} dz^i \wedge d\bar{z}^i$ and define linear maps L_0, Λ_0 of $\Lambda^*(\boldsymbol{C}^{n+1})$ into itself by

$$L_0\alpha = \Omega_0 \wedge \alpha$$

and

$$\Lambda_0\alpha = *_0^{-1} L_0 *_0 \alpha \qquad \text{for } \alpha \in \Lambda^*(\boldsymbol{C}^{n+1}) .$$

Then the operators L_0 and Λ_0 also commute with the action of $U(n+1)$ on $\Lambda^*(\boldsymbol{C}^{n+1})$. We have the following

Lemma 7.2. *For* $\alpha \in \Lambda^*(\boldsymbol{C}^{n+1})$, *we have*

(1) $\partial_0 e(W_0^*)\alpha + e(W_0^*)\partial_0\alpha = 0$

$\bar{\partial}_0 e(\bar{W}_0^*)\alpha + e(\bar{W}_0^*)\bar{\partial}_0\alpha = 0$

$\partial_0 e(\bar{W}_0^*)\alpha + e(\bar{W}_0^*)\partial_0\alpha = \sqrt{-1}L_0\alpha$

$\bar{\partial}_0 e(W_0^*)\alpha + e(W_0^*)\bar{\partial}_0\alpha = -\sqrt{-1}L_0\alpha$

(2) $\partial_0^* i(W_0)\alpha + i(W_0)\partial_0^*\alpha = 0$

$\bar{\partial}_0^* i(\bar{W}_0)\alpha + i(\bar{W}_0)\bar{\partial}_0^*\alpha = 0$

$\partial_0^* i(\bar{W}_0)\alpha + i(\bar{W}_0)\bar{\partial}_0^*\alpha = -\sqrt{-1}\Lambda_0\alpha$

$\bar{\partial}_0^* i(W_0)\alpha + i(W_0)\bar{\partial}_0^*\alpha = \sqrt{-1}\Lambda_0\alpha$.

Put $P_{k,l}^{p,q}$ be the set of $\alpha \in \Lambda^{p,q}(\boldsymbol{C}^{n+1})$ of the form

$$\alpha = \sum_{\substack{0 \leq i_1 < \cdots < i_p \leq n \\ 0 \leq j_1 < \cdots < j_q \leq n}} \alpha_{i_1 \cdots i_p, j_1 \cdots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} ,$$

where $\alpha_{i_1\cdots i_p, j_1\cdots j_q}$ are polynomials of degree k with respect to $z^0, z^1, \cdots, z^n$ and of degree l with respect to $\bar{z}^0, \bar{z}^1, \cdots, \bar{z}^n$. Then, using Lemma 7.1, we have in the same way as for Proposition 6.2

Proposition 7.3. *Let $\alpha \in P^{p,q}_{k,l}$. Then we have*

(1) $i(W_0)\partial_0\alpha + \partial_0 i(W_0)\alpha = (k+p)\alpha$

(2) $i(\bar{W}_0)\bar{\partial}_0\alpha + \bar{\partial}_0 i(\bar{W}_0)\alpha = (l+q)\alpha$

(3) $e(W_0^*)\partial_0^*\alpha + \partial_0^* e(W_0^*)\alpha = -(n+1-p+l)\alpha$

(4) $e(\bar{W}_0^*)\bar{\partial}_0^*\alpha + \bar{\partial}_0^* e(\bar{W}_0^*)\alpha = -(n+1-q+k)\alpha\,.$

By this proposition, we have the following direct sum decompositions;

$$\begin{aligned} P^{p,q}_{k,l} &= (\mathrm{Ker}\ \partial_0 \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ i(W_0) \cap P^{p,q}_{k,l}) && (k+p \neq 0) \\ &= (\mathrm{Ker}\ \bar{\partial}_0 \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ i(\bar{W}_0) \cap P^{p,q}_{k,l}) && (l+q \neq 0) \\ &= (\mathrm{Ker}\ \partial_0^* \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ e(W_0^*) \cap P^{p,q}_{k,l}) && (n+1-p+l \neq 0) \\ &= (\mathrm{Ker}\ \bar{\partial}_0^* \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ e(\bar{W}_0^*) \cap P^{p,q}_{k,l}) && (n+1-q+k \neq 0)\,. \end{aligned}$$

By Lemma 7.1, we have also the following direct sum decompositions;

$$\begin{aligned} (7.1) \qquad P^{p,q}_{k,l} &= (\mathrm{Ker}\ \partial_0 \cap \mathrm{Ker}\ \bar{\partial}_0 \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ \partial_0 \cap \mathrm{Ker}\ i(\bar{W}_0) \cap P^{p,q}_{k,l}) \\ &\quad \oplus (\mathrm{Ker}\ i(W_0) \cap \mathrm{Ker}\ \bar{\partial}_0 \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ i(W_0) \cap \mathrm{Ker}\ i(\bar{W}_0) \cap P^{p,q}_{k,l}) \\ &\qquad\qquad (k+p \neq 0,\ l+q \neq 0) \\ &= (\mathrm{Ker}\ \partial_0^* \cap \mathrm{Ker}\ \bar{\partial}_0^* \cap P^{p,q}_{k,k}) \oplus (\mathrm{Ker}\ \partial_0^* \cap \mathrm{Ker}\ e(\bar{W}_0^*) \cap P^{p,q}_{k,l}) \\ &\quad \oplus (\mathrm{Ker}\ e(W_0^*) \cap \mathrm{Ker}\ \bar{\partial}_0^* \cap P^{p,q}_{k,l}) \oplus (\mathrm{Ker}\ e(W_0^*) \cap \mathrm{Ker}\ e(\bar{W}_0^*) \cap P^{p,q}_{k,l}) \\ &\qquad\qquad (p-l \neq n+1,\ q-k \neq n+1)\,. \end{aligned}$$

REMARKS. We have

(1) $P^{0,q}_{0,l} = \mathrm{Ker}\ \partial_0 \cap P^{0,q}_{0,l} = \mathrm{Ker}\ i(W_0) \cap P^{0,q}_{0,l}\,,$

$P^{p,0}_{k,0} = \mathrm{Ker}\ \bar{\partial}_0 \cap P^{p,0}_{k,0} = \mathrm{Ker}\ i(\bar{W}_0) \cap P^{p,0}_{k,0}\,,$

$P^{n+1,q}_{k,0} = \mathrm{Ker}\ \partial_0^* \cap P^{n+1,q}_{k,0} = \mathrm{Ker}\ e(W_0) \cap P^{n+1,q}_{k,0}\,,$

$P^{p,n+1}_{0,l} = \mathrm{Ker}\ \bar{\partial}_0^* \cap P^{p,n+1}_{0,l} = \mathrm{Ker}\ e(\bar{W}_0) \cap P^{p,n+1}_{0,l}\,.$

(2) $P^{0,0}_{0,0} = \{\text{constant functions on } \boldsymbol{C}^{n+1}\}\,,$

$P^{n+1,n+1}_{0,0} = \{a\, dz^0 \wedge d\bar{z}^0 \cdots \wedge dz^n \wedge d\bar{z}^n : a \in \boldsymbol{C}\}\,.$

Furthermore, in the same way as for Lemma 6.4, we have

$$\begin{aligned} (7.2) \qquad P^{p,q}_{k,l} &= \mathrm{Ker}\ \partial_0^* \cap \mathrm{Ker}\ \bar{\partial}_0^* \cap \mathrm{Ker}\ \square_0 \cap \mathrm{Ker}\ \Lambda_0 \cap P^{p,q}_{k,l} \\ &\quad \oplus (e(W_0^*)P^{p-1,q}_{k,l-1} + e(\bar{W}_0^*)P^{p,q-1}_{k-1,l} + \Omega_0 P^{p-1,q-1}_{k,l} + r^2 P^{p,q}_{k-1,l-1})\,, \end{aligned}$$

where $\square_0 = \bar{\partial}_0^*\bar{\partial}_0 + \bar{\partial}_0\bar{\partial}_0^*$.

Now, put

$$H_{k,l}^{p,q} = \operatorname{Ker} \partial_0^* \cap \operatorname{Ker} \bar{\partial}_0^* \cap \operatorname{Ker} \square_0 \cap \operatorname{Ker} \Lambda_0 \cap P_{k,l}^{p,q} .$$

DEFINITION. A harmonic polynomial form on C^{n+1} is an element of the space $H_{k,l}^{p,q}$.

The space $H_{k,l}^{p,q}$ is an $SU(n+1)$-invariant subspace in the space $P_{k,l}^{p,q}$. Assume $p\neq 0, q\neq 0$ and $p+q\leqq n+1$. Then by (7.2), the module $H_{k,l}^{p,q}$ contains the highest weight vector in the module $P_{k,l}^{p,q}$. The irreducible subspace in $H_{k,l}^{p,q}$ which contains the highest weight vector of the module $P_{k,l}^{p,q}$ shall be denoted by $E_{k,l}^{p,q}$. Then the irreducible module $E_{k,l}^{p,q}$ has the following highest weight;

$$l\Lambda_1+\Lambda_q+k\Lambda_n+\Lambda_{n+1-p} . \tag{7.3}$$

The following Lemma 7.5 can be verified easily from the formulas in (3.5) and Lemma 7.1.

Lemma 7.4. *Let $\alpha\in H_{k,l}^{p,q}$. Then we have $\partial_0\alpha\in H_{k-1,l}^{p+1,q}$, $\bar{\partial}_0\alpha\in H_{k,l-1}^{p,q+1}$, $i(W_0)\alpha\in H_{k,l+1}^{p-1,q}$ and $i(\bar{W}_0)\alpha\in H_{k+1,l}^{p,q-1}$.*

Combining this lemma with (7.1), we have

$$\begin{aligned} H_{k,l}^{p,q} = {} & (\operatorname{Ker} \partial_0 \cap \operatorname{Ker} \bar{\partial}_0 \cap H_{k,l}^{p,q}) \oplus (\operatorname{Ker} \partial_0 \cap \operatorname{Ker} i(\bar{W}_0) \cap H_{k,l}^{p,q}) \\ & \oplus (\operatorname{Ker} i(W_0) \cap \operatorname{Ker} \bar{\partial}_0 \cap H_{k,l}^{p,q}) \oplus (\operatorname{Ker} i(W_0) \cap \operatorname{Ker} i(\bar{W}_0) \cap H_{k,l}^{p,q}) \\ & (k+p\neq 0,\ l+q\neq 0) . \end{aligned} \tag{7.2}$$

Proposition 7.5. *Suppose $p+q\leqq n$. Then the module $H_{k,l}^{p,q}$ contains the irreducible $SU(n+1)$-modules with the following highest weights;*

(1) $l\Lambda_1+k\Lambda_n$ $(p=q=0)$

(2) $l\Lambda_1+k\Lambda_n+\Lambda_q$, $(l-1)\Lambda_1+k\Lambda_n+\Lambda_{q+1}$ $(p=0,\ n>q>0)$

(3) $l\Lambda_1+k\Lambda_n+\Lambda_{n+1-p}$, $l\Lambda_1+(k-1)\Lambda_n+\Lambda_{n-p}$ $(n>p>0,\ q=0)$

(4) $l\Lambda_1+k\Lambda_n+\Lambda_q+\Lambda_{n+1-p}$, $(l-1)\Lambda_1+(k-1)\Lambda_n+\Lambda_{q+1}+\Lambda_{n+1-p}$, $l\Lambda_1+(k-1)\Lambda_n+\Lambda_q+\Lambda_{n-p}$, $(l-1)\Lambda_1+(k-1)\Lambda_n+\Lambda_{q+1}+\Lambda_{n-p}$ $(p>0,\ q>0,\ 1<p+q<n)$

(5) $l\Lambda_1+k\Lambda_n+\Lambda_q+\Lambda_{n+1-p}$, $(l-1)\Lambda_1+k\Lambda_n+\Lambda_{q+1}+\Lambda_{n+1-p}$, $l\Lambda_1+(k-1)\Lambda_n+\Lambda_q+\Lambda_{n-p}$ $(p\neq 0,\ q\neq 0,\ p+q=n)$

(6) $l\Lambda_1+(k+1)\Lambda_n$ $(p=0,\ q=n)$

(7) $(l+1)\Lambda_1+k\Lambda_n$ $(p=n,\ q=0)$

Proof. Assume $p\neq 0$, $q\neq 0$ and $p+q\leqq n+1$. Then in the same way as for the proof of Theorem 6.8, we get $\partial_0 E_{k,l}^{p,q}=0$ and $\bar{\partial}_0 E_{k,l}^{p,q}=0$. We shall show that the module $i(W_0)i(\bar{W}_0)E_{k,l}^{p,q}$ is not reduced to zero. Let $\alpha\in E_{k,l}^{p,q}$. Assume $i(W_0)i(\bar{W}_0)\alpha=0$. Then we have $i(W_0)\alpha\in \operatorname{Ker} i(\bar{W}_0)$. On the other hand, we get $\bar{\partial}_0 i(W_0)\alpha=-i(W_0)\bar{\partial}_0\alpha=0$. Therefore we have $i(W_0)\alpha\in \operatorname{Ker} \bar{\partial}_0$. Thus we have

$i(W_0)\alpha=0$, i.e., $\alpha\in\operatorname{Ker} i(W_0)$. Since $\alpha\in\operatorname{Ker}\partial_0$, we have $\alpha=0$. Thus we see that the module $i(W_0)i(\bar{W}_0)E_{k,l}^{p,q}$ is isomorphic to the module $E_{k,l}^{p,q}$. Moreover the modules $i(W_0)E_{k,l}^{p,q}$ and $i(\bar{W}_0)E_{k,l}^{p,q}$ are also isomorphic to $E_{k,l}^{p,q}$. Combining these facts with Lemma 7.4, we get Proposition 7.5.

q.e.d.

Let π be the natural projection of $C^{n+1}-\{0\}$ onto $P^n(C)$ and π_s its restriction to $S^{2n+1}\subset C^{n+1}-\{0\}$. For $p\in S^{2n+1}$, we denote by $T_p(S^{2n+1})$ the tangent space at p. Put $F_p=\operatorname{Ker}((\pi_s)_*)_p$. Let $F_p^\perp$ be the orthogonal complementary subspace to F_p in $T_p(S^{n+1})$;

$$T_p(S^{2n+1})=F_p\oplus F_p^\perp .$$

We introduce the Riemannian metric on $P^n(C)$ so that the restriction of $(\pi_s)_*$ to $F_p^\perp$ is an isometry onto $T_{\pi(p)}(P^n(C))$. Let J_0 and J be the standard complex structures on $C^{n+1}$and $P^n(C)$ respectively. Then for $v\in F_p^\perp$, we have

$$J(\pi_s)_*v=(\pi_s)_*J_0v .$$

We denote by $\Lambda_{S^1}^{p,q}(C^{n+1})$ the set of all $\alpha\in\Lambda^{p,q}(C^{n+1})$ such that $g^*\alpha=\alpha$ for any $g\in S^1=\left\{\begin{pmatrix} e^{2\pi i\theta} & & 0\\ & \ddots & \\ 0 & & e^{2\pi i\theta}\end{pmatrix}\in U(n+1)\right\}$ and we define a linear map

$$\phi\colon \Lambda_{S^1}^{p,q}(C^{n+1})\to\Lambda^{p,q}(P^n(C))$$

by

$$\phi(\alpha)(X_1,\cdots,X_{p+q})=\alpha(\tilde{X}_1,\cdots,\tilde{X}_{p+q}) ,$$

where X_i are tangent vectors at $\pi(p)$ to $P^n(C)$ and $\tilde{X}_i$ are the lift to S^{2n+1} by the isomorphism $F_p^\perp\approx T_{\pi(p)}(P^n(C))$. Then the map ϕ is well defined and we have

Proposition 7.6. (1) *$\phi(\sum_{k+p=l+q}P_{k,l}^{p,q})$ is dense in $\Lambda^{p,q}(P^n(C))$.*

(2) *Let Ω be the fundamental form on $P^n(C)$. Then we have $\phi(\Omega_0)=\Omega$.*

(3) $\phi(W_0^*)=\phi(\bar{W}_0^*)=0$.

(4) $\phi(r^2)=1$.

(5) *ϕ: $\operatorname{Ker}\partial_0^*\cap\operatorname{Ker}\bar\partial_0^*\cap P_{k,l}^{p,q}\to P^n(C)$ is injective.*

(6) *For $\alpha\in H_{k,l}^{p,q}$, $\phi(\alpha)$ is a primitive form on $P^n(C)$ if and only if $i(W_0)i(\bar{W}_0)\alpha=0$.*

Proof. We shall show here (5) and (6), the other statements being easily proved. Let $\alpha\in\operatorname{Ker}\partial_0^*\cap\operatorname{Ker}\bar\partial_0^*\cap P_{k,l}^{p,q}$. Assume $\phi(\alpha)=0$. Then we have $W_0^*\wedge\bar{W}_0^*\wedge\alpha=0$, i.e., $\bar{W}_0^*\wedge\alpha\in\operatorname{Ker} e(W_0^*)$. On the other hand we have $\partial_0^*(\bar{W}_0^*\wedge\alpha)=-\bar{W}_0^*\wedge\partial_0^*\alpha=0$. Therefore we have $\bar{W}_0^*\wedge\alpha\in\operatorname{Ker}\partial_0^*$. Thus we have $\bar{W}_0^*\wedge\alpha=0$. In the same way as above, we see $\alpha=0$. Thus we have proved (5). Let $W_1,\cdots,W_n$ be a unitary base complementary to W_0 at $p\in S^{2n+1}$

in C^{n+1}. For $\alpha \in H_{k,l}^{p,q}$, we have

$$0 = \sum_{i=0}^{n} i(W_i)i(\bar{W}_i)\alpha = \sum_{i=1}^{n} i(W_i)i(\bar{W}_i)\alpha + i(W_0)i(\bar{W}_0)\alpha .$$

It is clear that $\phi(\alpha)$ is primitive form if and only if the first term vanishes. Thus we have proved (6).

q.e.d.

According to (6) in Proposition 7.6, for an element of the last three direct summands in (7.2), its ϕ-image is a primitive form. On the other hand for $\alpha \in \operatorname{Ker} \bar{\partial}_0 \cap \operatorname{Ker} \partial_0 \cap H_{k,l}^{p,q}$, we put

$$\alpha_1 = (n+2-p-q)\alpha + \sqrt{-1}\Omega_0 i(W_0)i(\bar{W}_0)\alpha ,$$
$$\alpha_2 = i(W_0)i(\bar{W}_0)\alpha .$$

Then we have

Lemma 7.7. *$\phi(\alpha_1)$ and $\phi(\alpha_2)$ are primitive forms.*

Proof. $\phi(\alpha_2)$ is a primitive form follows from (6) in Proposition 7.6. On the other hand we, have

$$\begin{aligned}\Lambda\phi(\alpha_1) &= \phi(\Lambda_0\alpha_1 + \sqrt{-1}i(W_0)i(\bar{W}_0)\alpha_1) \\ &= \phi\{(n+2-p-q)\Lambda_0\alpha - \sqrt{-1}\Lambda_0\Omega_0 i(W_0)i(\bar{W}_0)\alpha \\ &\quad + \sqrt{-1}(n+2-p-q)i(W_0)i(\bar{W}_0)\alpha - i(W_0)i(\bar{W}_0)\Omega_0 i(W_0)i(\bar{W}_0)\alpha\} \\ &= \phi\{-\sqrt{-1}(n+3-p-q)i(W_0)i(\bar{W}_0)\alpha \\ &\quad + \sqrt{-1}(n+2-p-q)i(W_0)i(\bar{W}_0)\alpha + \sqrt{-1}i(W_0)i(\bar{W}_0)\alpha\} \\ &= 0 .\end{aligned}$$

q.e.d.

Lemma 7.8. *$\alpha \neq 0$ if and only if $\alpha_2 \neq 0$.*

Proof. Assume $\alpha_2=0$. Then $i(W_0)i(\bar{W}_0)\alpha=0$. Thus we have $i(\bar{W}_0)\alpha \in \operatorname{Ker} i(W_0)$. Since $\partial_0 i(\bar{W}_0)\alpha = -i(\bar{W}_0)\partial_0\alpha=0$, we have $i(W_0)\alpha=0$. In the same way, we see $\alpha=0$.

q.e.d.

Lemma 7.9. *If $\alpha \neq 0$ and $k+p \neq 0$, then $\alpha_1 \neq 0$.*

Proof. Assume $\alpha \neq 0$. We shall show $\partial_0\alpha_1 \neq 0$.

$$\begin{aligned}\sqrt{-1}\partial_0\alpha_1 &= \partial_0\Omega_0 i(W_0)i(\bar{W}_0)\alpha \\ &= \Omega_0\partial_0 i(W_0)i(\bar{W}_0)\alpha \\ &= -\Omega_0 i(W_0)\partial_0 i(\bar{W}_0)\alpha + (k+p)\Omega_0 i(\bar{W}_0)\alpha \\ &= (k+p)\Omega_0 i(\bar{W}_0)\alpha .\end{aligned}$$

By our assumption, we have $i(\bar{W}_0)\alpha \neq 0$. Since $i(\bar{W}_0)\alpha$ is a $(p, q-1)$-form and

$p+(q-1)\leqq n$, we have $\Omega_0 i(\bar{W}_0)\alpha\neq 0$. Thus we have shown $\partial_0\alpha_1\neq 0$.

q.e.d.

Corollary 7.10. *We have a direct sum decomposition,*

$$\phi(P^{p;q}_{k,l}) = \sum_{\substack{i\geqq 0\\ j\geqq 0}} \Omega^i\phi(H^{p-i;q-i}_{k-j,l-j}) \qquad (k+p=l+q)\,.$$

Corollary 7.11. $\phi\colon \sum_{\substack{i\geqq 0\\ k+p=l+q}} \Omega_0^i H^{p-i,q-i}_{k,l} \to \Lambda^{p,q}(P^n(C))$

is injective and its image is dense.

Combining this with Theorem 5.2, we have

Proposition 7.12. *Assume $k+p=l+q$. Then we see that $H^{p;q}_{k,l}$ contains only irreducible $SU(n+1)$-modules appearing in Proposition 7.5 with multiplicity one and no other submodules occur in $H^{p;q}_{k,l}$.*

Now, put

$$\begin{aligned} {}^{\prime}_{\prime}H^{p;q}_{k,l} &= \{\alpha\in H^{p;q}_{k,l}\colon \partial_0\alpha = 0,\ \bar{\partial}_0\alpha = 0\}\,, \\ {}^{\prime}_{\prime\prime}H^{p;q}_{k,l} &= \{\alpha\in H^{p;q}_{k,l}\colon \partial_0\alpha = 0,\ i(\bar{W}_0)\alpha = 0\}\,, \\ {}^{\prime\prime}_{\prime}H^{p;q}_{k,l} &= \{\alpha\in H^{p;q}_{k,l}\colon i(W_0)\alpha = 0,\ \bar{\partial}_0\alpha = 0\}\,, \end{aligned}$$

and

$${}^{\prime\prime}_{\prime\prime}H^{p;q}_{k,l} = \{\alpha\in H^{p;q}_{k,l}\colon i(W_0)\alpha = 0,\ i(\bar{W}_0)\alpha = 0\}\,.$$

Then by (7.2), we have

$$H^{p;q}_{k,l} = {}^{\prime}_{\prime}H^{p;q}_{k,l}\oplus{}^{\prime}_{\prime\prime}H^{p;q}_{k,l}\oplus{}^{\prime\prime}_{\prime}H^{p;q}_{k,l}\oplus{}^{\prime\prime}_{\prime\prime}H^{p;q}_{k,l} \qquad (k+p\neq 0,\ q+l\neq 0)\,. \tag{7.5}$$

Theorem 7.13. *Assume $k+p=l+q$ and $0\leqq p+q\leqq n$. Then we have* (1) ${}^{\prime}_{\prime}H^{0;q}_{k,l}={}^{\prime}_{\prime}H^{p;0}_{k,l}={}^{\prime}_{\prime\prime}H^{0;q}_{k,l}={}^{\prime\prime}_{\prime}H^{0;q}_{k,l}={}^{\prime\prime}_{\prime\prime}H^{p;n-p}_{k,l}=\{0\}$. (2) *The $SU(n+1)$-modules ${}^{\prime}_{\prime}H^{p;q}_{k,l}$, ${}^{\prime}_{\prime\prime}H^{p;q}_{k,l}$, ${}^{\prime\prime}_{\prime}H^{p;q}_{k,l}$, ${}^{\prime\prime}_{\prime\prime}H^{p;q}_{k,l}$, except for the modules listed in* (1), *are irreducible modules with the following highest weights;*

$p\neq 0,\ q\neq 0$	${}^{\prime}_{\prime}H^{p;q}_{k,l}$	$k(\Lambda_1+\Lambda_n)+(p-q)\Lambda_1+\Lambda_q+\Lambda_{n+1-p}$
	${}^{\prime}_{\prime\prime}H^{p;q}_{k,l}$	$(k-1)(\Lambda_1+\Lambda_n)+(p-q+1)\Lambda_1+\Lambda_q+\Lambda_{n-p}$
	${}^{\prime\prime}_{\prime}H^{p;q}_{k,l}$	$k(\Lambda_1+\Lambda_n)+(p-q-1)\Lambda_1+\Lambda_{q+1}+\Lambda_{n+1-p}$
	${}^{\prime\prime}_{\prime\prime}H^{p;q}_{k,l}$	$(k-1)(\Lambda_1+\Lambda_n)+(p-q)\Lambda_1+\Lambda_{q+1}+\Lambda_{n-p}$
$p=0$, $q\neq 0,\ n$	${}^{\prime\prime}_{\prime}H^{0;q}_{k,l}$	$k(\Lambda_1+\Lambda_n)+(-q)\Lambda_1+\Lambda_q$
	${}^{\prime}_{\prime\prime}H^{0;q}_{k,l}$	$k(\Lambda_1+\Lambda_n)+(-q-1)\Lambda_1+\Lambda_{q+1}$

$p\neq 0, n$ $q=0$	${}^{\prime}_{\prime\prime}H^{p;0}_{k,l}$	$k(\Lambda_1+\Lambda_n)+p\Lambda_1+\Lambda_{n+1-p}$
	${}^{\prime\prime}_{\prime\prime}H^{p;0}_{k,l}$	$(k-1)(\Lambda_1+\Lambda_n)+(p+1)\Lambda_1+\Lambda_{n-p}$
$p=0,\ q=n$	${}^{\prime\prime}_{\prime}H^{0;n}_{k,l}$	$(k+1)(\Lambda_1+\Lambda_n)+(-n-1)\Lambda_1$
$p=n,\ q=0$	${}^{\prime}_{\prime\prime}H^{n;0}_{k,l}$	$k(\Lambda_1+\Lambda_n)+(n+1)\Lambda_1$
$p=q=0$	${}^{\prime\prime}_{\prime\prime}H^{0;0}_{k,l}$	$k(\Lambda_1+\Lambda_n)$

(3) $\phi({}^{\prime\prime}_{\prime\prime}H^{p;q}_{k,l})\subset \operatorname{Ker}\delta\cap\Lambda^{p,q}(P^n(\boldsymbol{C}))$

$\phi({}^{\prime}_{\prime\prime}H^{p;q}_{k,l})\subset \operatorname{Ker}\partial\cap\operatorname{Ker}\bar\partial^*\cap\Lambda^{p,q}(P^n(\boldsymbol{C}))\qquad (p\neq 0)$

$\phi({}^{\prime\prime}_{\prime}H^{p;q}_{k,l})\subset \operatorname{Ker}\bar\partial\cap\operatorname{Ker}\partial^*\cap\Lambda^{p,q}(P^n(\boldsymbol{C}))\qquad (q\neq 0)$.

Proof. It is clear that $E^{p;q}_{k,l}\subset{}^{\prime\prime}_{\prime}H^{p;q}_{k,l}$, $i(\bar W_0)E^{p;q-1}_{k+1,l}\subset{}^{\prime}_{\prime\prime}H^{p;q}_{k,l}$, $i(W_0)E^{p-1;q}_{k,l+1}\subset{}^{\prime\prime}_{\prime}H^{p;q}_{k,l}$ and $i(W_0)i(\bar W_0)E^{p-1;q-1}_{k+1,l+1}\subset{}^{\prime\prime}_{\prime\prime}H^{p;q}_{k,l}$. Combining this fact with (7.5) and Proposition 7.12, we get (1) and (2). Moreover, we get (3) by comparing the irreducible $SU(n+1)$-modules appearing in $H^{p;q}_{k,l}$.

q.e.d.

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Added in proof. After submitting this paper, we knew the following two papers:
[16] A. Lévy-Bruhl-Laperrière: *Spectre de de Rham-Hodge sur les formes de degré 1 des sphères de* $R^n(n\geqq 6)$, Bull, Sc. Math., 2e série, **99**, (1975), 213–240.
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