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On Systems of Structures of a Completely Regular Space

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Considering the set of all structures [1] over a completely regular space R, we propose to investigate the connections between the algebraic properties of the set and the topological properties of R. To this purpose we introduce in the following an order into the set in question and characterize among others the space R which has either the maximum or the minimum for certain subsystems of the ordered system thus obtained, and determine further the cardinal numbers of the ordered systems in the case when R is metrizable.

For the notations we follow those used by Tukey [1].

§ 1. Definitions and remarks.

DEFINITION. Let gR be a structure over a completely regular space R with a basis $\{\mathfrak{U}_\alpha | A\}$ and let $g'R$ be another structure over the space R with a basis $\{\mathfrak{V}_\beta | B\}$, then we write $gR \geqq g'R$, if for any $\mathfrak{V}_\beta \in \{\mathfrak{V}_\beta | B\}$ there exists $\mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}$ such that $\mathfrak{V}_\beta > \mathfrak{U}_\alpha$, i. e., if the identical mapping of R on itself is a uniformly continuous mapping of gR onto $g'R$.

It is clear that the set of all structures over a completely regular space R is considered as an ordered system in this order. Let us denote by $D(R)$ this ordered system and let $D_t(R)$ be the subsystem of $D(R)$ which consists of all totally bounded structures over R. Let $\mathfrak{m}$ be an infinite cardinal number. We say that a structure gR is an $\mathfrak{m}$-structure, if the uniformity of gR contains a basis $\{\mathfrak{U}_\alpha | A\}$ with $|A| \leqq \mathfrak{m}$ [2], and we denote by $D_{\mathfrak{m}}(R)$ the set of all $\mathfrak{m}$-structures over R.

Evidently $D(R), D_t(R)$ and $D_{\mathfrak{m}}(R)$ are not only ordered systems, but also directed systems and $D(R)$ and $D_t(R)$ have always the maximal element. Moreover an $\aleph_0$-structure is a metric space.

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1) John W. Tukey, Convergence and uniformity in topology (1940). We will use "structure" to mean "struct" used by Tukey and in this note we will refer this book by the letter T.

2) The cardinal number of a set A will be denoted by $|A|$ and small German letters will be used for infinite cardinal numbers.

By Raouf Doss' papers [3], it is known that we can characterize the space which satisfies the condition: $D(R)=D_t(R)$ as well as the space whose $D(R)$ contains only one element.

In the following we will use "space" to mean "completely regular space".

§ 2. In this paragraph we shall characterize the space for which $D_{\mathfrak{m}}(R)$ has the minimum.

Lemma 1. Let gR be a structure of a space R and let $\{\mathfrak{U}_\alpha | A\}$ be some basis for the uniformity of gR and let $\mathfrak{F}=\{F_\alpha | A\}$ be a family of closed sets having the following properties:

1) if $\alpha, \beta \in A$ and if $\alpha < \beta$ [4], then $F_\alpha \supset F_\beta$,

2) the intersection of all F_α is void, i.e., $\prod_{\alpha \in A} F_\alpha = 0$, then there exists a $|A|$-structure $g'R$ such that $\mathfrak{F}$ is a Cauchy family of $g'R$ and such that $g'R \leqq gR$.

Proof. For each $\alpha \in A$, let $\mathfrak{V}_\alpha = (R-F_\alpha)\circ\mathfrak{U}_\alpha$ [5]. $\{\mathfrak{V}_\alpha | A\}$ will be seen to be a basis for the uniformity of the structure over R satisfying the statements of the lemma.

We shall show first that $\{\mathfrak{V}_\alpha | A\}$ satisfies the conditions of a basis for some uniformity. If $\alpha < \beta$, then $\mathfrak{V}_\alpha \overset{\triangle}{>} \mathfrak{V}_\beta$ [6], because $\mathfrak{U}_\alpha \overset{\triangle}{>} \mathfrak{U}_\beta$ and $F_\alpha \supset F_\beta$. For two α, β, there exists $\gamma \in A$ such that $\mathfrak{U}_\alpha \overset{\triangle}{>} \mathfrak{U}_\gamma$ and $\mathfrak{U}_\beta \overset{\triangle}{>} \mathfrak{U}_\gamma$, hence $\mathfrak{V}_\alpha \overset{\triangle}{>} \mathfrak{V}_\gamma$ and $\mathfrak{V}_\beta \overset{\triangle}{>} \mathfrak{V}_\gamma$.

Next we shall show that $\{\mathfrak{V}_\alpha | A\}$ agrees with the topology. For any point $a \in R$ there exists F_α from the condition 2) such that $F_\alpha \not\ni a$. Since F_α is closed, there exists β such that $S(a, \mathfrak{U}_\beta) \wedge F_\alpha = 0$. Then if $\gamma > \beta, \alpha$, $S(a, \mathfrak{U}_\gamma) \wedge F_\gamma = 0$ and hence $a \notin S(F_\gamma, \mathfrak{U}_\gamma)$, whence $S(a, \mathfrak{V}_\gamma) = S(a, \mathfrak{U}_\gamma)$. Now if U is an open set of R containing a, there exists a δ such that $S(a, \mathfrak{U}_\delta) \subset U$. If $\gamma > \alpha, \beta, \delta$, then $S(a, \mathfrak{V}_\gamma) = S(a, \mathfrak{U}_\gamma) \subset S(a, \mathfrak{U}_\delta) \subset U$.

Let $g'R$ be the structure with the basis $\{\mathfrak{V}_\alpha | A\}$. Then it is obvious

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3) Raouf Doss, On uniform spaces with a unique structure, Amer. J. Math. 71 (1949). Raouf Doss, On continuous function in uniform spaces, Ann. of Math. 48 (1947).

4) We say that $\alpha < \beta$ if $\mathfrak{U}_\beta$ is a $\triangle$-refinement of $\mathfrak{U}_\alpha$, i.e., $\mathfrak{U}_\alpha \overset{\triangle}{>} \mathfrak{U}_\beta$.

5) For a covering $\mathfrak{U}$ of R and for a subset H of R, the restriction $H\circ\mathfrak{U}$ of $\mathfrak{U}$ to H is a covering of R consisting of the set $S(R-H, \mathfrak{U})$ and those sets of $\mathfrak{U}$ included in H. Cf. T. p. 45.

6) If $\mathfrak{U} \overset{\triangle}{>} \mathfrak{V}$ and $A \supset B$, then $B\circ\mathfrak{U} \overset{\triangle}{>} A\circ\mathfrak{V}$.

that $gR \geqq g'R$ and that $\mathfrak{F}$ is a Cauchy family of $g'R$, because for any α, $\mathfrak{V}_\alpha \leqq \mathfrak{U}_\alpha$ and for any point $a \in F_\alpha$ $F_\alpha \subset S(a, \mathfrak{U}_a) \bigcup S(F_\alpha, \mathfrak{U}_a) = S(a, \mathfrak{V}_a)$. Thus the lemma has been proved.

Lemma 2. Let R be a locally bicompact space whose character is $\leqq \mathfrak{m}$ and let A be a set of cardinal $\leqq \mathfrak{m}$. Then for any totally bounded $\mathfrak{m}$-structure gR with a basis $\{\mathfrak{V}_\alpha | A\}$ and for any open basis $\mathfrak{U} = \{U_\alpha\}$, where U_α is bicompact and $\alpha \in A$, if the family $\mathfrak{U}$ is additive, then the collection

$$\mathfrak{P} = \{U_\beta \circ \mathfrak{U}_\alpha | \mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\} \ \& \ U_\beta \in \mathfrak{U}\}$$

is a basis for the uniformity of the minimal structure of $D_t(R)$ and $|\mathfrak{P}| \leqq \mathfrak{m}$.

Proof. We see first that $\mathfrak{P}$ is a basis for some uniformity, because if $\alpha, \beta, \gamma \in A$ and $\alpha < \gamma$, $\beta < \gamma$ and if $U_\delta, U_\varepsilon \in \mathfrak{U}$, then $U_\delta \circ \mathfrak{U}_\alpha \bigwedge \mathfrak{U}_3 \circ \mathfrak{U}_\beta \overset{\triangle}{>} (U_\delta \bigvee U_\varepsilon) \circ \mathfrak{U}_\gamma$ and $(U_\delta \bigvee U_\varepsilon) \circ \mathfrak{U}_\gamma \in \mathfrak{P}$, since $\mathfrak{U}$ is additive.

We shall show next that the uniformity agrees with the topology. For any point a of R and for an open set G containing a, there exists $\mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}$ such that $S(a, \mathfrak{U}_a) \subset G$. Without loss of generality, we may suppose that $G \in \mathfrak{U}$. Hence $G \circ \mathfrak{U}_\alpha \in \mathfrak{P}$, and then $S(a, G \circ \mathfrak{U}_\alpha) = S(a, \mathfrak{U}_a) \subset G$. Evidently $U_\beta \circ \mathfrak{U}_\alpha$ is a finite covering, since $\mathfrak{U}_\alpha$ may be considered as a finite covering. Therefore $\mathfrak{P}$ is a basis for some uniformity of totally bounded structure.

Let gR be this structure. We see that gR is the minimum of $D(R)$ and $D_t(R)$; that is, if $U_\beta \circ \mathfrak{U}_\alpha \in \mathfrak{P}$, then $U_\beta \circ \mathfrak{U}_\alpha \in \mathfrak{Q}$ for any uniformity $\mathfrak{Q}$ which agrees with the topology. To prove this let $U_\beta \circ \mathfrak{U}_\alpha$ be $\{S(R - U_\beta, \mathfrak{U}_a), U_1, U_2, \dots, U_n\}$ and let $F = R - S(R - U_\beta, \mathfrak{U}_a)$. Then $F \subset U_\beta$, hence $F = \overline{F} \subset \overline{U}_\beta$, thus F is bicompact. For any $a \in F$, there exists $U_{i(a)}$ such that $a \in U_{i(a)}$, hence there exists two coverings $\mathfrak{V}_{\gamma(a)}, \mathfrak{V}_{\gamma'(a)}$ such that $\mathfrak{V}_{\gamma(a)}, \mathfrak{V}_{\gamma'(a)} \in \mathfrak{Q}$, $\mathfrak{V}_{\gamma'(a)} \overset{\triangle}{<} \mathfrak{V}_{\gamma(a)}$ and $S(a, \mathfrak{V}_{\gamma(a)}) \subset U_{i(a)}$. Then $S^2(a, \mathfrak{V}_{\gamma'(a)}) \subset S(a, \mathfrak{V}_{\gamma(a)})$ and $\{S(a, \mathfrak{V}_{\gamma'(a)}) | a \in F\}$ is an open covering of F. Since F is bicompact, there exist a finite number of points $a_1, a_1, \dots, a_m$ such that $\bigcup_{i=1}^{m} S(a_i, \mathfrak{V}_{\gamma'(a_i)}) \supset F$. Then evidently $\mathfrak{V} = \bigwedge_{i=1}^{m} \mathfrak{V}_{\gamma'(a_i)} \in \mathfrak{Q}$. It is to prove that $\mathfrak{V} < U_\beta \circ \mathfrak{U}_\alpha$. To show this, let $V = \bigcap_{i=1}^{m} V_{\gamma'(a_i)}$ be an element of $\mathfrak{V}$ where $V_{\gamma'(a_i)} \in \mathfrak{V}_{\gamma'(a_i)}$. If $V \not\subset S(R - U_\beta, \mathfrak{U}_a)$, then $V \bigwedge F \neq 0$, hence there exists a point a and a point $a_j \in \{a_i | i = 1, 2, \dots, m\}$ such that $a \in V \bigwedge F$ and $a \in S(a_j, \mathfrak{V}_{\gamma'(a_j)})$. Since $V \subset V_{\gamma'(a_j)}$, we have $a \in V_{\gamma'(a_j)} \bigwedge S(a_j, \mathfrak{V}_{\gamma'(a_j)})$, therefore $V_{\gamma'(a_j)} \subset S(S(a_j, \mathfrak{V}_{\gamma'(a_j)}), \mathfrak{V}_{\gamma'(a_j)}) \subset S(a_j, \mathfrak{V}_{\gamma(a_j)})$, whence $V \subset U_{i(a_j)} \in U_\beta \circ \mathfrak{U}_\alpha$. Thus we have $\mathfrak{V} < U_\beta \circ \mathfrak{U}_\alpha$.

Evidently we have

Lemma 3. If $\mathfrak{U} \overset{\Delta\Delta\Delta}{>} \mathfrak{W}$ for two open coverings $\mathfrak{U}$ and $\mathfrak{W}$ of a space R and if $|\mathfrak{V}| \geq \mathfrak{m}$ for any open refinement $\mathfrak{V}$ of $\mathfrak{U}$, then there exists a subset A of R with potency $\geq \mathfrak{m}$ such that for any point p of R there exists at most an $a \in A$ such that $S(p, \mathfrak{W})$ and $S(a, \mathfrak{W})$ have points in common.

Theorem 1. *The following conditions on a space R are equivalent*:

1) *$D_{\mathfrak{m}}(R)$ has the minimum,*
2) *$D_{\mathfrak{m}}(R)$ contains the minimum of $D(R)$ and $D_t(R)$,*
3) *R is locally bicompact and its character $\leq \mathfrak{m}$.*

Proof. In view of Lemma 2, 3) implies 2). It is evident that 2) implies 1). Thus we have only to prove that 1) implies 3). Let gR be the minimum of $D_{\mathfrak{m}}(R)$, and let $\{\mathfrak{U}_\alpha | A\}$ be a basis for the uniformity of gR, the potency of A being $\leq \mathfrak{m}$. Then we show first that for any $\mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}$ there exists an open covering $\mathfrak{U}_\alpha'$ such that $\mathfrak{U}_\alpha' < \mathfrak{U}_\alpha$ and $|\mathfrak{U}_\alpha'| < \mathfrak{m}$. Suppose on the contrary that there exists some $\mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}$ such that for any $\mathfrak{U} < \mathfrak{U}_\alpha$, $|\mathfrak{U}| \geq \mathfrak{m}$. Then by Lemma 3 there exists an open covering $\mathfrak{W} \in \{\mathfrak{U}_\alpha | A\}$ and a subset F satisfying the statement of the lemma. Since $|A| \leq |F|$, there exists a 1-1 correspondence f between A and a subset of F. For any finite set $\delta = (\alpha_1, \alpha_2, \dots, \alpha_n)$, let $F_\delta = f(A) - \bigcup_{i=1}^{n} f(\alpha_i)$, $\mathfrak{U}_\delta = \bigwedge_{i=1}^{n} \mathfrak{U}_{\alpha_i}$ and let Γ be the set of all finite sets of A. Then for the basis $\{\mathfrak{U}_\delta | \Gamma\}$ for the uniformity of gR and for $\{F_\delta | \Gamma\}$, the statement of Lemma 1 is satisfied. Hence there exists a $g'R \in D(R)$ such that $g'R \leq gR$ and such that $\{F_\delta | \Gamma\}$ is a Cauchy family of $g'R$. But $\{F_\delta | \Gamma\}$ is not a Cauchy family of gR, since for all F_δ and for all $a \in R$, $S(a, \mathfrak{W})$ does not contain F_δ. Hence $g'R \lneqq gR$. This contradicts the minimality of gR.

Now let $\mathfrak{G} = \{U | U \in \mathfrak{U}_\alpha', \alpha \in A\}$. Then $\mathfrak{G}$ is an open basis of R and $|\mathfrak{G}| \leq \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$, hence the character of $R \leq \mathfrak{m}$.

It remains now to show that R is locally bicompact. For if $\{F_\alpha | A\}$ and $\{F_\alpha' | A\}$ are two Cauchy families of gR and if both of them do not converge, where we may suppose that both of them satisfy the conditions 1) and 2) of Lemma 1, then $\{F_\alpha | A\}$ and $\{F_\alpha' | A\}$ are equivalent. Since the family $\{F_\alpha \cup F_\alpha' | A\}$ and gR satisfy the statement of the lemma, there exists $g'R \in D_{\mathfrak{m}}(R)$ such that $g'R \leq gR$, and such that $\{F_\alpha \cup F_\alpha' | A\}$ is a Cauchy family of $g'R$. And since gR is the minimum of $D_{\mathfrak{m}}(R)$, $gR = g'R$, so that $\{F_\alpha \cup F_\alpha' | A\}$ is a Cauchy family of gR, therefore $\{F_\alpha | A\}$ and $\{F_\alpha' | A\}$ are equivalent.

Then the completion $\overline{gR}$ of gR is bicompact and $\overline{gR}-gR$ can contain at most one element, hence R is locally bicompact.

Corollary. *For a metrizable space R, $D\aleph_0(R)$ has the minimum if and only if R is locally bicompact and separable.*

Corollary. *The following conditions on a space R are equivalent:*

1) *$D(R)$ has the minimum,*
2) *$D_t(R)$ has the minimum,*
3) *$D(R)$ is a complete lattice,*
4) *$D_t(R)$ is a complete lattice,*
5) *R is locally bicompact.*

§3. Let $D_c(R)$ be the set of all complete structures over R and let $D_{c\mathfrak{m}}(R)=D_c(R)\wedge D_{\mathfrak{m}}(R)$.

Theorem 2. *The following conditions on a space R are equivalent:*

1) *$D_{c\mathfrak{m}}(R)$ has the minimum,*
2) *$D_{c\mathfrak{m}}(R)=D_{\mathfrak{m}}(R)\neq 0$,*
3) *$D_{c\mathfrak{m}}(R)$ contains only one element,*
4) *R is bicompact and the character of $R\leqq\mathfrak{m}$.*

Proof. Clearly, 3) implies 1), and 4) implies both 2) and 3). Thus we have only to show that 1) implies 4) and that 2) implies 4).

(i) Let gR be the minimum of $D_{c\mathfrak{m}}(R)$ and let $\{\mathfrak{U}_\alpha|A\}$ be a basis for the uniformity. Suppose that gR is not totally bounded. Then there exists a $\mathfrak{U}\in\{\mathfrak{U}_\alpha|A\}$ such that every refinements of $\mathfrak{U}$ are not finite. By Lemma 3, there exist $\mathfrak{V}\in\{\mathfrak{U}_\alpha|A\}$ and a countable subset $\{a_n|n=1,2,3,\dots\}$ of R such that they satisfy the statement of the lemma. Now let $F_n=\{a_i|i=n,n+1,\dots\}$ and let $P_n=\{a_{2n-1},a_{2n}\}$ for every $n=1,2,\dots$. Moreover for each $\mathfrak{U}_\alpha<\mathfrak{V}$ let $\mathfrak{V}_{\alpha n}=\{S(P_i,\mathfrak{U}_\alpha)|i=n,n+1,\dots\}\vee\{U|U\in\mathfrak{U}_\alpha \ \&\ U\wedge F_{2n-1}=0\}$. Then by the same method of the proof of Lemma 1, there exists a structure $g'R$ whose uniformity contains $\{\mathfrak{V}_{\alpha n}|\alpha\in A' \ \&\ n=1,2,\dots\}$ as a basis and which is $\mathfrak{m}$-structure, where A' consists of all $\alpha\in A$ for which $\mathfrak{U}_\alpha<\mathfrak{V}$, and that $g'R\lneqq gR$. Now we shall show that $g'R\in D_{c\mathfrak{m}}(R)$. For this, let $\mathrm{x}((\alpha n)|A'\times N)$[7] be a Cauchy phalanx of $g'R$. Then for each (α,n) there exists a point $a\in R$ such that $\mathrm{x}((\alpha n)|A'\times N)$ is decided for $S(a,\mathfrak{V}_{\alpha n})$. Then $S(a,\mathfrak{V}_{\alpha n})=S(a,\mathfrak{U}_\alpha)$ or $S(a,\mathfrak{V}_{\alpha n})=S(a,\mathfrak{U}_\alpha)\vee S(P_i,\mathfrak{U}_\alpha)$. Let $S(a_{x_i},\mathfrak{U}_\alpha)$, $i=1,2,3$ be such sets that $\mathrm{x}((\alpha n)|A'\times N)$ is not decided against $S(a_{x_i},\mathfrak{U}_\alpha)$ for $i=1,2,3$ and that $\mathrm{x}((\alpha n)|A'\times N)$ is decided

7) Let $A'\times N$ be the (direct) product of the directed system A' and the linear ordered set N consisting of all positive integers. Cf. T. p. 7.

against $R-\bigcup_{i=1,2,3} S(a_{x_i}, \mathfrak{U}_\alpha)$, where $a_{x_1}, a_{x_2}, a_{x_3}$ are not always different points. Let $\mathfrak{V}'_{\alpha n}=\{S(a_{x_i}, \mathfrak{U}_\alpha)\,|\,i=1,2,3\} \cup \{R-\bigcup_{i=1,2,3} S(a_{x_i}, \mathfrak{U}_\alpha)\}$, then $\{\mathfrak{V}'_{\alpha n}\,|\,A'\times N\}$ is a collection of finite coverings. Now, let $\mathfrak{B}$ be $\{B\,|$ $\mathrm{x}((\alpha n)\,|\,A'\times N)$ is decided against $B\}$. Then $\{\mathfrak{V}'_{\alpha n}\}$ and $\mathfrak{B}$ satisfy the condition of the Tukey's lemma [8]. Hence there exists a phalanx $\mathrm{x}'((\alpha n)\,|\,A'\times N)$ such that for each $\alpha\in A'$ it is decided for a set of $\{S(a_{x_i}, \mathfrak{U}_\alpha)\,|\,i=1,2,3\}$. Hence $\mathrm{x}'((\alpha n)\,|\,A'\times N)$ and $\mathrm{x}((\alpha n)\,|\,A'\times N)$ are a Cauchy pair of $g'R$ and $\mathrm{x}'((\alpha n)\,|\,A'\times N)$ is a Cauchy phalanx of gR. Since gR is complete, $\mathrm{x}'((\alpha n)\,|\,A'\times N)$ converges to a point $a\in R$, $\mathrm{x}((\alpha n)\,|\,A'\times N)$ converges to the same point a. Thus we prove that $g'R\in D_{c\mathfrak{m}}(R)$. But this contradicts the hypothesis that gR is the minimum of $D_{c\mathfrak{m}}(R)$. Hence gR is totally bounded and complete, i.e., R is bicompact and hence the character of $R\leqq\mathfrak{m}$, since $|A|\leqq\mathfrak{m}$.

(ii) We proceed to prove that 2) implies 4). For this we suppose that $D_{c\mathfrak{m}}(R)=D_{\mathfrak{m}}(R)\neq 0$ and R is not bicompact. Now choose a $gR\in D_{c\mathfrak{m}}(R)$, then gR cannot be totally bounded, hence by the same method of the above proof, there exists a covering $\mathfrak{U}\in\{\mathfrak{U}_\alpha\}$, where $\{\mathfrak{U}_\alpha\}$ is a basis for the uniformity of gR, and there exists a countable subset $\{a_n\,|\,n=1,2,\ldots\}$ of R, such that they satisfy the statement of Lemma 3. Let $F_n=\{a_i\,|\,i=n,n+1,\ldots\}$ and let $\mathfrak{V}_{\alpha n}=(R-F_n)\,\mathfrak{U}_\alpha$. Then it is obvious that $\{\mathfrak{V}_{\alpha n}\}$ is a basis for some structure $g'R$ and that $g'R\in D_{\mathfrak{m}}(R)-D_{c\mathfrak{m}}(R)$, since $\{a_i\}$ is a Cauchy sequence of $g'R$ and has no limit point. Hence $D_{c\mathfrak{m}}(R)\neq D_{\mathfrak{m}}(R)$. Thus 2) implies that R is bicompact, and that the character of R is $\leqq\mathfrak{m}$, since $D_{\mathfrak{m}}(R)\neq 0$.

Corollary. *The following conditions on a metrizable space R are equivalent*:

1) $D_{c\aleph_0}(R)$ *has the minimum,*
2) $D_{c\aleph_0}(R)=D_{\aleph_0}(R)$,
3) $D_{c\aleph_0}(R)$ *contains only one element,*
4) R *is bicompact.*

Corollary. *The following conditions on a space R are equivalent*:

1) $D_c(R)$ *has the minimum,*
2) $D_c(R)=D(R)$,
3) $D_c(R)$ *contains only one element,*
4) R *is bicompact.*

§ 4. In this paragraph we shall characterize the space R whose

8) Cf. T. p. 33.

$D_{\mathfrak{m}}(R)$ has the maximum.

Lemma 4. Let R be a fully normal space and let R' be the set of all cluster points of R. If R' satisfies the following conditions:

1) the character relative to $R \leqq \mathfrak{m}$ [9],

2) if $\mathfrak{U}$ is an open covering of R, there exists an open refinement $\mathfrak{V}$ of $\mathfrak{U}$ such that $|\{V | V \frown R' \neq 0 \ \&\ V \in \mathfrak{V}\}| \lneqq \mathfrak{n}$, where $\mathfrak{n}$ is the smallest cardinal number satisfying the condition $\mathfrak{m}^{\mathfrak{n}} \gneqq \mathfrak{m}$, then the a-structure aR [10] is contained in $D_{\mathfrak{m}}(R)$.

Proof. Let $\{\mathfrak{U}_\alpha | A\}$ be the uniformity of aR and let $\mathfrak{B} = \{U\}$ be the basis of R' relative to R such that $|\mathfrak{B}| \leqq \mathfrak{m}$. Then, by the conditions 1) and 2), for each α there exists an open refinement $\mathfrak{V}_\alpha$ of $\mathfrak{U}_\alpha$ such that it satisfies the condition 2) and such that if $U \in \mathfrak{V}_\alpha$ and $U \frown R' \neq 0$, then $U \in \mathfrak{B}$, and if $U \in \mathfrak{V}_\alpha$, $U \frown R' = 0$, then U contains only one point. It is obvious that the collection $\{\mathfrak{V}_\alpha | A\}$ contains open coverings of cardinal $\leqq \mathfrak{m}$, hence we can rewrite it by $\{\mathfrak{V}_\alpha | A'\}$ where $|A'| \leqq \mathfrak{m}$. Now for any $\mathfrak{V}_\alpha \in \{\mathfrak{V}_\alpha | A'\}$, there exists a normal covering $\mathfrak{V}_{\alpha, n}$ such that $\mathfrak{V}_\alpha = \mathfrak{V}_{\alpha, 1}$. Let Γ be the set of all finite subsets of A', and for any $\sigma = (\alpha_1, \alpha_2, \dots, \alpha_i) \in \Gamma$, let $\mathfrak{W}_\sigma = \mathfrak{V}_{\alpha_1, i} \wedge \mathfrak{V}_{\alpha_2, i} \wedge \dots \wedge \mathfrak{V}_{\alpha_i, i}$. Then if $\sigma \supset \sigma'$, $\mathfrak{W}_\sigma \overset{*}{<} \mathfrak{W}_{\sigma'}$ and so $\{\mathfrak{W}_\sigma | \Gamma\}$ is a basis for some uniformity. Obviously $\{\mathfrak{W}_\sigma | \Gamma\}$ agrees with the topology, the uniformity is that of the a-structure, and $|\Gamma| \leqq \mathfrak{m}$. Hence $aR \in D_{\mathfrak{m}}(R)$.

Lemma 5. For a fully normal space R, if $gR \in D_{\mathfrak{m}}(R)$ and if every (bounded) continuous real-valued functions of R are uniformly continuous with respect to gR and to the usual metric of the space of real numbers, then the set R' of all cluster points of R satisfies the conditions 1) and 2) of Lemma 4, where we use $\mathfrak{m}$ in place of $\mathfrak{n}$ in 2).

Proof. Assuming that gR satisfies the conditions of Lemma 5, we suppose that there exists an open covering $\mathfrak{U}$ such that it does not satisfy the condition 2) of Lemma 4 with $\mathfrak{n} = \mathfrak{m}$. Then it is obvious that there exists a set F with the potency $\geqq \mathfrak{m}$ and an open covering $\mathfrak{V}$ such that they satisfy the statement of Lemma 3 and such that $F \subset R'$. Now let $\{\mathfrak{U}_\alpha | A\}$ be a basis for gR and then we may assume that $|A| \leqq \mathfrak{m}$. Since $|F| \geqq \mathfrak{m}$, there exists a 1-1 mapping ρ of A onto

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9) For a space R and a subset M of R, we say that the collection $\mathfrak{B}$ of open sets of R is an open basis of M relative to R if for any point a of M and for any open set G of R containing a, there exists an open set $U \in \mathfrak{B}$ such that $a \in U \subset G$, and a cardinal number will be called the character of M relative to R, if it is the minimum of cardinal numbers of open basis of M relative to R.

10) We denote the structure with the uniformity made up of all normal coverings by aR.

the subset of F. For each $\alpha \in A$ there exists an $\alpha' \in A$ such that $\alpha' > \alpha$ and $S(\rho(\alpha), \mathfrak{U}_{\alpha'}) \subset S(\rho(\alpha), \mathfrak{V})$. Then for each $\alpha \in A$, there exists a $b_\alpha \in R$ such that $b_\alpha \in S(\rho(\alpha), \mathfrak{U}_\alpha')$ and $b_\alpha \neq \rho(\alpha)$ since $A \subset R'$. Now for each $\alpha \in A$, let f_α be a continuous function such that $f_\alpha(\rho(\alpha_\alpha))=1$, $f_\alpha(b_\alpha)=0$, $f_\alpha(x)=0$ for every $x \notin S(\rho(\alpha), \mathfrak{U}_\alpha')$, $|f_\alpha(x)| \leq 1$ for every $x \in R$. Moreover let $f=\sum_\alpha f_\alpha$, then f is a bounded continuous function of R, since for each $a \in R$, $S(a, \mathfrak{V})$ can meet at most one $S(\rho(\alpha), \mathfrak{V})$, where $\alpha \in A$. But f is not uniformly continuous with respect to gR, since for each $\alpha \in A$, $|f(\rho(\alpha))-f(b_\alpha)|=1$, $S(\rho(\alpha), \mathfrak{U}_\alpha') \in b_\alpha$, and $\{\alpha'\}$ is cofinal in A. This contradicts our hypothesis. Hence the condition 2) of Lemma 4 with $\mathfrak{n}=\mathfrak{m}$ is satisfied. Finally, it is evident by the condition 2) that the character of R' relative to R is $\leq \mathfrak{m}$, for $|A| \leq \mathfrak{m}$. Thus Lemma 5 is proved.

Lemma 6. The following conditions on a space R are equivalent:

1) $D_{\mathfrak{m}}(R)$ has the maximum,

2) $D_{\mathfrak{m}}(R)$ contains the maximum of $D(R)$, i.e., $aR \in D_{\mathfrak{m}}(R)$.

Proof. We prove only that 1) implies 2). Assume 1). Let gR be the maximum of $D(R)$, $\{\mathfrak{U}_\alpha | A\}$ the basis for the uniformity, $\mathfrak{U}$ an arbitrary normal covering and finally let $\{\mathfrak{U}_n\}$ be a normal sequence such that $\mathfrak{U} > \mathfrak{U}_1$. Then it is obvious that $\{\mathfrak{U}_\alpha \wedge \mathfrak{U}_n | \alpha \in A \ \& \ n=1, 2, \dots\}$ is a basis for some structure $g'R$ and that the $g'R$ is an $\mathfrak{m}$-structure. Since $g'R \geq gR$ $g'R=gR$. Then evidently $\mathfrak{U} > \mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}$, which implies that $gR=aR$.

Theorem 3. *Under the assumption of the generalized continuum hypothesis* [10], *if* $\mathfrak{m}$ *is regular* [11], *then the following conditions on a fully normal space* R *are equivalent*:

1) $D_{\mathfrak{m}}(R)$ *has the maximum,*

2) $D_{\mathfrak{m}}(R)$ *contains the maximum of* $D(R)$,

3) *there exists* $gR \in D_{\mathfrak{m}}(R)$ *such that every (bounded) continuous real-valued functions are uniformly continuous with respect to* gR *and to the usual metric of the space of real numbers,*

4) *the set* R' *of all cluster points of* R *has the character relative to* $R \leq \mathfrak{m}$, *and for each subset* F *of* R' *whose cardinal number is* $\geq \mathfrak{m}$, F' *is not void.*

Proof. In view of the above lemmas, 1) implies 2), 3) implies 4) and 4) implies 1) and it is evident that 2) implies 3).

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11) In this case, if $\mathfrak{n} < \mathfrak{m}$, then $\mathfrak{m}^{\mathfrak{n}} = \mathfrak{m}$ and hence if a set A has the potency $\mathfrak{m}$, then the set of all subsets of A with potency $< \mathfrak{m}$ has the potency $\mathfrak{m}$.

Corollary[12]. *The following conditions on a metrizable space R are equivalent*:

1) $D_{\aleph_0}(R)$ *has the maximum,*

2) $D_{\aleph_0}(R)$ *contains the maximum of* $D(R)$,

3) *there exists a metric of R such that every (bounded) continuous real-valued functions are uniformly continuous with respect to the metric and to the usual metric of the space of real numbers,*

4) *the set R' of all cluster points of R is bicompact.*

Corollary. *The following conditions on a fully normal space R are equivalent*:

1) $D_{\mathfrak{m}}(R)$ *has the maximum,*

2) $D_{c\mathfrak{m}}(R)$ *has the maximum.*

Proof. This is evident by the fact that for a fully normal space R, $D_c(R)\neq 0$ and that if $gR \in D_c(R)$ and if $g'R \geqq gR$, then $g'R \in D_c(R)$.

Corollary. *The following conditions on a metrizable space R are equivalent*:

1) $D_{\aleph_0}(R)$ *has the maximum,*

2) $D_{c\aleph_0}(R)$ *has the maximum.*

§ 5. In this paragraph we are concerned with the space whose $D_{t\mathfrak{m}}(R)$ has the maximum, where $D_{t\mathfrak{m}}(R)=D_t(R) \frown D_{\mathfrak{m}}(R)$.

Theorem 4. *The following conditions on a space R are equivalent*:

1) $D_{t\mathfrak{m}}(R)$ *has the maximum,*

2) *there is a basis for an open sets $\mathfrak{B}=\{U_\alpha | A\}$ such that i) $|A| \leqq \mathfrak{m}$ and ii) if F and C are two closed sets and completely separated*[13], *then there exist two finite open subcollections $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathfrak{B}$ such that $\bigcup_{U \in \mathfrak{U}} U \supset F$ and $\bigcup_{V \in \mathfrak{V}} V \supset C$ and that $\overline{\bigcup_{U \in \mathfrak{U}} U} \frown \overline{\bigcup_{V \in \mathfrak{V}} V}=0$.*

Proof. It is obvious that maximum of $D_{t\mathfrak{m}}(R)$ is the afR[14], so that 1) implies that $afR \in D_{\mathfrak{m}}(R)$. Let $\{\mathfrak{U}_\alpha | A\}$ be the basis for the uniformity of afR, where $|A| \leqq \mathfrak{m}$, then $\mathfrak{B}=\{U | U \in \mathfrak{U}_\alpha \ \& \ \mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}\}$ is a basis for open sets of R. If F and C are both closed and completely separated, then $\{R-F, R-C\}$ is a finite covering, hence there

12) J. Nagata, On the uniform topology of bicompactifications, Journal of the Institute of Polytechnics, Osaka City University (1950).

13) We say that A and B are completely separated when there exists a continuous function f such that $f(x)=0$ $(x \in A)$, $f(x)=1$ $(x \in B)$ and $0 \leqq f(x) \leqq 1$ for all $x \in R$.

14) We denote the structure with the basis for the uniformity made up of all normal finite coverings by afR.

exists a finite covering $\mathfrak{U}_\alpha \in \{\mathfrak{U}_\alpha | A\}$ such that $\{R-F, R-C\} \overset{\Delta\Delta}{>} \mathfrak{U}_\alpha$, then $S(F, \mathfrak{U}_\alpha) \wedge S(C, \mathfrak{U}_\alpha)=0$, $\mathfrak{U}=\{U | U \in \mathfrak{U}_\alpha \ \&\ U \wedge F \neq 0\} \subset \mathfrak{B}$ and $\mathfrak{B}=\{V | V \in \mathfrak{U}_\alpha \ \&\ V \wedge F \neq 0\} \subset \mathfrak{B}$. Since $\mathfrak{U}$ and $\mathfrak{B}$ are finite, thus we can see that 1) implies 2).

Conversely, assume 2). Let $\mathfrak{B}'=\{\dot{\bigcup\limits_{i=1}^{n}} U_{\alpha_i} | U_{\alpha_i} \in \mathfrak{B} \ \&\ n=1, 2, \dots\}$ and let $\mathfrak{B}''=\{R-\bar{U} | U \in \mathfrak{B}'\}$, then it is evident that $|\mathfrak{B}''| \leq \mathfrak{m}$. Now we shall prove that, if $\{U, V\}$ is a binary normal open covering of R, then there exists a binary open covering $\{U', V'\}$ such that $U \supset U'$, $V \supset V'$ and $V', U' \in \mathfrak{B}''$. Because, $R-U$ and $R-V$ are completely separated[15], hence there exist two open sets G and O such that $R-G$, $R-U$ and $R-O$, $R-V$ and $\bar{G}, \bar{O}$ are three completely separated pairs. Hence there exist two open sets G' and O' such that $R-U \subset G' \subset G$, $R-V \subset O' \subset O$ and that $O', G' \in \mathfrak{B}'$. Let $U'=R-\bar{G}'$ and let $V'=R-\bar{O}'$, then U', $V' \in \mathfrak{B}''$, $U' \subset U$, $V' \subset V$ and $\bar{G}', \bar{V}'$ are completely separated. Hence $\{U, V\} > \{U', V'\}$ and $\{U', V'\}$ is a normal open covering. Now let $\mathfrak{U}=\{U_1, U_2, \dots, U_n\}$ be an arbitrary normal open covering, then there exists a normal open covering $\mathfrak{B}$ such that $\mathfrak{U} \overset{\Delta\Delta}{>} \mathfrak{B}$. Then $\{R-S(R-U_i, \mathfrak{B}) | i=1, 2, \dots, n\}$ is a closed covering and $U_i \supset R-S(R-U_i, \mathfrak{B})$ and that $R-U_i$ and $R-S(R-U_i, \mathfrak{B})$ are completely separated, hence there exists a normal covering $\{O_i, G_i\}$ such that $O_i, G_i \in \mathfrak{B}''$ and $\{O_i, G_i\} < \{U_i, S(R-U_i, \mathfrak{B})\}$. Hence $\mathfrak{U} > \bigwedge\limits_{i=1}^{n} \{U_i, S(R-U_i, \mathfrak{B})\} > \bigwedge\limits_{i=1}^{n} \{O_i, G_i\}$. Now, let $\mathfrak{F}'=\{\mathfrak{U}_\alpha | \alpha \in A\}$ be the collection of all binary normal coverings whose elements are contained in $\mathfrak{B}''$, and for each $\alpha \in A$, let $\{\mathfrak{U}_{\alpha, n} | n=1, 2, \dots\}$ be a normal sequence such that $\mathfrak{U}_{\alpha, 1}=\mathfrak{U}_\alpha$ and such that every $\mathfrak{U}_{\alpha, n}$ are finite coverings. Let $\mathfrak{F}^*=\{\mathfrak{U}_{\alpha_1, n} \wedge \mathfrak{U}_{\alpha_2, n} \wedge \dots \wedge \mathfrak{U}_{\alpha_n, n} | n=1, 2, \dots \ \&\ \alpha \in A\}$ then $\mathfrak{F}^*$ is a basis for the uniformity of afR and $|\mathfrak{F}^*| \leq \mathfrak{m}$. Since for any normal finite covering $\mathfrak{U}$ there exists $\mathfrak{U}_i$ $i=1, 2, \dots, n$ such that $\mathfrak{U} > \bigwedge\limits_{i=1}^{n} \mathfrak{U}_i$ and $\mathfrak{U}_i \in \mathfrak{F}'$, there exists $\mathfrak{B} \in \mathfrak{F}^*$ such that $\mathfrak{B} < \mathfrak{U}$, and it is obvious that $|\mathfrak{F}^*| \leq \mathfrak{m}$.

Corollary. *If for a normal space R $Dt_{\mathfrak{m}}(R)$ has the maximum, then R satisfies the following conditions:*

1) *the character* $\leq \mathfrak{m}$,

2) *if* $A \subset R$ *and of* $|A| \geq \mathfrak{m}$,

then A' *is not void.*

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15) It is obvious that a binary open covering $\{U, V\}$ of R is normal if and only if R-U and R-V are completely separated. Cf. T. p. 53.

Proof. Assume that $Dt_{\mathfrak{m}}(R)$ has the maximum. Then we have already seen that the character of R is $\leqq \mathfrak{m}$. Now let $\mathfrak{B}$ be an open basis satisfying the condition 2) of Theorem 4, and let $\mathfrak{B}'=\{U_{\alpha_1}\cup U_{\alpha_2}\cup\ldots\cup U_{\alpha_n}\,|\,n=1,2,\ldots\ \&\ U_{\alpha_i}\in\mathfrak{B}\}$. Suppose that there exists a subset F of R such that $|F|\geqq\mathfrak{m}$ and $F'=0$. Then for every subset F_δ of F, let $F_{\delta'}=F-F_\delta$, then F_δ and F_δ' are both closed and $F_\delta\cap F_\delta'=0$. Since R is normal, there exist by Theorem 4 open sets O_δ and G_δ such that $O_\delta\supset F_\delta$, $G_\delta\supset F_{\delta'}$, $\bar{O}_\delta\cap\bar{G}_\delta=0$ and $O_\delta, G_\delta\in\mathfrak{B}'$. Now we see easily that $\{O_\delta\}\subset\mathfrak{B}'$ and $|\{O_\delta\}|\geqq 2^{\mathfrak{m}}$ since if $\delta\neq\varepsilon$, then $O_\delta\neq O_\varepsilon$. For if $a\in F_\delta-F_\varepsilon$, then $a\in O_\delta$, $a\in G_\varepsilon$, hence $a\notin O_\varepsilon$, i.e. $O_\delta\neq O_\varepsilon$. But $|\mathfrak{B}'|\leqq\mathfrak{m}$, which is a contradiction.

Corollary. *For a metrizable space R, R is bicompact if and only if $Dt_{\aleph_0}(R)$ has the maximum.*

From Corollary and Theorem 3, we have,

Corollary. *Under the assumption of the generalised continuum hypothesis, if $\mathfrak{m}$ is regular and if for a fully normal space R, $Dt_{\mathfrak{m}}(R)$ has the maximum, then $D_{\mathfrak{m}}(R)$ has the maximum.*

§6. In the last paragraph, we shall consider cardinal numbers of $D(R)$ and its subsystems in the case when R is metrizable.

Theorem 5. *If R is not bicompact and metrizable, then $D_{\aleph_0}(R)$ has the potency $\geqq 2^{\aleph_0}$.*

Proof. Since R is not bicompact, there exists a countable set $\{a_n\}$ without cluster points, hence for any point a of R, there exists an open neighbourhood $U(a)$ of a such that $U(a)\cap\{a_n\}$ contains at most one point. Thus $\mathfrak{U}=\{U(a)\,|\,a\in R\}$ is an open normal covering, hence there exists an open covering $\mathfrak{B}$ such that $\mathfrak{B}\overset{\Delta}{<}\mathfrak{U}$ and then there exists a non totally bounded $\aleph_0$-structure mR whose basis for the uniformity is $\{\mathfrak{U}_n\}$ and $\mathfrak{U}_1=\mathfrak{B}$. Now decompose the set $A=\{a_n\}$ into disjoint countable sets $A_i=\{a_{in}\,|\,n=1,2,\ldots\}$ and let A_{in} be $\{a_{ij}\,|\,j=n, n+1,\ldots\}$. We shall define a $\aleph_0$-structure $m(\mathfrak{p})R$ for each sequence $\mathfrak{p}=\{p_n\}$ of two numbers 0 and 1. Let $I(\mathfrak{p})$ be the set of all p_i such that $p_i=1$ and let $\mathfrak{B}_n=\{S(A_{in},\mathfrak{U}_n)\,|\,p_i\in I(\mathfrak{p})\}\cup\{U\,|\,U\cap A_{in}=0,\ U\in\mathfrak{U}_n\ \&\ p_i\in I(\mathfrak{p})\}$. Then $\{\mathfrak{B}_n\}$ is a basis of some uniformity of some $\aleph_0$-structure which we shall denote by $m(\mathfrak{p})R$, (cf. the proof of Lemma 1). If $\mathfrak{p}=\{p_n\}$ and $\mathfrak{q}=\{q_n\}$ are different sequences, then $m(\mathfrak{p})R$ and $m(\mathfrak{q})R$ are different. For e.g. for an $i, p_i=1$ and $q_i=0$, then $\{a_{in}\,|\,i=1,2,\ldots\}$ are a Cauchy sequence of $m(\mathfrak{p})R$, but not of $m(\mathfrak{q})R$. Hence $D_{\aleph_0}(R)$ has the potency $\geqq 2^{\aleph_0}$.

Theorem 6. *If R is separable and not bicompact, then both $D(R)$*

and $D_t(R)$ have the potency $2^{2^{\aleph_0}}$, and both $D_{\aleph_0}(R)$ and $Dt_{\aleph_0}(R)$ have the potency $2^{\aleph_0}$.

Proof. First we shall prove that $D(R)$ had the potency $\leqq 2^{2^{\aleph_0}}$, i. e., that $D_t(R)$ has the potency $\leqq 2^{2^{\aleph_0}}$ and tnat $D_{\aleph_0}(R) \leqq 2^{\aleph_0}$. Each normal sequence of open coverings is associated with a pseudo-metric [16] and if for two normal sequences of coverings $\{\mathfrak{U}_n\}$ and $\{\mathfrak{V}_n\}$ there exists $\mathfrak{U}_n \in \{\mathfrak{U}_n\}$ such that for any m, $\mathfrak{V}_m$ is not a refinement of $\mathfrak{U}_n$, then the corresponding pseudo-metrics are different [16]. Since a pseudo -metric is a continuous function on the Cartesian product $R \times R$, the set of all pseudo-metrics has the potency $\leqq 2^{\aleph_0}$ and a structure of R is associated with some set of normal coverings respectively, hence $D(R)$ has the potency $\leqq 2^{2^{\aleph_0}}$ and $D_{\aleph_0}(R) \leqq 2^{\aleph_0}$.

Next we shall show that $D_t(R) \geqq 2^{2^{\aleph_0}}$. Let $\beta(R)$ be the Čech's bicompactification of R, then $\beta(R)-R$ has the potency $\geqq 2^{2^{\aleph_0}}$ [19] hence the set of all equivalent classes of Cauchy families of the maximal structure afR of $D_t(R)$ without limit point has the potency $\geqq 2^{2^{\aleph_0}}$. Now by Lemma 1, for two non equivalent Cauchy families $\{F_\alpha | A\}$ and $\{F_\alpha | A\}$ of afR without limit point, we can find a totally bounded structure fR such that they are equivalent Cauchy families of fR, and for two different pairs of not equivalent families we obtain two different totally bounded structures respectively, hence $D_t(R)$ has the potency $\geqq 2^{2^{\aleph_0}}$.

Finally, we shall show that $Dt_{\aleph_0}(R)$ has the potency $\geqq 2^{\aleph_0}$. Since R is not bicompact, there exists a countable set N without cluster point. Let N be subdivided into countable subsets $N^i_j (i, j=1, 2, \dots)$ such that $N^i_j = \{a^i_{j_n} | n=1, 2, \dots\}$ and let $N^i = \sum_{j=1}^{\infty} N^i_j$, then $N = \sum_{i=1}^{\infty} N^i$. Now, let $\mathfrak{p} = \{p_n\}$ be a sequence of positive integers. Corresponding to $\mathfrak{p}$, let the elemeuts of $\bigcup_{j=p_i+1}^{\infty} N^j$ be ordered as follows, $a^i_{p_i+1, 1} < a^i_{p_i+1, 2} < a^i_{p_i+2, 1} < a^i_{p_i+1, 3} < a^i_{p_i+2, 2} < a^i_{p_i+3, 1} \dots$. Let this sequence be $N^i_{p_i+1} = N^i_{p_i+1}(\mathfrak{p})$. Then let $\{N^i_j\}$ be arranged as follows:

$$(*) \qquad N^1_1, N^1_2, \dots N^1_{p_1}, N^1_{p_1+1}, N^2_1, \dots, N^2_{p_2}, N^2_{p_2+1}, N^3_1, \dots$$

where N^i_j is considered as a sequence ordered by the indices.

Then let P_i be the i-th term of (*) and let us put $P_i = \{a_{i,j} | j=$

16) Cf. T. p. 51.

17) E. Čech, On bicompact space, Ann. of Math 38 (1938). Dedřich Pospíšil, Remark on bicompact space, Ann. of Math. 38 (1938).

$1, 2, \ldots\}$. Moreover for each n, let N be subdivided into $a_{i,j}(i, j \leqslant n)$, $\{a_{i,j} | j=n+1, n+2 \ldots\}$ $(i \leqq n)$ and $N-\sum_{i=1}^{n} P_i$, and let $P_{i,n}=\{a_{i,n+j} | j=1, 2, \ldots\}$, $P_{n+1,n}=N-\sum_{i=1}^{n} P_i$. Then there exist open sets U_{ij} $(i, j \leqq n)$ and V_{in} $(i \leqq n+1)$ such that any two of them are disjoint and $U_{ij} \ni a_{ij}$, $V_{in} \supset P_{i,n}$. Let $\mathfrak{B}_n=\{U_{ij} | i, j \leqq n\} \cup \{V_{in} | i \leqq n+1\} \cup \{R-N\}$, then $\mathfrak{B}_n$ is an open covering, hence there exists a normal sequence: $\mathfrak{B}_n=\mathfrak{B}_n^1 \overset{*}{>} \mathfrak{B}_n^2 \overset{*}{>} \ldots$ where every $\mathfrak{B}_n^i$ are finite. Now, let mR be an arbitrary but fixed element of $Dt_{\aleph_0}(R)$, whose uniformity cnotained a normal sequence $\{\mathfrak{U}_i | i=1, 2, \ldots\}$ as a basis and let $\mathfrak{A}_n=\mathfrak{B}_n \wedge \mathfrak{U}_1^n \wedge \ldots \wedge \mathfrak{B}_n^n$, then $\{\mathfrak{A}_n\}$ is a normal sequence and the basis of the uniformity of some totally bounded $\aleph_0$-structure. Let us set

$$\mathfrak{A}_n'=\{S(P_{i,n}, \mathfrak{A}_n) | i \leqq n+1\} \cup \{U | U \in \mathfrak{A}_n \ \& \ \text{for } i \leqq n+1 \ \ U \frown P_{i,n}=0\}.$$

then $\{\mathfrak{A}_n'\}$ is a normal sequence and a basis of the uniformity of some totally bounded $\aleph_0$-structure, which we denote by $m(\mathfrak{p})R$.

First, we shall show that $\{\mathfrak{A}_n'\}$ is a normal sequence. If $a \notin S(P_{i,n+1}, \mathfrak{A}_{n+1})$ for all $i \leqq n+2$ then there exists $U \in \mathfrak{A}_n$ such that $U \supset S(a, \mathfrak{A}_{n+1})=S(a, \mathfrak{A}_{n+1}')$, and $\mathfrak{A}_n' > \mathfrak{A}_n$, hence there exists $V \in \mathfrak{A}_n'$ such that $V \supset U \supset S(a, \mathfrak{A}_{n+1}')$. If $a \in S(P_{i,n+1}, \mathfrak{A}_{n+1})$ for some i, then $a \notin S(P_{j,n+1}, \mathfrak{A}_{n+1})$ for $j \neq i$, since $\mathfrak{B}_{n+1} > \mathfrak{B}_{n+1}^{n+1} > \mathfrak{A}_{n+1}$ and $\mathfrak{B}_{n+1}$ cannot contain an element which intersects with two sets $P_{i,n+1}, P_{j,n+1}$. Now $S(a, \mathfrak{A}_{n+1}')= S(a, \mathfrak{A}_{n+1}) \cup S(P_{i,n+1}, \mathfrak{A}_{n+1})$ and $P_{i,n+1} \subset P_{i,n}$ or $P_{i,n+1} \subset P_{n+1,n}$ hence $\mathfrak{A}_n' \ni S(P_{i,n}, \mathfrak{A}_n) \supset S(a, \mathfrak{A}_{n+1}')$ or $\mathfrak{A}_n' \ni S(P_{n+1,n}, \mathfrak{A}_n) \supset S(a, \mathfrak{A}_{n+1}')$.

Next, it is evident by the same way as in Lemma 1 that $m(\mathfrak{p})R$ agrees with the topology since $\prod_{n} \sum_{i \leqq n+1} P_{in}=0$.

Finally if $\mathfrak{q}=\{p_n\} \neq \mathfrak{q}=\{q_n\}$, then $m(\mathfrak{p})R \neq m(\mathfrak{q})R$. In fact if for some i $p_i \neq q_i$ and if $p_i > q_i$, then $N_{q_{i+1}}^i(\mathfrak{q})$ is a Cauchy sequence of $m(\mathfrak{q})R$, but $N_{q_{i+1}}^i(\mathfrak{q})=N_{q_i+1}^i \cup \ldots \cup N_{p_i}^i \cup N_{p_{i+1}}^i$, $N_{q_{i+1}}^i, \ldots, N_{p_i}^i, N_{p_{i+1}}^i$ are respectively Cauchy sequences of $m(\mathfrak{p})R$ and two of them are not equivalent, hence $N_{q_{i+1}}^i(\mathfrak{q})$ is not a Cauchy sequence of $m(\mathfrak{p})R$, hence $m(\mathfrak{p})R \neq m(\mathfrak{q})R$.

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