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A REMARK ON THE LAPLACE-BELTRAMI OPERATORS ATTACHED TO HERMITIAN SYMMETRIC PAIRS

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In this note, we shall give an alternative proof of the theorem due to Okamoto and Ozeki, which says that the Laplace-Beltrami operator attached to a hermitian homogeneous vector bundle over a hermitian symmetric space is a multiple of the Casimir operator plus a constant (Theorem 4.1 [2]). It plays an important role in the works [1], [2].

1. Let (G, K) be a hermitian symmetric pair of non-compact type, i.e., G is a connected non-compact semi-simple Lie group with a finite center and K its maximal compact subgroup. Denoting by \mathfrak{g} , \mathfrak{k} the Lie algebras of G , K , we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Its complexification will be denoted by $\mathfrak{g}^C = \mathfrak{k}^C \oplus \mathfrak{p}^C$, and we have a hermitian inner product $(x, y) = -B(x, \tau y)$ for x, y in \mathfrak{g}^C , where B is the Killing form of \mathfrak{g}^C , τ the conjugation with respect to the compact real form dual to \mathfrak{g} . We may take a Cartan subalgebra \mathfrak{h} of \mathfrak{g} in \mathfrak{k} , and have the root space decomposition $\mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ where Δ denotes the root system for the pair $(\mathfrak{g}^C, \mathfrak{h}^C)$. Let $\Delta_{\mathfrak{k}}$ (resp. $\Delta_{\mathfrak{p}}$) be the set of the compact (resp. non-compact) roots. Then there exists a positive root system Δ^+ such that the subspace

$$\mathfrak{p}_{\pm} = \sum_{\beta \in \Delta_{\mathfrak{p}}^+} \mathfrak{g}^{\pm\beta}$$

is, respectively, a K -submodule of \mathfrak{p}^C , where $\Delta_{\mathfrak{p}}^+ = \Delta^+ \cap \Delta_{\mathfrak{p}}$. Hereafter, we shall fix the above linear order on Δ .

This linear order determines an invariant complex structure on $X = G/K$ such as \mathfrak{p}_- can be identified with the anti-holomorphic tangent space at the origin $o = eK$ in X . Identifying the dual of \mathfrak{p}_- with \mathfrak{p}_+ via the Killing form B , we may

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consider $G \times_{K\mathfrak{p}_+}$ as the anti-holomorphic cotangent bundle, where $G \times_{K\mathfrak{p}_+}$ denotes the homogeneous vector bundle over X associated to the K -module \mathfrak{p}_+ . To an irreducible unitary K -module V_Λ with highest weight Λ with respect to the above linear order, the holomorphic vector bundle $E_\Lambda = G \times_K V_\Lambda$ is associated, which has a hermitian metric on each fibre. The space of the differential forms of type $(0, q)$ with coefficients in E_Λ may then be considered as the space of all C^∞ -sections of the homogeneous vector bundle over X associated to the K -module $V_\Lambda \otimes \Lambda^q \mathfrak{p}_+$, which will be denoted by $C^{0,q}(E_\Lambda)$. We shall also identify $C^{0,q}(E_\Lambda)$ with the space of $V_\Lambda \otimes \Lambda^q \mathfrak{p}_+$ -valued C^∞ -functions s on G such that $s(gk) = k^{-1}s(g)$ for $g \in G, k \in K$. The Cauchy-Riemann operator

$$\bar{\partial} : C^{0,q}(E_\Lambda) \rightarrow C^{0,q+1}(E_\Lambda)$$

is then expressed as follows. When we choose a basis of \mathfrak{p}^c such that

$$B(e_\beta, e_{-\beta}) = 1, e_\beta \in \mathfrak{g}^\beta \quad (1)$$

for each non-compact root $\beta \in \Delta_{\mathfrak{p}}$, we then have

$$\bar{\partial} = \sum_{\beta \in \Delta_{\mathfrak{p}}^+} \nu(e_{-\beta}) \otimes \varepsilon(e_\beta) \quad (2),$$

where $\nu(e_{-\beta})$ denotes the action of $e_{-\beta}$ as a left invariant vector field and $\varepsilon(e_\beta)$ the exterior multiplication of e_β on $\Lambda \mathfrak{p}_+$. In the choice above, we shall note that the Casimir operator has a form of

$$\Omega = \Omega_{\mathfrak{k}} + \sum_{\beta \in \Delta_{\mathfrak{p}}} e_{-\beta} e_\beta \quad (3),$$

where $\Omega_{\mathfrak{k}}$ denotes the part consisting of the basis of \mathfrak{k}^c . Introducing, as usual, an invariant kählerian metric on X via the hermitian metric $(\ , \)$ on \mathfrak{g}^c , we have the formal adjoint ϑ of $\bar{\partial}$ and the Laplace-Beltrami operator

$$\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}.$$

Theorem (Okamoto-Ozeki [2]). *Under the above situation, \square acts on each $C^{0,q}(E_\Lambda)$ as*

$$\square = \frac{1}{2} \{(\Lambda + 2\rho, \Lambda) \mathbf{1} - \nu(\Omega)\},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, $\mathbf{1}$ denotes the identity operator, and $(\Lambda + 2\rho, \Lambda)$ the inner product on the weights induced from the Killing form.

2. A proof of the above Theorem is as follows. We shall first see that $\square + \frac{1}{2} \nu(\Omega)$ is at most a first order operator on $C^{0,q}(E_\Lambda)$. Because of the invariance

of the differential operators, it suffices to see that it holds for any real cotangent vector ξ at the origin $o \in X$

$$\sigma_{\xi}^2(\square) = -\frac{1}{2} \sigma_{\xi}^2(\nu(\Omega)),$$

where $\sigma_{\xi}^2(\cdot)$ denotes the symbol map at the origin with respect to ξ . When we consider \mathfrak{p} as the real cotangent space at $o \in X$, ξ as the element of $\mathfrak{p} \subset \mathfrak{p}^c$, we then have

$$\sigma_{\xi}^1(\bar{\partial})v = \varepsilon(\xi_+)v \quad \text{for } v \in V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+,$$

where $\sigma(\cdot)$ denotes the symbol map at the origin and ξ_+ is the image of ξ by the orthogonal projection $\mathfrak{p}^c \rightarrow \mathfrak{p}_+$. In fact, take a real valued function f on X and a section $s \in C^{0,q}(E_{\Lambda})$ such that $f(o) = 0$, $(df)(o) = \xi$, $s(e) = v$, where e is the unit of G . Then we have $\sigma_{\xi}^1(\bar{\partial})v = (\bar{\partial}(fs))(e) = \varepsilon(\bar{\partial}f(o))v$, and through the above identification, $\bar{\partial}f(o) = \xi_+$. Since the symbol of the formal adjoint ϑ is

$$\sigma_{\xi}^1(\vartheta) = -\varepsilon(\xi_+)^*$$

where $\varepsilon(\xi_+)^*$ denotes the adjoint of $\varepsilon(\xi_+)$ with respect to the hermitian inner product on $V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+$, we have

$$\begin{aligned} \sigma_{\xi}^2(\square) &= \sigma_{\xi}^1(\bar{\partial}) \sigma_{\xi}^1(\vartheta) + \sigma_{\xi}^1(\vartheta) \sigma_{\xi}^1(\bar{\partial}) \\ &= -(\varepsilon(\xi_+) \varepsilon(\xi_+)^* + \varepsilon(\xi_+)^* \varepsilon(\xi_+)) \\ &= -(\xi_+, \xi_+) \mathbf{1} \\ &= -\frac{1}{2}(\xi, \xi) \mathbf{1}. \end{aligned}$$

On the other hand, we see easily that $\sigma_{\xi}^2(\nu(\Omega)) = (\xi, \xi) \mathbf{1}$, which implies the assertion.

3. We shall next see that every invariant first order operators on $C^{0,q}(E_{\Lambda})$ is, in effect, of order zero, i.e., a vector bundle map induced from some K -module endomorphism on $V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+$. Consider the symbol map $\sigma_{\xi}^1(D)$ of an invariant first order operator D as a bilinear map

$$\sigma_{\xi}^1(D): (V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+) \times \mathfrak{p} \rightarrow V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+,$$

and extend it to \mathfrak{p}^c on the part \mathfrak{p} complex-linearly. We then have a K -module homomorphism

$$\sigma_{\xi}^1(D): (V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+) \otimes \mathfrak{p}^c \rightarrow V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+.$$

It suffices to see $\sigma_{\xi}^1(D) = 0$. In fact, the highest weight of an irreducible component in $(V_{\Lambda} \otimes \Lambda^q \mathfrak{p}_+) \otimes \mathfrak{p}^c$ is of a form

$$\Lambda + \beta_1 + \cdots + \beta_q \pm \beta,$$

while that of one in $V_\Lambda \otimes \Lambda^q \mathfrak{p}_+$ is

$$\Lambda + \beta'_1 + \cdots + \beta'_q,$$

for some β_i, β'_i, β in $\Delta_\mathfrak{p}^+$. Therefore, if $\sigma^!(D) \neq 0$, then there must exist positive non-compact roots $\beta_1, \dots, \beta_q, \beta'_1, \dots, \beta'_q, \beta$ such that

$$\beta_1 + \cdots + \beta_q \pm \beta = \beta'_1 + \cdots + \beta'_q.$$

On the other hand, it is known that, when \mathfrak{g} is simple, there exists a simple root $\alpha_0 \in \Delta_\mathfrak{p}^+$ such as the following holds; expressing a positive non-compact root as a linear combination of the simple roots in Δ^+ , the coefficient of α_0 has to be 1. Hence, under our assumptions, the above equality is impossible, which implies the assertion.

4. Put $A_\Lambda^q = \square + \frac{1}{2} \nu(\Omega)$ on $C^{0,q}(E_\Lambda)$. Then we know through 2, 3 that A_Λ^q is induced from a K -module endomorphism of $V_\Lambda \otimes \Lambda^q \mathfrak{p}_+$, and it is easy to see

$$\bar{\partial} A_\Lambda^q = A_\Lambda^{q+1} \bar{\partial}$$

from the property of the Casimir operator. If we assume that A_Λ^q is a scalar operator $c_\Lambda^q \mathbf{1}$, then A_Λ^{q+1} is also $c_\Lambda^q \mathbf{1}$. In fact, for any $v \in V_\Lambda \otimes \Lambda^{q+1} \mathfrak{p}_+$ one can choose a section s in $C^{0,q}(E_\Lambda)$ such that $(\bar{\partial}s)(e) = v$. Since A_Λ^{q+1} is of order zero, $A_\Lambda^{q+1}(\bar{\partial}s)(e) = (A_\Lambda^{q+1} \bar{\partial}s)(e)$, where A_Λ^{q+1} denotes the K -module endomorphism of $V_\Lambda \otimes \Lambda^{q+1} \mathfrak{p}_+$ inducing A_Λ^{q+1} . Hence $A_\Lambda^{q+1} v = (\bar{\partial} A_\Lambda^q s)(e) = c_\Lambda^q (\bar{\partial}s)(e) = c_\Lambda^q v$, which shows $A_\Lambda^{q+1} = c_\Lambda^q \mathbf{1}$. On the other hand, $A_\Lambda^0 = c_\Lambda \mathbf{1}$ for some constant c_Λ , because V_Λ is irreducible. Thus we have

$$A_\Lambda^q = c_\Lambda \mathbf{1} \quad \text{for every } q.$$

It remains to determine the above constant c_Λ . For this purpose, it suffices to see the action of A_Λ^0 on $C^{0,q}(E_\Lambda)$. For a highest weight vector $v_\Lambda \in V_\Lambda$, take a local holomorphic section s near the origin $o \in X$ such that $s(e) = v_\Lambda$. We then have

$$A_\Lambda^0 s = \frac{1}{2} \nu(\Omega) s.$$

By the formula (3), we have

$$\nu(\Omega) s = \nu(\Omega_k) s + \sum_{\beta \in \Delta_\mathfrak{p}^+} (2\nu(e_\beta) \nu(e_{-\beta}) s - \nu([e_\beta, e_{-\beta}]) s).$$

It is well known that $\nu(\Omega_k) s = (\Lambda + 2\rho_k, \Lambda) s$ where ρ_k is a half of the sum of the

positive compact roots. On the other hand, $\nu(e_{-\beta})s=0$ for $\beta \in \Delta_n$ in view of (2), and $(\nu([e_\beta, e_{-\beta}]s))(e) = -(\beta, \Lambda)v_\Lambda$ because of $[e_\beta, e_{-\beta}] \in \mathfrak{h}^C$ and the choice (1). Therefore it holds

$$(\nu(\Omega)s)(e) = (\Lambda + 2\rho, \Lambda)s(e).$$

Thus we have $c_\Lambda = \frac{1}{2}(\Lambda + 2\rho, \Lambda)$, which completes a proof of the Theorem.

REMARK. This way of determination of the Laplace-Beltrami operator can be also applied for a hermitian symmetric pair of compact type, and we have a quite similar formula except for a switch of sign, which is due to B. Kostant.

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