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A REMARK ON THE LAPLACE-BELTRAMI OPERATORS ATTACHED TO HERMITIAN SYMMETRIC PAIRS

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In this note, we shall give an alternative proof of the theorem due to Okamoto and Ozeki, which says that the Laplace-Beltrami operator attached to a hermitian homogeneous vector bundle over a hermitian symmetric space is a multiple of the Casimir operator plus a constant (Theorem 4.1 [2]). It plays an important role in the works [1], [2].

1. Let (G, K) be a hermitian symmetric pair of non-compact type, i.e., G is a connected non-compact semi-simple Lie group with a finite center and K its maximal compact subgroup. Denoting by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of G, K, we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} .$$

Its complexification will be denoted by $\mathfrak{g}^C = \mathfrak{k}^C \oplus \mathfrak{p}^C$, and we have a hermitian inner product $(x, y) = -B(x, \tau y)$ for x, y in $\mathfrak{g}^C$, where B is the Killing form of $\mathfrak{g}^C$, τ the conjugation with respect to the compact real form dual to $\mathfrak{g}$. We may take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{k}$, and have the root space decomposition $\mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ where Δ denotes the root system for the pair $(\mathfrak{g}^C, \mathfrak{h}^C)$. Let $\Delta_{\mathfrak{k}}$ (resp. $\Delta_{\mathfrak{p}}$) be the set of the compact (resp. non-compact) roots. Then there exists a positive root system Δ^+ such that the subspace

$$\mathfrak{p}_\pm = \sum_{\beta\in\Delta_{\mathfrak{p}}^+} \mathfrak{g}^{\pm\beta}$$

is, respectively, a K-submodule of $\mathfrak{p}^C$, where $\Delta_{\mathfrak{p}}^+ = \Delta^+ \cap \Delta_{\mathfrak{p}}$. Hereafter, we shall fix the above linear order on Δ.

This linear order determines an invariant complex structure on $X = G/K$ such as $\mathfrak{p}_-$ can be identified with the anti-holomorphic tangent space at the origin $o = eK$ in X. Identifying the dual of $\mathfrak{p}_-$ with $\mathfrak{p}_+$ via the Killing form B, we may

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consider $G \times {}_K\mathfrak{p}_+$ as the anti-holomorphic cotangent bundle, where $G \times {}_K\mathfrak{p}_+$ denotes the homogeneous vector bundle over X associated to the K-module $\mathfrak{p}_+$. To an irreducible unitary K-module V_Λ with highest weight Λ with respect to the above linear order, the holomorphic vecter bundle $E_\Lambda = G \times {}_K V_\Lambda$ is associated, which has a hermitian metric on each fibre. The space of the differential forms of type $(0, q)$ with coefficients in E_Λ may then be considered as the space of all C^∞-sections of the homogeneous vector bundle over X associated to the K-module $V_\Lambda \otimes \Lambda^q\mathfrak{p}_+$, which will be denoted by $C^{0,q}(E_\Lambda)$. We shall aslo identify $C^{0,q}(E_\Lambda)$ with the space of $V_\Lambda \otimes \Lambda^q\mathfrak{p}_+$-valued C^∞-functions s on G such that $s(gk) = k^{-1}s(g)$ for $g \in G$, $k \in K$. The Cauchy-Riemann operator

$$\bar{\partial} : C^{0,q}(E_\Lambda) \to C^{0,q+1}(E_\Lambda)$$

is then expressed as follows. When we choose a basis of $\mathfrak{p}^C$ such that

$$B(e_\beta, e_{-\beta}) = 1, \ e_\beta \in \mathfrak{g}^\beta \qquad (1)$$

for each non-compact root $\beta \in \Delta_\mathfrak{p}$, we then have

$$\bar{\partial} = \sum_{\beta \in \Delta_\mathfrak{p}^+} \nu(e_{-\beta}) \otimes \varepsilon(e_\beta) \qquad (2),$$

where $\nu(e_{-\beta})$ denotes the action of $e_{-\beta}$ as a left invariant vector field and $\varepsilon(e_\beta)$ the exterior multiplication of e_β on $\Lambda\mathfrak{p}_+$. In the choice above, we shall note that the Casimir operator has a form of

$$\Omega = \Omega_k + \sum_{\beta \in \Delta_\mathfrak{p}} e_{-\beta}\, e_\beta \qquad (3),$$

where Ω_k denotes the part consisting of the basis of $\mathfrak{k}^C$. Introducing, as usual, an invariant kählerian metric on X via the hermitian metric $(\ ,\ )$ on $\mathfrak{g}^C$, we have the formal adjoint ϑ of $\bar{\partial}$ and the Laplace-Beltrami operator

$$\Box = \bar{\partial}\vartheta + \vartheta\bar{\partial}.$$

Theorem (Okamoto-Ozeki [2]). *Under the above situation, $\Box$ acts on each $C^{0,q}(E_\Lambda)$ as*

$$\Box = \frac{1}{2}\{(\Lambda + 2\rho, \Lambda)\,\mathbf{1} - \nu(\Omega)\},$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$, $\mathbf{1}$ denotes the identity operator, and $(\Lambda + 2\rho, \Lambda)$ the inner product on the weights induced from the Killing form.

2. A proof of the above Theorem is as follows. We shall first see that $\Box + \frac{1}{2}\nu(\Omega)$ is at most a first order operator on $C^{0,q}(E_\Lambda)$. Because of the invariance of the differential operators, it suffices to see that it holds for any real cotangent vector ξ at the origin $o\in X$

$$\sigma_\xi^2(\square) = -\frac{1}{2}\sigma_\xi^2(\nu(\Omega)),$$

where $\sigma_\xi^2(\ .\ )$ denotes the symbol map at the origin with respect to ξ. When we consider $\mathfrak{p}$ as the real cotangent space at $o\in X$, ξ as the element of $\mathfrak{p}\subset\mathfrak{p}^C$, we then have

$$\sigma_\xi^1(\bar\partial)v = \varepsilon(\xi_+)v \qquad \text{for } v\in V_\Lambda\otimes\Lambda^q\mathfrak{p}_+,$$

where $\sigma(\ .\ )$ denotes the symbol map at the origin and ξ_+ is the image of ξ by the orthogonal projection $\mathfrak{p}^C\to\mathfrak{p}_+$. In fact, take a real valued function f on X and a section $s\in C^{0,q}(E_\Lambda)$ such that $f(o)=0$, $(df)(o)=\xi$, $s(e)=v$, where e is the unit of G. Then we have $\sigma_\xi^1(\bar\partial)v=(\bar\partial(fs))(e)=\varepsilon(\bar\partial f(o))v$, and through the above identification, $\bar\partial f(o)=\xi_+$. Since the symbol of the formal adjoint ϑ is

$$\sigma_\xi^1(\vartheta) = -\varepsilon(\xi_+)^*$$

where $\varepsilon(\xi_+)^*$ denotes the adjoint of $\varepsilon(\xi_+)$ with respect to the hermitian inner product on $V_\Lambda\otimes\Lambda^q\mathfrak{p}_+$, we have

$$\begin{aligned}\sigma_\xi^2(\square) &= \sigma_\xi^1(\bar\partial)\,\sigma_\xi^1(\vartheta)+\sigma_\xi^1(\vartheta)\,\sigma_\xi^1(\bar\partial)\\ &= -\,(\varepsilon(\xi_+)\,\varepsilon(\xi_+)^*+\varepsilon(\xi_+)^*\varepsilon(\xi_+))\\ &= -\,(\xi_+,\xi_+)\,\mathbf{1}\\ &= -\frac{1}{2}(\xi,\xi)\,\mathbf{1}.\end{aligned}$$

On the other hand, we see easily that $\sigma_\xi^2(\nu(\Omega))=(\xi,\xi)\mathbf{1}$, which implies the assertion.

3. We shall next see that every invariant first order operators on $C^{0,q}(E_\Lambda)$ is, in effect, of order zero, i.e., a vector bundle map induced from some K–module endomorphism on $V_\Lambda\otimes\Lambda^q\mathfrak{p}_+$. Consider the symbol map $\sigma_\xi^1(D)$ of an invariant first order operator D as a bilinear map

$$\sigma_\cdot^1(D)\colon (V_\Lambda\otimes\Lambda^q\mathfrak{p}_+)\times\mathfrak{p}\to V_\Lambda\otimes\Lambda^q\mathfrak{p}_+,$$

and extend it to $\mathfrak{p}^C$ on the part $\mathfrak{p}$ complex-linearly. We then have a K–module homomorphism

$$\sigma_\cdot^1(D) : (V_\Lambda\otimes\Lambda^q\mathfrak{p}_+)\otimes\mathfrak{p}^C\to V_\Lambda\otimes\Lambda^q\mathfrak{p}_+.$$

It suffices to see $\sigma_\cdot^1(D)=0$. In fact, the highest weight of an irreducible component in $(V_\Lambda\otimes\Lambda^q\mathfrak{p}_+)\otimes\mathfrak{p}^C$ is of a form

$$\Lambda+\beta_1+\cdots+\beta_q\pm\beta,$$

while that of one in $V_\Lambda\otimes\Lambda^q\mathfrak{p}_+$ is

$$\Lambda+\beta_1'+\cdots+\beta_q',$$

for some β_i, β_i', β in $\Delta_\mathfrak{p}^+$. Therefore, if $\sigma_\cdot^1(D)\neq 0$, then there must exist positive non-compact roots $\beta_1,\cdots,\beta_q,\beta_1',\cdots,\beta_q',\beta$ such that

$$\beta_1+\cdots+\beta_q\pm\beta=\beta_1'+\cdots+\beta_q'.$$

On the other hand, it is known that, when $\mathfrak{g}$ is simple, there exists a simple root $\alpha_0\in\Delta_\mathfrak{p}^+$ such as the following holds; expressing a positive non-compact root as a linear combination of the simple roots in Δ^+, the coefficient of α_0 has to be 1. Hence, under our assumptions, the above equality is impossible, which implies the assertion.

4. Put $A_\Lambda^q=\square+\frac{1}{2}\nu(\Omega)$ on $C^{0,q}(E_\Lambda)$. Then we know through **2**, **3** that A_Λ^q is induced from a K-module endomorphism of $V_\Lambda\otimes\Lambda^q\mathfrak{p}_+$, and it is easy to see

$$\bar\partial A_\Lambda^q=A_\Lambda^{q+1}\bar\partial$$

from the property of the Casimir operator. If we assume that A_Λ^q is a scaler operator $c_\Lambda^q\,\mathbf{1}$, then A_Λ^{q+1} is also $c_\Lambda^q\,\mathbf{1}$. In fact, for any $v\in V_\Lambda\otimes\Lambda^{q+1}\mathfrak{p}_+$ one can choose a section s in $C^{0,q}(E_\Lambda)$ such that $(\bar\partial s)\,(e)=v$. Since A_Λ^{q+1} is of order zero, $\underline{A}_\Lambda^{q+1}(\bar\partial s)(e)=(A_\Lambda^{q+1}\bar\partial s)(e)$, where $\underline{A}_\Lambda^{q+1}$ denotes the K-module endomorphism of $V_\Lambda\otimes\Lambda^{q+1}\mathfrak{p}_+$ inducing A_Λ^{q+1}. Hence $\underline{A}_\Lambda^{q+1}v=(\bar\partial A_\Lambda^q s)(e)=c_\Lambda^q(\bar\partial s)(e)=c_\Lambda^q v$, which shows $A_\Lambda^{q+1}=c_\Lambda^q\,\mathbf{1}$. On the other hand, $A_\Lambda^0=c_\Lambda\,\mathbf{1}$ for some constant c_Λ, because V_Λ is irreducible. Thus we have

$$A_\Lambda^q=c_\Lambda\,\mathbf{1}\qquad\text{for every } q.$$

It remains to determine the above constant c_Λ. For this purpose, it suffices to see the action of A_Λ^0 on $C^{0,q}(E_\Lambda)$. For a highest weight vector $v_\Lambda\in V_\Lambda$, take a local holomorphic section s near the origin $o\in X$ such that $s(e)=v_\Lambda$. We then have

$$A_\Lambda^0\,s=\frac{1}{2}\,\nu(\Omega)s.$$

By the formula (3), we have

$$\nu(\Omega)s=\nu(\Omega_k)s+\sum_{\beta\in\Delta_\mathfrak{p}^+}(2\nu(e_\beta)\nu(e_{-\beta})s-\nu([e_\beta,e_{-\beta}])s).$$

It is well known that $\nu(\Omega_k)\,s=(\Lambda+2\rho_k,\Lambda)s$ where ρ_k is a half of the sum of the positive compact roots. On the other hand, $\nu(e_{-\beta})s=0$ for $\beta\in\Delta_n$ in view of (2), and $(\nu([e_\beta, e_{-\beta}]s)(e)=-(\beta, \Lambda)v_\Lambda$ because of $[e_\beta, e_{-\beta}]\in\mathfrak{h}^C$ and the choice (1). Therefore it holds

$$(\nu(\Omega)s)(e) = (\Lambda+2\rho, \Lambda)s(e).$$

Thus we have $c_\Lambda=\frac{1}{2}(\Lambda+2\rho, \Lambda)$, which completes a proof of the Theorem.

REMARK. This way of determination of the Laplace-Beltrami operator can be also applied for a hermitian symmetric pair of compact type, and we have a quite similar formula except for a switch of sign, which is due to B. Kostant.

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