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<td><strong>Author(s)</strong></td>
<td>Imayoshi, Yōichi</td>
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Osaka University
UNIVERSAL COVERING SPACES OF CERTAIN QUASI-PROJECTIVE ALGEBRAIC SURFACES

YOICHI IMAYOSHI

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Introduction. In this paper we investigate some function-theoretic properties of universal covering spaces of certain quasi-projective algebraic surfaces.

Let \( \hat{X} \) be a two-dimensional complex manifold and let \( C \) be a one-dimensional analytic subset of \( \hat{X} \) or an empty set. Let \( R \) be a Riemann surface. We assume that a proper holomorphic mapping \( \pi: \hat{X} \rightarrow R \) satisfies the following two conditions: (i) \( \pi \) is of maximal rank at every point of \( \hat{X} \), and (ii) by setting \( X=\hat{X}-C \) and \( \pi=\pi|X \), the fiber \( S_p=\pi^{-1}(p) \) over each point \( p \) of \( R \) is an non-singular irreducible analytic subset of \( X \) and is of fixed finite type \((g, n)\) with \( 2g-2+n>0 \) as a Riemann surface, where \( g \) is the genus of \( S_p \) and \( n \) is the number of punctures of \( S_p \). We call such a triple \((X, \pi, R)\) a holomorphic family of Riemann surfaces of type \((g, n)\) over \( R \). We also say that \( X \) has a holomorphic fibration \((X, \pi, R)\) of type \((g, n)\).

We assume throughout this paper \( R \) is a non-compact Riemann surface of finite type and its universal covering space is the unit disc \( \mathbb{D}=\{|t|<1\} \) in the complex \( t \)-plane.

P.A. Griffiths [2] got the following uniformization theorem of quasi-projective algebraic surfaces. Let \( \hat{X} \) be a two-dimensional, irreducible, smooth, quasi-projective algebraic varietie over the complex numbers. Then for every point \( x \) in \( \hat{X} \), there exists a Zariski neighborhood \( X \) of \( x \) in \( \hat{X} \) such that \( X \) has a holomorphic fibration \((X, \pi, R)\) as above. Then the universal covering space \( \tilde{X} \) of \( X \) is topologically a cell. Griffiths proved that \( \tilde{X} \) is biholomorphically equivalent to a bounded domain of holomorphy in \( \mathbb{C}^2 \) using the theory of simultaneous uniformization of Riemann surfaces due to Bers. (cf. Bers [1].) The function-theoretic properties of such interesting domains \( \tilde{X} \) are little studied. (cf. Shabat [10].)

At the begining, in §1, we recall some notations and results of [3], [4] and [5] which will be used later. Let \( \mathcal{M} \) be the homotopic monodromy group of \((X, \pi, R)\), which will be defined in §1. Then we get the following theorems in §2, §3, §4 and §5.
Theorem 1. The universal covering space \( \hat{X} \) of \( X \) is not biholomorphically equivalent to the two-dimensional unit ball \( B_2 = \{ |z|^2 + |w|^2 < 1 \} \).

Corollary. The universal covering space \( \hat{X} \) of \( X \) is not biholomorphically equivalent to any two-dimensional strongly pseudoconvex domains.

Theorem 2. The homotopic monodromy group \( \mathcal{M} \) is a finite group if and only if all the fibers \( S_p \) are conformally equivalent.

Theorem 3. The homotopic monodromy group \( \mathcal{M} \) is a finite group if and only if \( \hat{X} \) is biholomorphically equivalent to the two-dimensional polydisc \( D^2 = (|z| < 1) \times (|w| < 1) \).

Theorem 4. If \( (X, \pi, R) \) is of type \((g, 0)\) with \( g > 1 \), then \( \hat{X} \) is biholomorphic to the polydisc \( D^2 \) if and only if the analytic automorphism group \( \text{Aut}(\hat{X}) \) of \( \hat{X} \) is not a discrete group.

In the last § 6, we give some examples of these quasi-projective algebraic surfaces \( X \) and some related problems.

1. Preliminaries. We shall briefly explain some notations and results in [3], [4] and [5] which will be used later.

Let \( G \) be a finitely generated Fuchsian group of the first kind with no elliptic elements acting on the upper half-plane \( U \) such that the quotient space \( S = U/G \) is a finite Riemann surface of type \((g, n)\). Let \( Q_{\text{norm}}(G) \) be the set of all quasi-conformal automorphisms \( w \) of \( U \) leaving \( 0, 1, \infty \) fixed and satisfying \( wGw^{-1} \subset SL'(2; R) \), where \( SL'(2; R) \) is the set of all real M"obius transformations. Two elements \( w_1 \) and \( w_2 \) of \( Q_{\text{norm}}(G) \) are equivalent if \( w_1 = w_2 \) on the real axis \( \mathbb{R} \). The Teichmüller space \( T(G) \) of \( G \) is the set of all equivalence classes \([w]\) obtained by classifying \( Q_{\text{norm}}(G) \) by the above equivalence relation.

Let \( w_\mu \) be the element of \( Q_{\text{norm}}(G) \) with a Beltrami coefficient \( \mu \in L^\infty(U, G) \), and let \( W^\mu \) be a quasiconformal automorphism of the Riemann sphere \( \hat{C} \) such that \( W^\mu \) has the Beltrami coefficient \( \mu \) on the upper half-plane \( U \), and is conformal on the lower half-plane \( L \), and

\[
W^\mu(z) = \frac{1}{z + i} + O(|z + i|)
\]

as \( z \) tends to \(-i\). This mapping \( W^\mu \) is uniquely determined by \([w_\mu]\) up to the equivalence relation, that is, \( w_\mu = w_\nu \) on \( \mathbb{R} \) if and only if \( W^\mu = W^\nu \) on \( L \). Let \( \phi_\mu \) be the Schwarzian derivative of \( W^\mu \). Then \( \phi_\mu \) is an element of the space \( B_2(L, G) \) of bounded holomorphic quadratic differentials for \( G \) on \( L \). Bers proved that the mapping sending \([w_\mu]\) into \( \phi_\mu \) is a biholomorphic mapping of \( T(G) \) onto a holomorphically convex bounded domain of \( B_2(L, G) \), which
is denoted by the same notation $T(G)$. The space $B_2(L, G)$ is a $(3g-3+n)$-dimensional complex vector space. We associate with each $\phi$ of $B_2(L, G)$ a uniquely determined solution $W_\phi = \frac{w_1}{w_2}$ of the Schwarzian differential equation on $L$

$$(w''/w')' - \frac{1}{2} (w''/w')^2 = \phi,$$

where $w_1$ and $w_2$ are the solutions of the linear differential equation on $L$

$$2w'' + \phi w = 0$$

normalized by the conditions $w_1 = w_2 = 1$ and $w_1' = w_2' = 0$ at $z = -i$. The homomorphism $G \to SL(2, \mathbb{C})$ induced by $\phi$, which carries $g$ into $\hat{g}$ in such a way that $W_\phi \circ g = \hat{g} \circ W_\phi$, is denoted by $X_\phi$. Since each point $\phi$ of $T(G)$ is a Schwarzian derivative of some $W^\mu$ with $\mu \in L^\infty(U, G)$, we have $W_\phi = W^\mu$ on $L$. Hence $W_\phi$ is conformal on $L$ and has a quasiconformal extension of $\mathbb{C}$ onto itself, which is denoted by the same notation. If we set $G_\phi = X_\phi(G) = W_\phi \circ G \circ W_\phi^{-1}$ and $D_\phi = W_\phi(U)$, then $G_\phi$ is a quasi-Fuchsian group and the definitions are legitimate since $D_\phi$ is the complement of the closure of $W_\phi(L)$ and since $W_\phi|L$ depends only on $\phi$. The Koebe's one-quarter theorem implies that $D_\phi \subset \{ |w| < 2 \}$ for every $\phi$ of $T(G)$.

Let $(X, \pi, R)$ be a holomorphic family of Riemann surfaces of type $(g, n)$ with $2g-2+n > 0$ and let $\rho: D \to R$ be the universal covering with the covering transformation group $\Gamma$. Then there exists a holomorphic mapping $\Phi: D \to T(G)$ such that the quotient space $D_{\phi(t)}/G_{\phi(t)}$ is conformally equivalent to $S_{\phi(t)}$ for every $t \in D$. We abbreviate $G_{\phi(t)}$ to $G_t$ and $D_{\phi(t)}$ to $D_t$. We set

$$\bar{X} = \{ (t, w) | t \in D, w \in D_t \}.$$

This set $\bar{X}$ is topologically equivalent to the two-dimensional polydisc $D^2$. Since $D_t \subset \{ |w| < 2 \}$ for every $t \in D$, the set $\bar{X}$ is a bounded domain in $C^2$. We can also show that $\bar{X}$ is a domain of holomorphy. Let $F_t$ be the conformal mapping of $D_t/G_t$ onto $S_{\phi(t)}$ induced by $\Phi(t)$ for every $t \in D$ and let $\Pi$ be the holomorphic mapping of $\bar{X}$ onto $X$ sending $(t, w)$ into $F_t(w)$. Then $\Pi: \bar{X} \to X$ is the universal covering of $X$ constructed by Griffiths [2].

Let $G$ be the covering transformation group of the universal covering $\Pi: \bar{X} \to X$. We can explicitly express the elements of $G$ as follows. For each element $\gamma \in \Gamma$, the homotopic monodromy $M_\gamma$ of $\gamma$ is the element of the Teichmüller modular group $Mod(G)$ of $G$ with the property $\Phi \circ \gamma = M_\gamma \circ \Phi$. The subgroup $M_\gamma \subset G$ of $Mod(G)$ is called the homotopic monodromy group of $(X, \pi, R)$. Denote by $N(G)$ the set of all quasiconformal automorphisms $\omega$ of $U$ with $\omega \circ G \circ \omega^{-1} = G$. Take an element $\omega_\gamma$ of $N(G)$ which induces $M_\gamma$, that is, $\langle \omega_\gamma \rangle = M_\gamma$. We may assume that $\omega_{\gamma \delta} = \omega_\gamma \circ \omega_\delta$ for all $\gamma, \delta \in \Gamma$. 
For each \( t \in D \), let \([w_\mu]\) be the point of \( T(G) \) with a Beltrami coefficient \( \mu_t \) corresponding to the holomorphic quadratic differential \( \Phi(t) \) in \( B_2(L, G) \). For each \( g \in G \), we set \[ w_\mu = \lambda \circ w_\mu, \circ (\omega_r \circ g)^{-1} \in Q_{\text{norm}}(G), \]
where \( \lambda \) is a real Möbius transformation. If we set

\[ (\gamma, g)(t, w) = (\gamma(t), W^t \circ (\omega_r \circ g) \circ (W^t)^{-1}(w)), \]
then the mapping \((\gamma, g)\) is an analytic automorphism of \( \hat{X} \) for all \( \gamma \in \Gamma, g \in G \).

Now the covering transformation group \( \mathcal{G} \) is identical with the set \( \Gamma \times G \).

By definition, we have the relation

\[ (\gamma, g) \circ (\delta, h) = (\gamma \circ \delta, \omega_{\delta^{-1}} \circ g \circ \omega_{\gamma \circ h}) \]
for all \( \gamma, \delta \in \Gamma \) and \( g, h \in G \), that is, \( \mathcal{G} \) is a semi-direct product of \( \Gamma \) by \( G \).

It is noted that \((\gamma, g) = (\delta, h)\) if and only if \( \gamma = \delta \) and \( g = h \).

Now, we have the following fundamental theorem. (See [3] and [4].)

**Theorem.** Let \((X, \pi, R)\) be a holomorphic family of Riemann surfaces of type \((g, n)\) with \( 2g - 2 + n > 0 \). Take a puncture \( p_0 \) of \( R \). Let \( t_0 \) be a parabolic fixed point with \( \rho(t_0) = p_0 \) and let \( \gamma_0 \) be a generator of the stabilizer of \( t_0 \) in \( \Gamma \). Then there exists an element \( \phi_0 \) in the closure of \( T(G) \) such that the holomorphic mapping \( \Phi(t) : D \to T(G) \) converges to \( \phi_0 \) uniformly as \( t \) tends to \( t_0 \) through any cusped region at \( t_0 \) in \( D \). The homotopic monodromy \( M_{\gamma_0} \) is of finite order if and only if \( \phi_0 \in \partial T(G) \), and is of infinite order if and only if \( \phi_0 \in \partial T(G) \), where \( \partial T(G) \) is the boundary of \( T(G) \) in \( B_2(L, G) \). In the latter case, the boundary group \( G_{\phi_0} \) corresponding to \( \phi_0 \in \partial T(G) \) is a regular b-group.

2. **Proof of Theorem 1.** Assume that there exists a biholomorphic mapping \( F : \hat{X} \to B_2 \). Let \( p_0 \) be a puncture of \( R \) and let \( t_0 \) be a parabolic fixed point with \( \rho(t_0) = p_0 \). By the above Theorem, there is an element \( \phi_0 \) of the closure of \( T(G) \) such that holomorphic mapping \( \Phi(t) \) converges to \( \phi_0 \) uniformly as \( t \) tends to \( t_0 \) through any cusped region \( \Delta \) at \( t_0 \) in \( D \). Let \( G_{\phi_0} \) be the Kleinian group corresponding to \( \phi_0 \), which is a quasi-Fuchsian group or a regular b-group. Take a component \( \Omega \) of \( G_{\phi_0} \) which is not equal to the invariant component of \( G_{\phi_0} \) corresponding to the lower half-plane \( L \).

Let \( K \) be an arbitrary compact subset of \( \Omega \). Then \( K \subset D_t = D_{\phi(t)} \) for any \( \Delta \in t \) sufficiently near \( t_0 \). Hence, by the diagonal method, we can take a sequence \( \{t_n\}_{n=1}^\infty \) in \( \Delta \) such that \( t_n \to t_0 \) as \( n \to \infty \) and such that \( F(t_n, w) = (F_1(t_n, w), F_2(t_n, w)) \) converges to a holomorphic mapping \( f(w) = (f_1(w), f_2(w)) : \Omega \to \partial B_2 \) uniformly on any compact subset of \( \Omega \) as \( n \to \infty \). Since

\[ |f_1(z)|^2 + |f_2(z)|^2 = 1, \]
we have
which implies that \( \frac{\partial^2}{\partial z \partial \bar{z}} (|f_2(z)|^2 + |f_3(z)|^2) = \frac{\partial f_1(z)}{\partial z} \left| \frac{\partial f_2(z)}{\partial z} \right|^2 + \left| \frac{\partial f_2(z)}{\partial z} \right|^2 = 0 \),

which implies that \( \frac{\partial f_1}{\partial z} = \frac{\partial f_2}{\partial \bar{z}} = 0 \) on \( \Omega \). Hence \( f = (f_1, f_2) \) is a constant mapping. We may assume that \( f \) is a constant mapping with the value \((1, 0) \in \partial B_2\).

Denote by \( G_\Omega \) the stabilizer of \( \Omega \) in \( G_{\phi_0} \). Let \( G_\Omega = \chi_{\phi_0}^{-1}(G_{\Omega}) \), \( g_t = \chi_{\phi_0}(g) \) for \( g \in G, t \in D \), and \( g_t = \chi_{\phi_0}(g) \) for \( g \in G \). Set \( A_g = F^o(1, g) \circ F^{-1} \in \text{Aut}(B_2) \) for each \( g \in G \), where \( 1 \) is the identity element of \( \Gamma \). Since \( g_t \rightarrow g \) as \( t \rightarrow t_0 \) through \( \Delta \) for all \( g \in G \), and since \( g_t(\Omega) = \Omega \) for all \( g \in G_\Omega \), the boundary point \((1, 0) \) of \( B_2 \) is a fixed point of \( A_g \) for all \( g \in G_\Omega \).

We set

\[
S = \{(u, v) \in \mathbb{C}^2 | \text{Im}(u) > |v|^2 \},
\]

where \( \text{Im}(u) \) is the imaginary part of \( u \). This set \( S \) is a Siegel domain of the second kind. We put

\[
z_1 = \frac{u-i}{u+i}, \quad z_2 = \frac{2v}{u+i}.
\]

Then the mapping \( T: S \rightarrow B_2 \) sending \((u, v)\) into \((z_1, z_2)\) is biholomorphic and it carries the boundary point \((\infty, 0)\) of \( S \) into the boundary point \((1, 0)\) of \( B_2 \). It is known that an analytic automorphism \( \Psi \in \text{Aut}(S) \) of \( S \) has a fixed point \((\infty, 0)\) if and only if

\[
\Psi(u, v) = (|a|^2 u + 2ia bv + c + i |b|^2, av + b),
\]

where \( a \) is a non-zero complex number, \( b \) is a complex number and \( c \) is a real number. (See Pyatetskii-Shapiro [8, Chap. 1, §2, Thm. 1].)

Let \( A_g^* = T^{-1} \circ A_g \circ T \in \text{Aut}(S) \) for each \( g \in G \). Then the point \((\infty, 0)\) is a fixed point of \( A_g^* \) for all \( g \in G_\Omega \). Hence,

\[
A_g^*(u, v) = (|a|^2 u + 2ia bv + c + i |b|^2, a^* v + b_g)
\]

for all \( g \in G_\Omega \).

If \( |a_{g_0}| \neq 1 \) for some \( g_0 \in G_\Omega \), there exists an element \( \Psi \in \text{Aut}(S) \) with \( \Psi(\infty, 0) = (\infty, 0) \) such that \( \Psi \circ A_g^{* o} \circ \Psi^{-1}(u, v) = (|a|^2 u, a_0 v) \), where \( a_0 \) is a non-zero complex number with \( |a_0| \neq 1 \). Take an element \( h \in G_0 \) such that \( g_0 \circ h \neq h \circ g_0 \). We set

\[
U(u, v) = \Psi \circ A_g^{* o} \circ \Psi^{-1}(u, v) = (|a|^2 u, a_0 v),
\]

\[
V(u, v) = \Psi \circ A_g^{* o} \circ \Psi^{-1}(u, v) = (|a|^2 u + 2ia bv + c + i |b|^2, av + b).
\]

Since \( g_0 \circ h \neq h \circ g_0 \), we have \( U \circ V = V \circ U \), which implies that \( b \neq 0 \) or \( c \neq 0 \). By direct computation, we have
\[ W_n(u, v) = V \circ U^* \circ V^{-1} \circ U^{-n}(u, v) \]
\[ = (u + 2i(1 - a_0^2)bv + (1 - |a_0^2|c + 2|b|^2 \Im(a_0^*) + i)|1 - a_0^*|b|^2, v + (1 - a_0^*)b) \]
for any integer \( n \). Since \( |a_0| \neq 1 \), we have
\[ W_n(u, v) \to W(u, v) = (u + 2ibv + c + i|b|^2, v + b) \]
as \( n \to \infty \) or \(-\infty\), which implies that \( (F^{-1} \circ T \circ \Psi^{-1})^{-1} \circ \mathcal{G} \circ (F^{-1} \circ T \circ \Psi^{-1}) \) is not discrete. Hence, \( \mathcal{G} \) is not discrete and we have a contradiction.

ii) If \(|a_g| = 1\) for all \( g \in G_0 \) and if \( a_{g_0} \neq 1 \) for some \( g_0 \in G_0 \), there exists an element \( \Psi \in \Aut(S) \) with \( \Psi(\infty, 0) = (\infty, 0) \) such that \( \Psi \circ A^*_g \circ \Psi^{-1}(u, v) = (u + c_0, a_0v) \), where \( a_0 \) is a complex number with \( |a_0| = 1 \) and \( a_0 \neq 1 \), and \( c_0 \) is a real number. Take an element \( h \in G_0 \) such that \( g_0h \neq h \circ g_0 \). We set
\[ U(u, v) = \Psi \circ A^*_g \circ \Psi^{-1}(u, v) = (u + c_0, a_0v), \]
\[ V(u, v) = \Psi \circ A^*_g \circ \Psi^{-1}(u, v) = (u + 2iav + c + i|b|^2, av + b), \]
where \( a \) is a complex number with \( |a| = 1 \), \( b \) is a complex number, and \( c \) is a real number. Since \( h \circ g_0 \neq g_0 \circ h \) for all integer \( n \), we have \( V \circ U = U \circ V \) which implies that \( b \neq 0 \) and \( a_0 \neq 1 \). If we set \( a_0 = e^{i\theta} \), then \( \theta \) is an irrational number. By direct calculation, we have
\[ W_n(u, v) = V \circ U^* \circ V^{-1} \circ U^{-n}(u, v) \]
\[ = (u + 2ibv + c + i|b|^2 \Im(a_0^*) + i|b(1 - a_0^*)|^2, v + b(1 - a_0^*)) \]
for any integer \( n \). Since \( \theta \) is an irrational number, there exists a sequence \( \{n_j\} \) of integers such that \( (a_0)^{n_j} \to 1 \) as \( j \to \infty \). Therefore, \( W_{n_j}(u, v) \to W(u, v) = (u, v) \) as \( j \to \infty \), which implies that \( (F^{-1} \circ T \circ \Psi^{-1})^{-1} \circ \mathcal{G} \circ (F^{-1} \circ T \circ \Psi^{-1}) \) is not discrete. Hence, \( \mathcal{G} \) is not discrete and we have a contradiction.

iii) If \( a_g = 1 \) for all \( g \in G_0 \), we have
\[ A^*_g(u, v) = (u + 2ib^g v + c + i|b|^2, v + b_g). \]
Therefore,
\[ A^*_g \circ A^*_g \circ (A^*_g)^{-1} \circ (A^*_g)^{-1}(u, v) = (u - 4 \Im(b_g b_h), v). \]
Hence, the commutator subgroup of the group \( \{A^*_g \mid g \in G_0\} \) is commutative, which implies that the commutator subgroup \([G_0, G_0] \) of \( G_0 \) is commutative. Hence we have a contradiction. This completes the proof of Theorem 1.

Now, let us assume that there exists a strongly pseudoconvex domain \( \Omega \) in \( C^2 \) which is biholomorphically equivalent to \( \tilde{X} \). Let \( F: \tilde{X} \to \Omega \) be a biholomorphic mapping. Since \( \mathcal{G}^* = F \circ \mathcal{G} \circ F^{-1} \) is an infinite subgroup of \( \Aut(\Omega) \) and acts on \( \Omega \) properly discontinuously, for any point \( \xi \) of \( \Omega \), there exists an infinite sequence \( \{T_n\} \) of \( \mathcal{G}^* \) such that \( T_n(\xi) \) tends to a boundary point \( \xi_0 \) of \( \Omega \).
as \( n \to \infty \). Therefore, the Proposition in Rosay [9] implies that \( \Omega \) is biholomorphically equivalent to the unit ball \( B_2 \). Hence, we have a contradiction and this completes the proof of Corollary.

3. Proof of Theorem 2. If all the fibers \( S_p \) are conformally equivalent, then the mapping \( \Phi: D \to T(G) \) is a constant mapping with a value \( q_0 \in T(G) \). By the relation \( M_\gamma \circ \Phi = \Phi \circ \gamma \), the point \( q_0 \) is a fixed point of all \( M_\gamma \in \mathcal{M} \). Since the modular group \( \text{Mod}(G) \) of \( G \) acts on \( T(G) \) properly discontinuously, the subgroup \( \mathcal{M} \) of \( \text{Mod}(G) \) also acts on \( T(G) \) properly discontinuously. Hence, \( \mathcal{M} \) is a finite group.

Conversely, assume that \( \mathcal{M} \) is finite, and let \( \Gamma_0 \) be the kernel of the monodromy map \( \gamma \mapsto M_\gamma \). Then \( \Gamma_0 \) has finite index in \( \Gamma \), so \( R_0 = D / \Gamma_0 \) is a Riemann surface of finite type. Since \( \Phi \circ \gamma = \Phi \) for all \( \gamma \in \Gamma_0 \), the holomorphic map \( \Phi: D \to T(G) \) factors through \( R_0 \). Since \( T(G) \) is bounded, every holomorphic map from \( R_0 \) to \( T(G) \) is constant, so \( \Phi \) is a constant map. Hence, all the fibers \( S_p \) are conformally equivalent and this completes the proof of Theorem 2.

4. Proof of Theorem 3. Assume that there exists a biholomorphic mapping \( F = (F_1, F_2): X \to D^2 \). If we set \( G* = F*G = F \circ G \circ F^{-1} \), then \( G* \) is a properly discontinuous subgroup of the analytic automorphism group \( \text{Aut}(D^2) \).

We recall that any analytic automorphism of \( D^2 = (|z_1| < 1) \times (|z_2| < 1) \) is either one of the following two types:

(I) \( (A, B)(z_1, z_2) = (A(z_1), B(z_2)) \),

(II) \( (A, B)(z_1, z_2) = (A(z_2), B(z_1)) \),

where \( A, B \in \text{Aut}(D) \). (See Narasimhan [7, Chap. 5, Prop. 3].) Note that \( (A, B)^2 \) is of type (I) for all \( (A, B) \in \text{Aut}(D^2) \).

We also recall the following results, which will be used frequently in this section. (See Lehner [6, Chap. 2, §9, Thm. 1 and Thm. 2, and Chap. 3, Thm. 2E].)

Two Möbius transformations are commutative if and only if they have the same set of fixed points provided that neither is the identity and provided that neither is a transformation of order two.

Let \( A \) be a hyperbolic or loxodromic transformation and let \( B \) be a Möbius transformation which has one and only one fixed point in common with \( A \). Then the sequence \( \{B \circ A^n \circ B^{-1} \circ A^{-n}\} \) of Möbius transformations converges to a Möbius transformation as \( n \to \infty \) or \( -\infty \).

By these results, we have the following assertion.

Let \( A, B \) be two Möbius transformations of infinite order with \( A \circ B \neq B \circ A \) such that they have a common fixed point. Then the group generated
by $A$, $B$ is not discrete.

Let $p_0$ be a puncture of $R$, $t_0$ be a parabolic fixed point with $\rho(t_0) = p_0$ and let $\gamma_0$ be a generator of the stabilizer of $t_0$ in $\Gamma$. Then Theorem of § 1 implies that there exists an element $\phi_0$ in the closure of $T(G)$ in $B_2(L, G)$ such that the mapping $\Phi(t): D \to T(G)$ converges to $\phi_0$ uniformly as $t \to t_0$ through any cusped region $\Delta$ at $t_0$ in $D$ and such that the Kleinian group $G_{\phi_0}$ corresponding to $\phi_0$ is a quasi-Fuchsian group or a regular $b$-group. Let $D_0 = \Omega(G_{\phi_0}) - \Delta(G_{\phi_0})$, where $\Omega(G_{\phi_0})$ is the region of discontinuity of $G_{\phi_0}$ and $\Delta(G_{\phi_0})$ is the invariant component of $G_{\phi_0}$ corresponding to the lower half-plane $L$. Then the quotient space

$$S_0 = (D_0 \cup \{\text{accidental parabolic fixed points of } G_{\phi_0}\})/G_{\phi_0}$$

is a Riemann surface of type $(g, n)$ with or without nodes. Let $\{p_1, \ldots, p_k\}$ be the set of nodes of $S_0$, which may be empty. If $\pi_0: U \to S = U/G$ is the canonical projection and if $\alpha: S \to S_0$ is the deformation as in § 3 of [4], then there exists a family $\{W_t\}_{t \in \Delta}$ of quasiconformal automorphisms on $\tilde{C}$ such that $W_t$ is conformal on $L$ and has a Schwarzian derivative $\Phi(t)$ for all $t \in \Delta$ and such that $W_t$ converges uniformly on any compact subset of $U_0 = U - \pi_0^{-1}(\{p_1, \ldots, p_k\})$ to a locally quasiconformal mapping $W_0: U_0 \to D_0$ as $t \to t_0$ through $\Delta$. (See § 4 in [4].) Then the locally quasiconformal mapping $W_0$ induces the above deformation $\alpha: S \to S_0$. 

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Figure 1
Let $\Sigma^0, \ldots, \Sigma^r$ be the parts of $S_0$, that is, the connected components of $S_0 - \{p_1, \ldots, p_k\}$ and let $\Sigma^i = \alpha^{-1}(\Sigma^i)$ for each $i=1, \ldots, r$. Take a sufficiently small neighborhood $\delta_j = \{(x_1, x_2) \in \mathbb{C}^2 | |x_1| < \varepsilon$ and $|x_2| < \varepsilon\}$ of a node $p_j$ in $S_0$ for each $j=1, \ldots, k$ and set $\delta_0 = \delta_1 \cup \cdots \cup \delta_k$. If we set $C'_j = \alpha^{-1}((|z_1| = \varepsilon) \times (|z_2| = \varepsilon))$ for each $j=1, \ldots, k$, then the domain bounded by $C'_j$ and $C''_j$ is an annulus on $S$. Let $\Sigma'_i$ be the connected component of $S - \alpha^{-1}(\delta_0)$ contained in $\Sigma^i$ for each $i=1, \ldots, r$. Then $\Sigma'_i$ is homeomorphic to $\Sigma_i$. (See Figure 1.)

Take a point $q_0$ on $S$, which is fixed as a base point. Let $(C, q)$ be a pair of a point $q$ on $S$ and a path $C$ from $q_0$ to $q$ on $S$. A pair $(C_0, q_1)$ is equivalent to a pair $(C_0, q_2)$ if and only if $q_1 = q_2$ and $C_0 \circ C_1^{-1}$ is homotopic to the point $q_0$. Then we can identify the universal covering space $U$ of $S$ with the set of all these equivalence classes $[C, q]$ and the covering transformation group of the universal covering $\pi_1(S)$: $U \rightarrow S$ is identified with the fundamental group $\pi_1(S, q_0)$ of $S$ with a base point $q_0$, that is,

$$G = \{[C_0]_\ast | [C_0] \in \pi_1(S, q_0)\},$$

where $[C_0]_\ast$ is a covering transformation sending $[C, q]$ into $[C_0 \circ C, q]$ for $[C, q] \in U$. Suppose that $q_0 \in C'_i$ throughout this section and set

$$G_1 = \{[C_0]_\ast | C_0 \in \pi_1(\Sigma_i, q_0)\},$$

$$U_1 = \{[C, q] | q \in \Sigma_i \text{ and } C \text{ is a path from } q_0 \text{ to } q \text{ on } \Sigma_i\}.$$

Then $U_1$ is a connected component of $\pi_0^{-1}(\Sigma_i)$, which is invariant under $G_1$. Since $\Sigma'_i$ is homeomorphic to $\Sigma_i$, we have $G_1 = \{[C_0]_\ast | C_0 \in \pi_1(\Sigma'_i, q_0)\}$. If we set $\Omega_i = W\phi(U_1)$, then $\Omega_i$ is a component of $G_\phi_0$ and the isomorphism $\chi_{\phi_0} : G \rightarrow G_\phi_0$ induces an isomorphism $\chi_{\phi_0} : G_1 \rightarrow G_{\Omega_i}$, where $G_{\Omega_i}$ is the stabilizer of $\Omega_i$ in $G_{\phi_0}$.

Let $(f_{\gamma_0})_\ast$ be an element of the modular group Mod$(S)$ of the Teichmüller space $T(S)$ corresponding to the homotopic monodromy $M_{\gamma_0} = \langle \omega_{\gamma_0} \rangle \in \text{Mod}(G)$ of $\gamma_0$. Since there exists a positive integer $m$ such that $(f_{\gamma_0})^m$ is homotopic to a product $d$ of $\nu$-th powers of Dhen twists on $S$ about Jordan curves mapped by $\alpha : S \rightarrow S_0$ into nodes, we may assume that the quasiconformal automorphism $\omega_0$ of $U$ with $\omega_0 \circ G \circ \omega_0^{-1} = G$ and $\langle \omega_{\nu} \rangle = \langle M_{\gamma_0} \rangle^m$ is induced by $d$. Since $d | \Sigma'_i$ is the identity mapping, $\omega_0 | U'_i$ is also the identity mapping, where $U'_i$ is the connected component of $\pi_0^{-1}(\Sigma'_i)$ which is contained in $U_i$. Note that $U'_i$ is invariant under $G_1$. Hence, we have $\omega_0 \circ g \circ \omega_0^{-1} = g$ for all $g \in G_1$.

Set $(A, B) = F \circ (\gamma_0^m, 1) \circ F^{-1}, (A_\gamma, B_\gamma) = F \circ (1, g) \circ F^{-1}$ for each $g \in G$, where $1$ is the identity of $G$. We may assume that $(A, B)$ is of type $(I)$.

By the same reasoning as in §2, we can choose an infinite sequence $\{t_n\}_{n=1}^\infty$ of $\Delta$ such that $t_n \rightarrow t_0$ as $n \rightarrow \infty$ and such that $F(t_n, w) = (F_1(t_n, w), F_2(t_n, w))$ converges to a holomorphic mapping $f(w) = (f_1(w), f_2(w)) : \Omega_i \rightarrow \partial D^2$ uniformly on
any compact subset of $\Omega_1$ as $n \to \infty$. Since $\partial D^2 = \{(|z_1| = 1) \times (|z_2| \leq 1)\} \cup \{(|z_1| \leq 1) \times (|z_2| = 1)\}$, we have $|f_1(w)| = 1$ or $|f_2(w)| = 1$ for each $w \in \Omega_1$. Hence, $|f_1| = 1$ or $|f_2| = 1$ on a non-empty open subset of $\Omega_0$, which implies that $f_1$ or $f_2$ is a constant function with a value in $\partial D$. So we suppose that $f_1$ is a constant function with a value $c_1 \in \partial D$. Now, we have the following lemma.

**Lemma 1.** The analytic automorphism $(A, B) = F \circ \gamma_{g_0}^0, 1) \circ F^{-1}$ of $D^2$ is equal to $(A, 1)$ and $A$ is of infinite order. For each $g \in G_1$, the analytic automorphism $(A_\rho, B_\rho) = F \circ (1, g) \circ F^{-1}$ of $D^2$ is of type (I) and $B_\rho$ is of infinite order provided that $g \neq 1$. Moreover, the group $G = \{A_\rho \mid g \in G_1\}$ is commutative.

Proof. Since $\omega_1^0 \circ g \circ \omega_1^{-1} = g$ for each $g \in G_1$, the relation (1) of §1 implies that $(1, g) \circ \gamma(\gamma_{g_0}^0, 1) = (\gamma_{g_0}^0, 1) \circ (1, g)$ for each $g \in G_1$. Hence, we have $(A_\rho, B_\rho) = (A, B) \circ (A_\rho, B_\rho)$ for each $g \in G_1$. If $(A_\rho, B_\rho), g \in G_1$, is of type (I), then $A_\rho \circ A = A \circ A_\rho$ and $B_\rho \circ B = B \circ B_\rho$. In general, denote by $\text{Fix}(T)$ the set of fixed points in $\mathcal{C}$ of an element $T \in \text{Aut}(D)$. Then, if neither $A$ nor $A_\rho$ is the identity, we have $\text{Fix}(A) = \text{Fix}(A_\rho)$. Similarly, if neither $B$ nor $B_\rho$ is the identity, then $\text{Fix}(B) = \text{Fix}(B_\rho)$.

Assume that neither $A$ nor $B$ is the identity. Take two non-commutative elements $g_0, h_0 \in G_1$ such that both $(A_{g_0}, B_{g_0})$ and $(A_{h_0}, B_{h_0})$ are of type (I). If at least one of $A_{g_0}$ and $A_{h_0}$ is the identity, then clearly $A_{g_0}$ and $A_{h_0}$ are commutative. If $A_{g_0} \neq 1$ and $A_{h_0} \neq 1$, then $\text{Fix}(A) = \text{Fix}(A_{g_0}) = \text{Fix}(A_{h_0})$, which implies that $A_{g_0}$ and $A_{h_0}$ are commutative. Hence, in any case, $A_{g_0}$ and $A_{h_0}$ are commutative. Similarly, it is shown that $B_{g_0}$ and $B_{h_0}$ are commutative. Hence, $(A_{g_0}, B_{g_0})$ and $(A_{h_0}, B_{h_0})$ are commutative and so are $g_0$ and $h_0$. We have a contradiction. Therefore, at least one of $A$, $B$ is equal to the identity. Since $\gamma$ is of infinite order, either $A$ or $B$ is of infinite order. Hence, we have the two cases: (i) $A$ is of infinite order and $B=1$, (ii) $A=1$ and $B$ is of infinite order. Assume that $A=1$ and $B$ is of infinite order. Then we have $A_{g_0} \circ A_{h_0} = A_{g_0} \circ A_{h_0}, B_{g_0} \circ B_{h_0} = B_{h_0} \circ B_{g_0}$ and we have that $A_{g_0}$ and $A_{h_0}$ are of infinite order because no powers of $g_0$ or $h_0$ commute. Set $g_0 := \chi_{a(t)}(g_0)$ for each $t \in D$. Then $(1, g_0)(t, w) = (t, g_0(t, w))$ for each $(t, w) \in \mathcal{X}$. The relation $F \circ (1, g_0) = (A_{g_0}, B_{g_0}) \circ F$ implies that

$$F_1(t, g_0(t, w)) = A_{g_0} \circ F_1(t, w),$$

$$F_2(t, g_0(t, w)) = B_{g_0} \circ F_2(t, w)$$

for each $(t, w) \in \mathcal{X}$. Let $g_{0, t_0} := \chi_{a(t)}(g_0)$. Since $F_1(t_0, w), F_2(t_0, w)$ and $g_{0, t_0}(w)$ converge uniformly on any compact subset of $\Omega_1$ to $f_1(w) = c_0, f_2(w)$ and $g_{0, t_0}(w)$, respectively, as $n \to \infty$ and since $g_{0, t_0}(\Omega_1) = \Omega_0$, we have $A_{g_0}(c_1) = c_1$ and $f_2 \circ g_{0, t_0} = B_{g_0} \circ f_2$. Similarly, we have $A_{h_0}(c_1) = c_1$ and $f_1 \circ h_{0, t_0} = B_{h_0} \circ f_2$. Since $A_{g_0}$ and $A_{h_0}$ are two non-commutative Mobius transformations of infinite order with a common fixed point $c_1$ and since $B_{g_0}$ and $B_{h_0}$ are commutative, the group
generated by \((A_{g_0}, B_{g_0})\) and \((A_{h_0}, B_{h_0})\) is not discrete. Hence, \(F \circ g \circ F^{-1}\) is not discrete, which implies that \(\mathcal{G}\) is not discrete and we have a contradiction. Therefore, \(A\) is of infinite order and \(B=1\). Moreover, it is shown that both \(B_{g_0}\) and \(B_{h_0}\) are of infinite order, \(A_{g_0}\) and \(A_{h_0}\) are commutative, and \(B_{g_0}\) and \(B_{h_0}\) are non-commutative.

Now, assume that \((A_g, B_g)\) is of type (II) for some \(g \in G_1\). Then we have
\[
(A_g, B_g) \circ (A, 1)(z_1, z_2) = (A_g(z_2), B_gA(z_1)),
\]
\[
(A, 1) \circ (A_g, B_g)(z_1, z_2) = (A \circ A_g(z_2), B_g(z_1)).
\]
Since \((A_g, B_g)\) commutes with \((A, 1)\), we have
\[
(A_g(z_2), B_g \circ A(z_1)) = (A \circ A_g(z_2), B_g(z_1))
\]
for each point \((z_1, z_2)\) of \(D^2\). Hence, \(A=1\), which contradicts \(A \neq 1\). Therefore, \((A_g, B_g)\) is of type (I) for all \(g \in G_1\).

Since \((A, B) = (A, 1)\), \((A_g, B_g)\) is of type (I) and \((A, 1)\) commutes with \((A_g, B_g)\), we have that \(A \circ A_g = A_g \circ A\) for all \(g \in G_1\). Hence, the group \(A = \{A_g | g \in G_1\}\) is commutative.

Moreover, \(B_g\) is of infinite order for all \(g \neq 1\) of \(G_1\) by the same argument as the one that \(A_{g_0}\) and \(A_{h_0}\) are of infinite order. This completes the proof of Lemma 1.

**Lemma 2.** The homotopic monodromy \(M_{\gamma_0}\) of \(\gamma_0\) is of finite order.

**Proof.** We use the notations in the proof of Lemma 1. Assume that \(M_{\gamma_0}\) is of infinite order. Then \(S_0\) is a Riemann surface of type \((g, n)\) with nodes \(p_1, \ldots, p_k\). Denote by \(C_j\) the Jordan curve \(\alpha^{-1}(p_j)\) on \(S\) for each \(j = 1, \ldots, k\).

i) Assume that at least one of \(C_1, \ldots, C_k\), say \(C_1\), is a non-dividing cycle on \(S\). Suppose that \(q_0 \in C_1 = \alpha^{-1}([z_1 = \varepsilon] \times (x_2 = 0))\) and take a closed path \(C_0\) starting at \(q_0\) on \(\Sigma_1\). (See Figure 2.)

![Figure 2.](image-url)

Since the Dehn twist \(d\) inducing the homotopic monodromy \((M_{\gamma_0})^w = \langle \omega_d \rangle\) is the identity mapping on \(S - \alpha^{-1}(\delta_0)\), we have \([d(C_0)] = [C_1] \circ [C_0]\) for
some integer $v_0$. Set $g_0 = [C_{12}]^{v_0} \in G_1$, $h_0 = [C_{12}] \in G$, $U_2 = h_0(U_1)$ and $G_2 = h_0 \circ G_1 \circ h_0^{-1}$. Then the relations $[d(C_{12})] = [C_{12}]^{v_0} \circ [C_{12}]$, $d \circ \pi_0 = \pi_0 \circ \omega_0$ and $\omega_1(U_1) = 1$ imply that $\omega_1 \circ h_0 = g_0 \circ h_0$ on $U_1$. Hence, we have $\omega_1 = g_0$ on $U_2$. If we set $\omega_2 = g_0^{-1} \circ \omega_0$, then $\omega_2|U_2 = 1$, $\langle \omega_2 \rangle = \langle \omega_0 \rangle$ in $\text{Mod}(G)$ and $\omega_2 \circ h_0 \circ \omega_2^{-1} = h$ for all $h \in G_2$. Moreover, the quasiconformal mapping $\omega_2$ induces an analytic automorphism $(1, g_0^{-1} \circ (\gamma_{12}^n, 1))$ of $X$. Hence, we have an element $(A_{12}^{-1} \circ A, B_{12}^{-1}) \in F \circ G \circ F^{-1}$. Note that, by Lemma 1, $B_{12}$ is of infinite order. By the same reasoning as in the proof of Lemma 1, the relation $\omega_2 \circ h_0 \circ \omega_2^{-1} = h$ for each $h \in G_2$ implies that $A_{12}^{-1} \circ A = 1$, $(A_{12}, B_{12})$ is of type (I) for all $h \in G_2$ and the group $\{B_{12} | h \in G_2\}$ is commutative.

If $(A_{12}, B_{12}) = F \circ (1, h_0) \circ F^{-1}$ is of type (I), then $\{B_{12} | g \in G_3\}$ and $\{B_{12} | h \in G_2\}$ are conjugate by $B_{12}$. Since the group $\{B_{12} | h \in G_2\}$ is commutative, the group $\{B_{12} | g \in G_3\}$ is also commutative and we have a contradiction.

Now, suppose that $(A_{12}, B_{12})$ is of type (II). We set $h_1 = h_0 \circ g_2$ and $U_3 = h_1(U_2)$ for each $g_2 \in G_1$. The relations $[d(C_{12})] = [C_{12}]^{v_0} \circ [C_{12}]$, $d \circ \pi_0 = \pi_0 \circ \omega_0$, and $\omega_1(U_2) = 1$ imply that $\omega_2 = g_0^{-1} \circ g_0 \circ h_1^{-1}$ on $U_3$. If we set $\omega_3 = (h_0 \circ g_0^{-1} \circ h_1^{-1} \circ g_0^{-1}) \circ \omega_0$, then we have $\omega_2|U_3 = 1$, $\langle \omega_2 \rangle = \langle \omega_0 \rangle$ and $\omega_3 \circ h_0 \circ \omega_3^{-1} = h$ for all $h \in h_1 \circ G_1 \circ h_1^{-2}$. The element $\omega_3 \in N(G)$ induces an analytic automorphism $(1, h_0 \circ g_0^{-1} \circ h_1^{-1} \circ g_0^{-1}) \circ (\gamma_{12}^n, 1)$ of $X$ and we have an element $(X_{13}, Y_{13}) \in F \circ G \circ F^{-1}$, where $X_{13} = (A_{12} \circ B_{12})^{-1} \circ B_{12}^{-1} \circ (A_{12} \circ B_{12})^{-1}$ and $Y_{13} = B_{12} \circ h_1 \circ B_{12}^{-1} \circ h_1$. Note that $(X_{13}, Y_{13})$ is of type (I). By the same argument as the proof of Lemma 1, we see that $(X_{13}, Y_{13}) = (X_{13}, Y_{13})$ with $X_{13} \neq 1$ or $(X_{13}, Y_{13}) = (1, 1)$ with $Y_{13} \neq 1$. Since $B_{12}$ is of infinite order, we have $X_{13} \neq 1$ and $Y_{13} \neq 1$. We set $h_2 = h_0 \circ g_2$. The same reasoning as above implies that the element $(h_0 \circ g_0^{-1} \circ h_1^{-1} \circ g_0^{-1}) \circ \omega_1 \circ N(G)$ induces an analytic automorphism $(1, h_0 \circ g_0^{-1} \circ h_1^{-1} \circ g_0^{-1}) \circ (\gamma_{12}^n, 1)$ of $X$ and we have an element $(X_{23}, Y_{23}) \in F \circ G \circ F^{-1}$, where $X_{23} = (A_{12} \circ B_{12})^{-1} \circ B_{12}^{-1} \circ (A_{12} \circ B_{12})^{-1}$. Now, we can prove that $\mathcal{A} = \{A_\ast | g \in G_3\}$ is a discrete subgroup of $\text{Aut}(D)$ as follows. Assume that $\mathcal{A}$ is not discrete. Then there exists a sequence $\{A_n\}$ of distinct elements of $\mathcal{A}$ such that $A_n \to 1$ as $n \to \infty$. Take an element $g_1 \in G_1$ with $g_0 \circ g_1 \neq g_0 \circ g_0$ and consider the sequences $\{(A_n, B_n) \circ (X_{13}, 1), (A_n, B_n)^{-1}\} = \{(A_n \circ X_1, A_n^{-1}, 1)\}$ and $\{(A_n, B_n) \circ (X_{23}, 1), (A_n, B_n)^{-1}\} = \{(A_n \circ X_2, A_n^{-1}, 1)\}$ in $\mathcal{G}$. They converge to $(X_{13}, 1)$ and $(X_{23}, 1)$ respectively as $n \to \infty$. Therefore, the discreteness of $\mathcal{G}$ implies that for any sufficiently large $n$, $A_n$ commutes with $X_1$ and $X_2$. Thus, $A_n \circ X_1 \circ A_n^{-1} = X_1$ and $A_n \circ X_2 \circ A_n^{-1} = X_2$ for any sufficiently large $n$, which implies that

$$\text{Fix}(A) = \text{Fix}(A_n) = (A_{12} \circ B_{12})(\text{Fix}(B_{12}^{-1})),$$

$$\text{Fix}(A) = \text{Fix}(A_n) = (A_{12} \circ B_{12}^{-1})(\text{Fix}(B_{12}^{-1})).$$

Hence, we have $B_{12}(\text{Fix}(B_{12}^{-1})) = \text{Fix}(B_{12}^{-1})$, which implies that the group generated by $(A_{12} \circ B_{12})$ and $(A_{12} \circ B_{12}^{-1})$ is not discrete and we have a contradiction. Therefore, $\mathcal{A}$ is an Abelian discrete subgroup of $\text{Aut}(D)$. Then $\mathcal{A}$ is generated by an element $A_{12}$ for some $g_\ast \in G_1$ with $g_\ast \neq 1$. Take an element $g_2 \in G_1$ with
Let $A_{g_i} = (A_{g_i})^n$ for some integer $n$ and let $g_3 = g_2 \circ g_3^{-n} \in G_1$. Then $g_3 \neq 1$ and $F_0(1, g_3) F_0^{-1} = (A_{g_3}, B_{g_3}) = (1, B_{g_3})$. Since $(A_{g_3}, B_{g_3})$ is of type (II), we have $F_0(1, h_1 \circ g_3 \circ h_1^{-1}) F_0^{-1} = (A_{h_1} \circ B_{g_3} \circ A_{h_1}^{-1}, 1)$, which is of type (I). Therefore, $(A_{g_3}, B_{g_3})$ and $(A_{h_1} \circ B_{g_3} \circ A_{h_1}^{-1}, 1)$ are commutative, which implies that $g_3$ and $h_1 \circ g_3 \circ h_1^{-1}$ are commutative. Since $g_3$ and $h_1$ are elements of the discrete subgroup $G$ with no elliptic elements of $\text{Aut}(U)$, it is shown that $g_3$ and $h_0 \circ g_1$ are commutative, where $g_1$ is an arbitrary element of $G_1$. Take an element $g_1 \in G_1$ with $g_1 \circ h_0 = h_0 \circ g_1$. Since $g_3$ and $h_0 \circ g_1$ are commutative and $g_3$ and $h_0 \circ g_1$ are also commutative, we have that $h_0 \circ g_1$ and $h_0 \circ g_1$ are commutative. Hence, $h_0$ and $g_3$ are commutative and we have a contradiction.

ii) Assume that all the Jordan curves $C_1, \ldots, C_k$ are dividing cycles on $S$. Take two connected components $\Sigma_1$ and $\Sigma_2$ of $S \setminus \{p_1, \ldots, p_k\}$ which have the common boundary curve $C_1$. Let $q_0 \in C_1'$, $q_0' \in C_1''$ and let $L$ be a simple path from $q_0$ to $q_0'$ on the annulus bounded by $C_1'$ and $C_1''$. (See Figure 3.)

![Figure 3](image_url)

Now, we set

$$U_1 = \{[C, q] | q \in \Sigma_1 \text{ and } C \text{ is a path from } q_0 \text{ to } q \text{ on } \Sigma_1 \},$$

$$U_2 = \{[L \circ C, q] | q \in \Sigma_2 \text{ and } C \text{ is a path from } q_0' \text{ to } q \text{ on } \Sigma_2 \},$$

$$G_1 = \{[C]_* | [C] \in \pi_1(\Sigma_1, q_0) \},$$

$$G_2 = \{[L \circ C \circ L^{-1}]_* | [C] \in \pi_1(\Sigma_2, q_1') \}.$$

Then $U_1$ and $U_2$ are invariant under $G_1$ and $G_2$, respectively. Since the Dehn twist $d$ inducing the homotopic monodromy $(M_{\eta})^n = \langle \eta \rangle$ is the identity on $S - \alpha^{-1}(\delta_0)$, it is shown that $d(L)$ is homotopic to $(C_1')^\nu \circ L$ for some integer $\nu$. Hence, if we set $g_0 = [C_1']^\nu \in G_1$, then we have $\omega_1 = g_0$ on $U_2$ and $\omega_0 \circ h_0 \circ \alpha_1^{-1} = g_0 \circ h_0 \circ g_0^{-1}$ for all $h \in G_2$. Note that $g_0 \in G_1 \cap G_2$. If we set $g_2 = g_0^{-1} \circ \omega_1$, then we have $\omega_2 \mid U_2 = 1$ and $\omega_2 \circ h_0 \circ \omega_2^{-1} = h$ for all $h \in G_2$, and $\langle \omega_2 \rangle = \langle \omega_1 \rangle$ in $\text{Mod}(G)$. Moreover, the quasiconformal mapping $\omega_2$ induces an analytic automorphism $(1, g_0)^{-1}(g_0, 1)$ of $\mathbb{X}$ and we have an element $(A_{g_2}^{-1} \circ A, B_{g_2}^{-1}) \in F_0 \circ G_0 \circ F_0^{-1}$. Note that $B_{g_2}$ is of infinite order. By the same reasoning as in the proof of Lemma 1, the relation $\omega_2 \circ h_0 \circ \omega_2^{-1} = h$ for each $h \in G_2$ implies that $A_{g_2}^{-1} \circ A = 1$, $(A_{g_2}, B_{g_2})$ is of type (I) for each $h \in G_2$, $A_4$ is of infinite order for each $h \neq 1$ of $G_2$ and
the group \( \{ B_h | h \in G_2 \} \) is commutative. Take a closed path \( C_0 \) starting at \( q_0 \) on \( \Sigma \) and set \( C_0 = L_0 C_0 L^{-1} \) and \( h_0 = [C_0] \in G_2 \). (See Figure 3.) Let \( \bar{U}_1 = h_0(U_1) \), \( \bar{G}_1 = h_0 \circ G_1 \circ h_0^{-1} \) and \( \bar{\omega}_1 = (g_0 \circ \omega_0 \circ g_0^{-1} \circ h_0^{-1})^{-1} \omega_1 \). Since \( \omega_1 = g_0 \circ \omega_0 \circ g_0^{-1} \circ h_0^{-1} \) on \( \bar{U}_1 \), we have \( \bar{\omega}_1(0) = \bar{\omega}_0(0) \) and \( \bar{\omega}_1(\infty) = \bar{\omega}_0(\infty) \), which implies \( \omega_1 = \omega_0 \) and since \( G \) is discrete, we have \( \text{Fix}(\omega_0) = \text{Fix}(\omega_0) \), that is, \( \omega_1 \). Since both \( h_n \) and \( h_0 \) are Mobius transformations of infinite order with a common fixed point \( \# = 0 \) and since \( G \) is discrete, we have \( \text{Fix}(h_0) = \text{Fix}(h_0) \), which implies that \( \text{Fix}(h_0) = \text{Fix}(h_0) \) for any non-zero integer \( n \). This is impossible. In fact, by conjugation, we can take another element \( h_0 \in G_2 \) with the same property as \( h_0 \) and \( h_0 \circ h_1 \neq h_1 \circ h_0 \). Since \( B_{h_0} \) and \( B_{h_1} \) are commutative, \( A_{h_0} \) and \( A_{h_1} \) are non-commutative. Hence, \( A_{h_0} \) and \( A_{h_1} \) are two non-commutative Mobius transformations of infinite order with a common fixed \( \epsilon_0 \), which implies that the group generated by \( (A_{h_0}, B_{h_0}) \) and \( (A_{h_1}, B_{h_1}) \) is not discrete and we have a contradiction. Therefore, \( \mathcal{A} \) is an Abelian discrete subgroup of \( \text{Aut}(D) \). Then \( \mathcal{A} \) is generated by an element \( A_{g_1} \) for some \( g_1 \in G_1 \) with \( g_1 \neq 1 \). Take an element \( g_2 \in G_1 \) with \( g_2 \circ g_1 \neq g_1 \circ g_2 \). Let \( A_{g_2} = (A_{g_2})^n \) for some integer \( n \) and let \( g_3 = g_2 \circ g_2^{-1} \in G_1 \). Then \( g_3 \neq 1 \) and \( (A_{g_2}, B_{g_2}) = (1, B_{g_2} \circ B_{g_2}^{-1}) \). If we set \( \bar{g} = h_0 \circ g_0 \circ h_0^{-1} \), then we have \( (A_{g_2}, B_{g_2}) = (1, B_{g_2} \circ B_{g_2}^{-1}) \). Then \( (A, 1) \) and \( (A_{g_2}, B_{g_2}) \) are commutative and so are \( (\gamma^n, 1) \) and \( (1, \gamma) \). Then, by the relation (1) of §1, we have \( \omega_1 \circ \bar{g} \circ \omega_1^{-1} = \bar{g} \). Since \( \omega_1 \circ h_0 \circ \omega_1^{-1} = g_0 \circ h_0 \circ g_0^{-1} \) and \( \omega_1 \circ g_0 \circ \omega_1^{-1} = g_0 \), we have \( g_0 \circ (g_0 \circ h_0 \circ g_0^{-1} \circ h_0) = (g_0 \circ h_0 \circ g_0^{-1} \circ h_0) \circ g_0 \). Similarly, it can be proved that \( g_3 \) and \( h_n = g_0 \circ h_0 \circ g_0^{-1} \circ h_0 \) are commutative for any integer \( n \), which implies that \( \text{Fix}(g_3) = \text{Fix}(h_n) \) for any non-zero integer \( n \). This is impossible. In fact, by conjugation, we may assume that \( h_n(\bar{z}) = k^n \bar{z} \) for some constant \( k > 1 \) and \( g_n(\bar{z}) = (az+b)/(cz+d) \) with \( ad-bc=1 \). Since \( G \) is discrete and since \( g_0 \) and \( h_0 \) are non-commutative, we have \( g_0(0) \neq 0 \) and \( g_0(\infty) \neq 0 \), which implies that \( b \neq 0 \) and \( c \neq 0 \). By direct computation, we have

\[
(h_n)^2 = \frac{(ad-k^n bc)z + (1-k^{-2n})ab}{(1-k^n)cdz + ad-k^{-2n}bc}.
\]

If \( a = 0 \), then the relation \( ad-bc=1 \) implies that \( bc = -1 \) and we have

\[
h_n(\bar{z}) = \frac{k^n \bar{z}}{(1-k^n)cdz + k^{-2n}}.
\]

Since both \( h_0 \) and \( h_n \) are Mobius transformations of infinite order with a common fixed point \( z = 0 \) and since \( G \) is discrete, we have \( \text{Fix}(h_0) = \text{Fix}(h_0) \), that
is, \( h_n(\infty) = \infty \). Hence, we have \((1-k^{2n})cd = 0 \). Since \( k > 1 \) and \( c \neq 0 \), we have \( d = 0 \) and \( tr(g_0) = 0 \). Hence, \( g_0 \) is an elliptic element and we have a contradiction. Therefore, we have \( a \neq 0 \). Similarly, it can be shown that \( b \neq 0 \), \( c \neq 0 \) and \( d \neq 0 \).

Now, by direct computation, the fixed points \( z_n \) of \( h_n \) are given by the formula

\[
 z_n = \frac{(k^{-2n} - k^{2n})bc \pm \sqrt{((2ad - (k^{2n} + k^{-2n})bc)^2 - 4}}}{2(1 - k^{2n})cd}.
\]

Then the two fixed points go to 0 and \( b/d \) as \( n \to +\infty \) and they go to \( \infty \) and \( a/c \) as \( n \to -\infty \). On the other hand, since \( F_{1} \circ (g_0) = F_{1} \circ (h_n) \) for any non-zero integer \( n \), we have a contradiction. This completes the proof of Lemma 2.

**Lemma 3.** If \( X \) is biholomorphic to the polydisc \( \mathcal{D}^2 \) and the homotopic monodromy \( M_{\gamma_0} \) of \( \gamma_0 \) is of finite order, then the homotopic monodromy group \( \mathcal{M} \) of \( (X, \pi, R) \) is a finite group.

**Proof.** Let \( M_{\gamma_0} = \langle \omega_{\gamma_0} \rangle \) for some \( \omega_{\gamma_0} \in N(G) \). Since \( (M_{\gamma_0})^{-1} = 1 \) for some integer \( m \), we may assume that \( \langle \omega_{\gamma_0} \rangle^{-1} \) is represented by the identity mapping on the upper half-plane \( U \).

We use the notations in the proof of Lemma 1. By Lemma 1, we may assume that \( F \circ (\gamma_0^n, 1) \circ F^{-1} \) is equal to \((A, 1)\) is of type (1). Take an element \( \delta \in \Gamma \) with \( \gamma_0 \circ \delta \neq \delta \circ \gamma_0 \). Set \( F \circ (\delta, 1) \circ F^{-1} = (X, Y) \). We may assume that \((X, Y)\) is of type (I) and we have \( F \circ (\delta \circ \gamma_0^n \circ \delta^{-1}, 1) \circ F^{-1} = (X_0 \circ A \circ X^{-1}, 1) \). If \( X \) is of finite order, then \((X_0 \circ A \circ X^{-1}, 1) = (A, 1) \) for some integer \( n \). Hence, we have \((\gamma_0^n, 1) = (\delta \circ \gamma_0^n \circ \delta^{-n}, 1) \), which implies that \( \gamma_0^n = \delta \circ \gamma_0 \circ \delta^{-n} \). Hence, \( \gamma_0 \) and \( \delta \) are commutative and we have a contradiction. Therefore, \( X \) is of infinite order. Similarly, it is shown that \( A \) and \( X \) are non-commutative. Since \( (\omega_{\gamma_0})^{-1} = 1 \), we have \( \omega_{\delta \circ \gamma_0^n \circ \delta^{-1}} = 1 \) and the relation (1) of §1 implies that \((\delta \circ \gamma_0^n \circ \delta^{-1}, 1) \) and \((1, g)\) are commutative. Hence, we have \((X_0 \circ A \circ X^{-1}, A, B_g) = (A \circ X \circ A \circ X^{-1}, B_g) \), that is, \((X_0 \circ A \circ X^{-1}) \circ A_g = A_g \circ (X_0 \circ A \circ X^{-1}) \) for all \( g \in G \). Assume that \( A_g \neq 1 \) for some \( g \in G \) with \( g \neq 1 \). Since \( \text{Fix}(A) = \text{Fix}(A_g) = \text{Fix}(X_0 \circ A \circ X^{-1}) = X(\text{Fix}(A)) \), \( A \) and \( X \) have a common fixed point. Hence, \( A \) and \( X \) are non-commutative M"obius transformations of infinite order with a common fixed point, which implies that the group generated by \((A, 1)\) and \((X, Y)\) is not discrete. Therefore, we have a contradiction. Hence, \( A_g = 1 \) for all \( g \in G \). Then we have the relations \( F_1 \circ (1, g) = F_1, F_2 \circ (1, g) = B_g \circ F_2 \) and \( g \circ E_2 = E_2 \circ B_g \) for each \( g \in G \), where \( F = (F_1, F_2) \) is the above biholomorphic mapping, \( E = (E_0, E_2) = F^{-1} \) and \( g_i = X_{\phi(t)}(g) \) for each \( t \in D \). The relation \( F_1 \circ (1, g) = F_1 \) for all \( g \in G \) implies that \( F_1 \) is a bounded holomorphic automorphic function on \( D_{\phi(t)} \) for \( G_{\phi(t)} \) for each \( t \in D \). Since \( D_{\phi(t)}/G_{\phi(t)} \) is of finite type, the function \( F_1 \) is a constant function with a value \( c_i \in D \) on \( D_{\phi(t)} \) for
each \( t \in D \). Set \( D(t) = (\varepsilon_1, \varepsilon_2) \times (|z_2| < 1) \) for each \( t \in D \). Then \( F_2 \) induces an injective holomorphic function \( (F_2)_t: D_\Phi(t) \to D(t) \) for each \( t \in D \). Moreover, \( E_t \) is a constant function with a value \( t \) on \( D(t) \) and \( F_2 \circ E = 1_{D^\Phi} \). Hence, \( (F_2)_t: D_\Phi(t) \to D(t) \) is conformal and it induces a conformal mapping of \( D_\Phi(t)/\Gamma \) onto \( D(t)/\mathcal{B} \) for each \( t \in D \), where \( \mathcal{B} = \{ B_p \mid g \in G \} \) is a finitely generated Fuchsian group with no elliptic elements. Since all the Riemann surfaces \( D(t)/\mathcal{B} \) are conformally equivalent, all the fibers \( S_p, p \in R \), are also conformally equivalent. Hence, Theorem 2 implies that the homotopic monodromy group \( \mathcal{M} \) of \( (X, \pi, R) \) is a finite group. This completes the proof of Lemma 3.

Now, we can prove Theorem 3. If the homotopic monodromy group \( \mathcal{M} \) of \( (X, \pi, R) \) is a finite group, then Theorem 2 implies that the mapping \( \Phi: D \to T(G) \) is a constant mapping with a value \( \phi_0 \). Hence, the universal covering space \( \tilde{X} \) of \( X \) is equal to \( D \times D_{\phi_0} \) which is biholomorphic to the polydisc \( D^2 \).

Conversely, if \( \tilde{X} \) is biholomorphic to \( D^2 \), then Lemmas 2 and 3 imply that \( \mathcal{M} \) is a finite group. This completes the proof of Theorem 3.

5. Proof of Theorem 4. If \( \tilde{X} \) is biholomorphic to the polydisc \( D^2 \), then it is clear that \( \text{Aut}(\tilde{X}) \) is not discrete. Conversely, assume that \( \text{Aut}(\tilde{X}) \) is not discrete. Since the fibers of \( (X, \pi, R) \) are compact, Theorem 3 in Shabat [10] implies that \( \text{Aut}(\tilde{X}) \) is transitive. Hence, by E. Cartan's Theorem, the homogeneous bounded domain \( \tilde{X} \) in \( C^2 \) is biholomorphic to the unit ball \( B_2 \) or the polydisc \( D^2 \). By Theorem 1, \( \tilde{X} \) is not biholomorphic to \( B_2 \). Therefore, \( \tilde{X} \) is biholomorphic to \( D^2 \). This completes the proof of Theorem 4.

6. Examples and problems. We give the following typical examples of \( (X, \pi, R) \).

**Example 1.** Let \( S \) be a Riemann surface of finite type \((g, n)\) with \( 2g-2+n > 0 \) and let \( R \) be an open Riemann surface of finite type whose universal covering space is the upper half-plane. Let \( X = R \times S \) and let \( \pi \) be the canonical projection of \( X \) onto \( R \). Then \( (X, \pi, R) \) is a holomorphic family of Riemann surfaces of type \((g, n)\) over \( R \). All the fibers are conformally equivalent to \( S \) and the homotopic monodromy group \( \mathcal{M} \) is trivial. It is clear that the universal covering space \( \tilde{X} \) of \( X \) is biholomorphic to the polydisc \( D^2 \). Theorem 1 implies that \( \tilde{X} \) is not biholomorphic to the unit ball \( B_2 \). Hence, Theorem 1 is a generalization of the famous theorem due to Poincaré which asserts that the polydisc \( D^2 \) is not biholomorphic to the unit ball \( B_2 \).

**Example 2.** We set
Let $\pi: X \to R$ be the canonical projection. Then $(X, \pi, R)$ is a holomorphic family of Riemann surfaces of type $(1, 1)$ over $R$ and its homotopic monodromy group $\mathcal{M}$ is a finite cyclic group. All the fibers $S_t$ are conformally equivalent and the universal covering space $\tilde{X}$ of $X$ is biholomorphic to the polydisc $D^2$.

**Example 3.** We set

$$R = \mathbb{C} - \{0, 1, 2, 3\},$$
$$X = \{(x, y, z, t) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{C} \mid y^2z^3 = x(x-zt)(x-z)(x-2z)(x-3z)\},$$

where $\mathbb{P}^2(\mathbb{C})$ is the two-dimensional complex projective space and $(x, y, z)$ are the homogeneous coordinates of $\mathbb{P}^2(\mathbb{C})$. Let $\pi: X \to R$ be the canonical projection. Then $(X, \pi, R)$ is a holomorphic family of Riemann surfaces of type $(2, 0)$ and its homotopic monodromy group $\mathcal{M}$ is an infinite group. All the fibers $S_t, t \in R$, are not conformally equivalent. Theorems 1 and 2 imply that the universal covering space $\tilde{X}$ of $X$ is not biholomorphic to $B_2$ or $D^2$. Moreover, Theorem 4 implies that $\text{Aut}(\tilde{X})$ is a discrete group.

Let $(X, \pi, R)$ be a holomorphic family of Riemann surfaces of type $(g, n)$ with $2g-2+n > 0$. Let us give the following problems.

**Problem 1.** Let $R$ be a closed Riemann surface of genus $g_0 > 1$. Then prove that the universal covering space $\tilde{X}$ of $X$ is not biholomorphic to the unit ball $B_2$. (cf. Shabat [10].)

**Problem 2.** Let $X$ be a Stein manifold. Then prove that the universal covering space $\tilde{X}$ of $X$ is biholomorphic to the polydisc $D^2$ if and only if $\text{Aut}(\tilde{X})$ is not a discrete group. (cf. Shabat [10].)

**Problem 3.** When $\text{Aut}(\tilde{X})$ is a discrete group, can we write down all the elements of $\text{Aut}(\tilde{X})$? Note that the covering transformation group $\mathcal{G}$ of $\Pi: \tilde{X} \to X$ is a subgroup of $\text{Aut}(\tilde{X})$ and its elements are known as in §1.

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**References**


Department of Mathematics  
College of General Education  
Osaka University  
Toyonaka, Osaka 560  
Japan

Current address:  
Department of Mathematics  
White Hall  
Cornell University  
Ithaca, New York 14853  
U.S.A.