Title	Universal covering spaces of certain quasiprojective algebraic surfaces
Author(s)	Imayoshi, Yôichi
Citation	Osaka Journal of Mathematics. 1983, 20(3), p. 581-598
Version Type	VoR
URL	https://doi.org/10.18910/6965
rights	
Note	

Imayoshi, Y.
Osaka J. Math.
20 (1983), 581-598

UNIVERSAL COVERING SPACES OF CERTAIN QUASI-PROJECTIVE ALGEBRAIC SURFACES

YOICHI IMAYOSHI

(Received October 23, 1981)

Introduction. In this paper we investigate some function-theoretic properties of universal covering spaces of certain quasi-projective algebraic surfaces.

Let $\hat{X}$ be a two-dimensional complex manifold and let C be a one-dimensional analytic subset of $\hat{X}$ or an empty set. Let R be a Riemann surface. We assume that a proper holomorphic mapping $\hat{\pi}$: $\hat{X}\rightarrow R$ satisfies the following two conditions: (i) $\hat{\pi}$ is of maximal rank at every point of $\hat{X}$, and (ii) by setting $X=\hat{X}-C$ and $\pi=\hat{\pi}|X$, the fiber $S_p=\pi^{-1}(p)$ over each point p of R is an non-singular irreducible analytic subset of X and is of fixed finite type (g, n) with $2g-2+n>0$ as a Riemann surface, where g is the genus of S_p and n is the number of punctures of S_p. We call such a triple (X, π, R) a holomorphic family of Riemann surfaces of type $(,g\ n)$ over R. We also say that X has a holomorphic fibration (X, π, R) of type (g, n).

We assume throughout this paper R is a non-compact Riemann surface of finite type and its universal covering space is the unit disc $D=(|t|<1)$ in the complex t-plane.

P.A. Griffiths [2] got the following uniformization theorem of quasi-projective algebraic surfaces. Let $\hat{X}$ be a two-dimensional, irreducible, smooth, quasi-projective algebraic varitey over the complex numbers. Then for every point x in $\hat{X}$, there exists a Zariski neighborhood X of x in $\hat{X}$ such that X has a holomorphic fibration (X, π, R) as above. Then the universal covering space $\tilde{X}$ of X is topologically a cell. Griffiths proved that $\tilde{X}$ is biholomorphically equivalent to a bounded domain of holomorphy in C^2 using the theory of simultaneous uniformization of Riemann surfaces due to Bers. (cf. Bers [1].) The function-theoretic properties of such interesting domains $\tilde{X}$ are little studied. (cf. Shabat [10].)

At the begining, in §1, we recall some notations and results of [3], [4] and [5] which will be used later. Let $\mathcal{M}$ be the homotopic monodromy group of (X, π, R), which will be defined in §1. Then we get the following theorems in §2, §3, §4 and §5.

Theorem 1. *The universal covering space $\tilde{X}$ of X is not biholomorphically equivalent to the two-dimensional unit ball $B_2=(|z|^2+|w|^2<1)$.*

Corollary. *The universal covering space $\tilde{X}$ of X is not biholomorphically equivalent to any two-dimensional strongly pseudoconvex domains.*

Theorem 2. *The homotopic monodromy group $\mathcal{M}$ is a finite group if and only if all the fibers S_p are conformally equivalent.*

Theorem 3. *The homotopic monodromy group $\mathcal{M}$ is a finite group if and only if $\tilde{X}$ is biholomorphically equivalent to the two-dimensional polydisc $D^2=(|z|<1)\times(|w|<1)$.*

Theorem 4. *If (X, π, R) is of type $(g, 0)$ with $g>1$, then $\tilde{X}$ is biholomorphic to the polydisc D^2 if and only if the analytic automrophism group* $\mathrm{Aut}(\tilde{X})$ *of $\tilde{X}$ is not a discrete group.*

In the last § 6, we give some examples of these quasi-projective algebraic surfaces X and some related problems.

1. Preliminaries. We shall briefly explain some notations and results in [3], [4] and [5] which will be used later.

Let G be a finitely generated Fuchsian group of the first kind with no elliptic elements acting on the upper half-plane U such that the quotient space $S=U/G$ is a finite Riemann surface of type (g, n). Let $Q_{\mathrm{norm}}(G)$ be the set of all quasi-conformal automorphisms w of U leaving $0, 1, \infty$ fixed and satisfying $wGw^{-1}\subset SL'(2; R)$, where $SL'(2; R)$ is the set of all real Möbius transformations. Two elements w_1 and w_2 of $Q_{\mathrm{norm}}(G)$ are equivalent if $w_1=w_2$ on the real axis $\boldsymbol{R}$. The Teichmüller space $T(G)$ of G is the set of all equivalence classes $[w]$ obtained by classifying $Q_{\mathrm{norm}}(G)$ by the above equivalence relation.

Let w_μ be the element of $Q_{\mathrm{norm}}(G)$ with a Beltrami coefficient $\mu\in L^\infty(U, G)_1$ and let W^μ be a quasiconformal automorphism of the Riemann sphere $\hat{\boldsymbol{C}}$ such that W^μ has the Beltrami coefficient μ on the upper half-plane U, and is conformal on the lower half-plane L, and

$$W^\mu(z)=\frac{1}{z+i}+O(|z+i|)$$

as z tends to $-i$. This mapping W^μ is uniquely determined by $[w_\mu]$ up to the equivalence relation, that is, $w_\mu=w_\nu$ on $\boldsymbol{R}$ if and only if $W^\mu=W^\nu$ on L. Let ϕ_μ be the Schwarzian derivative of W^μ. Then ϕ_μ is an element of the space $B_2(L, G)$ of bounded holomorphic quadratic differentials for G on L. Bers proved that the mapping sending $[w_\mu]$ into ϕ_μ is a biholomorphic mapping of $T(G)$ onto a holomorphically convex bounded domain of $B_2(L, G)$, which is denoted by the same notation $T(G)$. The space $B_2(L, G)$ is a $(3g-3+n)$-dimensional complex vector space. We associate with each ϕ of $B_2(L, G)$ a uniquely determined solution $W_\phi=w_1/w_2$ of the Schwarzian differential equation on L

$$(w''/w')'-\frac{1}{2}(w''/w')^2=\phi\,,$$

where w_1 and w_2 are the solutions of the linear differential equation on L

$$2w''+\phi w=0$$

normalized by the conditions $w_1=w_2'=1$ and $w_1'=w_2=0$ at $z=-i$. The homomorphism $G\to SL'(2, \boldsymbol{C})$ induced by ϕ, which carries g into $\hat{g}$ in such a way that $W_\phi\circ g=\hat{g}\circ W_\phi$, is denoted by χ_ϕ. Since each point ϕ of $T(G)$ is a Schwarzian derivative of some W^μ with $\mu\in L^\infty(U, G)_1$, we have $W_\phi=W^\mu$ on L. Hence W_ϕ is conformal on L and has a quasiconformal extension of $\hat{\boldsymbol{C}}$ onto itself, which is denoted by the same notation. If we set $G_\phi=\chi_\phi(G)=W_\phi\circ G\circ W_\phi^{-1}$ and $D_\phi=W_\phi(U)$, then G_ϕ is a quasi-Fuchsian group and the definitions are legitimate since D_ϕ is the complement of the closure of $W_\phi(L)$ and since $W_\phi|L$ depends only on ϕ. The Koebe's one-quarter theorem implies that $D_\phi\subset(|w|<2)$ for every ϕ of $T(G)$.

Let (X, π, R) be a holomorphic family of Riemann surfaces of type (g, n) with $2g-2+n>0$ and let $\rho\colon D\to R$ be the universal covering with the covering transformation group Γ. Then there exists a holomorphic mapping $\Phi\colon D\to T(G)$ such that the quotient space $D_{\Phi(t)}/G_{\Phi(t)}$ is conformally equivalent to $S_{\rho(t)}$ for every $t\in D$. We abbreviate $G_{\Phi(t)}$ to G_t and $D_{\Phi(t)}$ to D_t. We set

$$\tilde{X}=\{(t, w)\,|\,t\in D,\ w\in D_t\}\,.$$

This set $\tilde{X}$ is topologically equivalent to the two-dimensional polydisc D^2. Since $D_t\subset(|w|<2)$ for every $t\in D$, the set $\tilde{X}$ is a bounded domain in $\boldsymbol{C}^2$. We can also show that $\tilde{X}$ is a domain of holomorphy. Let F_t be the conformal mapping of D_t/G_t onto $S_{\rho(t)}$ induced by $\Phi(t)$ for every $t\in D$ and let Π be the holomorphic mapping of $\tilde{X}$ onto X sending (t, w) into $F_t(w)$. Then $\Pi\colon\tilde{X}\to X$ is the universal covering of X constructed by Griffiths [2].

Let $\mathcal{G}$ be the covering transformation group of the universal covering $\Pi\colon\tilde{X}\to X$. We can explicitly express the elements of $\mathcal{G}$ as follows. For each element $\gamma\in\Gamma$, the homotopic monodromy M_γ of γ is the element of the Teichmüller modular group $\mathrm{Mod}(G)$ of G with the property $\Phi\circ\gamma=M_\gamma\circ\Phi$. The subgroup $\mathcal{M}=\{M_\gamma\,|\,\gamma\in\Gamma\}$ of $\mathrm{Mod}(G)$ is called the homotopic monodromy group of (X, π, R). Denote by $N(G)$ the set of all quasiconformal automorphisms ω of U with $\omega\circ G\circ\omega^{-1}=G$. Take an element ω_γ of $N(G)$ which induces M_γ, that is, $\langle\omega_\gamma\rangle=M_\gamma$. We may assume that $\omega_{\gamma\circ\delta}=\omega_\gamma\circ\omega_\delta$ for all γ, $\delta\in\Gamma$.

For each $t \in D$, let $[w_{\mu_t}]$ be the point of $T(G)$ with a Beltrami coefficient μ_t corresponding to the holomorphic quadratic differential $\Phi(t)$ in $B_2(L, G)$. For each $g \in G$, we set $w_{\nu_t} = \lambda \circ w_{\mu_t} \circ (\omega_\gamma \circ g)^{-1} \in Q_{\text{norm}}(G)$, where λ is a real Möbius transformation. If we set

$$(\gamma, g)(t, w) = (\gamma(t), W^{\nu_t} \circ (\omega_\gamma \circ g) \circ (W^{\mu_t})^{-1}(w)),$$

then the mapping (γ, g) is an analytic automorphism of $\tilde{X}$ for all $\gamma \in \Gamma$, $g \in G$. Now the covering transformation group $\mathcal{G}$ is identical with the set $\Gamma \times G$. By definition, we have the relation

$$(1) \qquad (\gamma, g) \circ (\delta, h) = (\gamma \circ \delta, \omega_\delta^{-1} \circ g \circ \omega_\delta \circ h)$$

for all γ, $\delta \in \Gamma$ and g, $h \in G$, that is, $\mathcal{G}$ is a semi-direct product of Γ by G. It is noted that $(\gamma, g) = (\delta, h)$ if and only if $\gamma = \delta$ and $g = h$.

Now, we have the following fundamental theorem. (See [3] and [4].)

Theorem. *Let (X, π, R) be a holomorphic family of Riemann surfaces of type (g, n) with $2g-2+n>0$. Take a puncture p_0 of R. Let t_0 be a parabolic fixed point with $\rho(t_0)=p_0$ and let γ_0 be a generator of the stabilizer of t_0 in Γ. Then there exists an element ϕ_0 in the closure of $T(G)$ in $B_2(L, G)$ such that the holomorphic mapping $\Phi(t)$: $D \to T(G)$ converges to ϕ_0 uniformly as t tends to t_0 through any cusped region at t_0 in D. The homotopic monodromy M_{γ_0} is of finite order if and only if $\phi_0 \in T(G)$, and is of infinite order if and only if $\phi_0 \in \partial T(G)$, where $\partial T(G)$ is the boundary of $T(G)$ in $B_2(L, G)$. In the latter case, the boundary group G_{ϕ_0} corresponding to $\phi_0 \in \partial T(G)$ is a regular b-group.*

2. Proof of Theorem 1. Assume that there exists a biholomorphic mapping F: $\tilde{X} \to B_2$. Let p_0 be a puncture of R and let t_0 be a parabolic fixed point with $\rho(t_0)=p_0$. By the above Theorem, there is an element ϕ_0 of the closure of $T(G)$ such that holomorphic mapping $\Phi(t)$ converges to ϕ_0 uniformly as t tends to t_0 through any cusped region Δ at t_0 in D. Let G_{ϕ_0} be the Kleinian group corresponding to ϕ_0, which is a quasi-Fuchsian group or a regular b-group. Take a component Ω of G_{ϕ_0} which is not equal to the invariant component of G_{ϕ_0} corresponding to the lower half-plane L.

Let K be an arbitrary compact subset of Ω. Then $K \subset D_t = D_{\Phi(t)}$ for any $\Delta \in t$ sufficiently near t_0. Hence, by the diagonal method, we can take a sequence $\{t_n\}_{n=1}^{\infty}$ in Δ such that $t_n \to t_0$ as $n \to \infty$ and such that $F(t_n, w) = (F_1(t_n, w), F_2(t_n, w))$ converges to a holomorphic mapping $f(w) = (f_1(w), f_2(w))$: $\Omega \to \partial B_2$ uniformly on any compact subset of Ω as $n \to \infty$. Since

$$|f_1(z)|^2 + |f_2(z)|^2 = 1,$$

we have

$$\frac{\partial^2}{\partial z\partial\bar{z}}(|f_1(z)|^2+|f_2(z)|^2)=\left|\frac{\partial f_1}{\partial z}(z)\right|^2+\left|\frac{\partial f_2}{\partial z}(z)\right|^2=0\,,$$

which implies that $\frac{\partial f_1}{\partial z}=\frac{\partial f_2}{\partial z}=0$ on Ω. Hence $f=(f_1, f_2)$ is a constant mapping. We may assume that f is a constant mapping with the value $(1, 0)\in\partial B_2$.

Denote by G_Ω the stabilizer of Ω in G_{ϕ_0}. Let $G_0=\chi_{\phi_0}^{-1}(G_\Omega)$, $g_t=\chi_{\Phi(t)}(g)$ for $g\in G$, $t\in D$, and $g_{t_0}=\chi_{\phi_0}(g)$ for $g\in G$. Set $A_g=F\circ(1, g)\circ F^{-1}\in \mathrm{Aut}(B_2)$ for each $g\in G$, where 1 is the identity element of Γ. Since $g_t\to g$ as $t\to t_0$ through Δ for all $g\in G$, and since $g_{t_0}(\Omega)=\Omega$ for all $g\in G_0$, the boundary point $(1, 0)$ of B_2 is a fixed point of A_g for all $g\in G_0$.

We set

$$S=\{(u, v)\in \boldsymbol{C}^2 \,|\, \mathrm{Im}(u)>|v|^2\}\,,$$

where $\mathrm{Im}(u)$ is the imaginary part of u. This set S is a Siegel domain of the second kind. We put

$$z_1=\frac{u-i}{u+i}\,,\quad z_2=\frac{2v}{u+i}\,.$$

Then the mapping T: $S\to B_2$ sending (u, v) into (z_1, z_2) is biholomorphic and it carries the boundary point $(\infty, 0)$ of S into the boundary point $(1, 0)$ of B_2. It is known that an analytic automorphism $\Psi\in\mathrm{Aut}(S)$ of S has a fixed point $(\infty, 0)$ if and only if

$$\Psi(u, v)=(|a|^2u+2ia\bar{b}v+c+i|b|^2,\ av+b)\,,$$

where a is a non-zero complex number, b is a complex number and c is a real number. (See Pyatetskii-Shapiro [8, Chap. 1, § 2, Thm. 1].)

Let $A_g^*=T^{-1}\circ A_g\circ T\in\mathrm{Aut}(S)$ for each $g\in G$. Then the point $(\infty, 0)$ is a fixed point of A_g^* for all $g\in G_0$. Hence,

$$A_g^*(u, v)=(|a_g|^2u+2ia_g\bar{b}_gv+c_g+i|b_g|^2,\ a_gv+b_g)$$

for all $g\in G_0$.

i) If $|a_{g_0}|\neq 1$ for some $g_0\in G_0$, there exists an element $\Psi\in\mathrm{Aut}(S)$ with $\Psi(\infty, 0)=(\infty, 0)$ such that $\Psi\circ A_{g_0}^*\circ\Psi^{-1}(u, v)=(|a_0|^2u, a_0v)$, where a_0 is a non-zero complex number with $|a_0|\neq 1$. Take an element $h\in G_0$ such that $g_0\circ h\neq h\circ g_0$. We set

$$U(u, v)=\Psi\circ A_{g_0}^*\circ\Psi^{-1}(u, v)=(|a_0|^2u,\ a_0v)\,,$$
$$V(u, v)=\Psi\circ A_h^*\circ\Psi^{-1}(u, v)=(|a|^2u+2ia\bar{b}v+c+i|b|^2,\ av+b)\,.$$

Since $g_0\circ h\neq h\circ g_0$, we have $U\circ V\neq V\circ U$, which implies that $b\neq 0$ or $c\neq 0$. By direct computation, we have

$$W_n(u, v) = V\circ U^n\circ V^{-1}\circ U^{-n}(u, v)$$
$$= (u+2i(1-\bar{a}_0^n)\bar{b}v+(1-|a_0|^{2n})c+2|b|^2 \operatorname{Im}(a_0^n)+i|(1-a_0^n)b|^2, v+(1-a_0^n)b)$$

for any integer n. Since $|a_0| \neq 1$, we have

$$W_n(u, v) \to W(u, v) = (u+2i\bar{b}v+c+i|b|^2, v+b)$$

as $n\to\infty$ or $-\infty$, which implies that $(F^{-1}\circ T\circ\Psi^{-1})^{-1}\circ\mathcal{G}\circ(F^{-1}\circ T\circ\Psi^{-1})$ is not discrete. Hence, $\mathcal{G}$ is not discrete and we have a contradiction.

ii) If $|a_g|=1$ for all $g\in G_0$ and if $a_{g_0}\neq 1$ for some $g_0\in G_0$, there exists an element $\Psi\in\operatorname{Aut}(S)$ with $\Psi(\infty, 0)=(\infty, 0)$ such that $\Psi\circ A_{g_0}^*\circ\Psi^{-1}(u, v)=(u+c_0, a_0v)$, where a_0 is a complex number with $|a_0|=1$ and $a_0\neq 1$, and c_0 is a real number. Take an element $h\in G_0$ such that $g_0\circ h\neq h\circ g_0$. We set

$$U(u, v) = \Psi\circ A_{g_0}^*\circ\Psi^{-1}(u, v) = (u+c_0, a_0v),$$
$$V(u, v) = \Psi\circ A_h^*\circ\Psi^{-1}(u, v) = (u+2ia\bar{b}v+c+i|b|^2, av+b),$$

where a is a complex number with $|a|=1$, b is a complex number, and c is a real number. Since $h\circ g_0^n\neq g_0^n\circ h$ for all integer n, we have $V\circ U^n\neq U^n\circ V$ which implies that $b\neq 0$ and $a_0^n\neq 1$. If we set $a_0=e^{i\pi\theta}$, then θ is an irrational number. By direct calculation, we have

$$W_n(u, v) = V\circ U^n\circ V^{-1}\circ U^{-n}(u, v)$$
$$= (u+2i\bar{b}(1-\bar{a}_0^n)v+2|b|^2 \operatorname{Im}(a_0^n)+i|b(1-a_0^n)|^2, v+b(1-a_n^0))$$

for any integer n. Since θ is an irrational number, there exists a sequence $\{n_j\}$ of integers such that $(a_0)^{n_j}\to 1$ as $j\to\infty$. Therefore, $W_{n_j}(u, v)\to W(u, v)=(u, v)$ as $j\to\infty$, which implies that $(F^{-1}\circ T\circ\Psi^{-1})^{-1}\circ\mathcal{G}\circ(F^{-1}\circ T\circ\Psi^{-1})$ is not discrete. Hence, $\mathcal{G}$ is not discrete and we have a contradiction.

iii) If $a_g=1$ for all $g\in G_0$, we have

$$A_g^*(u, v) = (u+2i\bar{b}_g v+c_g+i|b_g|^2, v+b_g).$$

Therefore,

$$A_g^*\circ A_h^*\circ(A_g^*)^{-1}\circ(A_h^*)^{-1}(u, v) = (u-4\operatorname{Im}(\bar{b}_g b_h), v).$$

Hence, the commutator subgroup of the group $\{A_g^*\,|\,g\in G_0\}$ is commutative, which implies that the commutator subgroup $[G_0, G_0]$ of G_0 is commutative. Hence we have a contradiction. This completes the proof of Theorem 1.

Now, let us assume that there exists a strongly pseudoconvex domain Ω in C^2 which is biholomorphically equivalent to $\tilde{X}$. Let F: $\tilde{X}\to\Omega$ be a biholomorphic mapping. Since $\mathcal{G}^*=F\circ\mathcal{G}\circ F^{-1}$ is an infinite subgroup of $\operatorname{Aut}(\Omega)$ and acts on Ω properly discontinuously, for any point ζ of Ω, there exists an infinite sequence $\{T_n\}$ of $\mathcal{G}^*$ such that $T_n(\zeta)$ tends to a boundary point ζ_0 of Ω

as $n\to\infty$. Therefore, the Proposition in Rosay [9] implies that Ω is biholomorphically equivalent to the unit ball B_2. Hence, we have a contradiction and this completes the proof of Corollary.

3. Proof of Theorem 2. If all the fibers S_p are conformally equivalent, then the mapping $\Phi\colon D\to T(G)$ is a constant mapping with a value $q_0\in T(G)$. By the relation $M_\gamma\circ\Phi=\Phi\circ\gamma$, the point q_0 is a fixed point of all $M_\gamma\in\mathcal{M}$. Since the modular group $\mathrm{Mod}(G)$ of G acts on $T(G)$ properly discontinuously, the subgroup $\mathcal{M}$ of $\mathrm{Mod}(G)$ also acts on $T(G)$ properly discontinuously. Hence, $\mathcal{M}$ is a finite group.

Conversely, assume that $\mathcal{M}$ is finite, and let Γ_0 be the kernel of the monodromy map $\gamma\mapsto M_\gamma$. Then Γ_0 has finite index in Γ, so $R_0=D/\Gamma_0$ is a Riemann surface of finite type. Since $\Phi\circ\gamma=\Phi$ for all γ in Γ_0, the holomorphic map $\Phi\colon D\to T(G)$ factors through R_0. Since $T(G)$ is bounded, every holomorphic map from R_0 to $T(G)$ is constant, so Φ is a constant map. Hence, all the fibers S_p are conformally equivalent and this completes the proof of Theorem 2.

4. Proof of Theorem 3. Assume that there exists a biholomorphic mapping $F=(F_1, F_2)\colon \tilde{X}\to D^2$. If we set $\mathcal{G}^*=F^*(\mathcal{G})=F\circ\mathcal{G}\circ F^{-1}$, then $\mathcal{G}^*$ is a properly discontinuous subgroup of the analytic automorphism group $\mathrm{Aut}(D^2)$.

We recall that any analytic automorphism of $D^2=(|z_1|<1)\times(|z_2|<1)$ is either one of the following two types:

$$\text{(I)} \qquad (A, B)(z_1, z_2)=(A(z_1), B(z_2))\,,$$

$$\text{(II)} \qquad (A, B)(z_1, z_2)=(A(z_2), B(z_1))\,,$$

where $A, B\in\mathrm{Aut}(D)$. (See Narasimhan [7, Chap. 5, Prop. 3].) Note that $(A, B)^2$ is of type (I) for all $(A, B)\in\mathrm{Aut}(D^2)$.

We also recall the following results, which will be used frequently in this section. (See Lehner [6, Chap. 2, §9, Thm. 1 and Thm. 2, and Chap. 3, Thm. 2E].)

Two Möbius transformations are commutative if and only if they have the same set of fixed points provided that neither is the identity and provided that neither is a transformation of order two.

Let A be a hyperbolic or loxodromic transformation and let B be a Möbius transformation which has one and only one fixed point in common with A. Then the sequence $\{B\circ A^n\circ B^{-1}\circ A^{-n}\}$ of Möbius transformations converges to a Möbius transformation as $n\to\infty$ or $-\infty$.

By these results, we have the following assertion.

Let A, B be two Möbius transformations of infinite order with $A\circ B\neq B\circ A$ such that they have a common fixed point. Then the group generated by A, B is not discrete.

Let p_0 be a puncture of R, t_0 be a parabolic fixed point with $\rho(t_0)=p_0$ and let γ_0 be a generator of the stabilizer of t_0 in Γ. Then Theorem of § 1 implies that there exists an element ϕ_0 in the closure of $T(G)$ in $B_2(L, G)$ such that the mapping $\Phi(t)$: $D\to T(G)$ converges to ϕ_0 uniformly as $t\to t_0$ through any cusped region Δ at t_0 in D and such that the Kleinian group G_{ϕ_0} corresponding to ϕ_0 is a quasi-Fuchsian group or a regular b-group. Let $D_0=\Omega(G_{\phi_0})-\Delta(G_{\phi_0})$, where $\Omega(G_{\phi_0})$ is the region of discontinuity of G_{ϕ_0} and $\Delta(G_{\phi_0})$ is the invariant component of G_{ϕ_0} corresponding to the lower half-plane L. Then the quotient space

$$S_0=(D_0\cup\{\text{accidental parabolic fixed points of } G_{\phi_0}\})/G_{\phi_0}$$

is a Riemann surface of type (g, n) with or without nodes. Let $\{p_1, \cdots, p_k\}$ be the set of nodes of S_0, which may be empty. If π_0: $U\to S=U/G$ is the canonical projection and if α: $S\to S_0$ is the deformation as in § 3 of [4], then there exists a family $\{W_t\}_{t\in\Delta}$ of quasiconformal automorphisms on $\hat{C}$ such that W_t is conformal on L and has a Schwarzian derivative $\Phi(t)$ for all $t\in\Delta$ and such that W_t converges uniformly on any compact subset of $U_0=U-\pi_0^{-1}\circ\alpha^{-1}(\{p_1, \cdots, p_k\})$ to a locally quasiconformal mapping W_0: $U_0\to D_0$ as $t\to t_0$ through Δ. (See § 4 in [4].) Then the locally quasiconformal mapping W_0 induces the above deformation α: $S\to S_0$.

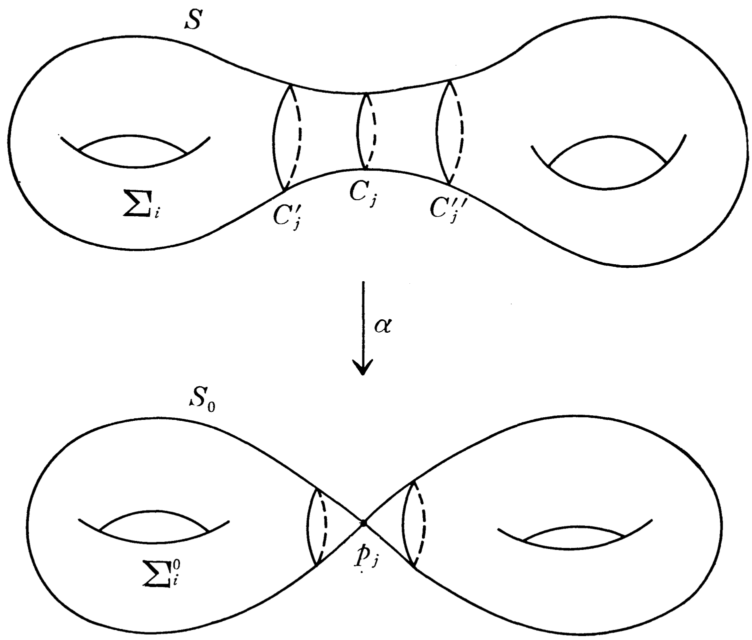

Figure 1

Let $\Sigma_1^0, \cdots, \Sigma_r^0$ be the parts of S_0, that is, the connected components of $S_0-\{p_1, \cdots, p_k\}$ and let $\Sigma_i=\alpha^{-1}(\Sigma_i^0)$ for each $i=1, \cdots, r$. Take a sufficiently small neighborhood $\delta_j=\{(z_1, z_2)\in \boldsymbol{C}^2 \mid z_1z_2=0, |z_1|<\varepsilon \text{ and } |z_2|<\varepsilon\}$ of a node p_j in S_0 for each $j=1, \cdots, k$ and set $\delta_0=\delta_1\cup\cdots\cup\delta_k$. If we set $C_j'=\alpha^{-1}((|z_1|=\varepsilon)\times(z_2=0))$ and $C_j''=\alpha^{-1}((z_1=0)\times(|z_2|=\varepsilon))$ for each $j=1, \cdots, k$, then the domain bounded by C_j' and C_j'' is an annulus on S. Let Σ_i' be the connected component of $S-\alpha^{-1}(\delta_0)$ contained in Σ_i for each $i=1, \cdots, r$. Then Σ_i' is homeomorphic to Σ_i. (See Figure 1.)

Take a point q_0 on S, which is fixed as a base point. Let (C, q) be a pair of a point q on S and a path C from q_0 to q on S. A pair (C_1, q_1) is equivalent to a pair (C_2, q_2) if and only if $q_1=q_2$ and $C_1\circ C_2^{-1}$ is homotopic to the point q_0. Then we can identify the universal covering space U of S with the set of all these equivalence classes $[C, q]$ and the covering transformation group of the universal covering $\pi_0\colon U\to S$ is identified with the fundamental group $\pi_1(S, q_0)$ of S with a base point q_0, that is,

$$G=\{[C_0]_* \mid [C_0]\in\pi_1(S, q_0)\}\ ,$$

where $[C_0]_*$ is a covering transformation sending $[C, q]$ into $[C_0\circ C, q]$ for $[C, q]\in U$. Suppose that $q_0\in C_1'$ throughout this section and set

$$G_1=\{[C_0]_* \mid C_0\in\pi_1(\Sigma_1, q_0)\}\ ,$$
$$U_1=\{[C, q] \mid q\in\Sigma_1 \text{ and } C \text{ is a path from } q_0 \text{ to } q \text{ on } \Sigma_1\}\ .$$

Then U_1 is a connected component of $\pi_0^{-1}(\Sigma_1)$, which is invariant under G_1. Since Σ_1' is homeomorphic to Σ_1, we have $G_1=\{[C_0]_* \mid C_0\in\pi_1(\Sigma_1', q_0)\}$. If we set $\Omega_1=W_0(U_1)$, then Ω_1 is a component of G_{Φ_0} and the isomorphism $\chi_{\Phi_0}\colon G\to G_{\Phi_0}$ induces an isomorphism $\chi_{\Phi_0}|G_1\colon G_1\to G_{\Omega_1}$, where G_{Ω_1} is the stabilizer of Ω_1 in G_{Φ_0}.

Let $(f_{\gamma_0})_*$ be an element of the modular group $\mathrm{Mod}(S)$ of the Teichmüller space $T(S)$ corresponding to the homotopic monodromy $M_{\gamma_0}=\langle\omega_{\gamma_0}\rangle\in\mathrm{Mod}(G)$ of γ_0. Since there exists a positive integer m such that $(f_{\gamma_0})^m$ is homotopic to a product d of ν-th powers of Dhen twists on S about Jordan curves mapped by $\alpha\colon S\to S_0$ into nodes, we may assume that the quasiconformal automorphism ω_1 of U with $\omega_1\circ G\circ\omega_1^{-1}=G$ and $\langle\omega_1\rangle=(M_{\gamma_0})^m$ is induced by d. Since $d|\Sigma_1'$ is the identity mapping, $\omega_1|U_1'$ is also the identity mapping, where U_1' is the connected component of $\pi_0^{-1}(\Sigma_1')$ which is contained in U_1. Note that U_1' is invariant under G_1. Hence, we have $\omega_1\circ g\circ\omega_1^{-1}=g$ for all $g\in G_1$.

Set $(A, B)=F\circ(\gamma_0^m, 1)\circ F^{-1}$, $(A_g, B_g)=F\circ(1, g)\circ F^{-1}$ for each $g\in G$, where 1 is the identity of Γ or G. We may assume that (A, B) is of type (I).

By the same reasoning as in § 2, we can choose an infinite sequence $\{t_n\}_{n=1}^{\infty}$ of Δ such that $t_n\to t_0$ as $n\to\infty$ and such that $F(t_n, w)=(F_1(t_n, w), F_2(t_n, w))$ converges to a holomorphic mapping $f(w)=(f_1(w), f_2(w))\colon \Omega_1\to\partial D^2$ uniformly on any compact subset of Ω_1 as $n\to\infty$. Since $\partial D^2=\{(|z_1|=1)\times(|z_2|\leqq 1)\}\cup\{(|z_1|\leqq 1)\times(|z_2|=1)\}$, we have $|f_1(w)|=1$ or $|f_2(w)|=1$ for each $w\in\Omega_1$. Hence, $|f_1|=1$ or $|f_2|=1$ on a non-empty open subset of Ω_1, which implies that f_1 or f_2 is a constant function with a value in ∂D. So we suppose that f_1 is a constant function with a value $c_1\in\partial D$. Now, we have the following lemma.

Lemma 1. *The analytic automorphism $(A, B)=F\circ(\gamma_0^m, 1)\circ F^{-1}$ of D^2 is equal to $(A, 1)$ and A is of infinite order. For each $g\in G_1$, the analytic automorphism $(A_g, B_g)=F\circ(1, g)\circ F^{-1}$ of D^2 is of type* (I) *and B_g is of infinite order provided that $g\neq 1$. Moreover, the group $\mathcal{A}=\{A_g\,|\,g\in G_1\}$ is commutative.*

Proof. Since $\omega_1\circ g\circ\omega_1^{-1}=g$ for each $g\in G_1$, the relation (1) of § 1 implies that $(1, g)\circ(\gamma_0^m, 1)=(\gamma_0^m, 1)\circ(1, g)$ for each $g\in G_1$. Hence, we have $(A_g, B_g)\circ(A, B)=(A, B)\circ(A_g, B_g)$ for each $g\in G_1$. If (A_g, B_g), $g\in G_1$, is of type (I), then $A_g\circ A=A\circ A_g$ and $B_g\circ B=B\circ B_g$. In general, denote by $\mathrm{Fix}(T)$ the set of fixed points in $\hat{C}$ of an element $T\in\mathrm{Aut}(D)$. Then, if neither A nor A_g is the identity, we have $\mathrm{Fix}(A)=\mathrm{Fix}(A_g)$. Similarly, if neither B nor B_g is the identity, then $\mathrm{Fix}(B)=\mathrm{Fix}(B_g)$.

Assume that neither A nor B is the identity. Take two non-commutative elements g_0, $h_0\in G_1$ such that both (A_{g_0}, B_{g_0}) and (A_{h_0}, B_{h_0}) are of type (I). If at least one of A_{g_0}, A_{h_0} is the identity, then clearly A_{g_0} and A_{h_0} are commutative. If $A_{g_0}\neq 1$ and $A_{h_0}\neq 1$, then $\mathrm{Fix}(A)=\mathrm{Fix}(A_{g_0})=\mathrm{Fix}(A_{h_0})$, which implies that A_{g_0} and A_{h_0} are commutative. Hence, in any case, A_{g_0} and A_{h_0} are commutative. Similarly, it is shown that B_{g_0} and B_{h_0} are commutative. Hence, (A_{g_0}, B_{g_0}) and (A_{h_0}, B_{h_0}) are commutative and so are g_0 and h_0. We have a contradiction. Therefore, at least one of A, B is equal to the identity. Since γ_0 is of infinite order, either A or B is of infinite order. Hence, we have the two cases: (i) A is of infinite order and $B=1$, (ii) $A=1$ and B is of infinite order. Assume that $A=1$ and B is of infinite order. Then we have $A_{g_0}\circ A_{h_0}\neq A_{h_0}\circ A_{g_0}$, $B_{g_0}\circ B_{h_0}=B_{h_0}\circ B_{g_0}$ and we have that A_{g_0} and A_{h_0} are of infinite order because no powers of g_0 or h_0 commute. Set $g_{0,t}=\chi_{\Phi(t)}(g_0)$ for each $t\in D$. Then $(1, g_0)(t, w)=(t, g_{0,t}(w))$ for each $(t, w)\in\tilde{X}$. The relation $F\circ(1, g_0)=(A_{g_0}, B_{g_0})\circ F$ implies that

$$F_1(t, g_{0,t}(w)) = A_{g_0}\circ F_1(t, w)\,,$$
$$F_2(t, g_{0,t}(w)) = B_{g_0}\circ F_2(t, w)$$

for each $(t, w)\in\tilde{X}$. Let $g_{0,t_0}=\chi_{\Phi_0}(g_0)$. Since $F_1(t_n, w)$, $F_2(t_n, w)$ and $g_{0,t_n}(w)$ converge uniformly on any compact subset of Ω_1 to $f_1(w)=c_1$, $f_2(w)$ and $g_{0,t_0}(w)$, respectively, as $n\to\infty$ and since $g_{0,t_0}(\Omega_1)=\Omega_1$, we have $A_{g_0}(c_1)=c_1$ and $f_2\circ g_{0,t_0}=B_{g_0}\circ f_2$. Similarly, we have $A_{h_0}(c_1)=c_1$ and $f_2\circ h_{0,t_0}=B_{h_0}\circ f_2$. Since A_{g_0} and A_{h_0} are two non-commutative Möbius transformations of infinite order with a common fixed point c_1 and since B_{g_0} and B_{h_0} are commutative, the group generated by (A_{g_0}, B_{g_0}) and (A_{h_0}, B_{h_0}) is not discrete. Hence, $F\circ\mathcal{G}\circ F^{-1}$ is not discrete, which implies that $\mathcal{G}$ is not discrete and we have a contradiction. Therefore, A is of infinite order and $B=1$. Moreover, it is shown that both B_{g_0} and B_{h_0} are of infinite order, A_{g_0} and A_{h_0} are commutative, and B_{g_0} and B_{h_0} are non-commutative.

Now, assume that (A_g, B_g) is of type (II) for some $g\in G_1$. Then we have

$$(A_g, B_g)\circ(A, 1)(z_1, z_2)=(A_g(z_2), B_g\circ A(z_1)),$$
$$(A, 1)\circ(A_g, B_g)(z_1, z_2)=(A\circ A_g(z_2), B_g(z_1)).$$

Since (A_g, B_g) commutes with $(A, 1)$, we have

$$(A_g(z_2), B_g\circ A(z_1))=(A\circ A_g(z_2), B_g(z_1))$$

for each point (z_1, z_2) of D^2. Hence, $A=1$, which contradicts $A\neq 1$. Therefore, (A_g, B_g) is of type (I) for all $g\in G_1$.

Since $(A, B)=(A, 1)$, (A_g, B_g) is of type (I) and $(A, 1)$ commutes with (A_g, B_g), we have that $A\circ A_g=A_g\circ A$ for all $g\in G_1$. Hence, the group $\mathcal{A}=\{A_g|g\in G_1\}$ is commutative.

Moreover, B_g is of infinite order for all $g\neq 1$ of G_1 by the same argument as the one that A_{g_0} and A_{h_0} are of infinite order. This completes the proof of Lemma 1.

Lemma 2. *The yomotopic monodromy M_{γ_0} of γ_0 is of finite order.*

Proof. We use the notations in the proof of Lemma 1. Assume that M_{γ_0} is of infinite order. Then S_0 is a Riemann surface of type (g, n) with nodes $p_1, \cdots, p_k$. Denote by C_j the Jordan curve $\alpha^{-1}(p_j)$ on S for each $j=1, \cdots, k$.

i) Assume that at least one of $C_1, \cdots, C_k$, say C_1, is a non-dividing cycle on S. Suppose that $q_0\in C_1'=\alpha^{-1}((|z_1|=\varepsilon)\times(z_2=0))$ and take a closed path C_0 starting at q_0 on Σ_1. (See Figure 2.)

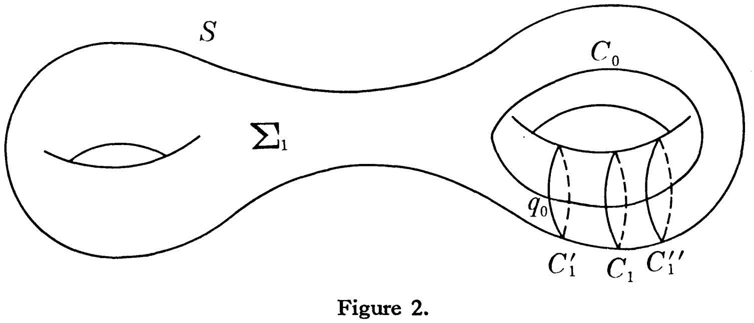

Figure 2.

Since the Dehn twist d inducing the homotopic monodromy $(M_{\gamma_0})^m=\langle\omega_1\rangle$ is the identity mapping on $S-\alpha^{-1}(\delta_0)$, we have $[d(C_0)]=[C_1']^{\nu_0}\circ[C_0]$ for some integer ν_0. Set $g_0=[C_1']_*^{\nu_0}\in G_1$, $h_0=[C_0]_*\in G$, $U_2=h_0(U_1)$ and $G_2=h_0\circ G_1\circ h_0^{-1}$. Then the relations $[d(C_0)]=[C_1']^{\nu_0}\circ[C_0]$, $d\circ\pi_0=\pi_0\circ\omega_1$ and $\omega_1|U_1=1$ imply that $\omega_1\circ h_0=g_0\circ h_0$ on U_1. Hence, we have $\omega_1=g_0$ on U_2. If we set $\omega_2=g_0^{-1}\circ\omega_1$, then $\omega_2|U_2=1$, $\langle\omega_2\rangle=\langle\omega_1\rangle$ in $\mathrm{Mod}(G)$ and $\omega_2\circ h\circ\omega_2^{-1}=h$ for all $h\in G_2$. Moreover, the quasiconformal mapping ω_2 induces an analytic automorphism $(1, g_0)^{-1}\circ(\gamma_0^m, 1)$ of $\tilde{X}$. Hence, we have an element $(A_{g_0}^{-1}\circ A, B_{g_0}^{-1})\in F\circ\mathcal{G}\circ F^{-1}$. Note that, by Lemma 1, B_{g_0} is of infinite order. By the same reasoning as in the proof of Lemma 1, the relation $\omega_2\circ h\circ\omega_2^{-1}=h$ for each $h\in G_2$ implies that $A_{g_0}^{-1}\circ A=1$, (A_h, B_h) is of type (I) for all $h\in G_2$ and the group $\{B_h|h\in G_2\}$ is commutative.

If $(A_{h_0}, B_{h_0})=F\circ(1, h_0)\circ F^{-1}$ is of type (I), then $\{B_g|g\in G_1\}$ and $\{B_h|h\in G_2\}$ are conjugate by B_{h_0}. Since the group $\{B_h|h\in G_2\}$ is commutative, the group $\{B_g|g\in G_1\}$ is also commutative and we have a contradiction.

Now, suppose that (A_{h_0}, B_{h_0}) is of type (II). We set $h_1=h_0\circ g_1$ and $U_3=h_1^2(U_1)$ for each $g_1\in G_1$. The relations $[d(C_0)]=[C_1']^{\nu_0}\circ[C_0]$, $d\circ\pi_0=\pi_0\circ\omega_1$ and $\omega_1|U=1$ imply that $\omega_1=g_0\circ h_1\circ g_0\circ h_1^{-1}$ on U_3. If we set $\omega_3=(h_1\circ g_0^{-1}\circ h_1^{-1}\circ g_0^{-1})\circ\omega_1$, then we have $\omega_3|U_3=1$, $\langle\omega_3\rangle=\langle\omega_1\rangle$ and $\omega_3\circ h\circ\omega_3^{-1}=h$ for all $h\in h_1^2\circ G_1\circ h_1^{-2}$. The elemnt $\omega_3\in N(G)$ induces an analytic auotmorphism $(1, h_1\circ g_0^{-1}\circ h_1^{-1}\circ g_0^{-1})\circ(\gamma_0^m, 1)$ of $\tilde{X}$ and we have an element $(X_1, Y_1)\in F\circ\mathcal{G}\circ F^{-1}$, where $X_1=(A_{h_0}\circ B_{g_1})\circ B_{g_0}^{-1}\circ(A_{h_0}\circ B_{g_1})^{-1}$ and $Y_1=B_{h_0}\circ A_{g_0}^{-1}\circ B_{h_0}^{-1}\circ B_{g_0}^{-1}$. Note that (X_1, Y_1) is of type (I). By the same argument as the proof of Lemma 1, we see that $(X_1, Y_1)=(X_1, 1)$ with $X_1\neq 1$ or $(X_1, Y_1)=(1, Y_1)$ with $Y_1\neq 1$. Since B_{g_0} is of infinite order, we have $X_1\neq 1$ and $Y_1=1$. We set $h_2=h_0\circ g_1^2$. The same reasoning as above implies that the element $(h_2\circ g_0^{-1}\circ h_2^{-1}\circ g_0^{-1})\circ\omega_1$ of $N(G)$ induces an element $(X_2, 1)$ of $F\circ\mathcal{G}\circ F^{-1}$, where $X_2=(A_{h_0}\circ B_{g_1}^2)\circ B_{g_0}^{-1}\circ(A_{h_0}\circ B_{g_1}^2)^{-1}$. Now, we can prove that $\mathcal{A}=\{A_g|g\in G_1\}$ is a discrete subgroup of $\mathrm{Aut}(D)$ as follows. Assume that $\mathcal{A}$ is not discrete. Then there exists a sequence $\{A_n\}$ of distinct elements of $\mathcal{A}$ such that $A_n\to 1$ as $n\to\infty$. Take an element $g_1\in G_1$ with $g_0\circ g_1\neq g_1\circ g_0$ and consider the sequences $\{(A_n, B_n)\circ(X_1, 1)\circ(A_n, B_n)^{-1}\}=\{(A_n\circ X_1\circ A_n^{-1}, 1)\}$ and $\{(A_n, B_n)\circ(X_2, 1)\circ(A_n, B_n)^{-1}\}=\{(A_n\circ X_2\circ A_n^{-1}, 1)\}$ in $\mathcal{G}$. They converge to $(X_1, 1)$ and $(X_2, 1)$ respectively as $n\to\infty$. Therefore, the discreteness of $\mathcal{G}$ implies that for any sufficiently large n, A_n commutes with X_1 and X_2. Thus, $A_n\circ X_1\circ A_n^{-1}=X_1$ and $A_n\circ X_2\circ A_n^{-1}=X_2$ for any sufficiently large n, which implies that

$$\mathrm{Fix}(A)=\mathrm{Fix}(A_n)=(A_{h_0}\circ B_{g_1})(\mathrm{Fix}(B_{g_0}^{-1}))\,,$$
$$\mathrm{Fix}(A)=\mathrm{Fix}(A_n)=(A_{h_0}\circ B_{g_1}^2)(\mathrm{Fix}(B_{g_0}^{-1}))\,.$$

Hence, we have $B_{g_1}(\mathrm{Fix}(B_{g_0}))=\mathrm{Fix}(B_{g_0})$, which implies that the group generated by (A_{g_0}, B_{g_0}) and (A_{g_1}, B_{g_1}) is not discrete and we have a contradiction. Therefore, $\mathcal{A}$ is an Abelian discrete subgroup of $\mathrm{Aut}(D)$. Then $\mathcal{A}$ is generated by an element A_{g_*} for some $g_*\in G_1$ with $g_*\neq 1$. Take an element $g_2\in G_1$ with

$g_*\circ g_2 \neq g_2\circ g_*$. Let $A_{g_2}=(A_{g_*})^n$ for some integer n and let $g_3=g_2\circ g_*^{-n}\in G_1$. Then $g_3\neq 1$ and $F\circ(1, g_3)\circ F^{-1}=(A_{g_3}, B_{g_3})=(1, B_{g_3})$. Since (A_{h_1}, B_{h_1}) is of type (II), we have $F\circ(1, h_1\circ g_3\circ h_1^{-1})\circ F^{-1}=(A_{h_1}\circ B_{g_3}\circ A_{h_1}^{-1}, 1)$, which is of type (I). Therefore, (A_{g_3}, B_{g_3}) and $(A_{h_1}\circ B_{g_3}\circ A\circ_{h_1}^{-1}, 1)$ are commutative, which implies that g_3 and $h_1\circ g_3\circ h_1^{-1}$ are commutative. Since g_3 and h_1 are elements of the discrete subgroup G with no elliptic elements of $\mathrm{Aut}(U)$, it is shown that g_3 and $h_1=h_0\circ g_1$ are commutative, where g_1 is an arbitrary element of G_1. Take an element $g_1\in G_1$ with $g_1\circ h_0\neq h_0\circ g_1$. Since g_3 and $h_0\circ g_1$ are commutative and g_3 and $h_0\circ g_1^2$ are also commutative, we have that $h_0\circ g_1$ and $h_0\circ g_1^2$ are commutative. Hence, h_0 and g_1 are commutative and we have a contradiction.

ii) Assume that all the Jordan curves $C_1, \cdots, C_k$ are dividing cycles on S. Take two connected components Σ_1 and Σ_2 of $S-\alpha^{-1}(\{p_1, \cdots, p_k\})$ which have the common boundary curve C_1. Let $q_0\in C_1'$, $q_0'\in C_1''$ and let L be a simple path from q_0 to q_0' on the annulus bounded by C_1' and C_1''. (See Figure 3.)

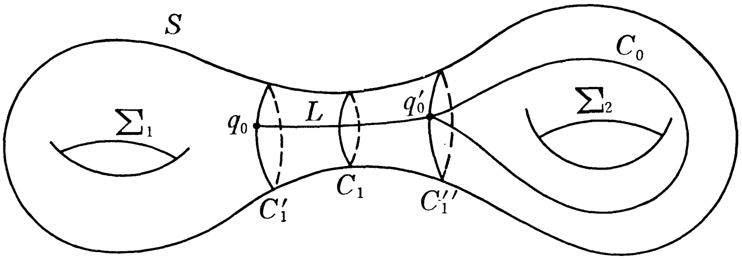

Figure 3.

Now, we set

$$U_1 = \{[C, q] \mid q\in\Sigma_1 \text{ and } C \text{ is a path from } q_0 \text{ to } q \text{ on } \Sigma_1\}\ ,$$
$$U_2 = \{[L\circ C, q] \mid q\in\Sigma_2 \text{ and } C \text{ is a path from } q_0' \text{ to } q \text{ on } \Sigma_2\}\ ,$$
$$G_1 = \{[C]_* \mid [C]\in\pi_1(\Sigma_1, q_0)\}\ ,$$
$$G_2 = \{[L\circ C\circ L^{-1}]_* \mid [C]\in\pi_1(\Sigma_2, q_1')\}\ .$$

Then U_1 and U_2 are invariant under G_1 and G_2, respectively. Since the Dehn twist d inducing the homotopic monodromy $(M_{\gamma_0})^m=\langle\omega_1\rangle$ is the identity on $S-\alpha^{-1}(\delta_0)$, it is shown that $d(L)$ is homotopic to $(C_1')^{\nu_0}\circ L$ for some integer ν_0. Hence, if we set $g_0=[C_1']_*^{\nu_0}\in G_1$, then we have $\omega_1=g_0$ on U_2 and $\omega_1\circ h\circ\omega_1^{-1}=g_0\circ h\circ g_0^{-1}$ for all $h\in G_2$. Note that $g_0\in G_1\cap G_2$. If we set $\omega_2=g_0^{-1}\circ\omega_1$, then we have $\omega_2|U_2=1$ and $\omega_2\circ h\circ\omega_2^{-1}=h$ for all $h\in G_2$, and $\langle\omega_2\rangle=\langle\omega_1\rangle$ in $\mathrm{Mod}(G)$. Moreover, the quasiconformal mapping ω_2 induces an analytic automorphism $(1, g_0)^{-1}\circ(\gamma_0^m, 1)$ of $\tilde{X}$ and we have an element $(A_{g_0}^{-1}\circ A, B_{g_0}^{-1})\in F\circ\mathcal{G}\circ F^{-1}$. Note that B_{g_0} is of infinite order. By the same reasoning as in the proof of Lemma 1, the relation $\omega_2\circ h\circ\omega_2^{-1}=h$ for each $h\in G_2$ implies that $A_{g_0}^{-1}\circ A=1$, (A_h, B_h) is of type (I) for each $h\in G_2$, A_h is of infinite order for each $h\neq 1$ of G_2 and the group $\{B_h | h \in G_2\}$ is commutative. Take a closed path C_0 starting at q_0' on Σ_2 and set $\tilde{C}_0 = L \circ C_0 \circ L^{-1}$ and $h_0 = [\tilde{C}_0]_* \in G_2$. (See Figure 3.) Let $\tilde{U}_1 = h_0(U_1)$, $\tilde{G}_1 = h_0 \circ_1 G_1 \circ h_0^{-1}$ and $\tilde{\omega}_1 = (g_0 \circ h_0 \circ g_0^{-1} \circ h_0^{-1})^{-1} \circ \omega_1$. Since $\omega_1 = g_0 \circ h_0 \circ g_0^{-1} \circ h_0^{-1}$ on $\tilde{U}_1$, we have $\tilde{\omega}_1 | \tilde{U}_1 = 1$, $\tilde{\omega}_1 \circ g \circ \tilde{\omega}_1^{-1} = g$ for all $g \in \tilde{G}_1$, and $\langle \tilde{\omega}_1 \rangle = \langle \omega_1 \rangle$ in Mod(G). The quasiconformal mapping $\tilde{\omega}_1$ induces an analytic automorphism $(1, g_0 \circ h_0 \circ g_0^{-1} \circ h_0^{-1})^{-1} \circ (\gamma_0^m, 1)$ of $\tilde{X}$ and we have an element $\Psi = (A_{h_0} \circ A_{g_0} \circ A_{h_0}^{-1} \circ A_{g_0}^{-1} \circ A,\ B_{h_0} \circ B_{g_0} \circ B_{h_0}^{-1} \circ B_{g_0}^{-1})$ of $F \circ \mathcal{G} \circ F^{-1}$. Since $A_{g_0}^{-1} \circ A = 1$ and since B_{g_0} and B_{h_0} are commutative, we have $\Psi = (A_{h_0} \circ A_{g_0} \circ A_{h_0}^{-1}, 1)$.

Now, assume that $\mathcal{A} = \{A_g | g \in G_1\}$ is not discrete. Then there exists a sequence $\{A_n\}$ of distinct elements of $\mathcal{A}$ such that $A_n \to 1$ as $n \to \infty$. Thus the sequence $\{(A_n, B_n) \circ (A_{h_0} \circ A \circ A_{h_0}^{-1}, 1) \circ (A_n, B_n)^{-1}\}$ tends to $(A_{h_0} \circ A \circ A_{h_0}^{-1}, 1)$ as $n \to \infty$, which implies that $A_n \circ (A_{h_0} \circ A \circ A_{h_0}^{-1}) \circ A_n^{-1} = A_{h_0} \circ A \circ A_{h_0}^{-1}$, that is, A_n and $A_{h_0} \circ A \circ A_{h_0}^{-1}$ are commutative for any sufficiently large integer n. Hence, we have $\mathrm{Fix}(A) = \mathrm{Fix}(A_n) = A_{h_0}(\mathrm{Fix}(A))$, which implies that A_{h_0} fixes every fixed point of A. By the same argument, we can take another element $h_1 \in G_2$ with the same property as h_0 and $h_0 \circ h_1 \neq h_1 \circ h_0$. Since B_{h_0} and B_{h_1} are commutative, A_{h_0} and A_{h_1} are non-commutative. Hence, A_{h_0} and A_{h_1} are two non-commutative Möbius transformations of infinite order with a common fixed c_0, which implies that the group generated by (A_{h_0}, B_{h_0}) and (A_{h_1}, B_{h_1}) is not discrete and we have a contradiction. Therefore, $\mathcal{A}$ is an Abelian discrete subgroup of Aut(D). Then $\mathcal{A}$ is generated by an element A_{g_1} for some $g_1 \in G_1$ with $g_1 \neq 1$. Take an element $g_2 \in G_1$ with $g_2 \circ g_1 \neq g_1 \circ g_2$. Let $A_{g_2} = (A_{g_0})^n$ for some integer n and let $g_3 = g_2 \circ g_1^{-n} \in G_1$. Then $g_3 \neq 1$ and $(A_{g_3}, B_{g_3}) = (1, B_{g_2} \circ B_{g_1}^{-n})$. If we set $\tilde{g} = h_0 \circ g_3 \circ h_0^{-1}$, then we have $(A_{\tilde{g}}, B_{\tilde{g}}) = (1, B_{h_0} \circ B_{g_3} \circ B_{h_0}^{-1})$. Then $(A, 1)$ and $(A_{\tilde{g}}, B_{\tilde{g}})$ are commutative and so are $(\gamma_0^m, 1)$ and $(1, \tilde{g})$. Then, by the relation (1) of § 1, we have $\omega_1 \circ \tilde{g} \circ \omega_1^{-1} = \tilde{g}$. Since $\omega_1 \circ h_0 \circ \omega_1^{-1} = g_0 \circ h_0 \circ g_0^{-1}$ and $\omega_1 \circ g_3 \circ \omega_1^{-1} = g_3$, we have $g_3 \circ (g_0 \circ h_0^{-1} \circ g_0^{-1} \circ h_0) = (g_0 \circ h_0^{-1} \circ g_0^{-1} \circ h_0) \circ g_3$. Similarly, it can be proved that g_3 and $h_n = g_0 \circ h_0^{-n} \circ g_0^{-1} \circ h_0^n$ are commutative for any integer n, which implies that $\mathrm{Fix}(g_3) = \mathrm{Fix}(h_n)$ for any non-zero integer n. This is impossible. In fact, by conjugation, we may assume that $h_0(z) = k^2 z$ for some constant $k > 1$ and $g_0(z) = (az+b)/(cz+d)$ with $ad - bc = 1$. Since G is discrete and since g_0 and h_0 are non-commutative, we have $g_0(0) \neq 0$ and $g_0(\infty) \neq \infty$, which implies that $b \neq 0$ and $c \neq 0$. By direct computation, we have

$$(h_n z) = \frac{(ad - k^{2n} bc)z + (1 - k^{-2n})ab}{(1 - k^{2n})cdz + ad - k^{-2n} bc}.$$

If $a = 0$, then the relation $ad - bc = 1$ implies that $bc = -1$ and we have

$$h_n(z) = \frac{k^{2n} z}{(1 - k^{2n})cdz + k^{-2n}}.$$

Since both h_0 and h_n are Möbius transformations of infinite order with a common fixed point $z = 0$ and since G is discrete, we have $\mathrm{Fix}(h_0) = \mathrm{Fix}(h_n)$, that is, $h_n(\infty)=\infty$. Hence, we have $(1-k^{2n})cd=0$. Since $k>1$ and $c\neq 0$, we have $d=0$ and $tr^2(g_0)=0$. Hence, g_0 is an elliptic element and we have a contradiction. Therefore, we have $a\neq 0$. Similarly, it can be shown that $b\neq 0$, $c\neq 0$ and $d\neq 0$.

Now, by direct computation, the fixed points z_n of h_n are given by the formula

$$z_n=\frac{(k^{-2n}-k^{2n})bc\pm\{(2ad-(k^{2n}+k^{-2n})bc)^2-4\}^{1/2}}{2(1-k^{2n})cd}.$$

Then the two fixed points go to 0 and b/d as $n\to+\infty$ and they go to ∞ and a/c as $n\to-\infty$. On the other hand, since $\mathrm{Fix}(g_3)=\mathrm{Fix}(h_n)$ for any non-zero integer n, we have a contradiction. This completes the proof of Lemma 2.

Lemma 3. *If $\tilde{X}$ is biholomorphic to the polydisc D^2 and the homotopic monodromy M_{γ_0} of γ_0 is of finite order, then the homotopic monodromy group $\mathcal{M}$ of (X, π, R) is a finite group.*

Proof. Let $M_{\gamma_0}=\langle\omega_{\gamma_0}\rangle$ for some $\omega_{\gamma_0}\in N(G)$. Since $(M_{\gamma_0})^m=1$ for some integer m, we may assume that $\langle(\omega_{\gamma_0})^m\rangle$ is represented by the identity mapping on the upper half-plane U.

We use the notations in the proof of Lemma 1. By Lemma 1, we may assume that $F\circ(\gamma_0^m, 1)\circ F^{-1}$ is equal to $(A, 1)$ and is of type (I). Take an element $\delta\in\Gamma$ with $\gamma_0\circ\delta\neq\delta\circ\gamma_0$. Set $F\circ(\delta, 1)\circ F^{-1}=(X, Y)$. We may assume that (X, Y) is of type (I) and we have $F\circ(\delta\circ\gamma_0^m\circ\delta^{-1}, 1)\circ F^{-1}=(X\circ A\circ X^{-1}, 1)$. If X is of finite order, then $(X^n\circ A\circ X^{-n}, 1)=(A, 1)$ for some integer n. Hence, we have $(\gamma_0^m, 1)=(\delta^n\circ\gamma_0^m\circ\delta^{-n}, 1)$, which implies that $\gamma_0^m=\delta^n\circ\gamma_0^m\circ\delta^{-n}$. Hence, γ_0 and δ are commutative and we have a contradiction. Therefore, X is of infinite order. Similarly, it is shown that A and X are non-commutative. Since $(\omega_{\gamma_0})^m=1$, we have $\omega_{\delta\circ\gamma_0^m\circ\delta^{-1}}=1$ and the relation (1) of § 1 implies that $(\delta\circ\gamma_0^m\circ\delta^{-1}, 1)$ and $(1, g)$ are commutative. Hence, we have $(X\circ A\circ X^{-1}\circ A_g, B_g)=(A_g\circ X\circ A\circ X^{-1}, B_g)$, that is, $(X\circ A\circ X^{-1})\circ A_g=A_g\circ(X\circ A\circ X^{-1})$ for all $g\in G$. Assume that $A_g\neq 1$ for some $g\in G$ with $g\neq 1$. Since $\mathrm{Fix}(A)=\mathrm{Fix}(A_g)=\mathrm{Fix}(X\circ A\circ X^{-1})=X(\mathrm{Fix}(A))$, A and X have a common fixed point. Hence, A and X are non-commutative Möbius transformations of infinite order with a common fixed point, which implies that the group generated by $(A, 1)$ and (X, Y) is not discrete. Therefore, we have a contradiction. Hence, $A_g=1$ for all $g\in G$. Then we have the relations $F_1\circ(1, g)=F_1$, $F_2\circ(1, g)=B_g\circ F_2$ and $g_t\circ E_2=E_2\circ B_g$ for each $g\in G$, where $F=(F_1, F_2)$ is the above biholomorphic mapping, $E=(E_1, E_2)=F^{-1}$ and $g_t=\chi_{\Phi(t)}(g)$ for each $t\in D$. The relation $F_1\circ(1, g)=F_1$ for all $g\in G$ implies that F_1 is a bounded holomorphic automorphic function on $D_{\Phi(t)}$ for $G_{\Phi(t)}$ for each $t\in D$. Since $D_{\Phi(t)}/G_{\Phi(t)}$ is of finite type, the function F_1 is a constant function with a value $c_t\in D$ on $D_{\Phi(t)}$ for each $t\in D$. Set $D(t)=(z_1=c_t)\times(|z_2|<1)$ for each $t\in D$. Then F_2 induces an injective holomorphic function $(F_2)_t$: $D_{\Phi(t)}\to D(t)$ for each $t\in D$. Moreover, E_1 is a constant function with a value t on $D(t)$ and E_2 induces an injective holomorphic function $(E_2)_t$: $D(t)\to D_{\Phi(t)}$ for each $t\in D$. Since $E\circ F=1_{\tilde{X}}$ and $F\circ E=1_{D^2}$, we have $(E_2)_t\circ(F_2)_t=1_{D_{\Phi(t)}}$ and $(F_2)_t\circ(E_2)_t=1_{D(t)}$. Hence, $(F_2)_t$: $D_{\Phi(t)}\to D(t)$ is conformal and it induces a conformal mapping of $D_{\Phi(t)}/G_{\Phi(t)}$ onto $D(t)/\mathscr{B}$ for each $t\in D$, where $\mathscr{B}=\{B_g\,|\,g\in G\}$ is a finitely generated Fuchsian group with no elliptic elements. Since all the Riemann surfaces $D(t)/\mathscr{B}$, $t\in D$, are conformally equivalent, all the fibers S_p, $p\in R$, are also conformally equivalent. Hence, Theorem 2 implies that the homotopic monodromy group $\mathscr{M}$ of (X, π, R) is a finite group. This completes the proof of Lemma 3.

Now, we can prove Theorem 3. If the homotopic monodromy group $\mathscr{M}$ of (X, π, R) is a finite group, then Theorem 2 implies that the mapping $\Phi: D\to T(G)$ is a constant mapping with a value ϕ_0. Hence, the universal covering space $\tilde{X}$ of X is equal to $D\times D_{\phi_0}$, which is biholomorphic to the polydisc D^2.

Conversely, if $\tilde{X}$ is biholomorphic to D^2, then Lemmas 2 and 3 imply that $\mathscr{M}$ is a finite group. This completes the proof of Theorem 3.

5. Proof of Theorem 4. If $\tilde{X}$ is biholomorphic to the polydisc D^2, then it is clear that $\mathrm{Aut}(\tilde{X})$ is not discrete. Conversely, assume that $\mathrm{Aut}(\tilde{X})$ is not discrete. Since the fibers of (X, π, R) are compact, Theorem 3 in Shabat [10] implies that $\mathrm{Aut}(\tilde{X})$ is transitive. Hence, by E. Cartan's Theorem, the homogeneous bounded domain $\tilde{X}$ in $\boldsymbol{C}^2$ is biholomorphic to the unit ball B_2 or the polydisc D^2. By Theorem 1, $\tilde{X}$ is not biholomorphic to B_2. Therefore, $\tilde{X}$ is biholomorphic to D^2. This completes the proof of Theorem 4.

6. Examples and problems. We give the following typical examples of (X, π, R).

EXAMPLE 1. Let S be a Riemann surface of finite type (g, n) with $2g-2+n>0$ and let R be an open Riemann surface of finite type whose universal covering space is the upper half-plane. Let $X=R\times S$ and let π be the canonical projection of X onto R. Then (X, π, R) is a holomorphic family of Riemann surfaces of type (g, n) over R. All the fibers are conformally equivalent to S and the homotopic monodromy group $\mathscr{M}$ is trivial. It is clear that the universal covering space $\tilde{X}$ of X is biholomorphic to the polydisc D^2. Theorem 1 implies that $\tilde{X}$ is not biholomorphic to the unit ball B_2. Hence, Theorem 1 is a generalization of the famous theorem due to Poincaré which asserts that the polydisc D^2 is not biholomorphic to the unit ball B_2.

EXAMPLE 2. We set

$$R = \boldsymbol{C} - \{0, 1\} ,$$
$$X = \{(x, y, t) \mid y^2 = x^3 + t,\ (x, y) \in \boldsymbol{C}^2,\ t \in R\} .$$

Let π: $X \to R$ be the canonical projection. Then (X, π, R) is a holomorphic family of Riemann surfaces of type (1, 1) over R and its homotopic monodromy group $\mathcal{M}$ is a finite cyclic group. All the fibers S_t are conformally equivalent and the universal covering space $\tilde{X}$ of X is biholomorphic to the polydisc D^2.

EXAMPLE 3. We set

$$R = \boldsymbol{C} - \{0, 1, 2, 3\} ,$$
$$X = \{(x, y, z, t) \in P_2(\boldsymbol{C}) \times R \mid y^2 z^3 = x(x - zt)(x - z)(x - 2z)(x - 3z)\} ,$$

where $P_2(\boldsymbol{C})$ is the two-dimensional complex projective space and (x, y, z) are the homogeneous coordinates of $P_2(\boldsymbol{C})$. Let π: $X \to R$ be the canonical projection. Then (X, π, R) is a holomrophic family of Riemann surfaces of type (2, 0) and its homotopic monodromy group $\mathcal{M}$ is an infinite group. All the fibers S_t, $t \in R$, are not confomally equivalent. Theorems 1 and 2 imply that the universal covering space $\tilde{X}$ of X is not biholomorphic to B_2 or D^2. Moreover, Theorem 4 implies that $\mathrm{Aut}(\tilde{X})$ is a discrete group.

Let (X, π, R) be a holomorphic family of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$. Let us give the following problems.

PROBLEM 1. Let R be a closed Riemann surface of genus $g_0 > 1$. Then prove that the universal covering space $\tilde{X}$ of X is not biholomorphic to the unit ball B_2. (cf. Shabat [10].)

PROBLEM 2. Let X be a Stein manifold. Then prove that the universal covering space $\tilde{X}$ of X is biholomorphic to the polydisc D^2 if and only if $\mathrm{Aut}(\tilde{X})$ is not a discrete group. (cf. Shabat [10].)

PROBLEM 3. When $\mathrm{Aut}(\tilde{X})$ is a discrete group, can we write down all the elements of $\mathrm{Aut}(\tilde{X})$? Note that the covering transformation group $\mathcal{G}$ of Π: $\tilde{X} \to X$ is a subgroup of $\mathrm{Aut}(\tilde{X})$ and its elements are known as in § 1.

References

[1] L. Bers: *On Hilbert's 22nd problem*, Proc. Sympos. Pure. Math. **28** (1976), 559–609.

[2] P.A. Griffiths: *Complex analytic properties of certain Zariski open sets on algebraic varieties*, Ann. of Math. **94** (1971), 21–55.

[3] Y. Imayoshi: *Holomorphic families of Riemann surfaces and Teichmüller spaces*, in "Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference" edited by I. Kra and B. Maskit, Ann. of Math. Studies 97, Princeton Univ. Press, (1981), 277–300.

[4] Y. Imayoshi: *Holomorphic families of Riemann surfaces and Teichmüller spaces II, Applications to uniformization of algebraic surfaces and compactification of two-dimensional Stein manifolds,* Tôhoku Math. J. **31** (1979), 469–489.

[5] Y. Imayoshi: *Holomorphic families of Riemann surfaces and Teichmüller spaces III, Bimeromorphic embedding of algebraic surfaces into projective spaces by automorphic forms,* Tôhoku Math. J. **33** (1981), 227–247.

[6] J. Lehner: Discontinuous groups and automorphic functions, American Mathematical Society, Rohde Island, 1964.

[7] R. Narasimhan: Several complex variables, University of Chicago Press, Chicago, 1971.

[8] I.I. Pyatetskii-Shapiro: Automorphic functions and the geometry of classical domains, Gordon and Breach Science Publishers, New York, 1969.

[9] J.P. Rosay: *Sur une caractérisation de la boule parmi les domaines de* $\boldsymbol{C}^n$ *par son groupe d'automorphismes,* Ann. Inst. Fourier **29** (1979), 91–97.

[10] G.B. Shabat: *The complex structure of domains covering algebraic surfaces,* Functional Anal. Appl. **11** (1976), 135–142.

Department of Mathematics
College of General Education
Osaka University
Toyonaka, Osaka 560
Japan

Current address:
Department of Mathematics
White Hall
Cornell University
Ithaca, New York 14853
U.S.A.