Convex polytopes of finite graphs and Ehrhart series

Submitted to
Graduate School of Information Science and Technology Osaka University

January 2018

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List of papers

[1] G. Hamano, Existence of a regular unimodular triangulation of the edge polytopes of finite graphs, *Comment. Math. Univ. St. Pauli,* **65** (2016), 85–96.

[2] G. Hamano, T. Hayashi, T. Hibi, K. Hirayama, H. Ohsugi, K. Sato, A. Shikama, and A. Tsuchiya, The smallest normal edge polytopes with no regular unimodular triangulations, *in* "Homological and Computational Methods in Commutative Algebra" (A. Conca et al. Eds), Springer INdAM Series, in press.

[3] G. Hamano, T. Hibi and H. Ohsugi, Ehrhart series of fractional stable set polytope of finite graphs, *Ann. Comb.,* to appear.

Abstract

In this thesis, we will discuss the results of two themes;

(1) Existence of a regular unimodular triangulation of the edge polytopes of finite graphs (Chapter 2),

(2) Ehrhart series of fractional stable set polytopes of finite graphs (Chapter 3).

As far as (1) is concerned, it is known that it is enough to consider a fundamental FHM graph. Ohsugi obtained a necessary and sufficient condition for an edge polytope to possess a regular unimodular triangulation [16]. However, this condition is not easy to apply to a given fundamental FHM graph by merely inspecting the graph. In Chapter 1, we present the basics of Gröbner bases and convex polytopes. Subsequently, in Section 2.3, we obtain four sufficient conditions for an edge polytope to possess a regular unimodular triangulation. This is applied to a given fundamental FHM graph by merely inspecting the graph and using the graph data. Moreover, we implement a program for the computer algebra system Magma [2] that determines whether a given fundamental FHM graph satisfies our sufficient conditions. In Section 2.5, we provide the details of the algorithm and the program.

As far as (2) is concerned, we will show that the Ehrhart ring of a fractional stable set polytope $\mathrm{FRAC}(G)$ of a finite simple graph is Gorenstein. In Section 3.1, we will be concerned with the convex polytope $\mathcal{P}(G) = 2\mathrm{FRAC}(G)$ and will show that the $\delta-$vector of $\mathcal{P}(G)$ is alternatingly increasing. In Section 3.3, using this result, we will show how to calculate the Ehrhart series of $\mathrm{FRAC}(G)$ and present some examples in the case of a complete graph. Finally, in Section 3.4, we will consider the dual polytope $\mathcal{Q}(G)^\vee$ of the convex polytope $\mathcal{Q}(G) := 3\mathrm{FRAC}(G) - (1, \ldots, 1)$. Therein, we will show the equivalence of the following four conditions:

(i) The graph G is a bipartite graph.

(ii) The dual polytope $\mathcal{Q}(G)^\vee$ has a unimodular triangulation.

(iii) The dual polytope $\mathcal{Q}(G)^\vee$ is normal.

(iv) The dual polytope $\mathcal{Q}(G)^\vee$ is a Gorenstein Fano polytope.

Acknowledgment

First of all, I am deeply grateful to professor Takayuki Hibi for introducing me to this area of research and giving me helpful suggestions. I would like to thank professor Hidefumi Ohsugi for his helpful advice in writing this thesis. Finally, I am thankful to my family for their support.

Contents

Chapter 1

Introduction

In this chapter, we review the basics of Gröbner bases and convex polytopes.

1.1 Gröbner bases

In this section, we review the basics of Gröbner bases, following Chapter 1 of [5] and Chapter 1 of [14].

Let $K[X] = K[x_1, \ldots, x_n]$ be a polynomial ring in n variables over the field K. We first define the monomial order on $K[X]$. A *monomial* in the variables $x_1, x_2, \ldots x_n$ is a product of the form

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

where all the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. The set of monomials M of $K[X]$ is in one-to-one correspondence to $\mathbb{Z}^n_{\geq 0}$ via

$$X^\alpha \longleftrightarrow \alpha \in \mathbb{Z}^n_{\geq 0}.$$

Therefore, we may regard M as $\mathbb{Z}^n_{\geq 0}$.

Definition 1.1.1 A *monomial ordering* on $K[x_1, \ldots, x_n]$ is any relation $>$ on $\mathbb{Z}^n_{\geq 0}$, or equivalently, any relation on the set of monomials $\{X^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$ satisfying:

(i) $>$ is a total (or liner) ordering on $\mathbb{Z}^n_{\geq 0}$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^n_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ is a well-ordering on $\mathbb{Z}^n_{\geq 0}$. That is, every non-empty subset of $\mathbb{Z}^n_{\geq 0}$ has a smallest element under $>$.

The following lemma will help us understand the well-ordering condition in part (iii) of the definition.

Lemma 1.1.2 *An order relation $>$ on $\mathbb{Z}^n_{\geq 0}$ is a well-ordering if and only if every strictly decreasing sequence in $\mathbb{Z}^n_{\geq 0}$*

$$\alpha(1) > \alpha(2) > \alpha(3) > \cdots$$

eventually terminates.

We usually use the monomial orders, *lexicographic*, *graded lex* and *graded reverse lexicographic order.*

Definition 1.1.3 (i) (*lexicographic order*) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. Let $X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in K[x_1, \ldots, x_n]$ and $X^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \in K[x_1, \ldots, x_n]$. The notation $\alpha >_{\text{lex}} \beta$ implies that the leftmost non-zero entry in the vector difference $\alpha - \beta \in \mathbb{Z}^n$ is positive. We will write $X^\alpha >_{\text{lex}} X^\beta$ if $\alpha >_{\text{lex}} \beta$.

(ii) (*graded lex order*) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. The notation $\alpha >_{\text{grlex}} \beta$ implies that

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i \quad \textit{or} \quad |\alpha| = |\beta| \;\; \textit{and} \;\; \alpha >_{\text{lex}} \beta.$$

(iii) (*graded reverse lex order*) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. The notation $\alpha >_{\text{rev}} \beta$ implies that

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i \quad \textit{or} \quad |\alpha| = |\beta|$$

and the rightmost non-zero entry in $\alpha - \beta \in \mathbb{Z}^n$ is negative.

We fix a monomial order $<$ on the polynomial ring $K[X] = K[x_1, x_2, \ldots, x_n]$. Given a non-zero polynomial

$$f = a_1 u_1 + a_2 u_2 + \cdots + a_t u_t$$

of $K[X]$, where $0 \neq a_i \in K$ and $u_1, u_2, \ldots, u_t$ are monomials with

$$u_1 > u_2 > \cdots > u_t,$$

the *support* of f is the set of monomials appearing in f. It is denoted by $\text{supp}(f)$. The *initial monomial* of f with respect to $<$ is the largest monomial belonging to $\text{supp}(f)$ with respect to $<$. It is denoted by $\text{in}_<(f)$. Thus,

$$\text{supp}(f) = \{u_1, u_2, \ldots, u_t\}$$

and

$$\text{in}_<(f) = u_1.$$

Let I be an ideal of the polynomial ring $K[X]$ with $I \neq \langle 0 \rangle$. The monomial ideal generated by $\{\text{in}_<(f) \mid 0 \neq f \in I\}$ is called the *initial ideal* of I with respect to $<$ and is denoted by $\text{in}_<(I)$. That is,

$$\text{in}_<(I) = \langle \{\text{in}_<(f) \mid 0 \neq f \in I\} \rangle.$$

However, even if $I = \langle \{f_\lambda\}_{\lambda \in \Lambda} \rangle$, it is not necessarily true that $\text{in}_<(I)$ coincides with $\langle \{\text{in}_<(f_\lambda)\}_{\lambda \in \Lambda} \rangle$.

Definition 1.1.4 We fix a monomial order $<$ on the polynomial ring $K[X] = K[x_1, \ldots, x_n]$. Let I be an ideal of the polynomial ring $K[X]$ with $I \neq \langle 0 \rangle$. Then a *Gröbner basis* of I with respect to $<$ is a finite set $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$ of nonzero polynomials belonging to I such that $\text{in}_<(I) = \langle \text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s) \rangle$.

For all ideals $I (\neq 0)$ of the polynomial ring $K[X]$, if $\mathcal{G} = \{g_1, g_2, \ldots, g_s\}$ is a Gröbner basis of I, then $\mathcal{G}$ is a system of generators of I.

Example 1.1.5 Let $n = 7$. Let $f = x_1x_4 - x_2x_3, g = x_4x_7 - x_5x_6$ and $I = \langle f, g \rangle$. Then, $\text{in}_{<_{\text{lex}}}(f) = x_1x_4$ and $\text{in}_{<_{\text{lex}}}(g) = x_4x_7$. Let $h = x_7f - x_1g = x_1x_5x_6 - x_2x_3x_7$. Since $h \in I$, we have that $\text{in}_{<_{\text{lex}}}(h) = x_1x_5x_6 \in \text{in}_{<_{\text{lex}}}(I)$. However, $x_1x_5x_6 \notin \langle x_1x_4, x_4x_7 \rangle$. Hence, $\langle x_1x_4, x_4x_7 \rangle \neq \text{in}_{<_{\text{lex}}}(I)$. Therefore, $\{f, g\}$ is not a Gröbner basis of I with respect to $<_{\text{lex}}$.

We will now review the Buchberger criterion and the Buchberger algorithm. We first introduce the division algorithm and the S-polynomial.

Theorem 1.1.6 (The division algorithm) *We work with a fixed monomial order $<$ on the polynomial ring $K[X] = K[x_1, \ldots, x_n]$ and non-zero polynomials $g_1, g_2, \ldots, g_s$ belonging to $K[X]$. Then, given a polynomial $0 \neq f \in K[X]$, there exist $f_1, f_2, \ldots, f_s$ and f' belonging to $K[X]$ with*

$$f = f_1g_1 + f_2g_2 + \cdots + f_sg_s + f'$$

such that the following conditions are satisfied:

- *If $f' \neq 0$ and $u \in \text{supp}(f')$, then none of the initial monomials $\text{in}_<(g_i), 1 \leq i \leq s$ divides u. That is, if $f' \neq 0$, then no monomial $u \in \text{supp}(f')$ belongs to the monomial ideal $\langle \text{in}_<(g_1), \text{in}_<(g_2), \ldots, \text{in}_<(g_s) \rangle$.*
- *If $f_i \neq 0$, then*

$$\text{in}_<(f) \geq \text{in}_<(f_ig_i).$$

Definition 1.1.7 Let $f, g \in K[X]$ be non-zero polynomials. Let $m(f, g)$ be the least common multiple of $\text{in}_<(f)$ and $\text{in}_<(g)$. Moreover, let c_f be the coefficient of $\text{in}_<(f)$ and c_g the coefficient of $\text{in}_<(g)$. Then, the *S-polynomial* of f and g is the combination

$$S(f,g) = \frac{m(f,g)}{c_f \text{in}_<(f)} f - \frac{m(f,g)}{c_g \text{in}_<(g)} g.$$

Theorem 1.1.8 (Buchberger Criterion) *Let $I \neq \langle 0 \rangle$ be an ideal of the polynomial ring $K[X]$ and $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a system of generators of I. Then, $\mathcal{G}$ is a Gröbner basis of I if and only if for all $i \neq j$, the remainder on division of $S(g_i, g_j)$ by $\mathcal{G}$ is 0.*

Example 1.1.9 We consider Example 1.1.5 using Theorem 1.1.8. We first use the lexicographic order $<_{\text{lex}}$. Since $\text{in}_{<_{\text{lex}}}(f) = x_1x_4$ and $\text{in}_{<_{\text{lex}}}(g) = x_4x_7$, $S(f,g) = x_1x_5x_6 - x_2x_3x_7$. The remainder of the division of $S(f,g)$ by $\{f,g\}$ is $x_1x_5x_6 - x_2x_3x_7 \neq 0$. Therefore, $\{f,g\}$ is not a Gröbner basis of I.

We now use the graded reverse lex order $<_{\text{rev}}$. Since $\text{in}_{<_{\text{rev}}}(f) = x_2x_3$ and $\text{in}_{<_{\text{rev}}}(g) = x_5x_6$, $S(f,g) = x_1x_4x_5x_6 - x_2x_3x_4x_7$. The remainder of the division of $S(f,g)$ by $\{f,g\}$ is 0. Therefore, $\{f,g\}$ is a Gröbner basis of I with respect to $<_{\text{rev}}$.

In general, when we fix a monomial order $<$ on the polynomial ring $K[X]$, a Gröbner basis is not unique. Therefore, we consider the *reduced Gröbner basis*, which is the standard Gröbner basis of I.

Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a Gröbner basis of an ideal I. We say that a Gröbner basis $\mathcal{G}$ of I is a *minimal Gröbner basis* if $\{\text{in}_<(g_1), \ldots, \text{in}_<(g_s)\}$ is a minimal system of monomial generators of $\text{in}_<(I)$ and if the coefficient of $\text{in}_<(g_i)$ coincides with 1 for all $1 \leq i \leq s$.

Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a minimal Gröbner basis of an ideal I. We say that a minimal Gröbner basis $\mathcal{G}$ of I is a *reduced Gröbner basis* if the coefficient of $\text{in}_<(g_i)$ is 1 for all $1 \leq i \leq s$ and if $i \neq j$, then none of the monomials belonging to $\text{supp}(g_j)$ is divided by $\text{in}_<(g_i)$.

Theorem 1.1.10 *We fix a monomial order $<$ on the polynomial ring $K[X]$ and let $I \neq \{0\}$ be a polynomial ideal. Then, I has a unique reduced Gröbner basis.*

If we are given a Gröbner basis $\mathcal{G}$ of an ideal I, it is easy to obtain the reduced Gröbner basis from $\mathcal{G}$. We first remove g_i from $\mathcal{G} = \{g_1, \ldots, g_s\}$ in order that $\text{in}_<(g_i)$ be divided by $\text{in}_<(g_j)$. Moreover, we can obtain a minimal Gröbner basis $\mathcal{G}'$ by making an adjustment in order that the initial coefficient be 1. Let $\mathcal{G}' = \{h_1, \ldots, h_t\}$. We calculate the remainder h_1' of the division of h_1 by $\mathcal{G}' \setminus \{h_1\}$ and let $\mathcal{G}'' = \{h_1', h_2, \ldots, h_t\}$ by replacing h_1 with h_1'. Similarly, we replace h_2 with h_2', where h_2' is the remainder of the division of h_2 by $\mathcal{G}'' \setminus \{h_2\}$ and let $\mathcal{G}''' = \{h_1', h_2', h_3, \ldots, h_t\}$. Repeating this procedure, we obtain a reduced Gröbner basis.

We now introduce the Buchberger Algorithm. Let $F := \{f_1, \ldots, f_s\}$ be a finite system of generators of an ideal I.

(i) For all $i > j$, we calculate the remainder h_{ij} of the division of $S(f_i, f_j)$ by F.

(ii) If $h_{ij} = 0$ for all i, j, then, by Theorem 1.1.8, F is a Gröbner basis.

(iii) If there exist $i > j$ such that $h_{ij} \neq 0$, then we let $F' = F \cup \{h_{ij}\}$, replace F with F' and go back to (i).

(iv) If the remainder of the division of the S-polynomials of all pairs of polynomials in F by F is 0, then F is a Gröbner basis of I.

1.2 Configuration matrices and toric ideals

Let $A = (a_{ij})_{1\leq i\leq d, 1\leq j\leq n}$ be a $d \times n$ matrix and

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{dj} \end{pmatrix}, \quad 1 \leq j \leq n,$$

the column vectors of A, where $a_{ij} \in \mathbb{Z}$.

The *inner product* of vectors $\mathbf{a} = [a_1, a_2, \ldots, a_d]^\perp$ and $\mathbf{b} = [b_1, b_2, \ldots, b_d]^\perp$ belonging to $\mathbb{R}^d$, where $\perp$ stands for the transpose, is defined by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{d} a_i b_i.$$

A matrix $A = (a_{ij})_{1\leq i\leq d, 1\leq j\leq n} \in \mathbb{Z}^{d\times n}$ is called a *configuration matrix* if there exists $\mathbf{c} \in \mathbb{R}^d$ such that

$$\mathbf{a}_j \cdot \mathbf{c} = 1, \quad 1 \leq j \leq n.$$

Given a configuration matrix $A \in \mathbb{Z}^{d\times n}$, we let $\mathrm{Ker}_{\mathbb{Z}} A$ be the set of column vectors $\mathbf{b} \in \mathbb{Z}^n$ with $A\mathbf{b} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector of $\mathbb{R}^d$. That is,

$$\mathrm{Ker}_{\mathbb{Z}} A = \{\mathbf{b} \in \mathbb{Z}^n \mid A\mathbf{b} = \mathbf{0}\}.$$

Lemma 1.2.1 *If a column vector* $\mathbf{b} = [b_1, b_2, \ldots, b_n]^\perp \in \mathbb{Z}^n$ *belongs to* $\mathrm{Ker}_{\mathbb{Z}} A$*, then*

$$b_1 + b_2 + \cdots + b_n = 0.$$

A *binomial* belonging to $K[X] = K[x_1, \ldots, x_n]$ is a polynomial of the form $u - v$, where u and v are monomials of the same degree belonging to $K[X]$. A *binomial ideal* is an ideal of $K[X]$ generated by binomials. Given a column vector

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

belonging to $\mathrm{Ker}_{\mathbb{Z}}A$, we introduce the binomial $f_{\mathbf{b}} \in K[X]$ defined by

$$f_{\mathbf{b}} = \prod_{b_i>0} x_i^{b_i} - \prod_{b_j<0} x_j^{-b_j}.$$

Since Lemma 1.2.1 ensures that the degree of $\prod_{b_i>0} x_i^{b_i}$ coincides with that of $\prod_{b_j<0} x_j^{-b_j}$, we have that $f_{\mathbf{b}}$ is, in fact, a binomial.

Let $A \in \mathbb{Z}^{d\times n}$ be a configuration matrix. The binomial ideal

$$I_A = \langle \{ f_{\mathbf{b}} \mid \mathbf{b} \in \mathrm{Ker}_{\mathbb{Z}}A \} \rangle$$

of $K[X]$ is called the *toric ideal* of A.

Let $t_1, t_2, \dots, t_d$ be variables and $A = (a_{ij})_{1\le i\le d, 1\le j\le n} \in \mathbb{Z}^{d\times n}$ be a configuration matrix. To each column vector

$$\mathbf{a}_j := \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{dj} \end{pmatrix},$$

we associate the monomial

$$\mathbf{t}^{\mathbf{a}_j} = t_1^{a_{1j}} t_2^{a_{2j}} \dots t_d^{a_{dj}},$$

allowing negative powers. If $f = f(x_1, x_2, \dots, x_n) \in K[X]$, then we define $\pi(f)$ by setting

$$\pi(f) = f(\mathbf{t}^{\mathbf{a}_1}, \mathbf{t}^{\mathbf{a}_2}, \dots, \mathbf{t}^{\mathbf{a}_n}).$$

That is , $\pi(f)$ is the rational function in $t_1, t_2, \dots, t_d$ that is obtained by substituting $\mathbf{t}^{\mathbf{a}_i}$ for each x_i in f. Let

$$K[A] = \{ \pi(f) \mid f \in K[X] \}.$$

Then, sum and product can be naturally defined in $K[A]$. We say that $K[A]$ is the *toric ring* of A.

Example 1.2.2 For the configuration matrix $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

the toric ring $K[\mathcal{A}]$ is

$$K[\mathcal{A}] = K[t_1t_3, t_1t_4, t_1t_5, t_2t_3, t_2t_4, t_2t_5]$$

and the toric ideal $I_{\mathcal{A}}$ is

$$I_{\mathcal{A}} = \langle x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5 \rangle.$$

1.3 Convex polytopes

In this section, we review the basics of convex polytopes, following Chapter 5 of [14]. A subset $P \subset \mathbb{R}^N$ is said to be *convex* if, for each pair of points $\alpha, \beta \in P$, the line segment

$$\{t\alpha + (1-t)\beta \mid t \in \mathbb{R},\ 0 \leq t \leq 1\}$$

that connects the two points is contained in P. We first define several convex sets, which will later play an important role.

Definition 1.3.1 For a finite subset $X = \{x_1, \ldots, x_n\}$ of $\mathbb{R}^N$. Let

$$\mathrm{CONV}(X) := \left\{ \sum_{i=1}^{n} r_i x_i \;\middle|\; 0 \leq r_i \in \mathbb{R},\ \sum_{i=1}^{n} r_i = 1 \right\}.$$

We call this the *convex hull* of X.

Definition 1.3.2 A non-empty subset P of $\mathbb{R}^N$ is called a *convex polytope* if there exists a finite subset $X \subset \mathbb{R}^N$ such that $P = \mathrm{CONV}(X)$. Moreover, the set

$$\mathbb{R}_{\geq 0}X := \left\{ \sum_{i=1}^{n} r_i x_i \;\middle|\; 0 \leq r_i \in \mathbb{R} \right\}$$

is called the *polyhedral convex cone* generated by X. A non-empty set $C \subset \mathbb{R}^N$ is called a *cone* if, for any finite subset X of C, we have $\mathbb{R}_{\geq 0}X \subset C$.

For a polyhedron $P \subset \mathbb{R}^N$ and a vector $\mathbf{w} \in \mathbb{R}^d$, the set

$$\mathrm{FACE}_{\mathbf{w}}(P) := \{\mathbf{u} \in P \mid \mathbf{w} \cdot \mathbf{u} \geq \mathbf{w} \cdot \mathbf{v} \text{ for all } \mathbf{v} \in P\}$$

is called a *face* of P (with respect to $\mathbf{w}$). A point α in a polyhedron P is called a *vertex* of P if $\{\alpha\}$ is a face of P. The *dimension* of a convex polytope $P \subset \mathbb{R}^N$ is the dimension of the subspace of $\mathbb{R}^N$ spanned by $\{x - \alpha \mid x \in P\} \subset \mathbb{R}^N$, where $\alpha \in P$ is any fixed point. We denote the dimension of P by $\dim P$. If the dimension of a face F of a convex polytope $P \subset \mathbb{R}^N$ equals $\dim P - 1$, then F is called a *facet* of P.

A convex polytope P is said to be *integral* if all the vertices of P are integer vectors. A convex polytope P is called a *simplex* if P has $\dim P + 1$ vertices. That is, a simplex is a convex polytope that has the least number of vertices among all convex polytopes with the same dimension. We note that every face of a simplex is a simplex.

Example 1.3.3 A simplex of dimension 0 is a point. A simplex of dimension 1 is a line segment. A simplex of dimension 2 is a triangle. A simplex of dimension 3 is a tetrahedron.

For a polytope $P \subset \mathbb{R}^N$, a finite set Δ of simplices is called a *triangulation* of P if the following conditions are satisfied:

(i) $P = \bigcup_{\sigma \in \Delta} \sigma$.

(ii) If two simplices σ, τ satisfy $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a face of both σ and τ.

(iii) For a simplex $\sigma \in \Delta$, if τ is a face of σ, then $\tau \in \Delta$.

In particular, if a finite set Δ of simplices satisfies conditions (ii) and (iii), then Δ is called a *simplicial complex*. The dimension of a simplicial complex Δ is the maximal dimension of a simplex in Δ.

Definition 1.3.4 (i) We say that an integral polytope $P \subset \mathbb{R}^d$ is a *Fano polytope* if the origin of $\mathbb{R}^d$ is the unique lattice point belonging to the interior of P.

(ii) Let P be a Fano polytope. We say that P is *Gorenstein* if the dual polytope of P is an integral polytope, where the dual polytope $P^\vee$ is defined by

$$P^\vee := \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for any } \mathbf{y} \in P\}.$$

1.4 Triangulation of the configuration matrix and Gröbner bases

Let $\mathcal{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$ be a configuration matrix. Let Δ be a collection of simplices whose vertices belong to $\mathcal{A}$. Then, Δ is called a *covering* of $\mathcal{A}$ if

$$\mathrm{CONV}(\mathcal{A}) = \bigcup_{F \in \Delta} F$$

holds. In addition, If a covering Δ of a configuration matrix $\mathcal{A}$ is a simplicial complex, then it is called a *triangulation* of $\mathcal{A}$.

For a configuration matrix $\mathcal{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$, let

$$\mathbb{Z}\mathcal{A} = \left\{ \sum_{i=1}^{n} z_i \mathbf{a}_i \;\middle|\; z_i \in \mathbb{Z} \right\} \subset \mathbb{Z}^d.$$

Let $\mathcal{B} \subset \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be the vertex set of a maximal simplex $\sigma \in \Delta$ in a covering (triangulation) Δ of $\mathcal{A}$.

Definition 1.4.1 We assume that the rank of a configuration matrix $\mathcal{A} \in \mathbb{Z}^{d \times n}$ is equal to d. Let δ be the greatest common divisor of all $d \times d$ minors of $\mathcal{A}$. Then, the *normalized volume* of σ is defined by

$$\mathrm{Nvol}(\sigma) = \frac{|\det(\mathcal{B})|}{\delta}.$$

Definition 1.4.2 A covering (triangulation) Δ of $\mathcal{A}$ is said to be *unimodular* if the normalized volume of any maximal simplex in Δ is equal to 1.

For a configuration matrix $\mathcal{A} = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in \mathbb{Z}^{d\times n}$ and a vector $\mathbf{w} = [w_1, \dots, w_n] \in \mathbb{Q}^n$, let $\Delta_{\mathbf{w}}$ be the set of all convex polytopes $\mathrm{CONV}(\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_r}\})$ satisfying the following condition:

$$\text{there exists } \mathbf{c} \in \mathbb{Q}^d \text{ such that } \begin{cases} \mathbf{a}_j \cdot \mathbf{c} = w_j & j \in \{i_1, \dots, i_r\}, \\ \mathbf{a}_j \cdot \mathbf{c} < w_j & j \notin \{i_1, \dots, i_r\}. \end{cases}$$

Definition 1.4.3 A triangulation Δ of a configuration matrix $\mathcal{A}$ is said to be *regular* if there exists $\mathbf{w} \in \mathbb{Q}^d$ such that $\Delta = \Delta_{\mathbf{w}}$.

We will now regard regular triangulations from an algebraic point of view. For an ideal $I \subset K[X]$,

$$\sqrt{I} := \{f \in K[X] \mid f^m \in I \text{ for a natural number } m\}$$

is called the *radical* of I. For a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, the *squarefree part* of m is defined by

$$\sqrt{m} = \prod_{a_i > 0} x_i.$$

Proposition 1.4.4 *Let $I \subset K[X]$ be the ideal generated by monomials $m_1, \dots, m_s$. Then, we have $\sqrt{I} = \langle \sqrt{m_1}, \dots, \sqrt{m_s} \rangle$.*

A monomial m is said to be a *squarefree* if m is equal to $\sqrt{m}$. By Proposition 1.4.4, for a monomial ideal I, $I = \sqrt{I}$ holds if and only if I has a minimal set of generators consisting of squarefree monomials.

We will now define the initial complex.

Definition 1.4.5 For a configuration matrix $\mathcal{A} = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in \mathbb{Z}^{d\times n}$ and a monomial order $<$,

$$\Delta(\mathrm{in}_<(I_{\mathcal{A}})) := \left\{ \mathrm{CONV}(\mathcal{B}) \;\middle|\; \mathcal{B} \subset \{\mathbf{a}_1, \dots, \mathbf{a}_n\},\ \prod_{\mathbf{a}_i \in \mathcal{B}} x_i \notin \sqrt{\mathrm{in}_<(I_{\mathcal{A}})} \right\}$$

is called the *initial complex.*

By the following theorem, for any configuration matrix and any monomial order, the initial complex is a triangulation.

Theorem 1.4.6 *Let $\mathcal{A} \in \mathbb{Z}^{d\times n}$ be a configuration matrix and let $<$ be a monomial order. If $\mathbf{w} \in \mathbb{Q}^n$ satisfies $\mathrm{in}_<(I_{\mathcal{A}}) = \mathrm{in}_{\mathbf{w}}(I_{\mathcal{A}})$, then $\Delta(\mathrm{in}_<(I_{\mathcal{A}})) = \Delta_{\mathbf{w}}$.*

We now introduce necessary and sufficient conditions for a regular triangulation to be unimodular.

Theorem 1.4.7 *For a configuration matrix $\mathcal{A} \in \mathbb{Z}^{d\times n}$ and a monomial order $<$, the regular triangulation $\Delta(\mathrm{in}_<(I_\mathcal{A}))$ is unimodular if and only if $\sqrt{\mathrm{in}_<(I_\mathcal{A})} = \mathrm{in}_<(I_\mathcal{A})$.*

Example 1.4.8 For the configuration matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

the toric ideal is $I_\mathcal{A} = \langle f \rangle$, where $f = x_1x_5^2 - x_2x_3x_4$. In this case, for any monomial order, $\{f\}$ is a minimal Gröbner basis. We note that there exist two types of initial ideals. We assume that the monomial orders $<_1$ and $<_2$ satisfy $\mathrm{in}_{<_1}(f) = x_1x_5^2$ and $\mathrm{in}_{<_2}(f) = x_2x_3x_4$, respectively. Then, we have $\mathrm{in}_{<_1}(I_\mathcal{A}) = \langle x_1x_5^2 \rangle$, $\sqrt{\mathrm{in}_{<_1}(I_\mathcal{A})} = \langle x_1x_5 \rangle$ and $\mathrm{in}_{<_2}(I_\mathcal{A}) = x_2x_3x_4 = \sqrt{\mathrm{in}_{<_2}(I_\mathcal{A})}$. Hence, $\Delta(\mathrm{in}_{<_1}(I_\mathcal{A}))$ is not unimodular and $\Delta(\mathrm{in}_{<_2}(I_\mathcal{A}))$ is unimodular.

Definition 1.4.9 A configuration matrix $\mathcal{A}$ is said to be *unimodular* if all triangulations of $\mathcal{A}$ are unimodular.

Theorem 1.4.10 *For a configuration matrix $\mathcal{A} \in \mathbb{Z}^{d\times n}$, the following conditions are equivalent.*

(i) *$\mathcal{A}$ is a unimodular configuration matrix.*

(ii) *Any regular triangulation of $\mathcal{A}$ is unimodular.*

(iii) *Any lexicographic triangulation of $\mathcal{A}$ is unimodular.*

(iv) *The normalized volume of any maximal simplex whose vertices belong to $\mathcal{A}$ is equal to* 1.

Moreover, if $\mathrm{rank}(\mathcal{A}) = d$*, then the following is equivalent to the above.*

(v) *All nonzero $d \times d$ minors of $\mathcal{A}$ have the same absolute value.*

We now introduce the normal configuration matrix. In general, a configuration matrix $\mathcal{A}$ satisfies $\mathbb{Z}_{\geq 0}\mathcal{A} \subset \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A}$. However, it does not, in general, satisfy $\mathbb{Z}_{\geq 0}\mathcal{A} \supset \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A}$. We say that $\mathcal{A}$ is *normal* if it satisfies $\mathbb{Z}_{\geq 0}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A}$. The formal definition of a normal ring is the following: "The integral domain $K[\mathcal{A}]$ is called normal if it is integrally closed in its field of fractions." With respect to the normality of the toric ring $K[\mathcal{A}]$, the existence of unimodular triangulations and unimodular coverings of $\mathcal{A}$ plays an important role.

Theorem 1.4.11 *If a configuration matrix $\mathcal{A}$ has a unimodular covering, then the toric ring $K[\mathcal{A}]$ is normal.*

Corollary 1.4.12 *Let $\mathcal{A}$ be a configuration matrix. If there exists a monomial order $<$ such that $\text{in}_<(I_\mathcal{A})$ is generated by squarefree monomials, then the toric ring $K[\mathcal{A}]$ is normal.*

The converse of Corollary 1.4.12 does not hold in general. However, the following proposition holds.

Proposition 1.4.13 *Let $\mathcal{A}$ be a configuration matrix. If there exists a minimal set of binomial generators of the toric ideal $I_\mathcal{A}$ of $\mathcal{A}$ that contains a binomial with no squarefree monomials, then $K[\mathcal{A}]$ is not normal.*

1.5 h-polynomials and δ-polynomials

Let Δ be a simplicial complex of dimension d. For i $(0 \leq i \leq d)$, let $f_i = f_i(\Delta)$ be the number of i-dimensional faces in Δ and let

$$f(\Delta) = (f_0, f_1, \ldots, f_d).$$

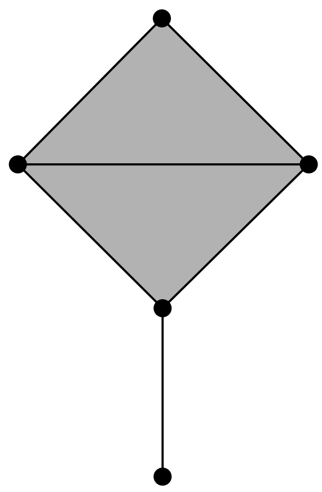
This is called the *f-vector* of Δ. We define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_{d+1})$ by setting

$$\sum_{i=0}^{d+1} f_{i-1}(x-1)^{d+1-i} = \sum_{i=0}^{d+1} h_i x^{d+1-i},$$

where $f_{-1} = 1$. We define the *h-polynomial* $h(\Delta, t)$ of Δ by

$$h(\Delta, t) = \sum_{i=0}^{d+1} h_i t^i.$$

Example 1.5.1 We consider the following simplicial complex Δ.



Simplicial complex Δ

Since $f_{-1} = 1, f_0 = 5, f_1 = 6, f_2 = 2$, we have

$$f(\Delta) = (5, 6, 2).$$

Moreover, since

$$\begin{aligned}
\sum_{i=0}^{3} f_{i-1}(x-1)^{3-i} &= f_{-1}(x-1)^3 + f_0(x-1)^2 + f_1(x-1) + f_2 \\
&= (x-1)^3 + 5(x-1)^2 + 6(x-1) + 2 \\
&= x^3 + 2x^2 - x \\
&= \sum_{i=0}^{3} h_i x^{3-i} \\
&= h_0 x^3 + h_1 x^2 + h_2 x + h_3,
\end{aligned}$$

we have $h_0 = 1, h_1 = 2, h_2 = -1$ and $h_3 = 0$. Hence,

$$h(\Delta) = (1, 2, -1, 0).$$

Therefore,

$$\begin{aligned}
h(\Delta, t) &= \sum_{i=0}^{3} h_i t^i \\
&= h_0 + h_1 t + h_2 t^2 + h_3 t^3 \\
&= 1 + 2t - t^2.
\end{aligned}$$

We now define the Ehrhart polynomial, Ehrhart series and δ-polynomial. Let P be a d-dimensional integral convex polytope in $\mathbb{R}^N$, i.e., all vertices of P are lattice points. For each $n \in \mathbb{N}$, let $nP = \{n\alpha \mid \alpha \in P\}$ and let nP° be the interior of nP. We define the functions $i(P, n) = \#(nP \cap \mathbb{Z}^N)$ and $i^*(P, n) = \#(nP^\circ \cap \mathbb{Z}^N)$. Thus, $i(P, n)$ (resp., $i^*(P, n)$) is the number of lattice points contained in nP (resp., nP°). It is known that $i(P, n)$ is indeed a polynomial in n of degree d. It is called the *Ehrhart polynomial* of P.

The generating functions of the Ehrhart polynomials $i(P, n)$, $i^*(P, n)$ are defined by

$$E(P, t) := 1 + \sum_{n=1}^{\infty} i(P, n) t^n, \qquad E^*(P, t) := \sum_{n=1}^{\infty} i^*(P, n) t^n$$

and are called the *Ehrhart series* of P.

Theorem 1.5.2 (Ehrhart's Law of Reciprocity) *We assume that $P \subset \mathbb{R}^N$ is a d-dimensional integral convex polytope. For the Ehrhart polynomial of P, we have*

$$i^*(P, n) = (-1)^d i(P, -n)$$

for every integer $n \geq 1$. *Moreover, for the Ehrhart series of* P, *we have*

$$E^*(P,t) = (-1)^{d+1} E\left(P, \frac{1}{t}\right).$$

The Ehrhart series $E(P,t)$ of a d-dimensional integral convex polytope P can be rewritten as

$$E(P,t) = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}}.$$

We define $\delta(P,t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$ and call this the δ-*polynomial* of P. The coefficient vector $\delta(P) = (\delta_0, \delta_1, \ldots, \delta_d)$ of $\delta(P,t)$ is called the δ-*vector* of P.

Let $P \subset \mathbb{R}^N$ be a d-dimensional rational convex polytope and m be the smallest natural number such that mP is an integral convex polytope. Then, the functions $i(P,n) = \#(nP \cap \mathbb{Z}^N)$ and $i^*(P,n) = \#(nP^\circ \cap \mathbb{Z}^N)$ are quasi-polynomials in n of degree d with period m.

Ehrhart's Law of Reciprocity (Theorem 1.5.2) is satisfied when P is a rational convex polytope. Let $E(P,t)$ be the Ehrhart series of a d-dimensional rational convex polytope P and m be the smallest natural number such that mP is an integral convex polytope. Then, $E(P,t)$ can be rewritten as

$$E(P,t) = \frac{\delta(P,t)}{(1-t^m)^{d+1}},$$

where $\delta(P,t)$ is a polynomial of degree less than $m(d+1)$ whose coefficients are integral. We call $\delta(P,t)$ the δ-polynomial of P. When $\delta(P,t) = \delta_0 + \delta_1 t + \cdots + \delta_s t^s$ ($\delta_s \neq 0, s < m(d+1)$), then $\delta(P) := (\delta_0, \ldots, \delta_s)$ is called the δ-vector of P.

We now define the Ehrhart ring of a d-dimensional rational convex polytope. Let $P \subset \mathbb{R}^N$ be a d-dimensional rational convex polytope and let $\{x_1, \ldots, x_v\}$ be the set of vertex of P. We define $\tilde{P} \subset \mathbb{R}^{N+1}$ by

$$\tilde{P} = \{(\alpha, 1) \in \mathbb{R}^{N+1} \mid \alpha \in P \subset \mathbb{R}^N\}.$$

Moreover, let

$$\mathcal{C}(\tilde{P}) := \left\{ \sum_{i=1}^{v} r_i(x_i, 1) \;\middle|\; 0 \leq r_i \in \mathbb{Q},\ 1 \leq i \leq v \right\} \subset \mathbb{Q}^{N+1},$$

where $\{(x_i, 1) \mid 1 \leq i \leq v\}$ is the set of vertices of $\tilde{P}$.

For $n \in \mathbb{Z}_{\geq 0}$, we set $\mathcal{C}(\tilde{P})_n := \{y \in \mathcal{C}(\tilde{P}) \mid y_{N+1} = n\}$. We note that $\mathcal{C}(\tilde{P})_n$ is a d-dimensional convex polytope that is isomorphic to nP by the correspondence

$$\mathcal{C}(\tilde{P})_n \ni (x, n) \longleftrightarrow x \in nP.$$

Then, the lattice point $(\alpha, n) \in \mathbb{Z}^{d \times n}$ belonging to $\mathcal{C}(\tilde{P})_n$ corresponds to the monomial $x^\alpha t^n$. Moreover, we define the linear space over the field K

$$[A_K(P)]_n := \langle x^\alpha t^n \mid (\alpha, n) \in \mathcal{C}(\tilde{P})_n \cap \mathbb{Z}^{N+1} \rangle.$$

The *Ehrhart ring* of P is defined by

$$A_K(P) := \bigoplus_{n\geq 0}[A_K(P)]_n.$$

We now review the definition of a Cohen-Macaulay ring and a Gorenstein ring. We consider a standard graded K-algebra $A = \oplus_{n\geq 0}A_n$ over the field K. By the Noether Normalization Lemma, there exist a finite number of homogeneous elements $\theta_1, \ldots, \theta_d$, which are algebraically independent over K, such that A is a finitely generated module over $K[\theta] = K[\theta_1, \ldots, \theta_d]$. Namely, there exist a finite number of homogeneous elements $\eta_1, \ldots, \eta_s$ such that

$$A = K[\theta]\eta_1 + \cdots + K[\theta]\eta_s.$$

Then, $\{\theta_1, \ldots, \theta_d\}$ is called a *system of parameters* of A. The number of parameters d is always equal to the Krull dimension.

Definition 1.5.3 A system of parameters $\{\theta_1, \ldots, \theta_d\}$ is called *regular* if A is a finitely generated free module over $K[\theta]$. If A possesses a regular system of parameters, then A is called a *Cohen-Macaulay ring*.

We assume that $A = \oplus_{n\geq 0}A_n$ is Cohen-Macaulay and define the Hilbert series of A by

$$H(A,t) := \sum_{n\geq 0}(\dim_K A_n)t^n.$$

Moreover, let $\{\theta_1, \ldots, \theta_d\}$ be a system of parameters of A, $d = \dim A$ and $e_k = \deg\theta_k$. Then, we have

$$H(A,t) = \frac{a(t)}{\prod_{k=1}^{d}(1-t^{e_k})}.$$

Definition 1.5.4 A finitely generated graded algebra $A = \oplus_{n\geq 0}A_n$ is *Gorenstein* if it has finite injective dimension as a module over itself.

Theorem 1.5.5 *We assume that a finitely generated graded algebra $A = \oplus_{n\geq 0}A_n$ is a d-dimensional Cohen-Macaulay integral domain and let $H(A,t) = (c_0 + c_1 t + \cdots + c_l t^l)/\prod_{k=1}^{d}(1-t^{e_k})$ $(c_l \neq 0)$ be the Hilbert series of A. Then, A is Gorenstein if and only if the coefficients $\{c_0, \ldots, c_l\}$ of the polynomial in the numerator of $H(A,t)$ are symmetric.*

Let P be a d-dimensional rational convex polytope and $A_K(P) := \oplus_{n\geq 0}[A_k(P)]_n$ be an Ehrhart ring of P. By the definition of the Ehrhart ring, since $\{x^\alpha t^n \mid (\alpha, n) \in \mathcal{C}(\tilde{P})_n \cap \mathbb{Z}^{N+1}\}$ is the basis of $A_K(P)$ as a liner space, the Hilbert series of $A_K(P)$ is equal to the Ehrhart series of P.

An Ehrhart ring $A_K(P)$ is a $(d+1)$-dimensional finitely generated graded algebra over K and, by Hochster's theorem, a Cohen-Macaulay integral domain. Therefore, the Hilbert series of $A_K(P)$ is

$$H(A_K(P), t) = \frac{a(t)}{\prod_{k=1}^{d+1}(1 - t^{e_k})}.$$

If P is an integral polytope, then $e_k = 1$ for all k. Thus, if P is not an integral polytope, then some e_k may not equal 1.

Moreover, by the definition of the δ-polynomial of P, if m is the minimal natural number such that mP is an integral convex polytope, then the Ehrhart polynomial of P is

$$E(P, t) = \frac{\delta(P, t)}{(1 - t^m)^{d+1}}.$$

Since $H(A_K(P), t) = E(P, t)$, we have

$$\frac{a(t)}{(1 + \cdots + t^{e_1 - 1}) \cdots (1 + \cdots + t^{e_{d+1} - 1})} = \frac{\delta(P, t)}{(1 + t + \cdots + t^{m-1})^{d+1}}.$$

Therefore, the coefficients of $a(t)$ are symmetric if and only if the coefficients of $\delta(P, t)$ are symmetric. Moreover, we have the following corollary.

Corollary 1.5.6 *Let P be a rational convex polytope. The Ehrhart ring $A_K(P)$ of P is Gorenstein if and only if the coefficients of the δ-polynomial of P are symmetric.*

Chapter 2

Regular unimodular triangulations of edge polytopes

Let G be a finite connected simple graph and P_G be the edge polytope of G. The combinatorial structure of P_G, especially the types of triangulations that P_G admits, is an interesting problem, which has been studied extensively (see [14, Chapter 5] and references therein). In [16], Ohsugi obtained a necessary and sufficient condition for P_G to possess a regular unimodular triangulation. Namely, there exists a monomial order such that the initial ideal of the toric ideal of the graph G is generated by squarefree monomials. However, this condition is not easy to apply to a given graph by merely inspecting the graph.

In this chapter, for a graph G, we will obtain several criteria for the existence of a regular unimodular triangulation of P_G in terms of simple data related to the graph. Moreover, we will present examples where we will apply our criteria to specific graphs and show that their edge polytopes possess a regular unimodular triangulation.

Furthermore, we will implement a program for the computer algebra system Magma [2] that determines whether a given fundamental FHM graph satisfies our criteria. Finally, we will show how the program can be used.

2.1 The edge polytope of a finite graph

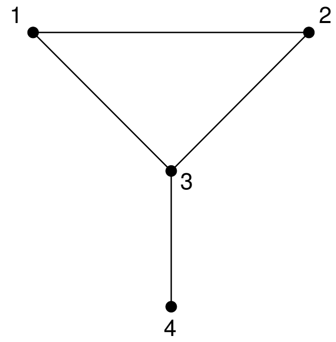
Let $G = (V, E)$ be a finite graph, where $V = \{1, 2, \ldots, d\}$ is the vertex set and $E = \{e_1, \ldots, e_n\}$ is the set of edges. A graph is called *simple* if it has no loops and no multiple edges. For each edge $e = \{i, j\} \in E$, we set

$$\rho(e) := \mathbf{e}_i + \mathbf{e}_j \in \mathbb{Z}^d,$$

where $\mathbf{e}_i$ is the i-th unit coordinate vector in $\mathbb{R}^d$.

Let $\mathcal{A}_G$ be a configuration matrix whose column vectors are $\{\rho(e) \mid e \in E\}$. We call the convex hull $\mathrm{CONV}(\mathcal{A}_G) \subset \mathbb{R}^d$ of $\mathcal{A}_G$ the *edge polytope* of G and denote it by P_G.

Example 2.1.1 We consider the following graph.



Graph G

The configuration matrix $\mathcal{A}_G$ of G is the following

$$\mathcal{A}_G = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now define certain concepts related to graphs. A sequence $\Gamma = (e_{j_1}, \ldots, e_{j_r})$ of edges of a finite graph G is called a *walk* of length r if Γ satisfies

$$e_{j_1} = \{i_1, i_2\}, e_{j_2} = \{i_2, i_3\}, \ldots, e_{j_r} = \{i_r, i_{r+1}\}.$$

In addition, if $i_s \neq i_t$ $(s \neq t,\ s, t = 1, \ldots, r+1)$, then Γ is called a *path* and if $i_{r+1} = i_1$, then Γ is called a *closed walk* of length r. A closed walk of even length is called an *even closed walk*. If $i_{r+1} = i_1$ and $i_1, \ldots, i_r$ $(r \geq 3)$ are distinct, then Γ is called a *cycle* of length r. A cycle of odd length is called an *odd cycle*. An edge that joins two vertices of a cycle that is not itself an edge of the cycle is called a *chord* of that cycle [8]. A cycle is called *minimal* if it possesses no chords.

A finite graph G is said to be *connected* if, for any two vertices i and j of G, there exists a walk from i to j. In what follows, we will always assume that G is a connected graph. If the vertex set V of a finite graph G is partitioned into $V = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$ and each edge of G joins a vertex in V_1 and a vertex in V_2, then G is called a *bipartite graph*. It is known that a finite graph is bipartite if and only if it has no odd cycles.

If two cycles C and C' in G have no common vertices, then they are called *disjoint*. For the disjoint cycles C and C', the *bridge* of C and C' is the edge of G joining a vertex in C and a vertex in C'.

Let $E = \{e_1, \ldots, e_m\}$ be the edge set of G. The *edge space* $\varepsilon(G)$ of G is the vector space over the two-element field $\mathbb{F}_2 = \{0, 1\}$ of all functions $E \to \mathbb{F}_2$. The *cycle space* is the subspace of $\varepsilon(G)$ spanned by all cycles in G.

A *Fulkerson—Hoffman—McAndrew (FHM) graph* (see [9]) is a finite connected simple graph such that any pair of disjoint odd cycles has a bridge. A *fundamental FHM graph* (see [9]) is an FHM graph that has at least one pair of disjoint odd cycles.

The following is a basic fact about FHM graphs.

Proposition 2.1.2 ([16, Proposition 3.4], [19], [18, Corollary 2.3] and [25]) *Let G be a finite connected simple graph.*

(i) *The edge polytope P_G is normal if and only if G is an FHM graph. In particular, if the edge polytope P_G possesses a regular unimodular triangulation, then G is an FHM graph.*

(ii) *If G possesses no pairs of disjoint odd cycles, then P_G possesses a regular unimodular triangulation. In particular, if G is a bipartite graph, then P_G possesses a regular unimodular triangulation.*

(iii) *There exists an example of an edge polytope P_G of a fundamental FHM graph G that possesses no regular unimodular triangulations.*

We show examples of 20 fundamental FHM graphs in Appendix A. One of the graphs in Appendix A, namely "Graph 2", satisfies condition (iii).

2.2 Theoretical lower bounds

In this section, we provide several preliminary lemmas and propositions that will help us determine the type of the graphs under consideration. The following lemma shows that we are not interested in any graph that has a vertex of degree 1.

Lemma 2.2.1 ([11], Lemma 2.1) *Let G be a graph having a vertex v with $\deg v = 1$ and let $G' = G \setminus v$. Then, the edge polytope P_G of G has a regular unimodular triangulation if and only if the edge polytope $P_{G'}$ has one.*

The following proposition shows that we are not interested in any graph with ≤ 5 vertices.

Proposition 2.2.2 ([22], Theorem 3.3) *For a finite connected graph of d vertices, if $d \leq 5$, then all triangulations of P_G are unimodular.*

Let G be a graph with d vertices and n edges. The following proposition shows that we are not interested in any graph with $n - d \leq 3$.

Proposition 2.2.3 ([19]) *Let G be a finite connected graph with d vertices and n edges. We assume that $n - d \leq 3$. Then, the edge polytope P_G has a regular unimodular triangulation if and only if P_G is normal.*

Proof. The "only if" part is clear. Therefore, we will prove the "if" part. By Lemma 2.2.1, we may assume that the degree of each vertex of G is at least 2. We will use the following fact:

・If a subgraph G' of G has d' vertices and n' edges, then $n' - d' \leq n - d$.

We assume that P_G is normal and has no regular unimodular triangulations. Then, using [16, Theorem 3.5], there exist two pairs of disjoint odd cycles $\Pi = (C_1, C_2)$ and $\Pi' = (C_3, C_4)$ such that there is an edge e of C_3 that joins C_1 and C_2. We assume that the sequence of vertices of C_3 is $v_1, v_2, \ldots, v_m, v_1$ and $e = \{v_1, v_m\}$. Let $G' = C_1 \cup C_2 \cup C_3$. Moreover, we assume that C_4 is a cycle in G'. Since C_3 and C_4 have no common vertices, the cycle C_4 is a subgraph of $C_1 \cup C_2$ and does not contain the vertices v_1 and v_m. This is a contradiction. Thus, C_4 is not contained in G'. We let $v(\geq 2)$ denote the number of maximal consecutive subsequences of vertices of C_1 or C_2 that appear in C_3. Let d' (resp., n') be the number of vertices (resp., edges) of the graph G'. Then, $v = n' - d' \leq n - d \leq 3$. Hence, $v \in \{2, 3\}$.

(Case 1) We assume that $n - d = n' - d'$. Since both G and G' are connected, the dimension of the cycle space of G and that of G' are both $n - d + 1 = n' - d' + 1$ (see, e.g., [8, Theorem 1.9.6]). Hence, the two cycle spaces are isomorphic and, in particular, the set of cycles of G coincides with that of G'. Thus, C_4 appears in G'. This is a contradiction. Hence, this case is not possible.

(Case 2) We assume that $n - d > n' - d'$. Then, $3 \geq n - d > n' - d' = v \geq 2$. Hence, $n - d = 3$ and $n' - d' = v = 2$. Let V' (resp., E') be the set of vertices (resp., edges) of G' and let G'' be a subgraph of G with edge set $E'' = E \setminus E'$. Moreover, let V'' be the set of vertices appearing in E''. Let $G_1, \ldots, G_s$ denote connected components of G''. Then, we have

$$1 = (n - d) - (n' - d') = \sum_{i=1}^{s} (|E_i| - |V_i| + |V' \cap V_i|), \tag{2.1}$$

where V_i is the vertex set of G_i and E_i is the edge set of G_i. Since G is connected, $|V' \cap V_i| \geq 1$ for each i. In addition, since G_i is connected, we have $|E_i| - |V_i| \geq -1$. We assume that $|E_i| - |V_i| + |V' \cap V_i| = 0$. Then, $|E_i| - |V_i| = -1$. Hence, G_i is a tree. It is known that any tree has at least two vertices of degree 1. Since the degree of each vertex of G is at least 2, we have $|V' \cap V_i| \geq 2$. Therefore, $|E_i| - |V_i| + |V' \cap V_i| \neq 0$. This is a contradiction. Thus, $|E_i| - |V_i| + |V' \cap V_i| \geq 1$. It follows from Equation (2.1) that $s = 1$ and $|E_1| - |V_1| + |V' \cap V_1| = 1$. Therefore, G'' is connected and either (i) $|E''| - |V''| = 0$ and $|V' \cap V''| = 1$ or (ii) $|E''| - |V''| = -1$ and $|V' \cap V''| = 2$.

We assume that $V' \cap V'' = \{v_1', v_2'\}$ and v_i' is a vertex of C_i for $i = 1, 2$. Since $|E''| - |V''| = -1$, G'' is a path. Then, any cycle in $C_1 \cup C_2 \cup G''$ contains one of the vertices v_1 and v_m. This contradicts the assumption that C_4 is a subgraph of $C_1 \cup C_2 \cup G''$. Thus, we may assume that $|V' \cap V''| \leq 2$ and $V' \cap V''$ is a subset of the vertex set of C_1. Then, C_4 is a subgraph of $C_1 \cup G''$ and has no vertices of C_2. Since G satisfies the odd-cycle condition, there exists an edge e' that is not an edge of G joining C_4 and C_2. Since $G'' \cup C_1$ and C_2 have no common vertices, e' is not an edge of $G'' \cup C_1$. Since C_4 and $C_2 \cup C_3$ have no common vertices, e' is not an edge of $C_2 \cup C_3$. Thus, e' is not an edge of $G = G'' \cup C_1 \cup C_2 \cup C_3$. This is a contradiction.

Thus, Case 1 and Case 2 are not possible. Hence, we arrive at the desired conclusion. □

Let G be a fundamental FMH graph with d vertices and n edges and having no vertices of degree 1. By the preceding, we are interested in fundamental FHM graphs G with $d \geq 6$ and $n - d \geq 4$.

2.3 Criteria for the existence of a regular unimodular triangulation

Let G be a finite connected graph and let C be an odd cycle contained in G. Let c be a chord of C. Then, c divides C into two cycles, where one is an odd cycle and the other is an even cycle. We call the even cycle *the even closed walk of the chord c in C*. In the even closed walk Γ of the chord c in C, we require that c be an even-numbered edge of Γ.

Let (C_1, C_2) be a pair of disjoint odd cycles in G (namely, the odd cycles C_1 and C_2 have no common vertices) and b be a bridge of this pair. Then, *the even closed walk of b in* (C_1, C_2) is the closed walk $(C_1, b, C_2, -b)$. Here, $-b$ denotes the oppositely directed edge of b and the cycle C_1 starts from the vertex $C_1 \cap b$ and ends at the same vertex. The same holds for C_2. We note that in the even closed walk Γ of the bridge b in (C_1, C_2), b appears twice as an even-numbered edge of Γ.

We will review the necessary and sufficient condition for P_G to have a regular unimodular triangulation ([16, Theorem 3.5]). We assume that G possesses p pairs of disjoint odd cycles $\Pi_1 = (C_1, C_1'), \ldots, \Pi_p = (C_p, C_p')$. For each i $(1 \leq i \leq p)$, let $\{b_j^i \mid 1 \leq j \leq q_i\}$ be the set of bridges of Π_i and the chords of C_i or C_i'. Let $\Gamma_j^i = (e_{i_1} e_{i_2} \ldots e_{i_{2r}})$ be the even closed walk of b_j^i, where the bridge or chord is even-numbered.

We now define the open half-space $H_{b_j^i}$ by

$$H_{b_j^i} := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{k=1}^{r} x_{i_{2k-1}} > \sum_{k=1}^{r} x_{i_{2k}} \right\}. \tag{2.2}$$

Furthermore, we set $W := \bigcap_{i=1}^{p} (\bigcup_{j=1}^{q_i} H_{b_j^i})$. The following result is our starting point.

Proposition 2.3.1 ([16, Theorem 3.5]) *The edge polytope P_G possesses a regular unimodular triangulation if and only if $W \neq \phi$.*

Let G be a fundamental FHM graph. In this section, we will provide four criteria for the edge polytope P_G to possess a regular unimodular triangulation in terms of simple graph data. Our criteria are based on the existence of special bridges in each pair of disjoint odd cycles. Let $\Pi_1, \ldots, \Pi_p$ be all the pairs of disjoint odd cycles in G as before and $\{b^1, \ldots, b^p\}$ be the set of bridges, where b^i is the bridge of Π_i. Let $\Gamma_i := (e_{i_1}, \ldots, e_{i_{2s+1}}, b^i, e_{j_1}, \ldots, e_{j_{2t+1}}, -b^i)$.

We define

$$\alpha_i := |\{b^1, \ldots, b^p\} \cap \{e_{i_2}, e_{i_4}, \ldots, e_{i_{2s}}, e_{j_2}, e_{j_4}, \ldots, e_{j_{2t}}\}|,$$

$$\beta_i := |\{b^1, \ldots, b^p\} \cap \{e_{i_1}, e_{i_3}, \ldots, e_{i_{2s+1}}, e_{j_1}, e_{j_3}, \ldots, e_{j_{2t+1}}\}|.$$

Furthermore, we set $a_i := 2 + \alpha_i - \beta_i$.

Theorem 2.3.2 *We use the same notation as above. The edge polytope of a fundamental FHM graph G possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ (b^i is the bridge of Π_i) that satisfies the following condition: For each i, $a_i \geq 0$ holds and the number of Γ_i such that $a_i = 0$ is at most two.*

Proof. We first rewrite W in Proposition 2.3.1, by the distributive law, as follows:

$$W = \bigcap_{i=1}^{p} \left(\bigcup_{j=1}^{q_i} H_{b^i_j} \right) = \bigcup_{j_1, \ldots, j_p} \left(H_{b^1_{j_1}} \cap \cdots \cap H_{b^p_{j_p}} \right),$$

where j_k satisfies $1 \leq j_k \leq q_k$. We set

$$C_b = C_{\left\{b^1_{j_1}, \ldots, b^p_{j_p}\right\}} := H_{b^1_{j_1}} \cap \cdots \cap H_{b^p_{j_p}}$$

and call C_b the open cone of $b = \{b^1_{j_1}, \ldots, b^p_{j_p}\}$. Thus, $W \neq \phi$ is equivalent to the existence of a set of bridges $b = \{b^1, \ldots, b^p\}$ (b^i is a bridge of Π_i) such that C_b is non-empty.

For each i, let Γ_i be the even closed walk of b^i and $f_i > 0$ be Inequality (2.2) defined by b^i. We denote by the same f_i an n-dimension vector that consists of the coefficients of the left-hand side (LHS) of the inequality $f_i > 0$. We note that if the bridge b^i is equal to an edge e_j, if the j-th component $f_i[j]$ of the vector f_i is -2 and if the other edge e_k is contained in Γ_i, then $f_i[k] = +1$ (resp., -1) if e_k is an odd (resp., even)-numbered edge of Γ_i. The other components of f_i are 0.

We define the standard weight vector $w \in \mathbb{R}^n$ of C_b as follows. If there exists i such that $f_i[k] = -2$, then we set $w[k] := -1$. The other components of w are 0. We note that a_i is equal to $f_i \cdot w$ (inner product) for each i.

(i) We assume that $a_i > 0$ for any i. Since $f_i \cdot w > 0$ for any i, we have $w \in W$.

(ii) We assume that $a_j = 0$ and $a_i > 0$ ($i \neq j$). Let b^j be a bridge of Γ_j and $b^j = e_l$. Let $w' := w + (-1/10\mathbf{e}_l)$, where $\mathbf{e}_l$ is a unit vector. We now consider $f_j \cdot w' = f_j \cdot w + f_j \cdot -1/10\mathbf{e}_l$. By assumption, $f_j \cdot w = a_j = 0$. Moreover, we obtain $f_j \cdot -1/10\mathbf{e}_l = 1/5$. Therefore, $f_j \cdot w' = f_j \cdot w + f_j \cdot -1/10\mathbf{e}_l = 1/5 > 0$.

Furthermore, let b^k be a bridge of Γ_i and $b^k = e_m$ and let $w' := w+(-1/10\mathbf{e}_m)$. We next consider $f_i \cdot w' = f_i \cdot w + f_i \cdot -1/10\mathbf{e}_m$. By assumption, $f_i \cdot w = a_i > 0$. Moreover, we obtain $f_i \cdot -1/10\mathbf{e}_m = 1/5$. Therefore, $f_i \cdot w' = f_i \cdot w + f_i \cdot -1/10\mathbf{e}_m > 0$.

(iii) We assume that $a_j = a_k = 0$ and $a_i > 0$ $(i \neq j, i \neq k)$. There exists at least an edge e_l in Γ_j that is not contained in Γ_k. Moreover, there exists at least an edge e_m in Γ_k that is not contained in Γ_j. Let v be a vector that satisfies the following condition: $v[l] = 1/10$ (resp., $-1/10$) if e_l is odd numbered (resp., even-numbered) in Γ_j. $v[m] = 1/10$ (resp., $-1/10$) if e_m is odd numbered (resp., even-numbered) in Γ_k. The other components of v are 0. Let $w' := w + v$. We now consider $f_i \cdot w' = f_i \cdot w + f_i \cdot v$. By assumption, $f_i \cdot w = a_i > 0$. Moreover, we obtain $f_i \cdot v \geq -3/10$. Since $a_i \in \mathbb{Z}_{>0}$, we have $f_i \cdot w = a_i \geq 1$. Therefore, $f_i \cdot w' = f_i \cdot w + f_i \cdot v \geq 7/10 > 0$. We next consider $f_j \cdot w' = f_j \cdot w + f_j \cdot v$. By assumption, $f_j \cdot w = a_j = 0$. Moreover, we obtain $f_j \cdot v = 1/10$ or $1/5$. Therefore, $f_j \cdot w' = f_j \cdot w + f_j \cdot v > 0$.

As above, we obtain $f_k \cdot w' = f_k \cdot w + f_k \cdot v > 0$.

□

We have the following corollaries.

Corollary 2.3.3 *We use the same notation as above. The edge polytope of a fundamental FHM graph G possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ (b^i is the bridge of Π_i) such that $a_i > 0$ for each i.*

Corollary 2.3.4 *The edge polytope of a fundamental FHM graph G possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ (b^i is the bridge of Π_i) that satisfies the following condition: For each even closed walk Γ_i of b^i, the number of the other bridges b^j contained in Γ_i is at most two and the number of Γ_i that contain exactly two other bridges is at most two.*

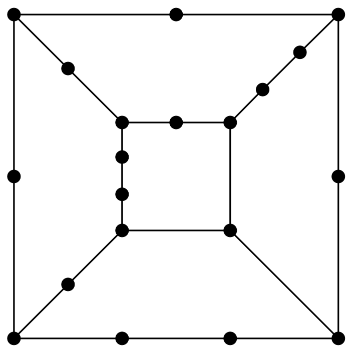
Corollary 2.3.5 *The edge polytope of a fundamental FHM graph G possesses a regular unimodular triangulation if it has a set of bridges $\{b^1, \ldots, b^p\}$ (b^i is the bridge of Π_i) that satisfies the following condition: Each even closed walk of the bridge b^i contains at most one different bridge b^j.*

We note that the strongest condition is Corollary 2.3.5, whereas the weakest is Theorem 2.3.2. However, Corollary 2.3.5 is the easiest to verify graphically.

Remark 2.3.6 (i) In Theorem 2.3.2, if there exist more than two i such that $a_i = 0$, the following holds. We assume that $a_i = 0$ for $i = i_1, \ldots, i_r$ $(r \geq 3)$ and $a_i > 0$ for the other i. Let $H \subset \mathbb{R}^n$ be the hyperplane defined by $\sum_{j=1}^{n} w[j]x_j = 0$. If the convex cone P generated by $f_{i_1}, \ldots, f_{i_r}$ in H is strongly convex, i.e., $P \cap -P = \{\mathbf{0}\}$, then W is non-empty. The proof is the same as that of Theorem 2.3.2. Namely, owing to this condition, we can vary w slightly to obtain a new weight w' such that $f_i \cdot w' > 0$ for any i. However, this condition is not clear at all by merely inspecting the graph.

(ii) More generally, let $C(f_1, \ldots, f_p)$ be an open cone in $\mathbb{R}^n$ defined by p linear homogeneous inequalities $f_i > 0$ $(1 \leq i \leq p)$. Then, $C(f_1, \ldots, f_p) \neq \phi$ holds if and only if the dual cone $C(f_1, \ldots, f_p)^\vee = \mathbb{R}_{\geq 0} f_1 + \cdots + \mathbb{R}_{\geq 0} f_p$ of $C(f_1, \ldots, f_p)$ is strongly convex (f_i is the coefficient vector of the LHS of the inequality). It is difficult to determine whether $C(f_1, \ldots f_p)^\vee$ is strongly convex or not by merely inspecting the graph.

(iii) The edge polytope of the following graph does not possess regular unimodular triangulations (Example 3.2 in [17]). Moreover, there exist three i such that $a_i = 0$. Therefore, we cannot improve the condition of Theorem 2.3.2 such that "the number of Γ_i such that $a_i = 0$ is at most three"



2.4 Applications

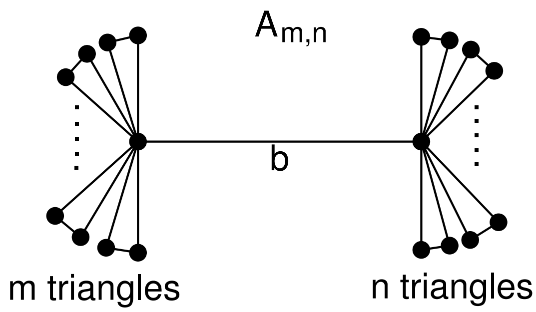
We first apply our criteria to the complete graph $G = K_6$ with six vertices. It is known that P_{K_d} possesses a regular unimodular triangulation for any d (see [30]). Moreover, the same is true of an edge polytope of a gap-free graph or a complete multipartite graph (see [6] and [20]).

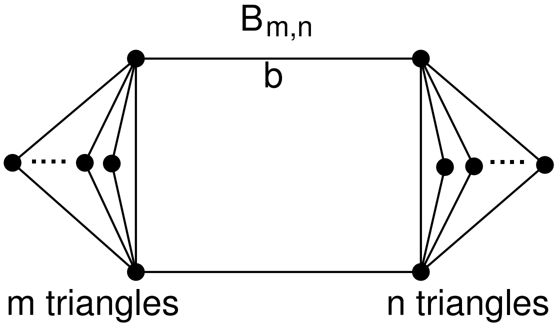
Remark 2.4.1 The complete graph K_6 satisfies the condition of Corollary 2.3.3. However, it does not satisfy the condition of Corollary 2.3.4.

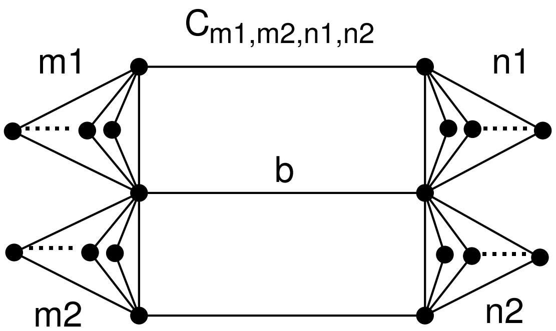
We finally provide several other examples that satisfy our criteria.

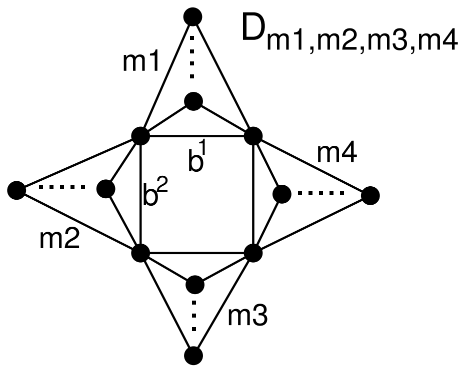
Example 2.4.2 The following five types of graphs satisfy the condition of Corollary 2.3.5. More precisely, in the graphs $A_{m,n}$, $B_{m,n}$ and C_{m_1,m_2,n_1,n_2}, all pairs of disjoint odd cycles (triangles) have a bridge b in common. Thus, there are no other bridges contained in the even closed walk of b.

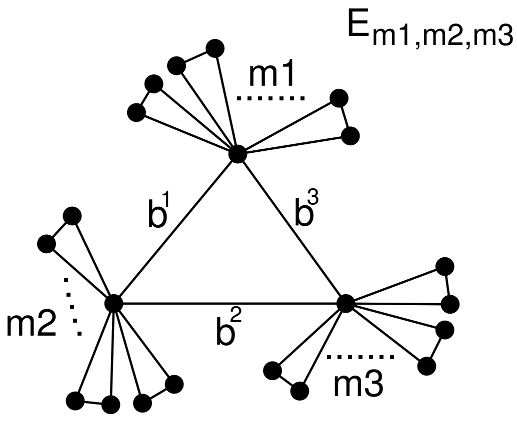
D_{m_1,m_2,m_3,m_4} has a set of bridges $\{b^1, b^2\}$, where any disjoint pair has a bridge in this set and the even closed walk of b^i $(i = 1, 2)$ contains (exactly) one other bridge. E_{m_1,m_2,m_3} has a set of three bridges $\{b^1, b^2, b^3\}$, where any disjoint pair has a bridge in this set and there are no other bridges contained in the even closed walk of b^i $(i = 1, 2, 3)$.



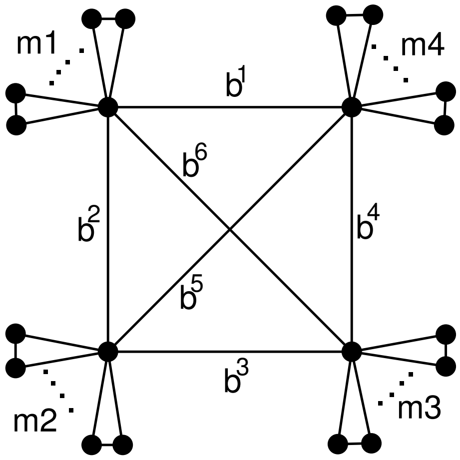




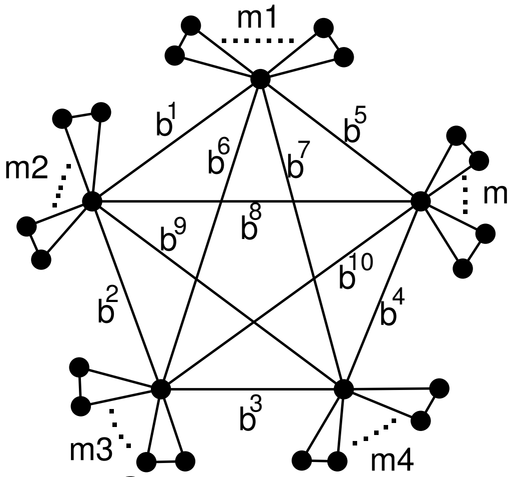




Example 2.4.3 The following two types of graphs satisfy the condition of Corollary 2.3.3 and not that of Corollary 2.3.4. F_{m_1,m_2,m_3,m_4} has a minimal set of six bridges $\{b^i \mid 1 \leq i \leq 6\}$, where any disjoint pair has a bridge in this set and G_{m_1,m_2,m_3,m_4,m_5} has a minimal set of ten bridges $\{b^i \mid 1 \leq i \leq 10\}$.

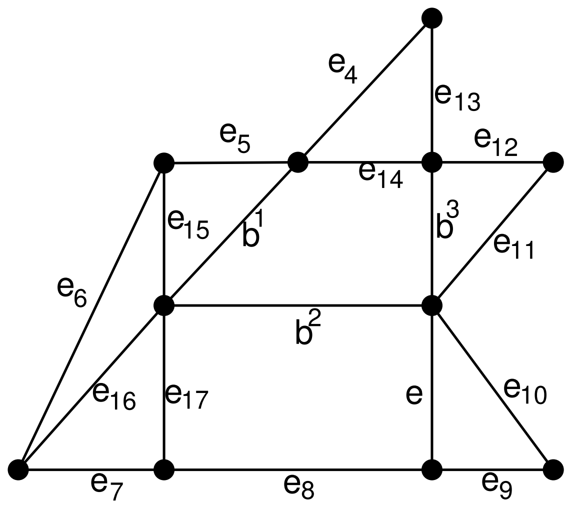


$F_{m1,m2,m3,m4}$



$G_{m1,m2,m3,m4,m5}$

Example 2.4.4 The following graph ("Graph 9" in Appendix A) satisfies the condition of Theorem 2.3.2. Moreover, there exist only two i such that $a_i = 0$. The graph has a minimal set of three bridges $\{b^i \,|\, 1 \leq i \leq 3\}$, where any disjoint pair has a bridge in this set. When $\Gamma_1 = (e_{15}, e_5, b^1, b^2, b^3, e_{12}, e_{11}, -b^2)$, then $\alpha_1 = 0$ and $\beta_1 = 2$. Therefore, $a_1 = 2 + \alpha_1 - \beta_1 = 2 + 0 - 2 = 0$. Moreover, when $\Gamma_2 = (e_{17}, e_7, e_6, e_5, b^1, b^2, b^3, e_{12}, e_{11}, -b^2)$, then $\alpha_2 = 0$ and $\beta_2 = 2$. Therefore, $a_2 = 2 + \alpha_2 - \beta_2 = 2 + 0 - 2 = 0$.



2.5 The algorithm and the program

We have implemented a program for the computer algebra system Magma [2] that determines whether a given fundamental FHM graph satisfies Theorem 2.3.2, Corollary 2.3.3, Corollary 2.3.4 and Corollary 2.3.5. In this section, we provide the details of the algorithm and the program. Let G be a FHM graph and P_G be an edge polytope of G. The program that determines whether P_G possesses a regular unimodular triangulation is based on Theorem 2.3.1. We can rewrite "the open cone C_b is not empty if and only of the dimension of the closed cone C_b is equal to the number of edges" as "P_G possesses a regular unimodular triangulation if and only if there exists at least one closed cone C_b such that the dimension of C_b is equal to the number of edges", where the closed cone is defined by replacing $>$ with $\geq$ in Inequality (2.2). The algorithm calculates the dimension of all closed cones. If there exists a cone such that the dimension of the cone is equal to the number of edges, then the program will output "W is not empty". If the dimension of all cones is less than the number of edges, then the program will output "W is empty".

We now show how the program "cycle12.c" can be used (the details are in Appendix B). The data of a graph $G = (V, E)$ is given by the vertex set V and the edge set E.

Example 2.5.1 The following is an example of the input data for "Graph 1" in Appendix A.

```
V1:= [1..9];
E1:= [{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{1,7},{2,7},{2,8},{7,8},{8,9},
{2,9},{7,9},{4,9},{5,9},{3,9},{6,9}];
```

The main commands are "main" and "initial". The command "main(V,E)" calculates the dimension of all closed cones of the graph $G = (V, E)$. If there exists a closed cone such that the dimension of the cone is equal to the number of edges, the program stops and outputs the coefficients of Inequality (2.2) of the cone and an inner point w in the cone. The inner point w is generated to solve Inequality (2.2) and calculate the minimal generators of the cone and their sum. If the dimension of all cones is less than the number of edges, then the program will output the message "W is empty".

The command "initial(V,E,w)" is used if W is non-empty. The command "initial(V,E,w)" outputs the initial ideal of the toric ideal with respect to weight monomial order on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ (n is the number of edge) defined from w and graded reverse lex order. Moreover, it outputs the set of maximal simplices of the regular unimodular triangulation.

Example 2.5.2 The following is the result for "Graph 5" in Appendix A.

```
main(V5,E5);
set of vertices =  [ 1 .. 10 ]
array of edges =  [
    { 1, 2 },
    { 2, 3 },
    { 3, 4 },
    { 4, 5 },
    { 5, 6 },
    { 6, 7 },
```

```
    { 7, 8 },
    { 1, 8 },
    { 1, 9 },
    { 8, 9 },
    { 9, 10 },
    { 2, 10 },
    { 7, 10 },
    { 5, 10 },
    { 2, 4 },
    { 5, 7 }
]
number of edges= 16
number of pairs= 12
array of the number of even closed walks=  [ 1, 6, 6, 3, 2, 4, 2, 1, 3, 3, 1, 4 ]
number of cones=  62208
1 16
[
    [ 0, -1, 1, -2, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1 ],
    [ -2, 1, -1, 0, -1, 1, -1, 1, 1, 0, -1, 0, 0, 1, 1, 0 ],
    [ -2, 1, -1, 1, -1, 1, 0, 1, 1, -1, 0, 1, -1, 0, 0, 0 ],
    [ -2, 1, -1, 1, 0, 0, 0, 1, 1, -1, 0, 1, 0, -1, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, -2, 1, -1, 1, 0, 0, 1, -1, 0, 1 ],
    [ -2, 0, 0, -1, 0, 0, 0, 1, 1, -1, 0, 1, -1, 0, 1, 1 ],
    [ 0, -1, 1, -2, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 1, 1 ],
    [ -2, 1, -1, 0, 0, 0, 0, 1, 1, -1, 0, 0, 0, 0, 1, 0 ],
    [ -2, 1, -1, 0, 0, 0, -1, 1, 1, 0, -1, 0, 1, 0, 1, 0 ],
    [ 0, -1, 1, -2, 0, 0, -1, 0, 0, 1, -1, 0, 0, 1, 1, 1 ],
    [ 0, 0, 0, 0, -1, 1, -2, 1, -1, 1, 0, 0, 0, 0, 0, 1 ],
    [ -1, 0, 0, 0, -1, 1, -2, 1, 0, 1, -1, 1, 0, 0, 0, 1 ]
]
16 non-empty
inner product = [ 2, 3, 1, 1, 2, 3, 2, 2, 3, 3, 2, 3 ]
inner product (absolute value) = [ 2, 3, 3, 3, 2, 3, 2, 2, 3, 3, 2, 3 ]
inner point weight=
[ 3, 3, 3, 3, 3, 3, 3, 7, 3, 3, 1, 3, 1, 1, 9, 7 ]
[ 3, 3, 3, 3, 3, 3, 3, 7, 3, 3, 1, 3, 1, 1, 9, 7 ]

----------------------------------------------------------------------------

initial(V5,E5,[ 3, 3, 3, 3, 3, 3, 3, 7, 3, 3, 1, 3, 1, 1, 9, 7 ]);
number of vertices=  10
number of edges =  16
initial ideal =  [
    x[8]*x[15]*x[16],
    x[2]*x[8]*x[9]*x[15],
    x[3]*x[5]*x[15]*x[16],
    x[2]*x[8]*x[11]*x[15],
    x[2]*x[8]*x[13]*x[15],
    x[6]*x[8]*x[10]*x[16],
    x[5]*x[7]*x[9]*x[15],
    x[8]*x[10]*x[13]*x[16],
    x[8]*x[12]*x[16],
```

```
    x[2]*x[4]*x[6]*x[8],
    x[1]*x[3]*x[10]*x[16],
    x[8]*x[11]*x[16],
    x[2]*x[4]*x[8]*x[11],
    x[2]*x[4]*x[8]*x[13],
    x[1]*x[3]*x[11]*x[16],
    x[3]*x[12]*x[16],
    x[14]*x[15],
    x[9]*x[12],
    x[7]*x[11],
    x[5]*x[13]
]
number of maximal simplices= 73
array of maximal simplices of regular unimodular triangulation=
[
    x[2]*x[3]*x[4]*x[6]*x[9]*x[10]*x[11]*x[13]*x[15]*x[16],
    x[2]*x[3]*x[4]*x[6]*x[9]*x[10]*x[11]*x[13]*x[14]*x[16],
    x[2]*x[3]*x[4]*x[6]*x[7]*x[9]*x[10]*x[13]*x[15]*x[16],
    x[2]*x[3]*x[4]*x[6]*x[7]*x[9]*x[10]*x[13]*x[14]*x[16],
    x[2]*x[3]*x[4]*x[5]*x[7]*x[8]*x[9]*x[10]*x[14]*x[16],
    x[2]*x[3]*x[4]*x[5]*x[6]*x[9]*x[10]*x[11]*x[14]*x[16],
    x[2]*x[3]*x[4]*x[5]*x[6]*x[7]*x[9]*x[10]*x[14]*x[16],
    x[1]*x[3]*x[4]*x[6]*x[8]*x[10]*x[11]*x[12]*x[13]*x[15],
    x[1]*x[3]*x[4]*x[6]*x[8]*x[10]*x[11]*x[12]*x[13]*x[14],
    x[1]*x[3]*x[4]*x[6]*x[8]*x[9]*x[10]*x[11]*x[13]*x[15],
    x[1]*x[3]*x[4]*x[6]*x[8]*x[9]*x[10]*x[11]*x[13]*x[14],
    x[1]*x[3]*x[4]*x[6]*x[7]*x[8]*x[10]*x[12]*x[13]*x[15],
    x[1]*x[3]*x[4]*x[6]*x[7]*x[8]*x[10]*x[12]*x[13]*x[14],
    x[1]*x[3]*x[4]*x[6]*x[7]*x[8]*x[9]*x[13]*x[14]*x[16],
    x[1]*x[3]*x[4]*x[6]*x[7]*x[8]*x[9]*x[10]*x[13]*x[15],
    x[1]*x[3]*x[4]*x[6]*x[7]*x[8]*x[9]*x[10]*x[13]*x[14],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[8]*x[10]*x[11]*x[12]*x[15],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[8]*x[10]*x[11]*x[12]*x[14],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[8]*x[9]*x[10]*x[11]*x[15],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[8]*x[9]*x[10]*x[11]*x[14],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[7]*x[8]*x[10]*x[12]*x[15],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[7]*x[8]*x[10]*x[12]*x[14],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[7]*x[8]*x[9]*x[14]*x[16],
    x[1]*x[3]*x[4]*x[5]*x[6]*x[7]*x[8]*x[9]*x[10]*x[14],
    x[1]*x[2]*x[4]*x[6]*x[10]*x[11]*x[12]*x[13]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[10]*x[11]*x[12]*x[13]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[9]*x[10]*x[11]*x[13]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[9]*x[10]*x[11]*x[13]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[7]*x[10]*x[12]*x[13]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[7]*x[10]*x[12]*x[13]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[7]*x[9]*x[10]*x[13]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[6]*x[7]*x[9]*x[10]*x[13]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[7]*x[8]*x[9]*x[10]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[6]*x[10]*x[11]*x[12]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[6]*x[10]*x[11]*x[12]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[6]*x[9]*x[10]*x[11]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[6]*x[9]*x[10]*x[11]*x[14]*x[16],
```

```
    x[1]*x[2]*x[4]*x[5]*x[6]*x[7]*x[10]*x[12]*x[15]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[6]*x[7]*x[10]*x[12]*x[14]*x[16],
    x[1]*x[2]*x[4]*x[5]*x[6]*x[7]*x[9]*x[10]*x[14]*x[16],
    x[1]*x[2]*x[3]*x[6]*x[8]*x[10]*x[11]*x[12]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[6]*x[8]*x[9]*x[10]*x[11]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[6]*x[7]*x[8]*x[10]*x[12]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[6]*x[7]*x[8]*x[9]*x[13]*x[14]*x[16],
    x[1]*x[2]*x[3]*x[6]*x[7]*x[8]*x[9]*x[10]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[5]*x[6]*x[8]*x[10]*x[11]*x[12]*x[14],
    x[1]*x[2]*x[3]*x[5]*x[6]*x[8]*x[9]*x[10]*x[11]*x[14],
    x[1]*x[2]*x[3]*x[5]*x[6]*x[7]*x[8]*x[10]*x[12]*x[15],
    x[1]*x[2]*x[3]*x[5]*x[6]*x[7]*x[8]*x[10]*x[12]*x[14],
    x[1]*x[2]*x[3]*x[5]*x[6]*x[7]*x[8]*x[9]*x[14]*x[16],
    x[1]*x[2]*x[3]*x[5]*x[6]*x[7]*x[8]*x[9]*x[10]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[10]*x[11]*x[12]*x[13]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[10]*x[11]*x[12]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[9]*x[10]*x[11]*x[13]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[9]*x[10]*x[11]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[7]*x[10]*x[12]*x[13]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[7]*x[10]*x[12]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[7]*x[9]*x[13]*x[15]*x[16],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[7]*x[9]*x[13]*x[14]*x[16],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[7]*x[9]*x[10]*x[13]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[6]*x[7]*x[9]*x[10]*x[13]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[7]*x[8]*x[10]*x[12]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[7]*x[8]*x[10]*x[12]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[7]*x[8]*x[9]*x[14]*x[16],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[7]*x[8]*x[9]*x[10]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[10]*x[11]*x[12]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[10]*x[11]*x[12]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[9]*x[10]*x[11]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[9]*x[10]*x[11]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[7]*x[10]*x[12]*x[15],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[7]*x[10]*x[12]*x[14],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[7]*x[9]*x[14]*x[16],
    x[1]*x[2]*x[3]*x[4]*x[5]*x[6]*x[7]*x[9]*x[10]*x[14]
]
```

The following table shows whether W is empty and, in case W is non-empty, the number of maximal simplices of the regular unimodular triangulation and the number of cones for 20 fundamental FHM graphs in Appendix A. In the case of "Graph 13", the program cannot determine whether W is empty and calculate the number of maximal simplices.

Table 2.1: Results for the 20 FHM graphs

Graph number	empty or non-empty	number of cones
Graph 1	non-empty	17280
Graph 2	empty	1
Graph 3	non-empty	32768
Graph 4	non-empty	32
Graph 5	non-empty	62208
Graph 6	non-empty	1761205026816
Graph 7	non-empty	18
Graph 8	non-empty	4
Graph 9	non-empty	16128
Graph 10	non-empty	6912
Graph 11	non-empty	8
Graph 12	non-empty	8124796533473280000
Graph 13	?	4918301009412067196928000000
Graph 14	non-empty	8
Graph 15	non-empty	3456
Graph 16	non-empty	81
Graph 17	non-empty	4
Graph 18	non-empty	4
Graph 19	non-empty	1024
Graph 20	non-empty	432

We next consider the command "bmain$k$" ($1 \leq k \leq 4$), which determines whether a given fundamental FHM graph G satisfies Theorem 2.3.2, Corollary 2.3.3, Corollary 2.3.4 and Corollary 2.3.5.

For example, "bmain1(V,E)" determines whether G possesses cones that satisfy Theorem 2.3.2. If G possesses cones that satisfy Theorem 2.3.2, then "bmain1(V,E)" outputs the inequalities of the cones and the weights that satisfy these inequalities.

Example 2.5.3

```
bmain1(V1,E1);
number of edges= 17
number of pairs= 8
array of the number of even closed walks=  [ 5, 4, 3, 3, 4, 4, 3, 2 ]
number of cones=  17280
[
    [ 0, -2, 1, -1, 1, 0, 0, 1, 1, -1, 0, 0, 0, 0, 0, 1, -1 ],
    [ 1, -2, 1, -1, 1, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 1, -1 ],
    [ 0, 0, 0, -1, 0, 0, 0, -1, 1, 1, -2, 0, 0, 1, 1, 0, 0 ],
    [ 1, -2, 1, 0, 0, 0, -1, 1, 0, 0, 0, 0, -1, 0, 1, 0 ],
    [ 0, -2, 1, 0, 0, 0, 0, 1, 1, -1, 0, 0, 0, -1, 0, 1, 0 ],
    [ 0, 0, 0, 0, 1, -2, 0, 1, -1, 1, 0, 0, 0, 0, -1, 0, 1 ],
    [ -1, 0, 0, 0, 1, -2, 1, 1, 0, 0, 0, 0, 0, 0, -1, 0, 1 ],
    [ 1, 0, 0, -1, 0, 0, -1, 1, 0, 0, 0, -2, 0, 1, 1, 0, 0 ]
]
17 non-empty
inner product = [ 2, 2, 2, 2, 2, 2, 2, 2 ]
inner product (absolute value) = [ 2, 2, 2, 2, 2, 2, 2, 2 ]
inner point weight=
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 5, 1, 9, 5 ]
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 5, 1, 9, 5 ]

------------------------------------------------------------------------------

bmain1(V2,E2);
number of edges= 15
number of pairs= 5
array of the number of even closed walks=  [ 1, 1, 1, 1, 1 ]
number of cones=  1
no bridge cones

------------------------------------------------------------------------------

bmain2(V3,E3);
number of edges= 15
number of pairs= 9
array of the number of even closed walks=  [ 3, 3, 2, 3, 3, 3, 2, 3, 2 ]
number of cones=  5832
[
    [ 0, 0, 1, -2, 1, -1, 0, 0, -1, 1, 0, -1, 1, 1, 0 ],
    [ 1, -2, 1, -1, 0, 0, 0, -1, 0, -1, 1, 1, 0, 1, 0 ],
    [ 1, -2, 1, 0, 0, 0, 0, 0, 0, -1, 1, 1, -1, 0, 0 ],
```

```
    [ 1, -1, 0, 0, 1, -2, -1, 0, 0, 1, 0, 1, 0, -1, 1 ],
    [ 1, -2, 1, 0, 0, -1, 0, 0, -1, 0, 1, 1, -1, 0, 1 ],
    [ 0, -1, 1, -2, 1, 0, -1, 0, 0, 0, 1, 0, 1, 1, -1 ],
    [ 0, 0, 1, -2, 1, 0, 0, 0, 0, 0, 0, -1, 1, 1, -1 ],
    [ 1, 0, 0, -1, 1, -2, 0, -1, 0, 1, -1, 0, 1, 0, 1 ],
    [ 1, 0, 0, 0, 1, -2, 0, 0, 0, 1, -1, 0, 0, -1, 1 ]
]
15 non-empty
standard weight vector=
[ 2, 1, 2, 1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
inner product = [ 3, 3, 2, 3, 3, 3, 2, 3, 2 ]
inner product (absplute value) = [ 3, 3, 2, 3, 3, 3, 2, 3, 2 ]
[ 2, 1, 2, 1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]

-----------------------------------------------------------------------------

bmain2(V4,E4);
number of edges= 12
number of pairs= 3
array of the number of even closed walks=  [ 4, 4, 2 ]
number of cones=  32
[
    [ 0, -2, 1, -1, 0, 0, 1, 1, 0, 1, 0, -1 ],
    [ 1, -2, 0, 0, 0, -1, 1, 0, 1, 1, -1, 0 ],
    [ 1, -2, 1, -1, 0, -1, 1, 0, 0, 1, 0, 0 ]
]
12 non-empty
standard weight vector=
[ 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
inner product = [ 2, 2, 2 ]
inner product (absplute value) = [ 2, 2, 2 ]
[ 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]

-----------------------------------------------------------------------------

bmain3(V5,E5);
number of edges= 16
number of pairs= 12
array of the number of even closed walks=  [ 1, 3, 3, 2, 2, 3, 2, 1, 2, 2, 1, 3 ]
number of cones=  2592
[
    [ 0, -1, 1, -2, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1 ],
    [ -2, 1, -1, 0, -1, 1, -1, 1, 1, 0, -1, 0, 0, 1, 1, 0 ],
    [ -2, 1, -1, 1, -1, 1, 0, 1, 1, -1, 0, 1, -1, 0, 0, 0 ],
    [ -2, 1, -1, 1, 0, 0, 0, 1, 1, -1, 0, 1, 0, -1, 0, 0 ],
    [ 0, 0, 0, 0, 0, 0, -2, 1, -1, 1, 0, 0, 1, -1, 0, 1 ],
    [ -2, 0, 0, -1, 0, 0, 0, 1, 1, -1, 0, 1, -1, 0, 1, 1 ],
    [ 0, -1, 1, -2, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 1, 1 ],
    [ -2, 1, -1, 0, 0, 0, 0, 1, 1, -1, 0, 0, 0, 0, 1, 0 ],
    [ -2, 1, -1, 0, 0, 0, -1, 1, 1, 0, -1, 0, 1, 0, 1, 0 ],
    [ 0, -1, 1, -2, 0, 0, -1, 0, 0, 1, -1, 0, 0, 1, 1, 1 ],
    [ 0, 0, 0, 0, -1, 1, -2, 1, -1, 1, 0, 0, 0, 0, 0, 1 ],
```

```
    [ -1, 0, 0, 0, -1, 1, -2, 1, 0, 1, -1, 1, 0, 0, 0, 1 ]
]
16 non-empty
inner product = [ 2, 3, 1, 1, 2, 3, 2, 2, 3, 3, 2, 3 ]
inner product (absolute value) = [ 2, 3, 3, 3, 2, 3, 2, 2, 3, 3, 2, 3 ]
inner point weight=
[ 3, 3, 3, 3, 3, 3, 3, 7, 3, 3, 1, 3, 1, 1, 9, 7 ]
[ 3, 3, 3, 3, 3, 3, 3, 7, 3, 3, 1, 3, 1, 1, 9, 7 ]

------------------------------------------------------------------------------

bmain3(V6,E6);
number of edges= 13
number of pairs= 5
array of the number of even closed walks=  [ 1, 2, 2, 1, 2 ]
number of cones=  8
[
    [ 1, -2, 1, -1, 0, 0, 0, 0, -1, 1, 0, 0, 1 ],
    [ -1, 0, 0, 0, 0, -1, 1, -2, 1, 1, 0, 1, 0 ],
    [ 1, -2, 1, -1, 0, 0, 0, 1, -1, 0, 1, -1, 1 ],
    [ 0, 0, -1, 1, -2, 1, -1, 0, 0, 0, 0, 1, 1 ],
    [ 0, -2, 1, -1, 0, -1, 1, -1, 0, 1, 1, 0, 1 ]
]
13 non-empty
inner product = [ 2, 2, 1, 2, 3 ]
inner product (absolute value) = [ 2, 2, 3, 2, 3 ]
inner point weight=
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2 ]
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2 ]

------------------------------------------------------------------------------

bmain4(V7,V7);
number of edges= 13
number of pairs= 4
array of the number of even closed walks=  [ 1, 2, 2, 1 ]
number of cones=  4
[
    [ -1, 0, 0, -1, 0, 0, 1, 1, 0, -2, 0, 1, 1 ],
    [ 1, 0, 0, 0, 1, -2, 1, -1, 0, 0, 1, -1, 0 ],
    [ 0, 1, -2, 1, 0, 0, 0, -1, 1, 0, 0, -1, 1 ],
    [ 0, -1, 0, 0, -1, 0, 0, 1, 1, -2, 1, 1, 0 ]
]
13 non-empty
standard weight vector=
[ 2, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2 ]
inner product= [ 2, 2, 2, 2 ]
inner product (absolute value) = [ 2, 2, 2, 2 ]
[ 2, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2 ]

------------------------------------------------------------------------------
```

```
bmain4(V8,E8);
number of edges= 18
number of pairs= 19
array of the number of even closed walks=  [ 2, 2, 1, 2, 3, 3, 2, 1, 2, 3, 2, 3,
1, 1, 4, 2, 3, 2, 2 ]
number of cones=  497664
no bridge cones
```

The following table shows whether the 20 fundamental FHM graphs in Appendix A satisfy Theorem 2.3.2, Corollary 2.3.3, Corollary 2.3.4 and Corollary 2.3.5, where "?" indicates that the program is inconclusive.

Table 2.2: Testing whether the 20 fundamental FHM graphs satisfy our criteria

Graph number	Theorem 2.3.2	Corollary 2.3.3	Corollary 2.3.4	Corollary 2.3.5
Graph 1	◯	◯	◯	◯
Graph 2	×	×	×	×
Graph 3	◯	◯	◯	◯
Graph 4	◯	◯	◯	◯
Graph 5	◯	◯	◯	◯
Graph 6	◯	◯	◯	◯
Graph 7	◯	◯	◯	◯
Graph 8	◯	◯	◯	×
Graph 9	◯	◯	◯	×
Graph 10	◯	◯	◯	◯
Graph 11	◯	◯	◯	◯
Graph 12	?	?	?	?
Graph 13	?	?	?	?
Graph 14	◯	◯	◯	◯
Graph 15	◯	◯	◯	◯
Graph 16	◯	◯	◯	◯
Graph 17	◯	◯	◯	◯
Graph 18	◯	◯	◯	◯
Graph 19	◯	◯	◯	◯
Graph 20	◯	◯	◯	◯

Chapter 3

Ehrhart series of fractional stable set polytopes of finite graphs

A fractional stable set polytope FRAC(G) of a simple graph G with d vertices is a rational polytope consisting of a set of non-negative vectors $(x_1, \ldots, x_d)$ satisfying $x_i + x_j \leq 1$ for every edge $\{i, j\}$ of G. In this chapter, we show that (i) The δ-vector of a lattice polytope 2FRAC(G) is alternatingly increasing. (ii) The Ehrhart ring of FRAC(G) is Gorenstein. (iii) The coefficients of the numerator of the Ehrhart series of FRAC(G) are symmetric, unimodal and can be computed by the δ-vector of 2FRAC(G).

Let G be a finite simple graph on the vertex set $[d] = \{1, 2, \ldots, d\}$ and let $E(G)$ be the edge set of G. Throughout this chapter, we always assume that G has no isolated vertices. Given a subset $W \subset [d]$, we associate the $(0, 1)$-vector $\rho(W) = \sum_{j \in W} \mathbf{e}_j \in \mathbb{R}^d$. Here, $\mathbf{e}_i$ is the i-th unit coordinate vector of $\mathbb{R}^d$. In particular, $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. A subset W is called *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in W$ with $i \neq j$. We note that the empty set and each single-element subset of $[d]$ are stable. Let $S(G)$ denote the set of all stable sets of G. The *stable set polytope* (*independent set polytope*) STAB(G) $\subset \mathbb{R}^d$ of a simple graph G is the $(0, 1)$-polytope that is the convex full of $\{\rho(W) \mid W \in S(G)\}$. Stable set polytopes are very important in many areas, e.g., optimization theory. The δ-vector of the stable set polytope of a perfect graph is studied in [1, 16]. Moreover, the *fractional stable set polytope* FRAC(G) of G is the d-polytope in $\mathbb{R}^d$ defined by

$$\mathrm{FRAC}(G) = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; \begin{array}{c} 0 \leq x_i \leq 1 \ (1 \leq i \leq d), \\ x_i + x_j \leq 1 \ (\{i, j\} \in E(G)) \end{array} \right\}.$$

In general, we have STAB(G) $\subset$ FRAC(G). Each vertex of FRAC(G) belongs to $\{0, 1/2, 1\}^d$ (see, e.g, [15]). It is known that FRAC(G) = STAB(G) if and only if G is bipartite. If G is bipartite, then STAB(G) has a unimodular triangulation and the δ-vector of STAB(G) is symmetric and unimodal (see [1, 4, 16]). We note that if G is bipartite, then STAB(G) is the *chain polytope* of a poset P of rank 1 whose comparability graph is G and is affinely equivalent to the *order polytope* of the poset

P (see [27]). Our purpose is to study the Ehrhart series of FRAC(G). The following two polytopes will play an important role:

$$\begin{aligned}\mathcal{P}(G) &= 2 \cdot \mathrm{FRAC}(G),\\ \mathcal{Q}(G) &= 3 \cdot \mathrm{FRAC}(G) - (1, \ldots, 1)\\ &= \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; \begin{array}{c} x_i \geq -1 \ (1 \leq i \leq d), \\ x_i + x_j \leq 1 \ (\{i, j\} \in E(G)) \end{array} \right\}.\end{aligned}$$

In [29], Steingrímsson called the lattice polytope $\mathcal{P}(G)$ the *extended* 2*-weak vertex-packing polytope* of G and studied the structure of $\mathcal{P}(G)$. In particular, he constructed a unimodular triangulation of $\mathcal{P}(G)$ and showed that the δ-vector of $\mathcal{P}(G)$ is obtained by a descent statistic on a subset of the hyperoctahedral group determined by G.

This chapter is organized as follows. In Section 3.1, we show that the δ-vector $(\delta_0, \ldots, \delta_{d-1})$ of $\mathcal{P}(G)$ is *alternatingly increasing* ([23, Definition 2.9]), i.e.,

$$\delta_0 \leq \delta_{d-1} \leq \delta_1 \leq \delta_{d-2} \leq \cdots \leq \delta_{\lfloor d/2 \rfloor - 1} \leq \delta_{d - \lfloor d/2 \rfloor} \leq \delta_{\lfloor d/2 \rfloor}.$$

In Section 3.2, we study the structure of $\mathcal{Q}(G)$ in order to show that the Ehrhart ring of FRAC(G) is Gorenstein. Using this result, in Section 3.3, we obtain a formula for the numerator of the Ehrhart series $E(\mathrm{FRAC}(G), t) := g(\mathrm{FRAC}(G), t)/(1 - t^2)^{d+1}$ via the δ-vector of $\mathcal{P}(G)$. Since the Ehrhart ring of FRAC(G) is Gorenstein and the δ-vector of $\mathcal{P}(G)$ is alternatingly increasing, the coefficients of $g(\mathrm{FRAC}(G), t)$ are symmetric and unimodal. Finally, in Section 3.4, we discuss the dual polytope $\mathcal{Q}(G)^\vee$ of $\mathcal{Q}(G)$.

3.1 The δ-vector of $\mathcal{P}(G)$

We first review the results in [29]. Let B_d denote the signed permutation words on $[d] = \{1, 2, \ldots, d\}$. For example, if $d = 2$,

$$B_2 = \{1\ 2, 2\ 1, \overline{1}\ 2, 2\ \overline{1}, 1\ \overline{2}, \overline{2}\ 1, \overline{1}\ \overline{2}, \overline{2}\ \overline{1}\},$$

where $\overline{1} = -1$ and $\overline{2} = -2$. An element $i \in [d]$ is called a *descent* in $\pi = a_1 \cdots a_d \in B_d$ if one of the following holds ([29, Definition 5]):

(i) $i < d$ and $a_i > a_{i+1}$;

(ii) $i = d$ and $a_i > 0$.

Let des(π) denote the number of descents in $\pi \in B_d$. For any subset S of B_d, the *descent polynomial* of S is $D(S, t) := \sum_{\pi \in S} t^{\mathrm{des}(\pi)}$. Let G be a simple graph on the vertex set $[d]$ and the edge set $E(G)$. We define a subset $\Pi(G)$ of B_d as follows ([29, Definition 11 and Theorem 12]):

$$\Pi(G) = \left\{ \pi \in B_d \;\middle|\; \begin{array}{c} \text{if } \{i, j\} \in E(G) \text{ and } +i \text{ appears in } \pi, \\ \text{then } -j \text{ must precede } +i \text{ in } \pi \end{array} \right\}.$$

Proposition 3.1.1 ([29]) *Let G be a finite simple graph. Then, the δ-polynomial of $\mathcal{P}(G)$ equals the descent polynomial $D(\Pi(G), t)$.*

Using this fact, we will show the following:

Theorem 3.1.2 *Let G be a simple graph with d vertices. Then, there exist symmetric and unimodal polynomials $a(t)$ and $b(t)$ of degree $d-1$ and $d-2$, respectively, such that $\delta(\mathcal{P}(G), t) = a(t) + tb(t)$. In particular, the δ-vector $(\delta_0, \delta_1, \dots, \delta_{d-1})$ of $\mathcal{P}(G)$ is alternatingly increasing, i.e.,*

$$\delta_0 \le \delta_{d-1} \le \delta_1 \le \delta_{d-2} \le \cdots \le \delta_{\lfloor d/2 \rfloor -1} \le \delta_{d-\lfloor d/2 \rfloor} \le \delta_{\lfloor d/2 \rfloor}.$$

Proof. Let Π_+ (resp., Π_-) denote the set of all $\pi \in \Pi(G)$ such that the last number of π is positive (resp., negative). Note that the first number of $\pi \in \Pi(G)$ is always negative since G has no isolated vertices.

Let $\pi \in \Pi_+$. Then, π has a representation

$$\pi = m_1^{(1)} \cdots m_{\alpha_1}^{(1)} p_1^{(1)} \cdots p_{\beta_1}^{(1)} m_1^{(2)} \cdots m_{\alpha_2}^{(2)} p_1^{(2)} \cdots p_{\beta_2}^{(2)} \cdots m_1^{(\gamma)} \cdots m_{\alpha_\gamma}^{(\gamma)} p_1^{(\gamma)} \cdots p_{\beta_\gamma}^{(\gamma)},$$

where $p_i^{(j)} > 0$ and $m_i^{(j)} < 0$. Let $S(\pi)$ denote the set of all signed permutation words on $[d]$ of the form

$$m_{\sigma_1(1)}^{(1)} \cdots m_{\sigma_1(\alpha_1)}^{(1)} p_{\tau_1(1)}^{(1)} \cdots p_{\tau_1(\beta_1)}^{(1)} \cdots m_{\sigma_\gamma(1)}^{(\gamma)} \cdots m_{\sigma_\gamma(\alpha_\gamma)}^{(\gamma)} p_{\tau_\gamma(1)}^{(\gamma)} \cdots p_{\tau_\gamma(\beta_\gamma)}^{(\gamma)},$$

where $\sigma_k \in \mathcal{S}_{\alpha_k}$ and $\tau_k \in \mathcal{S}_{\beta_k}$ are permutations. It is easy to see that $S(\pi) \subset \Pi_+$. Then,

$$D(S(\pi), t) = t^\gamma \prod_{j=1}^{\gamma} A_{\alpha_j}(t) A_{\beta_j}(t),$$

where $A_k(t) = \sum_{i=0}^{k-1} A(k,i) t^i$ is the Eulerian polynomial whose coefficient $A(k,i)$ is an *Eulerian number*. It is known that $(A(k,0), A(k,1), \dots, A(k,k-1))$ is symmetric and *unimodal*, i.e., $A(k,i) \le A(k,i+1)$ for $0 \le i \le \lfloor 2/k \rfloor$. The degree of $D(S(\pi), t)$ is $\gamma + \sum_{j=1}^{\gamma} (\alpha_j + \beta_j - 2) = d - \gamma$. Since $A_k(t)$ is symmetric and unimodal, so is $A_{\alpha_j}(t) A_{\beta_j}(t)$. Hence,

$$D(S(\pi), t) = s_\gamma t^\gamma + \cdots + s_{d-\gamma} t^{d-\gamma}$$

implies that $(s_\gamma, \dots, s_{d-\gamma})$ is symmetric and unimodal. Since

$$D(\Pi_+, t) = u_1 t + \cdots + u_{d-1} t^{d-1}$$

is a sum of such $D(S(\pi), t)$, $(u_1, \dots, u_{d-1})$ is symmetric and unimodal.

Let $\pi \in \Pi_-$. Then, π has a representation

$$\pi = m_1^{(1)} \cdots m_{\alpha_1}^{(1)} p_1^{(1)} \cdots p_{\beta_1}^{(1)} m_1^{(2)} \cdots m_{\alpha_2}^{(2)} p_1^{(2)} \cdots p_{\beta_2}^{(2)} \cdots m_1^{(\gamma)} \cdots m_{\alpha_\gamma}^{(\gamma)},$$

where $p_i^{(j)} > 0$ and $m_i^{(j)} < 0$. We define $S(\pi)$ as before. Then, we have

$$D(S(\pi), t) = t^{\gamma-1} A_{\alpha_\gamma}(t) \prod_{j=1}^{\gamma-1} A_{\alpha_j}(t) A_{\beta_j}(t).$$

The degree of $D(S(\pi), t)$ is $\gamma - 1 + \alpha_\gamma - 1 + \sum_{j=1}^{\gamma-1}(\alpha_j + \beta_j - 2) = d - \gamma$. Since

$$D(\Pi_-, t) = v_0 + v_1 t + \cdots + v_{d-1} t^{d-1}$$

is a sum of such $D(S(\pi), t)$, $(v_0, v_1, \ldots, v_{d-1})$ is symmetric and unimodal.

We now show that the δ-vector $(\delta_0, \ldots, \delta_{d-1}) = (v_0, u_1 + v_1, \ldots, u_{d-1} + v_{d-1})$ of $\mathcal{P}(G)$ is alternatingly increasing. We first note that $\delta_{d-1} - \delta_0 = u_{d-1} + v_{d-1} - v_0 = u_{d-1} \geq 0$. Moreover, for $i = 1, 2, \ldots, \lfloor d/2 \rfloor$, we have $\delta_i - \delta_{d-i} = u_i + v_i - u_{d-i} - v_{d-i} = v_i - v_{i-1} \geq 0$ and for $i = 1, 2, \ldots, \lfloor d/2 \rfloor - 1$, we have $\delta_{d-i-1} - \delta_i = u_{d-i-1} + v_{d-i-1} - u_i - v_i = u_{i+1} - u_i \geq 0$. Thus, the δ-vector of $\mathcal{P}(G)$ is alternatingly increasing. □

3.2 The Ehrhart ring of FRAC(G)

In this section, we will show that the Ehrhart ring of FRAC(G) is Gorenstein. In order to show this, we will use the following criterion [7, Theorem 1.1]:

Proposition 3.2.1 *Let $P \subset \mathbb{R}^d$ be a rational convex polytope of dimension d and let $\delta \geq 1$ denote the smallest integer for which $\delta(P - \partial P) \cap \mathbb{Z}^d \neq \emptyset$. We fix $\alpha \in \delta(P - \partial P) \cap \mathbb{Z}^d$ and let $Q = \delta P - \alpha \subset \mathbb{R}^d$. Then, the Ehrhart ring $A_K(P)$ of P is Gorenstein if and only if the following conditions are satisfied:*

(i) *The dual polytope $Q^\vee$ of Q is a lattice polytope.*

(ii) *Let $\widetilde{P} \subset \mathbb{R}^{d+1}$ denote the rational convex polytope that is the convex hull of the subset $\{(\beta, 0) \in \mathbb{R}^{d+1} \mid \beta \in P\} \cup \{(0, \ldots, 0, 1/\delta)\}$ in $\mathbb{R}^{d+1}$. Then $\widetilde{P}$ is facet-reticular, that is, if H is a hyperplane in $\mathbb{R}^{d+1}$ and if $H \cap \widetilde{P}$ is a facet of $\widetilde{P}$, then $H \cap \mathbb{Z}^{d+1} \neq \emptyset$.*

It is clear that there exist no lattices in the interior of $\mathcal{P}(G) = 2\mathrm{FRAC}(G)$ and there exists a lattice $(1, \ldots, 1)$ in the interior of $3\mathrm{FRAC}(G)$. Thus, it is enough to show that conditions (i) and (ii) in Proposition 3.2.1 are satisfied when $P = \mathrm{FRAC}(G)$, $\delta = 3, \alpha = (1, \ldots, 1)$ and $Q = \mathcal{Q}(G)$.

A criterion for a vector to be a vertex of FRAC(G) is given in [29, Theorem 15]:

Lemma 3.2.2 *Let G be a finite simple graph with d vertices. We assume that $\mathbf{v} = (v_1, \ldots, v_d) \in \{0, 1/2, 1\}^d$ belongs to* FRAC(G). *Let G_S be the subgraph of G induced by $S = \{i \in [d] \mid v_i = 1/2\}$. Then, $\mathbf{v}$ is a vertex of* FRAC(G) *if and only if either $S = \emptyset$ or each connected component of G_S contains an odd cycle.*

Using Lemma 3.2.2, we determine when $\mathcal{Q}(G)$ is a lattice polytope.

Proposition 3.2.3 *Let G be a finite simple graph without isolated vertices. Then, the following conditions are equivalent.*

(i) *The graph G is a bipartite graph.*

(ii) *The polytope* FRAC(G) *is a lattice polytope.*

(iii) *The polytope $\mathcal{Q}(G)$ is a lattice polytope.*

Proof. We first note that (ii) $\Rightarrow$ (iii) is trivial. If G is bipartite, then FRAC(G) = STAB(G) is a lattice polytope. Hence, (i) $\Rightarrow$ (ii) holds. We now show that (iii)$\Rightarrow$ (i). We assume that G contains an odd cycle C. Let H be a connected component of G that contains C and let $V(H)$ be the set of vertices of H. We define $\mathbf{v} = (v_1, \ldots, v_d)$ by $v_i = 1/2$ if $i \in V(H)$ and $v_i = 0$ if $i \notin V(H)$. Then, $\mathbf{v}$ is a $(0, 1/2)$-vector in FRAC(G). Moreover, since $\mathbf{v}$ satisfies the condition in Lemma 3.2.2, $\mathbf{v}$ is a vertex of FRAC(G). Then, $3\mathbf{v} - (1, \ldots, 1) \in \{-1, 1/2\}^d$ is a vertex of $\mathcal{Q}(G)$ that is not an integer vector. Hence, $\mathcal{Q}(G)$ is not a lattice polytope. □

We next show that $\mathcal{Q}(G)^\vee$ is a lattice polytope.

Proposition 3.2.4 *Suppose G is a finite simple graph without isolated vertices. Then, the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{Q}(G)$ and*

$$\{\mathbf{e}_i + \mathbf{e}_j \mid \{i, j\} \in E(G)\} \cup \{-\mathbf{e}_i \mid 1 \leq i \leq d\}$$

is the vertex set of $\mathcal{Q}(G)^\vee$. In particular, if G is a bipartite graph, then $\mathcal{Q}(G)$ is a Gorenstein Fano polytope.

Proof. It is known that the inequalities $x_i \geq 0$ $(1 \leq i \leq d)$ and $x_i + x_j \leq 1$ $(\{i, j\} \in E(G))$ define the facets of FRAC(G). Hence, the inequalities $x_i \geq -1$ $(1 \leq i \leq d)$ and $x_i + x_j \leq 1$ $(\{i, j\} \in E(G))$ define the facets of $\mathcal{Q}(G)$. Thus, a vector $(v_1, \ldots, v_d) \in \mathbb{R}^d$ belongs to the interior of $\mathcal{Q}(G)$ if and only if $v_i > -1$ $(1 \leq i \leq d)$ and $v_i + v_j < 1$ $(\{i, j\} \in E(G))$. It is clear that the origin of $\mathbb{R}^d$ belongs to the interior of $\mathcal{Q}(G)$. We assume that $(v_1, \ldots, v_d) \in \mathbb{Z}^d$ belongs to the interior of $\mathcal{Q}(G)$. Since v_i and $v_i + v_j$ are integers, we have $v_i \geq 0$ $(1 \leq i \leq d)$ and $v_i + v_j \leq 0$ $(\{i, j\} \in E(G))$. Hence, $v_i = 0$ for all i, i.e, $(v_1, \ldots, v_d) = \mathbf{0}$. It is known that there is a one-to-one correspondence between the facets of $\mathcal{Q}(G)$ and the vertices of $\mathcal{Q}(G)^\vee$. The set $\{\mathbf{e}_i + \mathbf{e}_j \mid \{i, j\} \in E(G)\} \cup \{-\mathbf{e}_i \mid 1 \leq i \leq d\}$ of coefficient vectors of the inequalities that define the facets is the set of vertices of $\mathcal{Q}(G)^\vee$. Thus, in particular, $\mathcal{Q}(G)^\vee$ is a lattice polytope. By Proposition 3.2.3, if G is a bipartite graph, then $\mathcal{Q}(G)$ is a lattice polytope. Hence, it is a Gorenstein Fano polytope. □

We are now in a position to show that the Ehrhart ring of FRAC(G) is Gorenstein.

Theorem 3.2.5 *Let G be a finite simple graph without isolated vertices. Then, the Ehrhart ring of* $\mathrm{FRAC}(G)$ *is Gorenstein.*

Proof. It is enough to show that conditions (i) and (ii) in Proposition 3.2.1 are satisfied when $P = \mathrm{FRAC}(G)$, $\delta = 3, \alpha = (1, \ldots, 1)$ and $Q = \mathcal{Q}(G)$. We first note that Proposition 3.2.4 ensures that $Q^\vee$ is a lattice polytope. Let $F = H \cap \widetilde{P}$ be a facet of $\widetilde{P}$, where H is a hyperplane in $\mathbb{R}^{d+1}$ and let $F' = F \cap \{x_{d+1} = 0\}$. Then, F' is a facet of P, whose supporting hyperplane is $H' = H \cap \{x_{d+1} = 0\}$. Therefore, H' is defined by either $x_i + x_j = 1$ ($\{i, j\} \in E(G)$) or $x_i = 0$ ($1 \leq i \leq d$). Hence, it is clear that there exists a lattice in H'. Thus, there exists a lattice in H and condition (ii) in Proposition 3.2.1 is satisfied. Therefore, the Ehrhart ring of P is Gorenstein by Proposition 3.2.1. □

3.3 The Ehrhart series of $\mathrm{FRAC}(G)$

In this section, we show that we can calculate the Ehrhart series and the Ehrhart quasi-polynomial of $\mathrm{FRAC}(G)$ from those of $\mathcal{P}(G)$. Let G be a simple graph on the vertex set $[d]$ without isolated vertices. Since the interior of $\mathcal{P}(G)$ possesses no lattices and the interior of $2\mathcal{P}(G)$ has a lattice, we have that $\deg \delta(\mathcal{P}(G), t) = d + 1 - 2 = d - 1$. Moreover, the degree of $E(\mathrm{FRAC}(G), t)$ is -3 as a rational function. Since $i(\mathrm{FRAC}(G), n)$ is a quasi-polynomial of period at most 2, there exist polynomials $i^{\mathrm{odd}}(\mathrm{FRAC}(G), n)$ and $i^{\mathrm{even}}(\mathrm{FRAC}(G), n)$ of degree d such that

$$i(\mathrm{FRAC}(G), n) = \begin{cases} i^{\mathrm{odd}}(\mathrm{FRAC}(G), n) & \text{if } n \text{ is odd,} \\ \\ i^{\mathrm{even}}(\mathrm{FRAC}(G), n) & \text{if } n \text{ is even.} \end{cases}$$

In particular, if G is bipartite, then $i^{\mathrm{odd}}(\mathrm{FRAC}(G), n) = i^{\mathrm{even}}(\mathrm{FRAC}(G), n)$.

Theorem 3.3.1 *Let G be a simple graph on the vertex set $[d]$ without isolated vertices and let $\delta(\mathcal{P}(G), t) = \delta_0 + \delta_1 t + \cdots + \delta_{d-1} t^{d-1}$. Then, we have*

$$\begin{aligned} E(\mathrm{FRAC}(G), t) &= \frac{\delta(\mathcal{P}(G), t^2) + t^{2d-1}\delta(\mathcal{P}(G), 1/t^2)}{(1-t^2)^{d+1}} \\ &= \frac{\delta_0 + \delta_{d-1}t + \delta_1 t^2 + \delta_{d-2}t^3 + \cdots + \delta_1 t^{2d-3} + \delta_{d-1}t^{2d-2} + \delta_0 t^{2d-1}}{(1-t^2)^{d+1}}, \end{aligned}$$

where $(\delta_0, \delta_{d-1}, \delta_1, \delta_{d-2}, \ldots, \delta_1, \delta_{d-1}, \delta_0)$ is symmetric and unimodal. In addition,

$$\begin{aligned} i^{\mathrm{odd}}(\mathrm{FRAC}(G), 2k+1) &= (-1)^d i^{\mathrm{even}}(\mathrm{FRAC}(G), -2k-4) \\ &= (-1)^d i(\mathcal{P}(G), -k-2). \end{aligned}$$

Proof. Let $W = \mathrm{FRAC}(G)$ and $P = \mathcal{P}(G)$. Then,

$$E(W,t) = \sum_{k\geq 0} i^{\mathrm{even}}(W,2k)t^{2k} + \sum_{k\geq 0} i^{\mathrm{odd}}(W,2k+1)t^{2k+1}.$$

Since $i^{\mathrm{even}}(W,2k) = i(2W,k) = i(P,k)$, we have

$$\sum_{k\geq 0} i^{\mathrm{even}}(W,2k)t^{2k} = \sum_{k\geq 0} i(P,k)(t^2)^k = \frac{\delta(P,t^2)}{(1-t^2)^{d+1}}.$$

Since the degree of $i^{\mathrm{odd}}(W,2k+1)$ is d, by [26, Corollary 4.3.1], we have

$$\sum_{k\geq 0} i^{\mathrm{odd}}(W,2k+1)t^{2k+1} = t\sum_{k\geq 0} i^{\mathrm{odd}}(W,2k+1)(t^2)^k = t\frac{a(t^2)}{(1-t^2)^{d+1}},$$

where $a(t)$ is a polynomial of degree $\leq d$. Thus,

$$E(W,t) = \frac{\delta(P,t^2)}{(1-t^2)^{d+1}} + t\cdot\frac{a(t^2)}{(1-t^2)^{d+1}} = \frac{\delta(P,t^2)+ta(t^2)}{(1-t^2)^{d+1}}.$$

Since the degree of $E(W,t)$ is -3 as a rational function, the degree of $\delta(P,t^2)+ta(t^2)$ is $2d-1$. Hence, $\deg a(t) = d-1$ $(= \deg \delta(P,t))$. Moreover, since the Ehrhart ring of W is Gorenstein, the coefficients of $\delta(P,t^2)+ta(t^2)$ are symmetric. Thus, $a(t) = t^{d-1}\delta(P,1/t)$ and $\delta(P,t^2)+ta(t^2) = \delta(P,t^2)+t^{2d-1}\delta(P,1/t^2)$. By the Ehrhart reciprocity, it follows that

$$\begin{aligned}\sum_{k\geq 0} i^{\mathrm{odd}}(W,2k+1)t^{2k+1} &= \frac{t^{2d-1}\delta(P,1/t^2)}{(1-t^2)^{d+1}}\\ &= \frac{(-1)-d+1}{t^3}\frac{\delta(P,1/t^2)}{(1-1/t^2)^{d+1}}\\ &= \frac{(-1)^{d+1}}{t^3}E(P,1/t^2)\\ &= \frac{(-1)^d}{t^3}\sum_{k\geq 1} i(P,-k)t^{2k}\end{aligned}$$

Thus, $i^{\mathrm{odd}}(W,2k+1) = (-1)^d i(P,-k-2) = (-1)^d i^{\mathrm{even}}(W,-2k-4)$, as desired. $\square$

Example 3.3.2 Let $W = \mathrm{FRAC}(K_d)$ and $P = \mathcal{P}(K_d)$, where K_d is a complete graph with d vertices. It is known [29, Example 27] that $\delta(P,t) = A_d(t)+dtA_{d-1}(t)$. Let

$$E(W,t) = \frac{b_0+b_1t+\cdots+b_{2d-1}t^{2d-1}}{(1-t^2)^{d+1}}.$$

Since

$$\begin{aligned}\delta(P,t) &= A_d(t) + dtA_{d-1}(t)\\ &= \sum_{i=0}^{d-1} A(d,i)t^i + dt\sum_{i=0}^{d-2} A(d-1,i)t^i\\ &= \sum_{i=0}^{d-1} A(d,i)t^i + d\sum_{i=1}^{d-1} A(d-1,i-1)t^i\\ &= 1 + \sum_{i=1}^{d-1}(A(d,i) + dA(d-1,i-1))t^i\end{aligned}$$

hold, the δ-vector $(\delta_0, \ldots, \delta_{d-1})$ of P satisfies $\delta_0 = 1$ and $\delta_i = A(d,i)+dA(d-1,i-1)$ for $i = 1, 2, \ldots, d-1$. By Theorem 3.3.1, we obtain $b_0 = 1$ and $b_i = A(d, \lfloor i/2 \rfloor) + dA(d-1, \lfloor (i-1)/2 \rfloor)$ for $i = 1, 2, \ldots, 2d-1$.

Example 3.3.3 Let $W_d = \text{FRAC}(C_d)$, where C_d is an odd cycle of length d. We computed the numerator $g(W_d, t)$ of $E(W_d, t) = g(W_d, t)/(1-t^2)^{d+1}$ for $d = 3, 5, 7, 9$ using software `Normaliz` ([3]).

$$\begin{aligned}g(W_3,t) &= 1 + 4t + 7t^2 + 7t^3 + 4t^4 + t^5.\\ g(W_5,t) &= 1 + 11t + 51t^2 + 131t^3 + 206t^4 + 206t^5 + 131t^6 + 51t^7 + 11t^8 + t^9.\\ g(W_7,t) &= 1 + 29t + 281t^2 + 1408t^3 + 4320t^4 + 8814t^5 + 12475t^6\\ &\quad + 12475t^7 + 8814t^8 + 4320t^9 + 1408t^{10} + 281t^{11} + 29t^{12} + t^{13}.\\ g(W_9,t) &= 1 + 76t + 1450t^2 + 12844t^3 + 67000t^4 + 230986t^5 + 561004t^6\\ &\quad + 996310t^7 + 1321369t^8 + 1321369t^9 + 996310t^{10} + 561004t^{11}\\ &\quad + 230986t^{12} + 67000t^{13} + 12844t^{14} + 1450t^{15} + 76t^{16} + t^{17}.\end{aligned}$$

3.4 The dual polytope of $\mathcal{Q}(G)$

In this section, we will discuss the dual polytope $\mathcal{Q}(G)^\vee$ of $\mathcal{Q}(G)$. We recall that

$$\mathcal{Q}(G)^\vee = \text{CONV}(\{\mathbf{e}_i + \mathbf{e}_j \mid \{i,j\} \in E(G)\} \cup \{-\mathbf{e}_i \mid 1 \leq i \leq d\})$$

if G has no isolated vertices. It is easy to see that $\mathcal{Q}(G)^\vee$ is Fano. A lattice polytope $P \subset \mathbb{R}^d$ is called *normal* if $\mathbb{Z}_{\geq 0}A = \mathbb{Q}_{\geq 0}A \cap \mathbb{Z}A$, where $A = \{(\alpha, 1) \in \mathbb{Z}^{d+1} \mid \alpha \in P \cap \mathbb{Z}^d\}$. It is known that a lattice polytope P is normal if P has a unimodular triangulation. See, e.g., [14].

Theorem 3.4.1 *Let G be a finite simple graph without isolated vertices. Then, the following conditions are equivalent.*

(i) *The graph G is a bipartite graph.*

(ii) *The dual polytope* $\mathcal{Q}(G)^\vee$ *has a unimodular triangulation.*

(iii) *The dual polytope* $\mathcal{Q}(G)^\vee$ *is normal.*

(iv) *The dual polytope* $\mathcal{Q}(G)^\vee$ *is a Gorenstein Fano polytope.*

Proof. We note that $(\mathcal{Q}(G)^\vee)^\vee = \mathcal{Q}(G)$. By Proposition 3.2.3, we have (i) $\Leftrightarrow$ (iv). Moreover, (ii) $\Rightarrow$ (iii) holds in general. Let $\mathcal{A}_G$ be the vertex-edge incidence matrix of G and let $\mathcal{A}'_G$ be the configuration matrix of $\mathcal{Q}(G)^\vee$, namely,

$$\mathcal{A}'_G = \begin{pmatrix} \mathbf{0} & \mathcal{A}_G & -E_d \\ 1 & 1\cdots 1 & 1\cdots 1 \end{pmatrix},$$

where E_d is the identity matrix.

(i)$\Rightarrow$(ii): Suppose that G is bipartite. It is known [24] that the vertex-edge incidence matrix of any bipartite graph is *totally unimodular*, i.e., the determinant of every square non-singular submatrix is ± 1. Hence, the submatrix $B = \begin{pmatrix} \mathcal{A}_G & -E_d \end{pmatrix}$ of $\mathcal{A}'_G$ is totally unimodular. Let Δ be a pulling triangulation of $\mathcal{Q}(G)^\vee$ such that the origin is a vertex of every maximal simplex in Δ. Such a triangulation is obtained by a Gröbner basis of the toric ideal of $\mathcal{A}'_G$ with respect to a reverse lexicographic order such that the smallest variable corresponds to the origin. See, e.g., [14]. Then, the normalized volume of each maximal simplex in Δ is equal to the absolute value of the corresponding maximal minor of B. Since B is totally unimodular, each maximal minor of B is ± 1. Hence the triangulation Δ is unimodular.

(iii)$\Rightarrow$(i): We assume that the graph G contains an odd cycle C. We will now show that $\mathcal{Q}(G)^\vee$ is not normal, that is, $\mathbb{Z}_{\geq 0}\mathcal{A}'_G \neq \mathbb{Q}_{\geq 0}\mathcal{A}'_G \cap \mathbb{Z}^{d+1}$. (It is easy to see that $\mathbb{Z}\mathcal{A}'_G = \mathbb{Z}^{d+1}$.) We may assume that $C = (1, 2, \ldots, 2k+1)$. Let

$$\begin{aligned} \mathbf{u} &= \frac{1}{2}\left(\mathbf{e}_{d+1} + (\mathbf{e}_1 + \mathbf{e}_{2k+1} + \mathbf{e}_{d+1}) + \sum_{i=1}^{2k} (\mathbf{e}_i + \mathbf{e}_{i+1} + \mathbf{e}_{d+1}) \right) \\ &= (k+1)\mathbf{e}_{d+1} + \sum_{i=1}^{2k+1} \mathbf{e}_i. \end{aligned}$$

Then, $\mathbf{u}$ belongs to $\mathbb{Q}_{\geq 0}\mathcal{A}'_G \cap \mathbb{Z}^{d+1}$. It is enough to show that $\mathbf{u} \notin \mathbb{Z}_{\geq 0}\mathcal{A}'_G$. We assume that

$$\mathbf{u} = \gamma \mathbf{e}_{d+1} + \sum_{\{i,j\} \in E(G)} \alpha_{ij}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{d+1}) + \sum_{i=1}^{d} \beta_i(-\mathbf{e}_i + \mathbf{e}_{d+1}), \tag{3.1}$$

for some $\alpha_{ij}, \beta_i \in \mathbb{Z}_{\geq 0}$. Then, the coefficient of $\mathbf{e}_i$ $(1 \leq i \leq 2k+1)$ in (3.1) is $1 = \sum_{\{i,j\} \in E(G)} \alpha_{ij} - \beta_i$ and that of $\mathbf{e}_i$ $(2k+2 \leq i \leq d)$ in (3.1) is $0 = \sum_{\{i,j\} \in E(G)} \alpha_{ij} - \beta_i$. Summing up the equations for $1 \leq i \leq d$, we obtain

$$2k+1 = 2\sum_{\{i,j\} \in E(G)} \alpha_{ij} - \sum_{i=1}^{d} \beta_i. \tag{3.2}$$

Moreover, the coefficient of $\mathbf{e}_{d+1}$ in (3.1) is

$$k+1=\gamma+\sum_{\{i,j\}\in E(G)}\alpha_{ij}+\sum_{i=1}^{d}\beta_i. \tag{3.3}$$

Since γ and $\sum_{i=1}^{d}\beta_i$ are non-negative, by Equations (3.2) and (3.3), we obtain

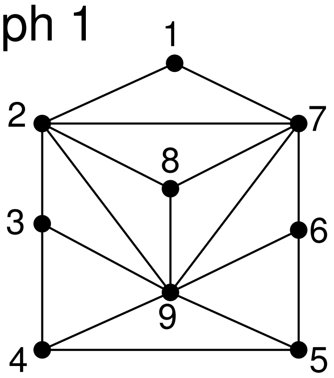
$$k+\frac{1}{2}\leq\sum_{\{i,j\}\in E(G)}\alpha_{ij}\leq k+1.$$

Since $\sum_{\{i,j\}\in E(G)}\alpha_{ij}\in\mathbb{Z}$, we have$\sum_{\{i,j\}\in E(G)}\alpha_{ij}=k+1$. Hence, by Equation (3.2), we have $\sum_{i=1}^{d}\beta_i=1$. Thus, by Equation (3.3), we have $\gamma+1=0$. This is a contradiction. $\square$

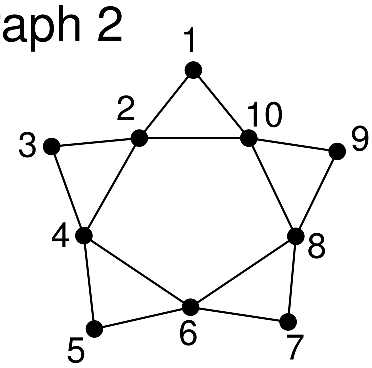
Appendix A

Examples of fundamental FHM graphs

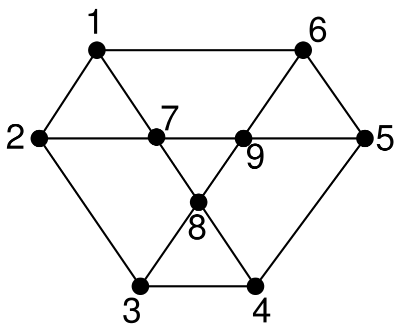
Graph 1



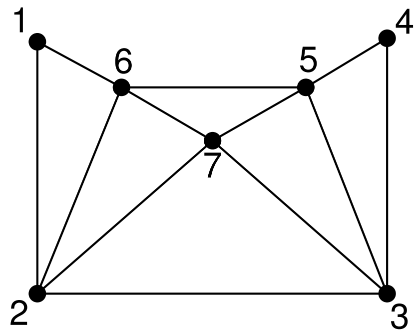
Graph 2



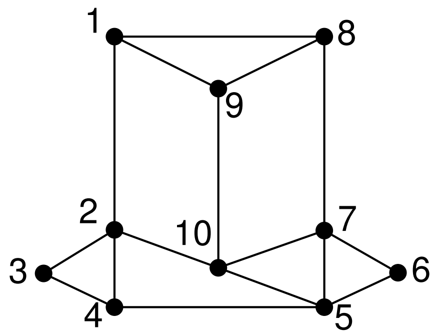
Graph 3



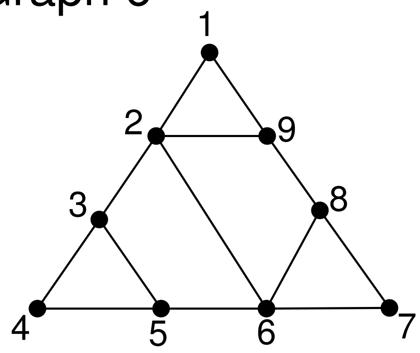
Graph 4



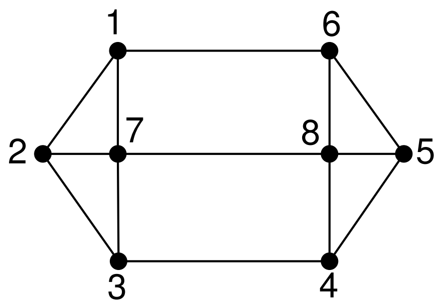
Graph 5



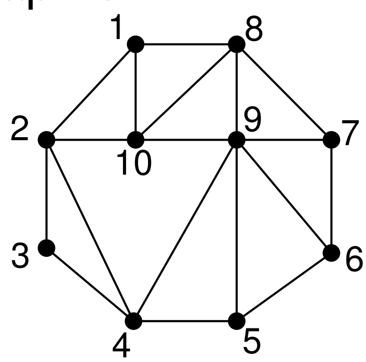
Graph 6



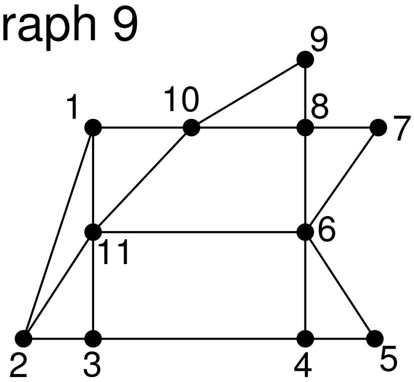
Graph 7



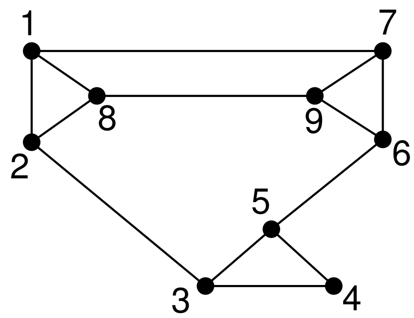
Graph 8



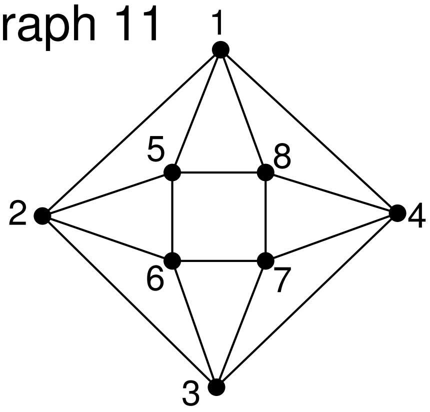
Graph 9



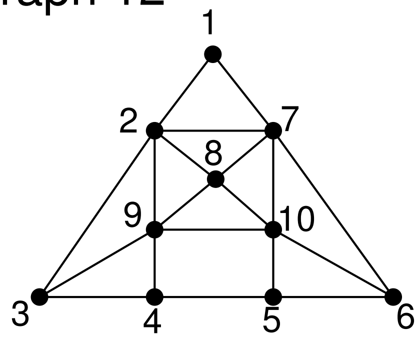
Graph 10



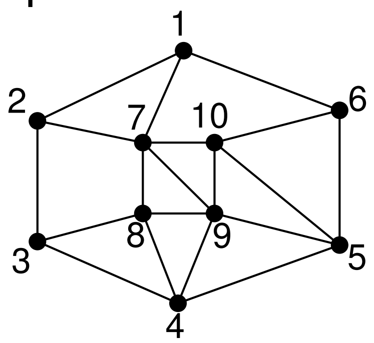
Graph 11



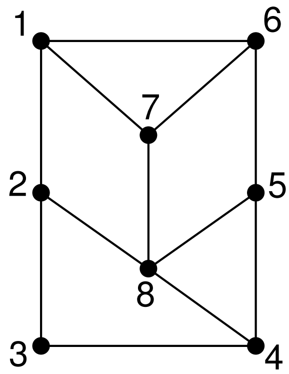
Graph 12



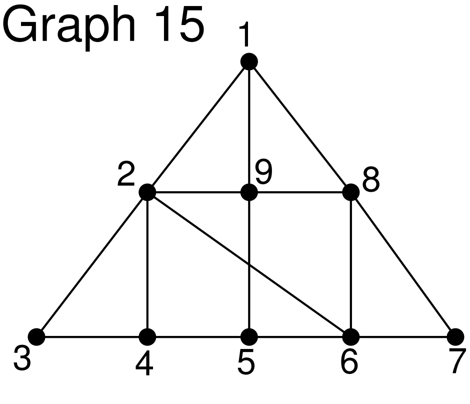
Graph 13



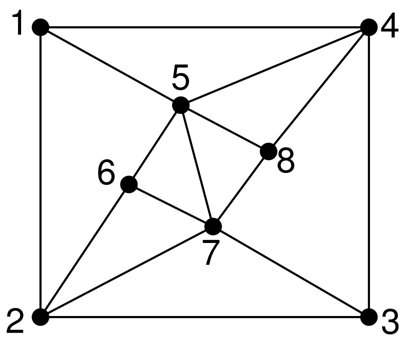
Graph 14



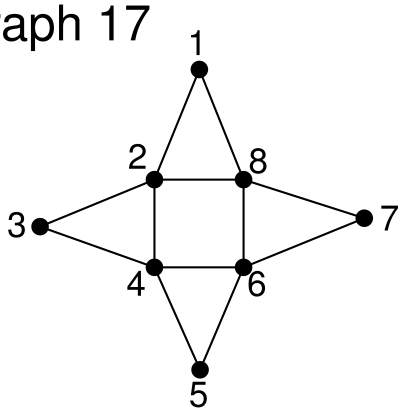
Graph 15



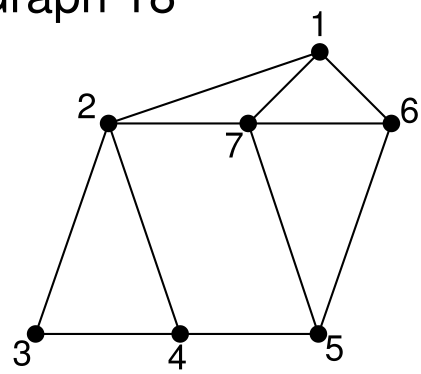
Graph 16



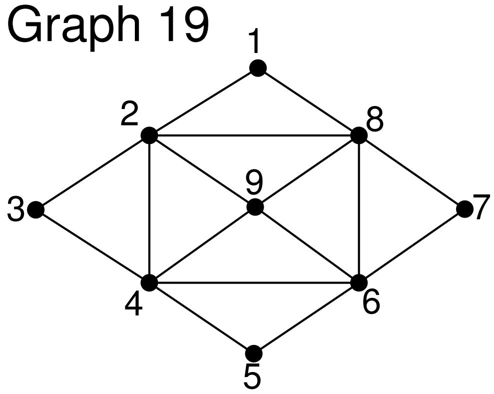
Graph 17



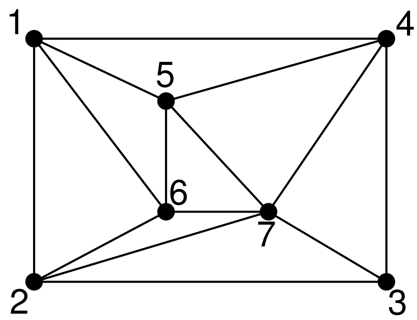
Graph 18



Graph 19



Graph 20



Appendix B

The program "cycle12.c"

```
V1:= [1..9];
E1:= [{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{1,7},{2,7},{2,8},{7,8},{8,9},{2,9},{7,9},{4,9},
{5,9},{3,9},{6,9}];

V2:= [1..10];
E2:= [{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,9},{9,10},{10,1},{2,10},{2,4},{4,6},
{6,8},{8,10}];

V3:= [1..9];
E3:= [{1,2},{2,3},{3,4},{4,5},{5,6},{6,1},{1,7},{2,7},{6,9},{9,5},{7,9},{7,8},{8,9},{3,8},
{4,8}];

V4:=[1..7];
E4:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,1},{3,5},{7,3},{5,7},{6,7},{2,6},{2,7}];

V5:=[1..10];
E5:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,1},{1,9},{8,9},{9,10},{2,10},{10,7},
{10,5},{2,4},{5,7}];

V6:=[1..9];
E6:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,9},{9,1},{2,9},{2,6},{8,6},{3,5}];

V7:=[1..8];
E7:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,1},{1,7},{2,7},{3,7},{7,8},{6,8},{8,5},{8,4}];

V8:=[1..10];
E8:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,1},{1,10},{9,8},{8,10},{10,9},
{9,7},{2,4},{4,9},{9,6},{9,5},{2,10}];

V9:=[1..11];
E9:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,9},{9,10},{10,1},{10,11},{1,11},
{11,3},{11,2},{10,8},{11,6},{6,4},{8,6}];

V10:=[1..9];
E10:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,1},{1,8},{2,8},{8,9},{9,7},{9,6},{5,3}];

V11:=[1..8];
E11:=[{1,2},{2,3},{3,4},{4,1},{1,5},{1,8},{5,8},{4,8},{4,7},{8,7},{3,6},{3,7},{6,7},
{2,6},{2,5},{5,6}];

V12:=[1..10];
E12:=[{1,2},{2,3},{3,4},{4,5},{5,6},{7,6},{1,7},{7,8},{8,9},{8,2},{9,4},{2,9},{10,5},
{10,6},{10,7},{10,9},{10,8},{2,7},{3,9}];

V13:=[1..10];
```

```
E13:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,1},{1,7},{2,7},{3,8},{4,8},{4,9},{5,9},{5,10},
{10,6},{7,10},{7,8},{8,9},{9,10},{7,9}];

V14:= [1..8];
E14:= [{1,2},{2,3},{3,4},{4,5},{5,6},{6,1},{1,7},{6,7},{7,8},{2,8},{8,5},{8,4}];

V15:=[1..9];
E15:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,1},{1,9},{2,9},{8,9},{2,4},{9,5},
{6,8},{2,6}];

V16:=[1..8];
E16:=[{1,2},{2,3},{3,4},{4,1},{5,6},{6,7},{7,8},{8,5},{1,5},{5,4},{4,8},{3,7},{2,6},
{2,7},{5,7}];

V17:=[1..8];
E17:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{8,1},{2,8},{8,6},{6,4},{4,2}];

V18:=[1..7];
E18:=[{1,2},{2,3},{3,4},{4,5},{5,6},{1,6},{1,7},{2,7},{6,7},{7,5},{2,4}];

V19:=[1..9];
E19:=[{1,2},{2,3},{3,4},{4,5},{5,6},{6,7},{7,8},{1,8},{2,8},{8,6},{6,4},{4,2},{2,9},{8,9},
{6,9},{4,9}];

V20:=[1..7];
E20:=[{1,2},{2,3},{3,4},{1,4},{1,5},{4,5},{5,6},{1,6},{5,7},{6,7},{4,7},{7,3},{2,7},{2,6}];


//////////////////////////////////////////////////////////////////////

cycle := function(V,E,p)
L := {{e,f}: e,f in E | #(e meet f) eq 1};
L := [SetToSequence(a): a in L];
repeat
n:= #(L[1]);
L := [a cat [f]: a in L, f in E | (a[n] diff (a[n] meet a[n-1])) subset f and f notin a];
if L eq [] then break;
end if;
until #(L[1]) eq p;
L1:= [a: a in L | a[1] diff (a[1] meet a[2]) eq a[p] diff (a[p] meet a[p-1])];
L2:= [a : a in L1 | #(&join(a)) eq p];
L3:= {SequenceToSet(a): a in L2};
L4:= [SetToSequence(b): b in L3];
m:= #(L4);
for a in [1..m] do
S:= [s: s in L2 | SequenceToSet(s) eq SequenceToSet(L4[a])];
L4[a]:= S[1];
end for;
return L4;

end function;

////////////////////////////////////////////////////

allodd:= function(V,E)
d:= #(V);
S:= [a: a in [3..d] | IsOdd(a)];
W := [];
for a in S do
W:= W cat cycle(V,E,a);
end for;
return W;
end function;

///////////////////////////////////////////////////
```

```
oddpair:= function(V,E)
C:= allodd(V,E);
D := {{e,f}: e,f in C | &join(e) meet &join(f) eq {}};
D1:= [SetToSequence(a): a in D];
return D1;
end function;

////////////////////////////////////////////////////

chord:= function(V,E,C)
Ver:= &join(C);
L:= [e: e in E | Max(e) in Ver and Min(e) in Ver and e notin C];
return L;
end function;

///////////////////////////////////////////////////

bridge := function(V,E,P)
Ver1:= &join(P[1]);
Ver2:= &join(P[2]);
L := [e: e in E| (Max(e) in Ver1 and Min(e) in Ver2) or (Max(e) in Ver2 and Min(e) in Ver1)];
return L;
end function;

/////////////////////////////////////////////////////////////

evenwalk1:= function(b,C)
p:= #(C);
b1:= Min(b);
b2:= Max(b);
D1:= [a: a in [1..p] | b1 in C[a]];
if AbsoluteValue(D1[1]-D1[2]) gt 1 then C1:= C;
else
k1:= Min(D1);
C1:= Rotate(C,p-k1);
end if;
D2:= [a: a in [1..p] | b2 in C1[a]];
b3:= Min(D2);
if IsOdd(b3) then
L:= [C1[i]: i in [1..b3]] cat [b];
else L := [C1[i]: i in [b3+1..p]] cat [b];
end if;
return L;
end function;

/////////////////////////////////////////////////////////////

evenwalk2:= function(b,F)
C1:= F[1]; C2:= F[2];
p1:= #(C1); p2:= #(C2);
D1:= [i: i in [1..p1] | #(b meet C1[i]) eq 1];
if AbsoluteValue(D1[1] - D1[2]) gt 1 then C11:= C1;
else a:= Min(D1);
    C11:= Rotate(C1,p1-a);
end if;
D2:= [j: j in [1..p2] | #(b meet C2[j]) eq 1];
if AbsoluteValue(D2[1]- D2[2]) gt 1 then C21:= C2;
else c:= Min(D2);
    C21:= Rotate(C2,p2-c);
end if;
C3:= C11 cat [b] cat C21 cat [b];
return C3;
end function;
```

```
/////////////////////////////////////////////////////////////////

times:= function(e,C)
m:= #(C);
L:= [a: a in C | a eq e];
return #(L);
end function;

//////////////////////////////////////////////////////////////

sign:= function(V,E,f)
n:= #(E);
p:= #(f);
F1:= [f[i]: i in [1..p] | IsEven(i)];
F2 := [f[i]: i in [1..p] | IsOdd(i)];
E1:= [j: j in [1..n] | E[j] in F1];
E2:= [j: j in [1..n] | E[j] in F2];
B := [0*i: i in [1..n]];
for i in [1..n] do
   if i in E1 then B[i]:= times(E[i],f); end if;
   if i in E2 then B[i]:= -1; end if;
end for;
B1:= [-x: x in B];
return B1;
end function;

////////////////////////////////////////////////////

bij:= function(V,E)
X:= [];
OP:= oddpair(V,E);
p:= #(OP);
for i in [1..p] do
C1:= OP[i][1]; C2:= OP[i][2];
T:= bridge(V,E,OP[i]);
S1:= chord(V,E,C1);
S2:= chord(V,E,C2);
A:= [evenwalk2(t,OP[i]): t in T];
B1 := [evenwalk1(s,C1): s in S1];
B2 := [evenwalk1(s,C2): s in S2];
X:= X cat [A cat B1 cat B2];
end for;

return X;
end function;

////////////////////////////////////////////////////

bij2:= function(V,E)
OP:= oddpair(V,E);
p:= #(OP);
X:= [];
for i in [1..p] do
C1:= OP[i][1]; C2:= OP[i][2];
T:= bridge(V,E,OP[i]);
//S1:= chord(V,E,C1);
//S2:= chord(V,E,C2);
A:= [evenwalk2(t,OP[i]): t in T];
//B1 := [evenwalk1(s,C1): s in S1];
//B2 := [evenwalk1(s,C2): s in S2];
X:= X cat [A];
end for;

return X;
end function;
```

```
//////////////////////////////////////////////////////////////

fhm := function(V,E)
P := oddpair(V,E);
if P eq [] then
return "no odd pairs";
end if;

P1:= [a: a in P | #(bridge(V,E,a)) eq 0];
if P1 eq [] then
return "fFHM"; ///
else return "not FHM";
end if;
end function;

/////////////////////////////////////////////////

main := function(V,E)
n:= #(E);
A:= bij(V,E);
p:= #(A);
B :=[ #(A[i]) : i in [1..p]];
b:= &*(B);
"set of vertices = ", V;
"array of edges = ", E;
"number of edges=", n;
"number of pairs=", p;
"array of the number of even closed walks= ", B;
"number of cones= ", b;
C:= [[sign(V,E,A[i][j]): j in [1..B[i]]]: i in [1..p]];
if p eq 1 then
print "only one pair";
end if;
D:= CartesianProduct(C);

counter := 0;
c:= 0;

for x in D do
y:= [x[i]: i in [1..p]];
z:= Dimension(ConeWithInequalities(y));
c:= c+1;
c,z;
if z eq n then
counter := 1;
break;
end if;
end for;
if counter eq 0 then return "empty";
else F:= ConeWithInequalities(y);
L := {};
for j in [1..p] do
for k in [1..n] do
if y[j][k] eq -2 then L := L join {k};
end if;
end for;
end for;
M1:= [0*i: i in [1..p]];
for j in [1..p] do
M1[j]:=  &+[-y[j][k]: k in L];
end for;
M2:= [0*i: i in [1..p]];
for j in [1..p] do
M2[j]:=  &+[AbsoluteValue(y[j][k]): k in L];
```

```
end for;
print y, Dimension(F), "non-empty";
print "inner product =", M1;
print "inner product (absolute value) =", M2;
F1:= MinimalRGenerators(F);
w:= &+(F1);
w1:= ElementToSequence(w);
m:= Min(w1);
if m gt 0 then w2:= w1;
else w2:= [x-m+1: x in w1];
end if;
"inner point weight= ";w2;
return w2;
end if;

end function;

/////////////////////////////////////////////////////////////

bmain4 := function(V,E)
n:= #(E);
A:= bij2(V,E);
p:= #(A);
B :=[ #(A[i]) : i in [1..p]];
b:= &*(B);
"number of edges=", n;
"number of pairs=", p;
"array of the number of even closed walks= ", B;
"number of cones= ", b;
C:= [[sign(V,E,A[i][j]): j in [1..B[i]]]: i in [1..p]];

///if p eq 1 then
///return "only one pair";
///end if;
D:= CartesianProduct(C);

counter := 0;
for x in D do
y:= [x[i]: i in [1..p]];
L := {};
for j in [1..p] do
for k in [1..n] do
if y[j][k] eq -2 then L := L join {k};
end if;
end for;
end for;
L := SetToSequence(L);
counter2 := 0;
for j in [1..p] do
d:= &+[AbsoluteValue(y[j][k]): k in L];
if d ge 4 then counter2 := 1;
break;
end if;
end for;
if counter2 eq 0 then counter:= 1;
break;
end if;
end for;
M := [0*k: k in [1..n]];
for j in L do
M[j]:= -1;
end for;
M:= [m+2 : m in M];
if counter eq 0 then return "no bridge cones";
else F:= ConeWithInequalities(y);
```

```
print y, Dimension(F), "non-empty";

"standard weight vector=";M;
M1:= [0*i: i in [1..p]];
for j in [1..p] do
M1[j]:=  &+[-y[j][k]: k in L];
end for;
M2:= [0*i: i in [1..p]];
for j in [1..p] do
M2[j]:=  &+[AbsoluteValue(y[j][k]): k in L];
end for;
print "inner product=", M1;
print "inner product (absolute value) =", M2;

return M;
end if;
end function;

////////////////////////////////////////////////////////////////////

bmain3 := function(V,E)
n:= #(E);
A:= bij2(V,E);
p:= #(A);
B :=[ #(A[i]) : i in [1..p]];
b:= &*(B);
"number of edges=", n;
"number of pairs=", p;
"array of the number of even closed walks= ", B;
"number of cones= ", b;
C:= [[sign(V,E,A[i][j]): j in [1..B[i]]]: i in [1..p]];

///if p eq 1 then
///return "only one pair";
///end if;
D:= CartesianProduct(C);

counter := 0;
for x in D do
y:= [x[i]: i in [1..p]];
L := {};
for j in [1..p] do
for k in [1..n] do
if y[j][k] eq -2 then L := L join {k};
end if;
end for;
end for;
L := SetToSequence(L);
M:= [0*i: i in [1..p]];
for j in [1..p] do
M[j]:=  &+[AbsoluteValue(y[j][k]): k in L];
end for;
counter2 := 0;
counter3:= 0;
for j in [1..p] do
d:= &+[AbsoluteValue(y[j][k]): k in L];
if d ge 5 then counter2 := 1;
break;
end if;
if d eq 4 then counter3:= counter3 +1;
end if;
end for;
if (counter2 eq 0) and (counter3 le 2) then counter:= 1;
break;
end if;
```

```
end for;
if counter eq 0 then return "no bridge cones";
else F:= ConeWithInequalities(y);
print y, Dimension(F), "non-empty";
M1:= [0*i: i in [1..p]];
for j in [1..p] do
M1[j]:=  &+[-y[j][k]: k in L];
end for;
M2:= [0*i: i in [1..p]];
for j in [1..p] do
M2[j]:=  &+[AbsoluteValue(y[j][k]): k in L];
end for;
print "inner product =", M1;
print "inner product (absolute value) =", M2;

F1:= MinimalRGenerators(F);
w:= &+(F1);
w1:= ElementToSequence(w);
m:= Min(w1);
if m gt 0 then w2:= w1;
else w2:= [x-m+1: x in w1];
end if;
"inner point weight= ";w2;
return w2;

end if;
end function;

//////////////////////////////////////////////////////

bmain2 := function(V,E)
n:= #(E);
A:= bij2(V,E);
p:= #(A);
B :=[ #(A[i]) : i in [1..p]];
b:= &*(B);
"number of edges=", n;
"number of pairs=", p;
"array of the number of even closed walks= ", B;
"number of cones= ", b;
C:= [[sign(V,E,A[i][j]): j in [1..B[i]]]: i in [1..p]];

///if p eq 1 then
///return "only one pair";
///end if;
D:= CartesianProduct(C);

counter := 0;
for x in D do
y:= [x[i]: i in [1..p]];
L := {};
for j in [1..p] do
for k in [1..n] do
if y[j][k] eq -2 then L := L join {k};
end if;
end for;
end for;
L := SetToSequence(L);
counter2 := 0;
for j in [1..p] do
d:= &+[-y[j][k]: k in L];
if d le 0 then counter2 := 1;
break;
end if;
end for;
```

```
if counter2 eq 0 then counter:= 1;
break;
end if;
end for;
L1:= [0*k: k in [1..n]];
for k in L do
L1[k]:= -1;
end for;
L2:= [x+2: x in L1];
if counter eq 0 then return "no bridge cones";
else F:= ConeWithInequalities(y);
print y, Dimension(F), "non-empty";
"standard weight vector=";L2;
M1:= [0*i: i in [1..p]];
for j in [1..p] do
M1[j]:=  &+[-y[j][k]: k in L];
end for;
M2:= [0*i: i in [1..p]];
for j in [1..p] do
M2[j]:=  &+[AbsoluteValue(y[j][k]): k in L];
end for;
print "inner product =", M1;
print "inner product (absplute value) =", M2;

return L2;
end if;
end function;

///////////////////////////////////////////////////////

bmain1 := function(V,E)
n:= #(E);
A:= bij2(V,E);
p:= #(A);
B :=[ #(A[i]) : i in [1..p]];
b:= &*(B);
"number of edges=", n;
"number of pairs=", p;
"array of the number of even closed walks= ", B;
"number of cones= ", b;
C:= [[sign(V,E,A[i][j]): j in [1..B[i]]]: i in [1..p]];

///if p eq 1 then
///return "only one pair";
///end if;
D:= CartesianProduct(C);

counter := 0;

for x in D do
y:= [x[i]: i in [1..p]];
L := {};
for j in [1..p] do
for k in [1..n] do
if y[j][k] eq -2 then L := L join {k};
end if;
end for;
end for;
L := SetToSequence(L);
M:= [0*i: i in [1..p]];
for j in [1..p] do
M[j]:=  &+[-y[j][k]: k in L];
end for;

counter2 := 0;
```

```
counter3:= 0;
for j in [1..p] do
d:= &+[-y[j][k]: k in L];
if d lt 0 then counter2 := 1;
break;
end if;
if d eq 0 then counter3:= counter3 +1;
end if;
end for;
if (counter2 eq 0) and (counter3 le 2) then counter:= 1;
break;
end if;
end for;

if counter eq 0 then return "no bridge cones";
else F:= ConeWithInequalities(y);
print y, Dimension(F), "non-empty";
M1:= [0*i: i in [1..p]];
for j in [1..p] do
M1[j]:=  &+[-y[j][k]: k in L];
end for;
M2:= [0*i: i in [1..p]];
for j in [1..p] do
M2[j]:=  &+[AbsoluteValue(y[j][k]): k in L];
end for;
print "inner product =", M1;
print "inner product (absolute value) =", M2;
F:= ConeWithInequalities(y);
F1:= MinimalRGenerators(F);
w:= &+(F1);
w1:= ElementToSequence(w);
m:= Min(w1);
if m gt 0 then w2:= w1;
else w2:= [x-m+1: x in w1];
end if;
"inner point weight= ";w2;
return w2;
end if;
end function;

////////////////////////////////////////////////////////////////////////

initial := function(V,E,w)
Q := RationalField();
d:= #(V);
n:= #(E);
print "number of vertices= ", d; print "number of edges = ", n;
P1<[x]>:= PolynomialRing(Q,n,"grevlexw",w);
P2<[t]>:= PolynomialRing(Q,d);
L := [(P2 ! 0) * k: k in [1..n]];
for i in [1..n] do
a:= Max(E[i]); b := Min(E[i]);
L[i]:= t[a]*t[b];
end for;
f:= hom <P1 -> P2 | L>;
K := AffineAlgebraMapKernel(f);
K1:= GroebnerBasis(K);
LM:= [LeadingMonomial(x): x in K1];
print "initial ideal = ", LM;
A:= [0,1];
X:= CartesianPower(A,n);
Y:= [z: z in X];
L := [];
for z in Y do
C:= [x[i]^(z[i]): i in [1..n]];
```

```
m:= &*(C);
///D:= [x: x in LM | LCM(x,m) eq m];
///if D eq [] and TotalDegree(m) eq d then L:= L cat [m];
///else L := L cat [];
///end if;
///end for;
D:= [x: x in LM | LCM(x,m) eq m];
if D eq [] then L:= L cat [m];
else L := L cat [];
end if;
end for;
M := [TotalDegree(x): x in L];
m:= Max(M);
if d ne m then print "something is wrong";
else M1:= [m: m in L | TotalDegree(m) eq d];
end if;
print "number of maximal simplices=", #(M1);
print "array of maximal simplices of the regular unimodular triangulation=";
return M1;
end function;

//////////////////////////////////////////

toric := function(V,E)
d:= #(V);
n:= #(E);
Q := RationalField();
P1<[x]>:= PolynomialRing(Q,n);
P2<[t]>:= PolynomialRing(Q,d);
L := [(P2 ! 0) * k: k in [1..n]];
for i in [1..n] do
a:= Max(E[i]); b := Min(E[i]);
L[i]:= t[a]*t[b];
end for;
f:= hom <P1 -> P2 | L>;
K := AffineAlgebraMapKernel(f);
return K;
end function;
```

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