

Title	Relations between homotopy and homology. I
Author(s)	Komatu, Atuo
Citation	Osaka Mathematical Journal. 1949, 1(2), p. 150-155
Version Type	VoR
URL	https://doi.org/10.18910/6978
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

Relations between Homotopy and Homology. I.

By Atuo KOMATU.

1. INTRODUCTION.

This paper is a continuation of the author's earlier investigation [1], studying the problem of essential dimensions¹⁾ of continuous transformations using the method of homology with local coefficients [2]. The exact homology sequence, recently clarified by J. L. KELLEY and E. PITCHER [3], can be applied to this method and give many new results some of which are already obtained by S. EILENBERG and S. MAC LANE [4], L. PONTRIAGIN [5] and G. W. WHITEHEAD [6].

Let K^n be the n -section of a complex K , then we have the following exact sequence with respect to the homotopy groups

$$\begin{array}{ccccccc} \pi_m(K^{n-1}) & \xrightarrow{i} & \pi_m(K^n) & \xrightarrow{r} & \pi_m(K^n \bmod K^{n-1}) & \xrightarrow{\partial_t} & \\ & & & & & & \\ & & \pi_{m-1}(K^{n-1}) & \xrightarrow{i} & \pi_{m-1}(K^n) & & \end{array}$$

The kernel-images in $\pi_m(K^n)$, $\pi_m(K^n \bmod K^{n-1})$, $\pi_{m-1}(K^{n-1})$ of this sequence are essentially the same as the groups $\nu_m(K^n)$, $\mu_m(K^n)$, $\lambda_{m-1}(K^{n-1})$, respectively, which were introduced by the author in [1].

2. THE CASE OF SIMPLY CONNECTED COMPLEX.

THEOREM 1. *Let α_n be the number of n -simplexes of a simply connected complex K . Then the relative homotopy group $\pi_n(K^n \bmod K^{n-1})$ ($n > 2$) is isomorphic with the weak direct sum (I, α_n) of α_n integer groups.*

PROOF. The proof is similar to that of theorem 2.1, [1].

COROLLARY 1.1.

$$\pi_n(K^n) \approx \mu_n(K^n) + \nu_n(K^n).$$

PROOF. The group $\mu_n(K^n) \approx \pi_n(K^n)/\nu_n(K^n)$ is a subgroup of the

¹⁾ For this definition see [1]. The essential dimension of a continuous mapping f of M in K is the least dimension of the image sets $g(M)$, where g is any continuous mapping of the same homotopy class with f .

free abelian group $\pi_n(K^n \text{ mod } K^{n-1})$, therefore $\mu_n(K^n)$ is a direct component of $\pi_n(K^n)$.

COROLLARY 1.2. $\lambda_n(K^n)$ is isomorphic with the direct sum of the subgroup $\lambda_n(K^n) \cap \nu_n(K^n)$ and the subgroup isomorphic with $\lambda_n(K^n)/\lambda_n(K^n) \cap \nu_n(K^n)$.

For $\lambda_n(K^n)/\lambda_n(K^n) \cap \nu_n(K^n)$ is a module, being isomorphic with a subgroup of $\pi_n(K^n \text{ mod } K^{n-1})$.

COROLLARY 1.3. The n -chain group with integer coefficients $L^n(K, I)$ of K is isomorphic with $\pi_n(K^n \text{ mod } K^{n-1})$.

THEOREM. 2. Let ∂_i be the homology boundary operator of $L^n(K, I)$ ($n > 3$), and ∂_t the homotopy boundary operator, then there holds the relation

$$\partial_i = r \partial_t$$

PROOF. It is sufficient to prove the case of one simplex $1. \sigma^n \in L^n(K, I)$, for $\partial_i, r, \partial_t$ are all homomorphic mappings of abelian groups.

$$\begin{aligned} \text{Let } \partial_i(\sigma^n) &= \sum_i \sigma_i^{n-1}, \\ \partial_t(\sigma^n) &= \alpha \in \lambda_{n-1}(K^{n-1}), \end{aligned}$$

where α is a homotopy class of the continuous mapping of an $(n-1)$ -sphere S^{n-1} on the sphere $\partial_i(\sigma^n) = \sum_i \sigma_i^{n-1}$ with mapping-degree $+1$. Then

$$r(\alpha) = \sum_i \sigma_i^{n-1}, \text{ i. e. } \partial_i = r \partial_t.$$

A chain $c^n \in \pi_n(K^n \text{ mod } K^{n-1})$ is a cycle, when $r \partial_i(c^n) = 0$, and is a spherical cycle, when $\partial_t(c^n) = 0$. A homology-boundary is a spherical cycle and the spherical homology group $\Sigma^n(K)$ is defined as the factor group of the group $\mu_n(K^n)$ of spherical cycles by the homology boundary $r(\lambda_n(K^n))$.

COROLLARY 2.1. $\Sigma^n(K) \approx \pi_n(K)/\nu_n(K) \approx \mu_n(K)$.

PROOF. The group of boundaries is $r(\lambda_n(K^n)) \approx \lambda_n(K^n)/\lambda_n(K^n) \cap \nu_n(K^n) = B^n(K)$. Therefore $\Sigma^n(K)$ is isomorphic with

$$\mu_n(K^n)/r(\lambda_n(K^n)) \approx \pi_n(K^n)/\nu_n(K^n)/\lambda_n(K^n)/\lambda_n(K^n) \cap \nu_n(K^n).$$

The last term of the above sequence of groups is easily verified to be isomorphic with $\pi_n(K)/\nu_n(K) \approx \mu_n(K)$.

COROLLARY 2.2. $H^n(K)/\Sigma^n(K) \approx \lambda_{n-1}(K^{n-1}) \cap \nu_{n-1}(K^{n-1})$.

LEMMA 2.1. $\nu_n(K^n) \approx \pi_n(K^{n-1})/\lambda_n(K^{n-1})$.

COROLLARY 2.3. If $\pi_i(K) = 0$ ($0 \leq i < n$), then

$$\begin{aligned} H^n(K, I) &\approx \Sigma^n(K) \approx \pi_n(K), \\ H^{n+1}(K, I) &\approx \Sigma^{n+1}(K) \approx \pi_{n+1}(K)/\nu_{n+1}(K). \end{aligned}$$

PROOF. By the result of W. HUREWICZ any compact set of K^{n-1} is homotopic to zero in K^n . Therefore $\pi_n(K^{n-1}) \approx \lambda_n(K^{n-1})$. And so by LEMMA 2.1, $\nu_n(K^n) = 0$, i. e. $\nu_n(K) = 0$. This proves the theorem by COROLLARIES 2.1, 2.2.

If we apply the Freudenthal's theory of "Einhängung" to the group $\nu_{n+1}(K) \approx \nu_{n+1}(K^{n+1})/\nu_{n+1}(K^{n+1}) \wedge \lambda_{n+1}(K^{n+1})$, we can deduce the results of G. W. WHITEHEAD. For instance we get the following relations:

$$\begin{aligned} \text{If } \pi_i(K) &= 0 \quad (0 < i < n), \\ \pi_n(K^n)/2\pi_n(K^n) &\approx \pi_{n+1}(K^n), \\ \pi_n(K^n)/(\lambda_n(K^n), 2\pi_n(K^n)) &\approx \pi_{n+1}(K^n)/\lambda_{n+1}(K^n), \\ \pi_n(K)/2\pi_n(K) &\approx \nu_{n+1}(K^{n+1}). \end{aligned}$$

3. THE CASE WHEN K IS NOT SIMPLY CONNECTED.

Let \bar{K} be the universal covering complex of K and \bar{K}^n the n -section of \bar{K} . \bar{K}^n ($n > 1$) is the universal covering complex of K^n . Let $\mathfrak{F} = \{x_\alpha\}$ be the fundamental group of K , then the n -simplex of K are represented in the form $\{x_\alpha \sigma_i^n\}$, where $\{\sigma_i^n\}$ are n -simplexes of K . The mapping $u: x_\alpha \sigma_i^n \rightarrow \sigma_i^n$ is the covering mapping of K onto K . Remembering that the homotopy groups of a complex are isomorphic with those of the covering complex, we can easily verify that the following two sequences

$$\begin{aligned} \pi_{n+1}(K^{n+1} \text{ mod } K^n) &\rightarrow \pi_n(K^n) \rightarrow \pi_n(K^n \text{ mod } \bar{K}^{n-1}), \\ \pi_{n+1}(K^{n+1} \text{ mod } K^n) &\rightarrow \pi_n(K^n) \rightarrow \pi_n(K^n \text{ mod } K^{n-1}) \end{aligned}$$

are equivalent as homomorphism sequences. In particular we have

²⁾ After this paper was submitted for publication, I have read G. W. WHITEHEAD'S paper [6] that recently came to Japan. Although the proof is only sketched, it seems to me that his method is different from that of mine. I could not read the paper of H. HOPF: Über die Bettischen Gruppen, die zu einer beliebigen Gruppen gehören. Comment. Math. Helv. 17, 1944.

$$\begin{aligned} \lambda_n(K^n) &\approx \lambda_n(\bar{K}^n), \\ \mu_n(\bar{K}^n) &\approx \mu_n(K^n), \\ \nu_n(\bar{K}^n) &\approx \nu_n(K^n). \end{aligned}$$

As is shown in § 2, $\pi_n(\bar{K}^n \text{ mod } \bar{K}^{n-1})$ is isomorphic with the chain group $L^n(\bar{K}, I)$, and its elements can be represented in the form $\sum a x_\alpha \sigma_i^n$, where a 's are integers. Clearly the elements of the form $\sum a \cdot 1 \sigma^n$, where 1 is the unit element of \mathfrak{F} , form a subgroup of $L^n(\bar{K}, I)$ which is isomorphic with the chain group $L^n(K, I)$. We suppose that $L^n(K, I)$ is imbedded in $\pi_n(\bar{K}^n \text{ mod } \bar{K}^{n-1}) \approx L_n(\bar{K}, I)$ by the above isomorphism.

We remark that $L_n(K, I)$ is a direct summand of $\pi_n(\bar{K}^n \text{ mod } \bar{K}^{n-1})$ and the natural homomorphism of the latter group onto the former is induced by the covering mapping $u: x_\alpha \sigma_i^n \rightarrow \sigma_i^n$. We denote by Γ^n the kernel of the last homomorphism.

Then we have the following important

THEOREM 3. *The homology boundary operator ∂_i of $L^n(K, I)$ ($n > 3$) can be decomposed into 3 successive operators, i. e.*

$$\partial_i = u r \partial_i.$$

PROOF. It is sufficient to prove the case of one simplex σ^n .

Let

$$\begin{aligned} \partial_i(\sigma^n) &= \sum_i \sigma_i^{n-1} \\ \partial_i(\sigma^n) &= \alpha \in \lambda_{n-1}(K^{n-1}) \approx \lambda_{n-1}(\bar{K}^{n-1}), \end{aligned}$$

where α is the homotopy class of continuous mapping f of S^{n-1} on the $(n-1)$ -sphere $\sum_i \sigma_i^{n-1}$ of K^{n-1} with mapping degree + 1, or the mapping \bar{f} of S^{n-1} on an $(n-1)$ -sphere $\sum_i x_\alpha \sigma_i^{n-1}$ of \bar{K}^{n-1} . The mapping f is equal to the mapping $u \bar{f}$. The image sphere $\sum_i x_\alpha \sigma_i^{n-1}$ is invariant by the relativisation r , as in theorem 2 and by the covering mapping u it reduces to the sphere $\sum_i \sigma_i^{n-1}$, i. e. $\partial_i(\sigma^n)$. Therefore for every chain c^n of $L^n(K, I)$

$$\partial_i(c^n) = u r \partial_i(c^n).$$

A chain $c^n \in L^n(K, I) \subset \pi_n(K^n \text{ mod } K^{n-1})$ is called spherical, when it satisfies $\partial_i(c^n + \gamma^n) = 0$ for some $\gamma^n \in \Gamma^n$, and is called simple, when

it satisfies $r \partial_t (c^n + \gamma^n) = 0$ for some $\gamma^n \ni 1^n$. Then we see easily that c^n is a spherical cycle or a simple cycle if and only if it is an image under u of a spherical cycle or a cycle of \bar{K} , respectively.

THEOREM 4. *Homology boundaries are spherical.*

PROOF. Let c^n be the boundary of a chain c^{n+1} , that is, $\partial_t (c^{n+1}) = u r \partial_t (c^{n+1}) = c^n$ or $r \partial_t (c^{n+1}) = c^n + \gamma^n$ for some $\gamma^n \in \Gamma^n$. Using relation $\partial_t r = 0$, we have then $\partial_t (c^n + \gamma^n) = \partial_t r \partial_t (c^{n+1}) = 0$.

By this theorem we can define the spherical homology group $\Sigma^n(K, I)$ and the simple homology group $\Theta^n(K, I)$ of K as subgroups of $H^n(K, I)$.

THEOREM 5.

$$\begin{aligned}\Sigma^n(K, I) &\approx \Sigma^n(\bar{K}, I) / \Sigma^n(K, I) \cap \Gamma^n, \\ \Theta^n(K, I) &\approx H^n(\bar{K}, I) / H^n(K, I) \cap \Gamma^n.\end{aligned}$$

PROOF. We shall prove only the former relation. The proof of the latter is similar.

Let c^n be the homology boundary of c^{n+1} and d^n, d^{n+1} , respectively, the image chains $u(c^n), u(c^{n+1})$ in K . Then for a suitable element $\gamma^{n+1} \in \pi_{n+1}(\bar{K}^{n+1} \bmod K^n)$

$$\begin{aligned}c^{n+1} &= d^{n+1} + \gamma^{n+1}, \\ u r \partial_t (d^{n+1}) &= u r \partial_t (c^{n+1} - \gamma^{n+1}) \\ &= u r \partial_t (c^{n+1}) - u r \partial_t (\gamma^{n+1}) = u(c^n) = d^n.\end{aligned}$$

Hence the mapping u defines a homomorphism of $\Sigma^n(\bar{K}, I)$ in $\Sigma^n(K, I)$.

Let d^n be a spherical cycle in K . With a suitable γ^n the sum $\gamma^n + d^n = c^n$ is a spherical cycle in \bar{K} , i. e.

$$\partial_t (\gamma^n + d^n) = 0,$$

and $u(c^n) = d^n$. Hence $u(\Sigma^n(\bar{K}, I)) = \Sigma^n(K, I)$.

Let d^n be a boundary in K and c^n the original element $u^{-1}(d^n)$ in $\Sigma^n(\bar{K}, I)$. These conditions are written

$$\begin{aligned}c^n &= d^n + \gamma^n, \gamma^n \in L(K^n, I), \\ (1) \quad \partial_t (c^n) &= 0, \\ (2) \quad u r \partial_t (d^{n+1}) &= d^n, \quad d^{n+1} \in L^{n+1}(K^{n+1}, I).\end{aligned}$$

From (2) for a suitable γ'^n

$$r \partial_t (d^{n+1}) = d^n + \gamma'^n,$$

hence (3) $\partial_t(d^n + \gamma^m) = 0$.

From (1) and (3)

$$\partial_t(\gamma^n - \gamma'^n) = 0, \text{ i. e. } \gamma^n - \gamma'^n \in \Sigma^n(\bar{K}, I) \cap \Gamma^n,$$

and

$$c^n = r \partial_t(d^{n+1}) + (\gamma^n - \gamma'^n).$$

Therefore the original element $c^n = u^{-1}(d^n)$ is contained in the subgroup $\Sigma^n(\bar{K}, I) \cap \Gamma^n$ of $\Sigma^n(\bar{K}, I)$.

LITERATURE.

1. A. KOMATU: Zur Topologie der Abbildungen von Komplexen, Jap. Jour. of Math., vol. 17, 1941.
2. N. E. STEENROD: Homology with local coefficients, Ann. of Math. 44, 1943.
3. J. L. KELLEY and E. PITCHER: Exact homomorphism sequences in homology theory, Ann. of Math. 48, 1947.
4. S. EILENBERG and S. MACLANE: Relations between homology and homotopy groups of spaces, Ann. of Math. 46, 1945.
5. L. PONTRJAGIN: Mappings of the three dimensional sphere into an n -dimensional complex, Comp. Rendus URSS, 34, 1942.
6. G. W. WHITEHEAD: On spaces with vanishing low-dimensional groups, Proc. Nat. Acad. Sci., 1948.

(Received February 18, 1949)