Kanenobu, T. and Marumoto, Y.
Osaka J. Math.
34 (1997), 525–540

UNKNOTTING AND FUSION NUMBERS OF RIBBON 2-KNOTS

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(Received August 5, 1996)

Introduction

Hosokawa and Kawauchi [6] proved that any 2-knot can give an unknotted surface after adding enough 1-handles in an appropriate manner. Hosokawa, Maeda, and Suzuki [7] then defined the *unknotting number* $u(K)$ of a 2-knot K to be the least number of such 1-handles.

A ribbon 2-knot is obtained from a trivial $(n+1)$-component 2-link by adding n 1-handles for some n. The *fusion number* $f(K)$ of a ribbon 2-knot K is the least number of n possible for K.

Let K be a ribbon 2-knot. Then Miyazaki [17, Lemma 1] proved:

Proposition 1.

$$u(K) \leq f(K).$$

Note that the fusion number is called the ribbon number in [17]. Let K^* denote the spun 2-knot of a 1-knot K. Let $T_{p,q}$ denote the torus knot of type (p,q). Miyazaki [17, Claim, Remark 2] also showed:

$$(1) \qquad u(T^*_{2,2m-1} \# T^*_{2,2m+1}) = 1, \qquad f(T^*_{2,2m-1} \# T^*_{2,2m+1}) = 2,$$

if $m \geq 2$. The purpose of this paper is to provide more examples of ribbon 2-knots that do not satisfy the equality in Proposition 1. First, we will prove:

Theorem 1. *For a nontrivial spun torus knot $T^*_{p,q}$ with $1 < p < q$, we have*

$$u(T^*_{p,q}) = 1, \quad f(T^*_{p,q}) = p - 1.$$

Using the composition of a spun torus knot and some copies of a ribbon 2-knot with fusion number one, we will show:

Theorem 2. *For any integers m and n with $0 < m \leq n$, there exists a ribbon 2-knot K such that $u(K) = m$ and $f(K) = n$.*

Sections 2 and 3 are devoted to prove $u(T^*_{p,q}) = 1$. We give two other characterizations of a ribbon 2-knot by making use of a handle decomposition of a cobordism between a ribbon 2-knot and a trivial 2-knot, from which we define the handle and dual-handle numbers of a ribbon 2-knot. These invariants coincide (Proposition 3), and are greater than or equal to the unknotting number (Theorem 3) and less than or equal to the fusion number (Theorem 4). We define a class of tubed 2-knots, including all spun torus knots (Lemma 3), and show that a tubed 2-knot has dual-handle number one (Theorem 5). From the fact that the torus knot has a $(1,1)$-decomposition, we see that the spun 2-knot of a 1-knot with $(1,1)$-decomposition has weak unknotting number one (Proposition 5). In Sections 4 and 5, we prove Theorems 1 and 2.

The fusion number plus one is an analogy of the bridge number for a 1-knot, both of which are greater than or equal to the Wirtinger rank of the knot group. In Section 6, we give a relation between the fusion number and the rank of the fundamental group of the finite cyclic branched covering space of a ribbon 2-knot (Proposition 6), which is also an analogy for the case of the bridge number. We use this to generalize the example (1) of Miyazaki (Proposition 7).

In Section 7, we give another example for the inequality in Proposition 1 using a satellite knot of a ribbon 2-knot.

1. Preliminaries

The boundary and the interior of a manifold M are denoted by ∂M and $\mathrm{int} M$. Both D^n and B^n denote the n-ball, and R^n and S^n denote the Euclidean n-space and n-sphere, respectively. I is the unit interval $[0,1]$.

Let K be an n-knot. We denote by πK the group of K, i.e., $\pi K = \pi_1(S^{n+2} - K)$. For a non-negative integer n, nK means the composition of n copies of K; if $n = 0$, the knot is a trivial one.

The *Wirtinger rank* W-rank(πK) of a knot group πK is the least number of meridional elements of K we need to generate πK. The *rank* rank$G$ of a group G is the least number of generators of G.

Let L be a 2-link in S^4. A 1-handle h on L is an embedding $h : I \times D^2 \to S^4$ with $h(I \times D^2) \cap L = h(\partial I \times D^2)$. The arc $h(I \times \{O\})$ is called the *core* of h, where O is the center of D^2. We define the surface $L + h$ obtained from L by adding a 1-handle h to be the surface $L + h = (L - h(\partial I \times D^2)) \cup h(I \times \partial D^2)$. We define a *ribbon* 2-knot K to be a 2-knot obtained by adding n 1-handles $\{h_i | 1 \leq i \leq n\}$ to a trivial 2-link with $n+1$ components $\bigcup_{i=0}^{n} S_i^2$ for some n,

$$K = \bigcup_{i=0}^{n} S_i^2 + \sum_{i=1}^{n} h_i.$$

See [28], where such a 2-knot is called a *fusion* of a trivial 2-link. If the ith 1-handle

h_i connects $S^2_{i(1)}$ with $S^2_{i(2)}$, then the group πK has a Wirtinger presentation

$$\langle x_0, x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n \rangle,$$

where x_i is a meridian of S^2_i and $r_i = x^{-1}_{i(1)} w_i^{-1} x_{i(2)} w_i$ with w_i a word in x_0, $x_1, \ldots, x_n$ determined by the core of h_i; see [27]. Then we have

$$f(K) \geq \text{W-rank}(\pi K) - 1. \tag{2}$$

Let K be a 2-knot. For any element $g \in \pi K$, there exists a 1-handle on K whose core represents g [2]. So there exist $g_1, g_2, \ldots, g_n \in \pi K$ such that

$$\pi K / \langle [g_1, x], [g_2, x], \ldots, [g_n, x] \rangle$$

is the infinite cyclic group, where x is a meridian and $[g_i, x] = g_i x g_i^{-1} x^{-1}$. We define the *weak unknottig number* $u_w(K)$ of K to be the least number of n possible for K; see [9]. Then

$$u(K) \geq u_w(K). \tag{3}$$

Let $\widetilde{X}_K$ be the infinite cyclic covering space of $S^4 - K$. Then the Alexander invariant of K, $H_1(\widetilde{X}_K)$, is a finitely generated Λ-module, where Λ is the polynomial ring $\mathbf{Z}[t, t^{-1}]$ [21, Chapter 7]. We define $e(K)$ to be the least number of generators of $H_1(\widetilde{X}_K)$ as a Λ-module. Then we have the following [9, Proposition 2]:

$$u_w(K) \geq e(K). \tag{4}$$

Concerning the composition of 2-knots, the following is easy to see:

$$u(K_1 \# K_2) \leq u(K_1) + u(K_2) \tag{5}$$

$$f(K_1 \# K_2) \leq f(K_1) + f(K_2). \tag{6}$$

2. Handle number of a ribbon 2-knot

Let (B^5_0, B^3_0) be a standard ball pair. We attach n 1-handles $\{H^1_i | 1 \leq i \leq n\}$ to $D^5_0 - D^3_0$. Let $V = D^5_0 \cup \bigcup_{i=1}^n H^1_i$. The fundamental group $\pi_1(V - D^3_0)$ is free on $n+1$ generators x, y_1, y_2, ..., y_n, where x is a meridian of D^3_0 and y_i, $1 \leq i \leq n$, is represented by a loop running through the ith 1-handle H^1_i just once and missing H^1_j for $j \neq i$. Let $v_i = v_i(x, y_1, y_2, \ldots, y_n)$ be an element of $\pi_1(V - D^3_0)$ which becomes y_i if we kill the element x:

$$v_i(1, y_1, y_2, \ldots, y_n) = y_i. \tag{7}$$

We attach 2-handles H_i^2 to $V - D_0^3$ along embedded simple closed curves α_i representing v_i in the boundary. Then $V \cup \bigcup_{i=1}^n H_i^2$ is a 5-ball and the sphere pair

$$(8) \qquad (\partial(V \cup \bigcup_{i=1}^{n} H_i^2), \partial D_0^3)$$

is a ribbon 2-knot. Conversely, any ribbon 2-knot is constructed in this manner. See [1, 5, 15].

The following is shown in [5, Theorem III.5.1; 14, Lemma 2.2].

Lemma 1. *In the above, we can take the attaching spheres α_i of the 2-handles H_i^2 so that the exponent sum of v_i on x is zero for each i.*

We define the *handle number* $\varphi(K)$ of a ribbon 2-knot K to be the least number of 1-handles for which the above construction yields K. The group of the ribbon 2-knot (8) is presented by

$$(9) \qquad \langle x, y_1, y_2, \ldots, y_n | v_1, v_2, \ldots, v_n \rangle,$$

and so we have

Proposition 2.

$$\operatorname{rank}(\pi K) - 1 \leq \varphi(K).$$

Theorem 3.

$$u(K) \leq \varphi(K).$$

Proof. Suppose that K is the 2-knot with $\varphi(K) = n$ described as (8). Let $v_i = u_1 x^{\epsilon_1} u_2 x^{\epsilon_2} \cdots u_k x^{\epsilon_k} u_{k+1}$, where $u_1, u_2, \ldots, u_{k+1}$ are words in $y_1, y_2, \ldots, y_n$. By Lemma 1, we may assume that $\sum_{j=1}^k \epsilon_j = 0$.

Let h_i be a 1-handle on ∂D_0^3 corresponding to y_i and $T_0 = \partial D_0^3 + \sum_{i=1}^n h_i$, a surface of genus n. Then the fundamental group $\pi_1(\partial V - T_0)$ is given by

$$\langle x, y_1, y_2, \ldots, y_n | [x, y_i] (1 \leq i \leq n) \rangle.$$

In this group, v_i becomes $u_1 u_2 \cdots u_{k+1}$, which is y_i by (7). Thus we can move isotopically the attaching spheres α_i to the standard ones. Therefore the surface T_0 is unknotted in the 4-sphere $\partial(V \cup \bigcup_{i=1}^n H_i^2)$. □

3. Dual-handle number of a ribbon 2-knot

Let K be a ribbon 2-knot given by (8). Attaching a 5-handle D_1^5, we obtain a handle decomposition of a 5-sphere:

$$S^5 = D_0^5 \cup \bigcup_{i=1}^{n} H_i^1 \cup \bigcup_{i=1}^{n} H_i^2 \cup D_1^5.$$

Turning this handle decomposition of S^5 upside down, we have

$$S^5 = D_1^5 \cup \bigcup_{i=1}^{n} \overline{H}_i^3 \cup \bigcup_{i=1}^{n} \overline{H}_i^4 \cup D_0^5,$$

where $\overline{H}_i^j$ is the handle of index j naturally obtained from H_i^{5-j} for $j = 3, 4$. Then K is on the boundary of D_1^5 and the 3-handles $\{\overline{H}_i^3\}$ are trivial. From this, we obtain the following [15, Corollary 1.10.1]; see [15, Theorem 1.10] for the detailed proof.

Lemma 2. *A 2-knot K in $S^4 = \partial D^5$ is a ribbon 2-knot if and only if there exist 3-handles $\{H_i^3 | 1 \leq i \leq n\}$ on D^5 with the following properties.*
(i) *The set of attaching spheres for the 3-handles forms a trivial 2-link in S^4 that does not intersect K.*
(ii) *K bounds a 3-ball in the boundary of the resulting manifold $D^5 \cup \bigcup_{i=1}^{n} H_i^3$.*

We define the *dual-handle number* $\varphi^*(K)$ of a ribbon 2-knot K to be the least number of the 3-handles for K as in Lemma 2. From the proof of Lemma 2, we have:

Proposition 3.
$$\varphi^*(K) = \varphi(K).$$

Theorem 4.
$$\varphi^*(K) \leq f(K).$$

Proof. Suppose that K is a ribbon 2-knot in $S^4 (= \partial D^5)$ obtained by adding n 1-handles $\{h_i | 1 \leq i \leq n\}$ to a trivial 2-link with $n+1$ components $\bigcup_{i=1}^{n} S_j^2$. We take a 2-sphere $\sum_j^2$ parallel to S_j^2 for each j, $1 \leq j \leq n$, so that $\bigcup_{j=1}^{n} \sum_j^2$ forms a trivial link in S^4 and does not intersect the knot K. Let W be the manifold obtained from D^5 by adding trivial 3-handles along the 2-spheres $\bigcup_{j=1}^{n} \sum_j^2$. Then each S_j^2, $1 \leq j \leq n$, bounds a 3-ball in ∂W that intersects K only at the 1-handles. By using these 3-balls, each S_j^2 may be moved toward S_0^2 along 1-handles, and the knot K becomes trivial in ∂W. □

Let K be a ribbon 2-knot. Then K is called a *tubed* 2-knot if there exists $S^1 \times B^3$ that is embedded in S^4 such that K is obtained from a trivial link $\bigcup_{i=0}^{n}\{x_i\} \times \partial B^3$, $x_i \in S^1$, by adding 1-handles $\{h_j | 1 \leq j \leq n\}$ in $S^1 \times B^3$ for some n.

Theorem 5. *If K is a tubed 2-knot, then $\varphi(K) \leq 1$.*

Proof. Let K be a tubed 2-knot as above. Assume that $S^4 = \partial D^5$, and put $S_i^2 = \{x_i\} \times \partial B^3$. Let W be the manifold obtained from D^5 by adding trivial 3-handles along the 2-sphere in $S^4 - S^1 \times B^3$ that is parallel to S_i^2. There is a component, say S_n^2, to which the only one 1-handle, say h_n, is added; we suppose $S_n^2 \cap \bigcup_{j=1}^{n} h_j(I \times D^2) = h_n(\{0\} \times D^2)$. Then S_n^2 bounds a 3-ball B_0^3 in ∂W such that $K \cap B_0^3 = S_n^2 - \mathrm{int} h_n(\{0\} \times D^2)$. Then there is an isotopy of ∂W carrying K to $\bigcup_{i=0}^{n-1} S_i + \sum_{j=1}^{n-1} h_j$. Repeating this argument inductively, we finally obtain a trivial knot, which is K on the boundary of D^5 with suitable number of 3-handles added. □

4. Spun torus knots

Let R_+^3 and R_-^3 denote the upper-half space $\{(x, y, z) | z \geq 0\}$ and lower-half space $\{(x, y, z) | z \leq 0\}$, respectively. We define the *spin* $\sigma X (\subset R^4)$ of any set X in R_+^3 to be

$$\sigma X = \{(x, y, z\cos\theta, z\sin\theta) | (x, y, z) \in X, 0 \leq \theta \leq 2\pi\}.$$

Let K be a 1-knot in R^3 such that $K \cap R_-^3$ is a properly embedded unknotted arc. Then the spun 2-knot K^* of K is given as the spin of $K \cap R_+^3$; see [21, p.85].

Lemma 3. *A spun torus knot is a tubed 2-knot.*

Proof. For the torus knot $T_{p,q}$, there is a regular projection $\psi : R^3 \to R^2$ with q overpasses $\alpha_1, \alpha_2, \ldots, \alpha_q$ and q underpasses $\beta_1, \beta_2, \ldots, \beta_q$ such that
(i) $\psi(T_{p,q})$ is in an annulus $A = S^1 \times I$ in R^2, $S^1 = I/(0 \sim 1)$,
(ii) $\psi(\alpha_i) = \{i/q\} \times I$,
(iii) $\psi(\beta_i)$ is a properly embedded arc in A.
Fig. 1 gives such a regular projection for $T_{4,5}$.

We denote the unions $\bigcup_{i=1}^{q} \alpha_i$ and $\bigcup_{i=1}^{q} \beta_i$ by α and β, respectively. Then $T_{p,q}$ is isotopic to the following knot in $A \times [0, 2]$ having the same diagram $\psi(T_{p,q})$:

$$\{(x, 2) | x \in \psi(\alpha)\} \cup \{(x, 1) | x \in \psi(\beta)\} \cup \{(x, t) | x \in \psi(\partial\alpha) = \psi(\partial\beta), 1 \leq t \leq 2\},$$

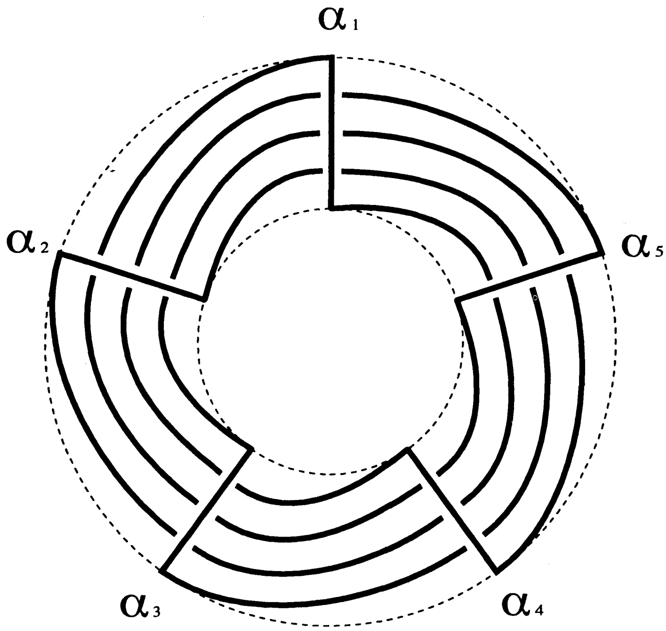

Fig. 1.

and thus the spun torus knot $T^*_{p,q}$ is the spin of the following arc in R^3_+:

$$\{(x,2)|x\in\psi(\alpha)\}\cup\{(x,1)|x\in\psi(\beta_1\cup\ldots\cup\beta_{q-1})\}$$
$$\cup\ \{(x,t)|x\in\psi(\partial\beta_1\cup\ldots\cup\partial\beta_{q-1}),1\le t\le 2\}$$
$$\cup\ \{(x,t)|x\in\psi(\partial\beta_q),0\le t\le 2\}.$$

We denote by $\bar{\alpha}_i$ the unknotted arc in R^3_+ defined by

$$\bar{\alpha}_i=\{(x,2)|x\in\psi(\alpha_i)\}\cup\{(x,t)|x\in\psi(\partial\alpha_i),0\le t\le 2\}.$$

Then the spin of $\bigcup_{i=1}^q\bar{\alpha}_i$ forms a trivial 2-link with q components. We see that the spun torus knot $T^*_{p,q}$ is a ribbon 2-knot obtained from this trivial link by adding 1-handles h_j, $1\le j\le q-1$, whose core is $\sigma\{(x,0)|x\in\psi(\beta_j)\}$; cf. [16]. This is a tubed 2-knot since the spin of the solid torus $A\times[0,2]$ is homeomorphic to $S^1\times B^3$. □

Proof of Theorem 1. By Theorems 3, 5 and Lemma 3, we have $u(T^*_{p,q})\le 1$. Since $T^*_{p,q}$ is nontrivial, we have $u(T^*_{p,q})=1$.

From the proof of Lemma 3, we have $f(T^*_{p,q})\le p-1$. From [22], we have W-rank$(\pi T_{p,q})\ge p-1$, which implies $f(T^*_{p,q})\ge p-1$ by (2). Thus we have $f(T^*_{p,q})=p-1$. □

The spun 2-knot of a 1-knot is a ribbon 2-knot. Let $b(K)$ denote the bridge number of a 1-knot K. Then from the proof of Lemma 3, we see:

Proposition 4. *For any 1-knot K,*

$$f(K^*) \le b(K) - 1.$$

Questions. (i) Does the equality in Proposition 4 always hold? Note also that $b(K) \ge \text{W-rank}(\pi K)$; cf. (2).

(ii) It is known [24] that $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$. How about the fusion number? That is, does the equality of (6) always hold?

A 1-knot K in S^3 has a *(1,1)-decomposition* if the pair (S^3, K) has a decomposition into a union $(V_1, \alpha_1) \cup (V_2, \alpha_2)$, where V_i is a solid torus and α_i is an unknotted arc properly embedded in V_i, $i = 1, 2$. The following is due to Makoto Sakuma.

Proposition 5. *For a nontrivial 1-knot K with $(1,1)$-decomposition, we have* $u_w(K^*) = 1$.

Proof. From the (1,1)-decomposition of K, the group πK, which is isomorphic to πK^*, is generated by two elements x, a, where x is a meridian. If h is a 1-handle on K^* corresponding to the element a, then the group of the embedded torus $K^* + h$ is given by $\pi K^* / \langle [a, x] \rangle$, which is abelian. The proof is complete. □

A torus knot has a (1,1)-decomposition, cf. [19], and so $u_w(T^*_{p,q}) = 1$. In fact, let

$$\langle a, b | a^p = b^q \rangle$$

be a presentation of $\pi T^*_{p,q}$. Then the element $a^r b^s$ describes a meridian, where $ps + qr = 1$; cf. [3, Proposition 3.28]. The torus obtained by adding a 1-handle on $T^*_{p,q}$ corresponding to the element a has the group given by

$$\pi T^*_{p,q} / \langle [a, a^r b^s] \rangle.$$

The new relation $[a, a^r b^s] = 1$ implies $[a, b^s] = 1$. Thus we have $[a, b] = [a, b^{ps+qr}] = [a, (b^s)^p (b^q)^r] = [a, (b^s)^p (a^p)^r] = 1$.

REMARK. We note that a nontrivial 1-knot with (1,1)-decomposition is a tunnel number one knot, which is known to be prime [20, 23]. Thus the example (1) of Miyazaki shows that the converse of Proposition 5 does not hold.

Question. Does it hold that $u_w(K) = u(K)$ for every 2-knot K? In particular, does it hold that $u(K^*) = 1$ for every nontrivial 1-knot K with $(1,1)$-decomposition?

5. Proof of Theorem 2

The spun torus knot $T^*_{p,p+1}$ has the Alexander polynomial

$$\Delta_{p,p+1}(t) = \frac{(t^{p(p+1)} - 1)(t-1)}{(t^p - 1)(t^{p+1} - 1)}.$$

By [11], there exists a ribbon 2-knot J with $f(J) = 1$ (and thus $u(J) = 1$) having the Alexander polynomial $\Delta_J(t) = \Delta_{2,3}(t)\Delta_{p,p+1}(t)$. We will show

$$u(T^*_{p,p+1}\#qJ) = q+1, \qquad f(T^*_{p,p+1}\#qJ) = p+q-1,$$

where $p \geq 2$ and $q \geq 0$.

From Theorem 1, using (5) and (6), we have $u(T^*_{p,p+1}\#qJ) \leq q+1$ and $f(T^*_{p,p+1}\#qJ) \leq p+q-1$. The Alexander invariant of $T^*_{p,p+1}\#qJ$ is

$$\Lambda/(\Delta_{p,p+1}) \oplus \underbrace{\Lambda/(\Delta_{2,3}\Delta_{p,p+1}) \oplus \ldots \oplus \Lambda/(\Delta_{2,3}\Delta_{p,p+1})}_{q},$$

and so using (3) and (4), we have $u(T^*_{p,p+1}\#qJ) \geq e(T^*_{p,p+1}\#qJ) = q+1$.

There is an epimorphism

$$\theta : \pi T^*_{p,p+1} \to \mathfrak{S}_{p+1}$$

onto the symmetric group of degree $p+1$ defined by $\theta(x_i) = (i, i+1)$ $(1 \leq i \leq p)$ and $\theta(x_{p+1}) = (p+1, 1)$, where x_i is a meridian of the overpass α_i of the regular projection of $T_{p,p+1}$ given in the proof of Lemma 3 (Fig. 1).

Let

$$\langle x, y | x = w^{-1}yw \rangle$$

be a Wirtinger presentation of πJ, where $w = w(x,y)$ is a word in x and y. Then the quotient group $\pi J/\langle x^2 = y^2 = 1\rangle$ is the dihedral group of order $2r$:

$$\langle x, y | x^2 = y^2 = (xy)^r = 1 \rangle,$$

where

$$r = \Delta_J(-1) = 3\Delta_{p,p+1}(-1) = \begin{cases} 3p & \text{if } p \text{ is odd,} \\ 3(p+1) & \text{if } p \text{ is even,} \end{cases}$$

cf. [3, p.243]. Thus there is an epimorphism $\pi J \to \mathfrak{S}_3$ defined by $x \mapsto (12)$ and $y \mapsto (13)$.

Let

$$\pi(qJ) = \langle x, y_1, y_2, \ldots, y_q | x = w(x, y_k)^{-1} y_k w(x, y_k)(1 \leq k \leq q)\rangle.$$

Extending the above epimorphisms, we have an epimorphism

$$\tilde{\theta} : \pi(T^*_{p,p+1} \# qJ) = \pi T^*_{p,p+1} \underset{x_1 = x}{*} \pi(qJ) \to \mathfrak{S}_{p+q+1}$$

defined by $\tilde{\theta}|\pi T^*_{p,p+1} = \theta$ and $\tilde{\theta}(y_k) = (1, p+k+1)$. Since we need at least $p+q$ transpositions to generate $\mathfrak{S}_{p+q+1}$, we have $f(T^*_{p,p+1} \# qJ) \geq p+q-1$; cf. [12, p.460]. This completes the proof of Theorem 2. □

6. Branched cyclic cover of a ribbon 2-knot

Let K be an n-knot in S^{n+2}. We denote by $B_m(K)$ the m-fold branched cyclic covering space of S^{n+2} branched over K. If K is a 1-knot with b bridges, then the Heegaard genus of $B_m(K)$ is less than or equal to $(m-1)(b-1)$, and so $\mathrm{rank}(\pi_1(B_m(K))) \leq (m-1)(b-1)$; see [3, Proposition 11.4; 21, 10C3]. Similarly we have

Proposition 6. *If K is a ribbon 2-knot, then*

$$\mathrm{rank}(\pi_1(B_m(K))) \leq (m-1)f(K).$$

Proof. Suppose that $f(K) = n$. By sliding 1-handles, πK has a Wirtinger presentation

$$\langle x (= x_0), x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n\rangle,$$

where $r_i = x^{-1} w_i^{-1} x_i w_i$ with w_i a word in x, x_1, $x_2, \ldots, x_n$. Let $M_m(K)$ be the m-fold cyclic covering space of $S^4 - K$ corresponding to the kernel of the map $\pi K \to \mathbf{Z}_m = \langle t | t^m\rangle$ defined by $x_i \mapsto t$. We denote by G_m the fundamental group $\pi_1(M_m(K))$, which is isomorphic to the kernel of this map. We find a set of generators for G_m using the Reidemeister-Schreier method; cf. [13, Sect. 2.3].

From the right coset decompositions $\pi K = G_m \cup G_m x \cup G_m x^2 \cup \ldots \cup G_m x^{m-1}$, we may take $U = \{1, x, x^2, \ldots, x^{m-1}\}$ as a Schreier system for G_m in πK. Then G_m is generated by

$$\{t x_i \overline{t x_i}^{-1} | t \in U, 0 \leq i \leq n\},$$

where $g \mapsto \overline{g}$ is the map defined by $G_m g \cap U = \{\overline{g}\}$. Since $x^j x_i \overline{x^j x_i}^{-1} = x^j x_i x^{-j-1}$, putting

$$a_{ij} = \begin{cases} x_i x^{-1} & \text{if } \ j = 0; \\ x^j x_i x^{-j-1} & \text{if } \ 1 \leq j \leq m-2; \\ x^{m-1} x_i & \text{if } \ j = m-1, \end{cases}$$

we see that G_m is generated by

$$\{a_{ij} | 1 \leq i \leq n, 0 \leq j \leq m-1\} \cup \{x^m\}.$$

Now we consider $\pi_1(B_m(K))$, which is given by $G_m/\langle x^m = 1 \rangle$. Since $x^m = w_i^{-1} x_i^m w_i$ in πK, $a_{i0} a_{i1} \ldots a_{i,m-1} = x_i^m = 1$ in $\pi_1(B_m(K))$. Thus $\pi_1(B_m(K))$ is generated by $\{a_{ij} | 1 \leq i \leq n, 1 \leq j \leq m-1\}$, completing the proof. □

See [18, 26] for the geometric interpretation of $B_2(K)$.

Using Proposition 6, we can prove the following, which is a generalization of (1).

Proposition 7.

$$u(kT_{2,3}^* \# lT_{2,5}^*) = \max\{k, l\}, \quad f(kT_{2,3}^* \# lT_{2,5}^*) = k + l.$$

Proof. We prove for $k \leq l$. From (5), we have $u(kT_{2,3}^* \# lT_{2,5}^*) \leq ku(T_{2,3}^* \# T_{2,5}^*)$ $+(l-k)u(T_{2,5}^*)$. From (1) and Theorem 1, $u(T_{2,3}^* \# T_{2,5}^*) = u(T_{2,5}^*) = 1$, and thus we obtain $u(kT_{2,3}^* \# lT_{2,5}^*) \leq l$.

On the other hand, the Alexander invariant of $kT_{2,3}^* \# lT_{2,5}^*$ is

$$\underbrace{\Lambda/(\Delta_{2,3}\Delta_{2,5}) \oplus \ldots \oplus \Lambda/(\Delta_{2,3}\Delta_{2,5})}_{k} \oplus \underbrace{\Lambda/(\Delta_{2,5}) \oplus \ldots \oplus \Lambda/(\Delta_{2,5})}_{l-k},$$

where $\Delta_{2,p} = (t^p + 1)/(t + 1)$, and so $e(kT_{2,3}^* \# lT_{2,5}^*) = l$. Therefore, by (3) and (4), we have $u(kT_{2,3}^* \# lT_{2,5}^*) = l$.

By Theorem 1 and (6), we have $f(kT_{2,3}^* \# lT_{2,5}^*) \leq k + l$. On the other hand, from $\pi_1(B_2(T_{2,p}^*)) = \pi_1(B_2(T_{2,p})) = \mathbf{Z}_p$ with $p(> 0)$ odd integer, we have

$$\pi_1(B_2(kT_{2,3}^* \# lT_{2,5}^*)) = \underbrace{\mathbf{Z}_3 * \ldots * \mathbf{Z}_3}_{k} * \underbrace{\mathbf{Z}_5 * \ldots * \mathbf{Z}_5}_{l}.$$

The rank of this group is $k + l$ [13, p.192, Corollary], and so by Proposition 6, we have $f(kT_{2,3}^* \# lT_{2,5}^*) \geq k + l$. This completes the proof. □

7. Satellite knot of a ribbon 2-knot

Let $R^3[t] = R^3 \times \{t\}$. Fig. 2 gives the *motion pictures* [4] of the spun trefoil $T^*_{2,3}$ in $R^4(\subset S^4)$, where D^2 is a disk such that $(T^*_{2,3} \cap R^3[0]) \cup D^2 = (T^*_{2,3} \cap R^3[2]) \cup D^2 = (T^*_{2,3} \cap R^3[1])$. More precisely,

$$T^*_{2,3} \cap R^3[t] = \begin{cases} T^*_{2,3} \cap R^3[0] & \text{for} \quad -1 < t < 1, \\ T^*_{2,3} \cap R^3[1] & \text{for} \quad t = \pm 1, \\ T^*_{2,3} \cap R^3[2] & \text{for} \quad 1 < |t| < 3, \\ D_1^2 \cup D_2^2 & \text{for} \quad t = \pm 3, \\ \emptyset & \text{for} \quad |t| > 3, \end{cases}$$

where D_1^2 and D_2^2 are disjoint disks bounding the trivial 2-component link $T^*_{2,3} \cap R^3[2]$.

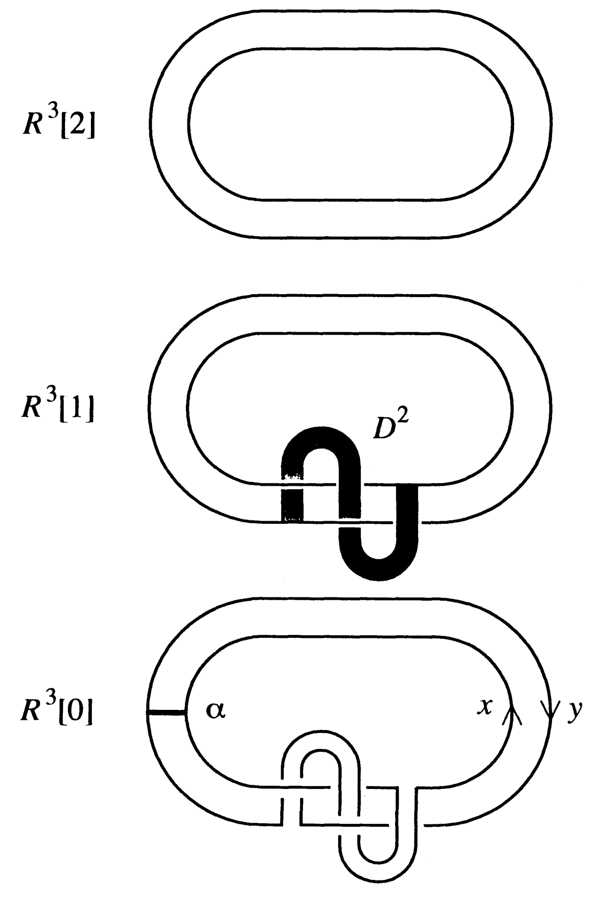

Fig. 2.

The group $\pi T^*_{2,3}$ has a presentation

$$\langle x, y | xyx = yxy \rangle.$$

Let V be a tubular neighborhood of the simple closed curve representing the element $y^{-1}x$ in the exterior of $T^*_{2,3}$. Let J be a 2-knot and $N(J)$ its tubular neighborhood. Then there is a homeomorphism $\xi : S^4 - \mathrm{int}V \to N(J)$. The 2-knot $\xi(K)$ is called the *satellite knot* of $(T^*_{2,3}, y^{-1}x)$ about J, which we denote by $\Sigma(J)$. See [8]. By the van Kampen theorem, the group $\pi\Sigma(J)$ is the free product of πJ and $\pi T^*_{2,3}$, amalgamating the subgroup generated by a meridian of πJ and the subgroup generated by $y^{-1}x$, both of which are infinite cyclic groups. Thus it contains subgroups isomorphic to $\pi T^*_{2,3}$ and πJ. The Alexander invariant of $\Sigma(J)$ is $\Lambda/(t^2 - t + 1)$; cf. [25].

When J is a spun trefoil, $\Sigma(J)$ may be presented as in Fig. 3:

$$\Sigma(J) \cap R^3[t] = \begin{cases} \Sigma(J) \cap R^3[0] & \text{for } -1 < t < 1, \\ \Sigma(J) \cap R^3[1] & \text{for } t = \pm 1, \\ \Sigma(J) \cap R^3[2] & \text{for } 1 < |t| < 3, \\ D_1^2 \cup D_2^2 \cup D_3^2 \cup D_4^2 & \text{for } t = \pm 3, \\ \emptyset & \text{for } |t| > 3, \end{cases}$$

where D_i^2, $i = 1, 2, 3, 4$, are disjoint disks bounding the trivial 4-component link $\Sigma(J) \cap R^3[2]$.

$\Sigma(J)$ is also regarded as a 2-knot constructed by attaching a 1-handle to a parallel link [10] of two components associated with J. The following is easy to see.

Lemma 4. *If J is a ribbon 2-knot, then $\Sigma(J)$ is also a ribbon 2-knot and $f(\Sigma(J)) \leq 2f(J) + 1$.*

Adding a 1-handle on $T^*_{2,3}$ whose core is α in Fig. 2, we obtain an unknotted surface. Similarly, adding a 1-handle on $\Sigma(J)$ whose core is β in Fig. 3, we obtain an unknotted surface. Note that this 1-handle corresponds to $y^{-1}x \in \pi T^*_{2,3} \subset \pi\Sigma(J)$. More generally, we have:

Lemma 5. *For any 2-knot J, $u(\Sigma(J)) = 1$.*

Now we consider $\Sigma(kT^*_{2,5})$, the satellite knot whose companion is the composition of k copies of the spun 2-knot of the $(2,5)$-torus knot. Let $K(k,l)$ be the composition of $\Sigma(kT^*_{2,5})$ and the composition of l copies of the spun trefoil: $K(k,l) = \Sigma(kT^*_{2,5}) \# lT^*_{2,3}$, $k, l \geq 0$. Then we have:

$R^3[2]$

$R^3[1]$

$R^3[0]$

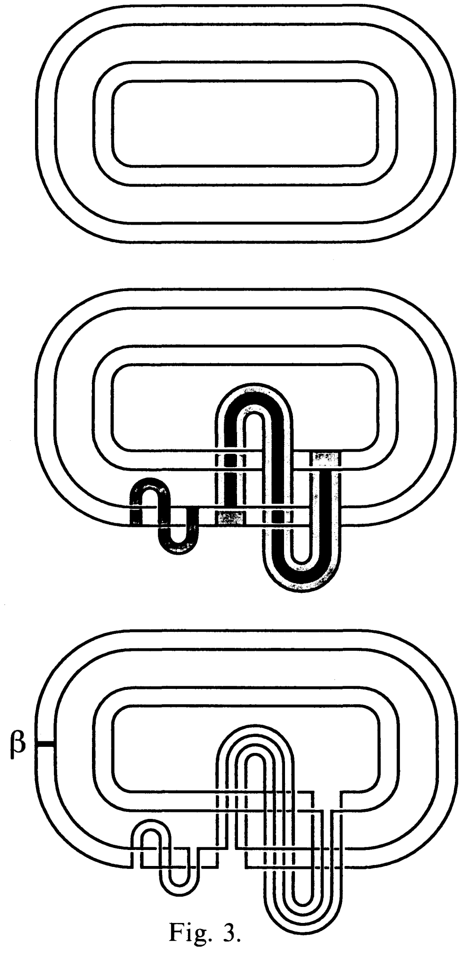

Fig. 3.

Proposition 8.

$$u(K(k,l)) = l+1, \quad f(K(k,l)) = 2k+l+1.$$

Proof. Since $u(\Sigma(kT^*_{2,5})) = u(T^*_{2,3}) = 1$, we have $u(K(k,l)) \leq l+1$ by (5). On the other hand, the Alexander invariant of $K(k,l)$ is the sum of the $l+1$ copies of $\Lambda/(t^2-t+1)$, and so $e(K(k,l)) = l+1$. Therefore, from (3) and (4), we have $u(K(k,l)) = l+1$.

Next we consider the fusion number. By Lemma 5 and (6), we have $f(K(k,l)) \leq 2k+l+1$. Using the presentation

$$\langle a, b_1, b_2, \ldots, b_k | ab_iab_ia = b_iab_iab_i \ (1 \leq i \leq k) \rangle$$

of $\pi(kT^*_{2,5})$, we have a presentation of $\pi\Sigma(kT^*_{2,5})$:

$$\begin{aligned}\langle x, y, a, b_1, b_2, \dots, b_k, \mid\ & xyx = yxy, a = y^{-1}x,\\ & ab_iab_ia = b_iab_iab_i\ (1 \le i \le k)\rangle.\end{aligned}$$

Thus $\pi K(k,l)$ has a presentation

$$\begin{aligned}\langle x, y, z_1, z_2, \dots, z_l, a, b_1, b_2, \dots, b_k \mid\ & xyx = yxy,\\ & xz_jx = z_jxz_j\ (1 \le j \le l), a = y^{-1}x,\\ & ab_iab_ia = b_iab_iab_i\ (1 \le i \le k)\rangle.\end{aligned}$$

There is a homomorphism from $\pi K(k,l)$ to $\mathfrak{S}_{2k+l+3}$, defined by

$$\begin{gathered} x \mapsto (12), \quad y \mapsto (23), \quad z_j \mapsto (2, 2k+j+3)\ (1 \le j \le l),\\ a \mapsto (123), \quad b_i \mapsto (3, 2i+2, 2i+3)\ (1 \le i \le k).\end{gathered}$$

Then we have $b_iyb_i^{-1} \mapsto (2, 2i+3)$ and $b_i^{-1}yb_i \mapsto (2, 2i+4)$, and so this homomorphism is surjective. Since we need at least $2k+l+2$ transpositions to generate $\mathfrak{S}_{2k+l+3}$, $f(K(k,l)) \ge 2k+l+1$. The proof is complete. □

Clearly many more examples can be constructed using satellite knots.

REMARK. The results of this paper may be generalized to higher dimensions.

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