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<th>Unknotted and fusion numbers of ribbon 2-knots</th>
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Introduction

Hosokawa and Kawauchi [6] proved that any 2-knot can give an unknotted surface after adding enough 1-handles in an appropriate manner. Hosokawa, Maeda, and Suzuki [7] then defined the unknotting number $u(K)$ of a 2-knot $K$ to be the least number of such 1-handles.

A ribbon 2-knot is obtained from a trivial $(n + 1)$-component 2-link by adding $n$ 1-handles for some $n$. The fusion number $f(K)$ of a ribbon 2-knot $K$ is the least number of $n$ possible for $K$.

Let $K$ be a ribbon 2-knot. Then Miyazaki [17, Lemma 1] proved:

**Proposition 1.**

$$u(K) \leq f(K).$$

Note that the fusion number is called the ribbon number in [17]. Let $K^*$ denote the spun 2-knot of a 1-knot $K$. Let $T_{p,q}$ denote the torus knot of type $(p,q)$. Miyazaki [17, Claim, Remark 2] also showed:

$$u(T_{2,2m-1}^* \# T_{2,2m+1}^*) = 1, \quad f(T_{2,2m-1}^* \# T_{2,2m+1}^*) = 2,$$

if $m \geq 2$. The purpose of this paper is to provide more examples of ribbon 2-knots that do not satisfy the equality in Proposition 1. First, we will prove:

**Theorem 1.** For a nontrivial spun torus knot $T_{p,q}^*$ with $1 < p < q$, we have

$$u(T_{p,q}^*) = 1, \quad f(T_{p,q}^*) = p - 1.$$ 

Using the composition of a spun torus knot and some copies of a ribbon 2-knot with fusion number one, we will show:

**Theorem 2.** For any integers $m$ and $n$ with $0 < m \leq n$, there exists a ribbon 2-knot $K$ such that $u(K) = m$ and $f(K) = n$. 
Sections 2 and 3 are devoted to prove $u(T_{p,q}^*) = 1$. We give two other characterizations of a ribbon 2-knot by making use of a handle decomposition of a cobordism between a ribbon 2-knot and a trivial 2-knot, from which we define the handle and dual-handle numbers of a ribbon 2-knot. These invariants coincide (Proposition 3), and are greater than or equal to the unknotting number (Theorem 3) and less than or equal to the fusion number (Theorem 4). We define a class of tubed 2-knots, including all spun torus knots (Lemma 3), and show that a tubed 2-knot has dual-handle number one (Theorem 5). From the fact that the torus knot has a $(1,1)$-decomposition, we see that the spun 2-knot of a 1-knot with $(1,1)$-decomposition has weak unknotting number one (Proposition 5). In Sections 4 and 5, we prove Theorems 1 and 2.

The fusion number plus one is an analogy of the bridge number for a 1-knot, both of which are greater than or equal to the Wirtinger rank of the knot group. In Section 6, we give a relation between the fusion number and the rank of the fundamental group of the finite cyclic branched covering space of a ribbon 2-knot (Proposition 6), which is also an analogy for the case of the bridge number. We use this to generalize the example (1) of Miyazaki (Proposition 7).

In Section 7, we give another example for the inequality in Proposition 1 using a satellite knot of a ribbon 2-knot.

1. Preliminaries

The boundary and the interior of a manifold $M$ are denoted by $\partial M$ and $\text{int} M$. Both $D^n$ and $B^n$ denote the $n$-ball, and $R^n$ and $S^n$ denote the Euclidean $n$-space and $n$-sphere, respectively. $I$ is the unit interval $[0, 1]$.

Let $K$ be an $n$-knot. We denote by $\pi K$ the group of $K$, i.e., $\pi K = \pi_1(S^{n+2} - K)$. For a non-negative integer $n$, $nK$ means the composition of $n$ copies of $K$; if $n = 0$, the knot is a trivial one.

The Wirtinger rank $\text{W-rank}(\pi K)$ of a knot group $\pi K$ is the least number of meridional elements of $K$ we need to generate $\pi K$. The rank $\text{rank} G$ of a group $G$ is the least number of generators of $G$.

Let $L$ be a 2-link in $S^4$. A 1-handle $h$ on $L$ is an embedding $h : I \times D^2 \to S^4$ with $h(I \times D^2) \cap L = h(\partial I \times D^2)$. The arc $h(I \setminus \{0\})$ is called the core of $h$, where $O$ is the center of $D^2$. We define the surface $L + h$ obtained from $L$ by adding a 1-handle $h$ to be the surface $L + h = (L - h(\partial I \times D^2)) \cup h(I \times \partial D^2)$. We define a ribbon 2-knot $K$ to be a 2-knot obtained by adding $n$ 1-handles $\{h_i|1 \leq i \leq n\}$ to a trivial 2-link with $n + 1$ components $\bigcup_{i=0}^{n} S^2_i$ for some $n$,

$$K = \bigcup_{i=0}^{n} S^2_i + \sum_{i=1}^{n} h_i.$$  

See [28], where such a 2-knot is called a fusion of a trivial 2-link. If the $i$th 1-handle
h_i connects $S^2_{i(1)}$ with $S^2_{i(2)}$, then the group $\pi K$ has a Wirtinger presentation

$$\langle x_0, x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_n \rangle,$$

where $x_i$ is a meridian of $S^2_i$ and $r_i = x_{i(1)}^{-1}w_i^{-1}x_{i(2)}w_i$ with $w_i$ a word in $x_0, x_1, \ldots, x_n$ determined by the core of $h_i$; see [27]. Then we have

$$f(K) \geq \text{W-rank}(\pi K) - 1.$$  

Let $K$ be a 2-knot. For any element $g \in \pi K$, there exists a 1-handle on $K$ whose core represents $g$ [2]. So there exist $g_1, g_2, \ldots, g_n \in \pi K$ such that

$$\pi K/\langle [g_1, x], [g_2, x], \ldots, [g_n, x] \rangle$$

is the infinite cyclic group, where $x$ is a meridian and $[g_i, x] = g_ixg_i^{-1}x^{-1}$. We define the weak unknotting number $u_w(K)$ of $K$ to be the least number of $n$ possible for $K$; see [9]. Then

$$u(K) \geq u_w(K).$$

Let $\tilde{X}_K$ be the infinite cyclic covering space of $S^4 - K$. Then the Alexander invariant of $K$, $H_1(\tilde{X}_K)$, is a finitely generated $\Lambda$-module, where $\Lambda$ is the polynomial ring $\mathbb{Z}[t, t^{-1}]$ [21, Chapter 7]. We define $e(K)$ to be the least number of generators of $H_1(\tilde{X}_K)$ as a $\Lambda$-module. Then we have the following [9, Proposition 2]:

$$u_w(K) \geq e(K).$$

Concerning the composition of 2-knots, the following is easy to see:

$$u(K_1 \# K_2) \leq u(K_1) + u(K_2)$$

$$f(K_1 \# K_2) \leq f(K_1) + f(K_2).$$

2. Handle number of a ribbon 2-knot

Let $(D^5_0, B^3_0)$ be a standard ball pair. We attach $n$ 1-handles $\{H^1_i | 1 \leq i \leq n\}$ to $D^5_0 - D^3_0$. Let $V = D^5_0 \cup \bigcup_{i=1}^n H^1_i$. The fundamental group $\pi_1(V - D^3_0)$ is free on $n + 1$ generators $x, y_1, y_2, \ldots, y_n$, where $x$ is a meridian of $D^3_0$ and $y_i, 1 \leq i \leq n$, is represented by a loop running through the $i$th 1-handle $H^1_i$ just once and missing $H^1_j$ for $j \neq i$. Let $v_i = v_i(x, y_1, y_2, \ldots, y_n)$ be an element of $\pi_1(V - D^3_0)$ which becomes $y_i$ if we kill the element $x$:

$$v_i(1, y_1, y_2, \ldots, y_n) = y_i.$$
We attach 2-handles $H_i^2$ to $V - D^3_0$ along embedded simple closed curves $\alpha_i$ representing $v_i$ in the boundary. Then $V \cup \bigcup_{i=1}^{n} H_i^2$ is a 5-ball and the sphere pair

$$\partial(V \cup \bigcup_{i=1}^{n} H_i^2), \partial D^3_0)$$

is a ribbon 2-knot. Conversely, any ribbon 2-knot is constructed in this manner. See [1, 5, 15].

The following is shown in [5, Theorem III.5.1; 14, Lemma 2.2].

**Lemma 1.** In the above, we can take the attaching spheres $\alpha_i$ of the 2-handles $H_i^2$ so that the exponent sum of $v_i$ on $x$ is zero for each $i$.

We define the **handle number** $\varphi(K)$ of a ribbon 2-knot $K$ to be the least number of 1-handles for which the above construction yields $K$. The group of the ribbon 2-knot (8) is presented by

$$\langle x, y_1, y_2, \ldots, y_n|v_1, v_2, \ldots, v_n \rangle,$$

and so we have

**Proposition 2.**

$$\text{rank}(\pi K) - 1 \leq \varphi(K).$$

**Theorem 3.**

$$u(K) \leq \varphi(K).$$

**Proof.** Suppose that $K$ is the 2-knot with $\varphi(K) = n$ described as (8). Let $v_i = u_1 x^{e_1} u_2 x^{e_2} \cdots u_k x^{e_k} u_{k+1}$, where $u_1, u_2, \ldots, u_{k+1}$ are words in $y_1, y_2, \ldots, y_n$. By Lemma 1, we may assume that $\sum_{j=1}^{k} e_j = 0$.

Let $h_i$ be a 1-handle on $\partial D^3_0$ corresponding to $y_i$ and $T_0 = \partial D^3_0 + \sum_{i=1}^{n} h_i$, a surface of genus $n$. Then the fundamental group $\pi_1(\partial V - T_0)$ is given by

$$\langle x, y_1, y_2, \ldots, y_n|[x, y_i]|(1 \leq i \leq n) \rangle.$$ 

In this group, $v_i$ becomes $u_1 u_2 \cdots u_{k+1}$, which is $y_i$ by (7). Thus we can move isotopically the attaching spheres $\alpha_i$ to the standard ones. Therefore the surface $T_0$ is unknotted in the 4-sphere $\partial(V \cup \bigcup_{i=1}^{n} H_i^2)$. \qed
3. Dual-handle number of a ribbon 2-knot

Let $K$ be a ribbon 2-knot given by (8). Attaching a 5-handle $D^5_i$, we obtain a handle decomposition of a 5-sphere:

$$S^5 = D^5_0 \cup \bigcup_{i=1}^{n} H^1_i \cup \bigcup_{i=1}^{n} H^2_i \cup D^5.$$

Turning this handle decomposition of $S^5$ upside down, we have

$$S^5 = D^5_1 \cup \bigcup_{i=1}^{n} \overline{H}_i^3 \cup \bigcup_{i=1}^{n} \overline{H}_i^4 \cup D^5,$$

where $\overline{H}_i^j$ is the handle of index $j$ naturally obtained from $H_i^{5-j}$ for $j = 3, 4$. Then $K$ is on the boundary of $D^5_1$ and the 3-handles $\{\overline{H}_i^3\}$ are trivial. From this, we obtain the following [15, Corollary 1.10.1]; see [15, Theorem 1.10] for the detailed proof.

**Lemma 2.** A 2-knot $K$ in $S^4 = \partial D^5$ is a ribbon 2-knot if and only if there exist 3-handles $\{H_i^3\}_{1 \leq i \leq n}$ on $D^5$ with the following properties.

(i) The set of attaching spheres for the 3-handles forms a trivial 2-link in $S^4$ that does not intersect $K$.

(ii) $K$ bounds a 3-ball in the boundary of the resulting manifold $D^5 \cup \bigcup_{i=1}^{n} H_i^3$.

We define the dual-handle number $\varphi^*(K)$ of a ribbon 2-knot $K$ to be the least number of the 3-handles for $K$ as in Lemma 2. From the proof of Lemma 2, we have:

**Proposition 3.**

$$\varphi^*(K) = \varphi(K).$$

**Theorem 4.**

$$\varphi^*(K) \leq f(K).$$

**Proof.** Suppose that $K$ is a ribbon 2-knot in $S^4(= \partial D^5)$ obtained by adding $n$ 1-handles $\{h_i\}_{1 \leq i \leq n}$ to a trivial 2-link with $n + 1$ components $\bigcup_{i=1}^{n} S^2_j$. We take a 2-sphere $\sum_{j=1}^{n} S^2_j$ parallel to $S^2_j$ for each $j$, $1 \leq j \leq n$, so that $\bigcup_{j=1}^{n} \sum_{j=1}^{2}$ forms a trivial link in $S^4$ and does not intersect the knot $K$. Let $W$ be the manifold obtained from $D^5$ by adding trivial 3-handles along the 2-spheres $\bigcup_{j=1}^{n} \sum_{j=1}^{2}$. Then each $S^2_j$, $1 \leq j \leq n$, bounds a 3-ball in $\partial W$ that intersects $K$ only at the 1-handles. By using these 3-balls, each $S^2_j$ may be moved toward $S^2_0$ along 1-handles, and the
knot $K$ becomes trivial in $\partial W$.

Let $K$ be a ribbon 2-knot. Then $K$ is called a **tubed** 2-knot if there exists $S^1 \times B^3$ that is embedded in $S^4$ such that $K$ is obtained from a trivial link $\bigcup_{i=0}^{n}{x_i} \times \partial B^3$, $x_i \in S^1$, by adding 1-handles $\{h_j | 1 \leq j \leq n\}$ in $S^1 \times B^3$ for some $n$.

**Theorem 5.** *If $K$ is a tubed 2-knot, then $\varphi(K) \leq 1$.*

**Proof.** Let $K$ be a tubed 2-knot as above. Assume that $S^4 = \partial D^5$, and put $S^2_1 = \{x_i\} \times \partial B^3$. Let $W$ be the manifold obtained from $D^5$ by adding trivial 3-handles along the 2-sphere in $S^4 - S^1 \times B^3$ that is parallel to $S^2_1$. There is a component, say $S^2_2$, to which the only one 1-handle, say $h_n$, is added; we suppose $S^2_2 \cap \bigcup_{j=1}^{n-1} h_j (I \times D^2) = h_n (\{0\} \times D^2)$. Then $S^2_2$ bounds a 3-ball $B^3_0$ in $\partial W$ such that $K \cap B^3_0 = S^2_2 - \text{int} h_n (\{0\} \times D^2)$. Then there is an isotopy of $\partial W$ carrying $K$ to $\bigcup_{i=0}^{n-1} S_i + \sum_{j=1}^{n-1} h_j$. Repeating this argument inductively, we finally obtain a trivial knot, which is $K$ on the boundary of $D^5$ with suitable number of 3-handles added.

### 4. Spun torus knots

Let $R^3_+$ and $R^3_-$ denote the upper-half space $\{(x, y, z) | z \geq 0\}$ and lower-half space $\{(x, y, z) | z \leq 0\}$, respectively. We define the *spin* $\sigma X(\subset R^4)$ of any set $X$ in $R^3_+$ to be

$$
\sigma X = \{(x, y, z \cos \theta, z \sin \theta) | (x, y, z) \in X, 0 \leq \theta \leq 2\pi\}.
$$

Let $K$ be a 1-knot in $R^3$ such that $K \cap R^3_+$ is a properly embedded unknotted arc. Then the spun 2-knot $K^*$ of $K$ is given as the spin of $K \cap R^3_+$; see [21, p.85].

**Lemma 3.** *A spun torus knot is a tubed 2-knot.*

**Proof.** For the torus knot $T_{p,q}$, there is a regular projection $\psi: R^3 \rightarrow R^2$ with $q$ overpasses $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $q$ underpasses $\beta_1, \beta_2, \ldots, \beta_q$ such that

(i) $\psi(T_{p,q})$ is in an annulus $A = S^1 \times I$ in $R^2$, $S^1 = I/(0 \sim 1)$,

(ii) $\psi(\alpha_i) = \{i/q\} \times I$,

(iii) $\psi(\beta_i)$ is a properly embedded arc in $A$.

Fig. 1 gives such a regular projection for $T_{4,5}$.

We denote the unions $\bigcup_{i=1}^{q} \alpha_i$ and $\bigcup_{i=1}^{q} \beta_i$ by $\alpha$ and $\beta$, respectively. Then $T_{p,q}$ is isotopic to the following knot in $A \times [0,2]$ having the same diagram $\psi(T_{p,q})$:

$$
\{(x,2) | x \in \psi(\alpha)\} \cup \{(x,1) | x \in \psi(\beta)\} \cup \{(x,t) | x \in \psi(\partial \alpha) = \psi(\partial \beta), 1 \leq t \leq 2\},
$$
and thus the spun torus knot $T^*_{p,q}$ is the spin of the following arc in $R^3_+$:

$$\{(x, 2)|x \in \psi(\alpha)\} \cup \{(x, 1)|x \in \psi(\beta_1 \cup \ldots \cup \beta_{q-1})\}$$
$$\cup \{(x, t)|x \in \psi(\partial \beta_1 \cup \ldots \cup \partial \beta_{q-1}), 1 \leq t \leq 2\}$$
$$\cup \{(x, t)|x \in \psi(\partial \beta_q), 0 \leq t \leq 2\}.$$  

We denote by $\tilde{\alpha}_i$ the unknotted arc in $R^3_+$ defined by

$$\tilde{\alpha}_i = \{(x, 2)|x \in \psi(\alpha_i)\} \cup \{(x, t)|x \in \psi(\partial \alpha_i), 0 \leq t \leq 2\}.$$  

Then the spin of $\bigcup_{i=1}^q \tilde{\alpha}_i$ forms a trivial 2-link with $q$ components. We see that the spun torus knot $T^*_{p,q}$ is a ribbon 2-knot obtained from this trivial link by adding 1-handles $h_j$, $1 \leq j \leq q - 1$, whose core is $\sigma\{(x, 0)|x \in \psi(\beta_j)\}$; cf. [16]. This is a tubed 2-knot since the spin of the solid torus $A \times [0, 2]$ is homeomorphic to $S^1 \times B^3$.

Proof of Theorem 1. By Theorems 3, 5 and Lemma 3, we have $u(T^*_{p,q}) \leq 1$. Since $T^*_{p,q}$ is nontrivial, we have $u(T^*_{p,q}) = 1$.

From the proof of Lemma 3, we have $f(T^*_{p,q}) \leq p - 1$. From [22], we have $\text{W-rank}(\pi T_{p,q}) \geq p - 1$, which implies $f(T^*_{p,q}) \geq p - 1$ by (2). Thus we have $f(T^*_{p,q}) = p - 1$.  

\[\square\]
The spun 2-knot of a 1-knot is a ribbon 2-knot. Let $b(K)$ denote the bridge number of a 1-knot $K$. Then from the proof of Lemma 3, we see:

**Proposition 4.** For any 1-knot $K$,

$$f(K^*) \leq b(K) - 1.$$ 

**Questions.** (i) Does the equality in Proposition 4 always hold? Note also that $b(K) \geq W$-rank$(\pi K)$; cf. (2).

(ii) It is known [24] that $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$. How about the fusion number? That is, does the equality of (6) always hold?

A 1-knot $K$ in $S^3$ has a (1,1)-decomposition if the pair $(S^3, K)$ has a decomposition into a union $(V_1, \alpha_1) \cup (V_2, \alpha_2)$, where $V_i$ is a solid torus and $\alpha_i$ is an unknotted arc properly embedded in $V_i$, $i = 1, 2$. The following is due to Makoto Sakuma.

**Proposition 5.** For a nontrivial 1-knot $K$ with (1,1)-decomposition, we have $u_w(K^*) = 1$.

**Proof.** From the (1,1)-decomposition of $K$, the group $\pi K$, which is isomorphic to $\pi K^*$, is generated by two elements $x, a$, where $x$ is a meridian. If $h$ is a 1-handle on $K^*$ corresponding to the element $a$, then the group of the embedded torus $K^* + h$ is given by $\pi K^*/\langle [a, x] \rangle$, which is abelian. The proof is complete.

A torus knot has a (1,1)-decomposition, cf. [19], and so $u_w(T_{p,q}^*) = 1$. In fact, let

$$\langle a, b | a^p = b^q \rangle$$

be a presentation of $\pi T_{p,q}^*$. Then the element $a^rb^s$ describes a meridian, where $ps + qr = 1$; cf. [3, Proposition 3.28]. The torus obtained by adding a 1-handle on $T_{p,q}^*$ corresponding to the element $a$ has the group given by

$$\pi T_{p,q}^*/\langle [a, a^rb^s] \rangle.$$ 

The new relation $[a, a^rb^s] = 1$ implies $[a, b^s] = 1$. Thus we have $[a, b] = [a, b^{ps+qr}] = [a, (b^s)^p(b^r)^r] = [a, (b^s)^p(a^p)^r] = 1$.

**Remark.** We note that a nontrivial 1-knot with (1,1)-decomposition is a tunnel number one knot, which is known to be prime [20, 23]. Thus the example (1) of Miyazaki shows that the converse of Proposition 5 does not hold.
Question. Does it hold that \( u_w(K) = u(K) \) for every 2-knot \( K \)? In particular, does it hold that \( u(K^*) = 1 \) for every nontrivial 1-knot \( K \) with \((1, 1)\)-decomposition?

5. Proof of Theorem 2

The spun torus knot \( T_{p,p+1}^* \) has the Alexander polynomial

\[
\Delta_{p,p+1}(t) = \frac{(tp(p+1) - 1)(t - 1)}{(tp - 1)(p+1 - 1)}.
\]

By [11], there exists a ribbon 2-knot \( J \) with \( f(J) = 1 \) (and thus \( u(J) = 1 \)) having the Alexander polynomial \( \Delta_J(t) = \Delta_{2,3}(t)\Delta_{p,p+1}(t) \). We will show

\[
u(T_{p,p+1}^* \# qJ) = q + 1, \quad f(T_{p,p+1}^* \# qJ) = p + q - 1,
\]

where \( p \geq 2 \) and \( q \geq 0 \).

From Theorem 1, using (5) and (6), we have \( u(T_{p,p+1}^* \# qJ) \leq q + 1 \) and \( f(T_{p,p+1}^* \# qJ) \leq p + q - 1 \). The Alexander invariant of \( T_{p,p+1}^* \# qJ \) is

\[
\Lambda/(\Delta_{p,p+1}) \Theta \Lambda/(\Delta_{2,3}\Delta_{p,p+1}) \Theta \ldots \Theta \Lambda/(\Delta_{2,3}\Delta_{p,p+1}),
\]

and so using (3) and (4), we have \( u(T_{p,p+1}^* \# qJ) \geq e(T_{p,p+1}^* \# qJ) = q + 1 \).

There is an epimorphism

\[
\theta : \pi T_{p,p+1}^* \rightarrow \mathfrak{S}_{p+1}
\]

onto the symmetric group of degree \( p + 1 \) defined by \( \theta(x_i) = (i, i + 1) \) \((1 \leq i \leq p)\) and \( \theta(x_{p+1}) = (p+1, 1) \), where \( x_i \) is a meridian of the overpass \( \alpha_i \) of the regular projection of \( T_{p,p+1} \) given in the proof of Lemma 3 (Fig. 1).

Let

\[
\langle x, y | x = w^{-1}yw \rangle
\]

be a Wirtinger presentation of \( \pi J \), where \( w = w(x,y) \) is a word in \( x \) and \( y \). Then the quotient group \( \pi J/\langle x^2 = y^2 = 1 \rangle \) is the dihedral group of order \( 2r \):

\[
\langle x, y | x^2 = y^2 = (xy)^r = 1 \rangle,
\]

where

\[
r = \Delta_J(-1) = 3\Delta_{p,p+1}(-1) = \begin{cases} 
3p & \text{if } p \text{ is odd,} \\
3(p + 1) & \text{if } p \text{ is even,}
\end{cases}
\]
Thus there is an epimorphism $\pi J \rightarrow \Sigma_3$ defined by $x \mapsto (12)$ and $y \mapsto (13)$.

Let

$$\pi(qJ) = \langle x, y_1, y_2, \ldots, y_q \mid x = w(x, y_k)^{-1}y_kw(x, y_k)(1 \leq k \leq q) \rangle.$$

Extending the above epimorphisms, we have an epimorphism

$$\tilde{\theta} : \pi(T_{p,p+1}^* \# qJ) = \pi T_{p,p+1}^* \ast_{x_1} \pi(qJ) \rightarrow \Sigma_{p+q+1}$$

defined by $\tilde{\theta}|_{T_{p,p+1}^*} = \theta$ and $\tilde{\theta}(y_k) = (1, p + k + 1)$. Since we need at least $p + q$ transpositions to generate $\Sigma_{p+q+1}$, we have $f(T_{p,p+1}^* \# qJ) \geq p + q - 1$; cf. [12, p.460]. This completes the proof of Theorem 2.

6. Branched cyclic cover of a ribbon 2-knot

Let $K$ be an $n$-knot in $S^{n+2}$. We denote by $B_m(K)$ the $m$-fold branched cyclic covering space of $S^{n+2}$ branched over $K$. If $K$ is a 1-knot with $b$ bridges, then the Heegaard genus of $B_m(K)$ is less than or equal to $(m - 1)(b - 1)$, and so $\text{rank}(\pi_1(B_m(K))) \leq (m - 1)(b - 1)$; see [3, Proposition 11.4; 21, 10C3]. Similarly we have

Proposition 6. If $K$ is a ribbon 2-knot, then

$$\text{rank}(\pi_1(B_m(K))) \leq (m - 1)f(K).$$

Proof. Suppose that $f(K) = n$. By sliding 1-handles, $\pi K$ has a Wirtinger presentation

$$\langle x = x_0, x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle,$$

where $r_i = x^{-1}w_i^{-1}x_iw_i$ with $w_i$ a word in $x$, $x_1$, $x_2$, $\ldots$, $x_n$. Let $M_m(K)$ be the $m$-fold cyclic covering space of $S^4 - K$ corresponding to the kernel of the map $\pi K \rightarrow \mathbb{Z}_m = \langle t \mid t^m \rangle$ defined by $x_i \mapsto t$. We denote by $G_m$ the fundamental group $\pi_1(M_m(K))$, which is isomorphic to the kernel of this map. We find a set of generators for $G_m$ using the Reidemeister-Schreier method; cf. [13, Sect. 2.3].

From the right coset decompositions $\pi K = G_m \cup G_m x \cup G_m x^2 \cup \ldots \cup G_m x^{m-1}$, we may take $U = \{1, x, x^2, \ldots, x^{m-1}\}$ as a Schreier system for $G_m$ in $\pi K$. Then $G_m$ is generated by

$$\{tx_ix_i^{-1} \mid t \in U, 0 \leq i \leq n\},$$
where $g \mapsto \bar{g}$ is the map defined by $G_m g \cap U = \{\bar{g}\}$. Since $x_i^j x_i x_i^{-j} = x_i x_i x_i^{-j-1}$, putting

$$a_{ij} = \begin{cases} 
  x_i x_i^{-1} & \text{if } j = 0; \\
  x_i^j x_i x_i^{-j} & \text{if } 1 \leq j \leq m - 2; \\
  x_i^{m-1} x_i & \text{if } j = m - 1,
\end{cases}$$

we see that $G_m$ is generated by

$$\{a_{ij} | 1 \leq i \leq n, 0 \leq j \leq m - 1\} \cup \{x^m\}.$$

Now we consider $\pi_1(B_m(K))$, which is given by $G_m / (x^m = 1)$. Since $x^m = w_i^{-1} x_i^m w_i$ in $\pi K$, $a_{i0} a_{i1} \ldots a_{i m-1} = x_i^m = 1$ in $\pi_1(B_m(K))$. Thus $\pi_1(B_m(K))$ is generated by $\{a_{ij} | 1 \leq i \leq n, 1 \leq j \leq m - 1\}$, completing the proof.

See [18, 26] for the geometric interpretation of $B_2(K)$.

Using Proposition 6, we can prove the following, which is a generalization of (1).

**Proposition 7.**

$$u(kT_{2,3}^* \# lT_{2,5}^*) = \max\{k, l\}, \quad f(kT_{2,3}^* \# lT_{2,5}^*) = k + l.$$

**Proof.** We prove for $k \leq l$. From (5), we have $u(kT_{2,3}^* \# lT_{2,5}^*) \leq ku(T_{2,3}^* \# T_{2,5}^*) + (l - k)u(T_{2,5}^*)$. From (1) and Theorem 1, $u(T_{2,3}^* \# T_{2,5}^*) = u(T_{2,5}^*) = 1$, and thus we obtain $u(kT_{2,3}^* \# lT_{2,5}^*) \leq l$.

On the other hand, the Alexander invariant of $kT_{2,3}^* \# lT_{2,5}^*$ is

$$\frac{\Lambda}{(\Delta_{2,3} \Delta_{2,5})} \oplus \ldots \oplus \frac{\Lambda}{(\Delta_{2,3} \Delta_{2,5})} \oplus \frac{\Lambda}{(\Delta_{2,5})} \oplus \ldots \oplus \frac{\Lambda}{(\Delta_{2,5})},$$

where $\Delta_{2,p} = (t^p + 1)/(t + 1)$, and so $e(kT_{2,3}^* \# lT_{2,5}^*) = l$. Therefore, by (3) and (4), we have $u(kT_{2,3}^* \# lT_{2,5}^*) = l$.

By Theorem 1 and (6), we have $f(kT_{2,3}^* \# lT_{2,5}^*) \leq k + l$. On the other hand, from $\pi_1(B_2(T_{2,p})) = \pi_1(B_2(T_{2,p})) = \mathbb{Z}_p$ with $p(> 0)$ odd integer, we have

$$\pi_1(B_2(kT_{2,3}^* \# lT_{2,5}^*)) = \mathbb{Z}_3 \ast \ldots \ast \mathbb{Z}_3 \ast \mathbb{Z}_5 \ast \ldots \ast \mathbb{Z}_5.$$

The rank of this group is $k + l$ [13, p.192, Corollary], and so by Proposition 6, we have $f(kT_{2,3}^* \# lT_{2,5}^*) \geq k + l$. This completes the proof.
7. Satellite knot of a ribbon 2-knot

Let $R^3[t] = R^3 \times \{t\}$. Fig. 2 gives the motion pictures [4] of the spun trefoil $T_{2,3}^*$ in $R^4(\subset S^4)$, where $D^2$ is a disk such that $(T_{2,3}^* \cap R^3[0]) \cup D^2 = (T_{2,3}^* \cap R^3[2]) \cup D^2 = (T_{2,3}^* \cap R^3[1])$. More precisely,

$$T_{2,3}^* \cap R^3[t] = \begin{cases} T_{2,3}^* \cap R^3[0] & \text{for } -1 < t < 1, \\ T_{2,3}^* \cap R^3[1] & \text{for } t = \pm 1, \\ T_{2,3}^* \cap R^3[2] & \text{for } 1 < |t| < 3, \\ D_1^2 \cup D_2^2 & \text{for } t = \pm 3, \\ \emptyset & \text{for } |t| > 3, \end{cases}$$

where $D_1^2$ and $D_2^2$ are disjoint disks bounding the trivial 2-component link $T_{2,3}^* \cap R^3[2]$.

Fig. 2.
The group $\pi T_{2,3}^*$ has a presentation

$$\langle x, y | xyx = yxy \rangle.$$ 

Let $V$ be a tubular neighborhood of the simple closed curve representing the element $y^{-1}x$ in the exterior of $T_{2,3}^*$. Let $J$ be a 2-knot and $N(J)$ its tubular neighborhood. Then there is a homeomorphism $\xi : S^4 - \text{int} V \to N(J)$. The 2-knot $\xi(K)$ is called the satellite knot of $(T_{2,3}^*, y^{-1}x)$ about $J$, which we denote by $\Sigma(J)$. See [8]. By the van Kampen theorem, the group $\pi \Sigma(J)$ is the free product of $\pi J$ and $\pi T_{2,3}^*$, amalgamating the subgroup generated by a meridian of $\pi J$ and the subgroup generated by $y^{-1}x$, both of which are infinite cyclic groups. Thus it contains subgroups isomorphic to $\pi T_{2,3}^*$ and $\pi J$. The Alexander invariant of $\Sigma(J)$ is $\Lambda/(t^2 - t + 1)$; cf. [25].

When $J$ is a spun trefoil, $\Sigma(J)$ may be presented as in Fig. 3:

$$\Sigma(J) \cap R^3[\tau] = \begin{cases} 
\Sigma(J) \cap R^3[0] & \text{for } -1 < \tau < 1, \\
\Sigma(J) \cap R^3[1] & \text{for } \tau = \pm 1, \\
\Sigma(J) \cap R^3[2] & \text{for } 1 < |\tau| < 3, \\
D_1^2 \cup D_2^2 \cup D_3^2 \cup D_4^2 & \text{for } \tau = \pm 3, \\
\emptyset & \text{for } |\tau| > 3,
\end{cases}$$

where $D_i^2$, $i = 1, 2, 3, 4$, are disjoint disks bounding the trivial 4-component link $\Sigma(J) \cap R^3[2]$. $\Sigma(J)$ is also regarded as a 2-knot constructed by attaching a 1-handle to a parallel link [10] of two components associated with $J$. The following is easy to see.

**Lemma 4.** If $J$ is a ribbon 2-knot, then $\Sigma(J)$ is also a ribbon 2-knot and \( f(\Sigma(J)) \leq 2f(J) + 1 \).

Adding a 1-handle on $T_{2,3}^*$ whose core is $\alpha$ in Fig. 2, we obtain an unknotted surface. Similarly, adding a 1-handle on $\Sigma(J)$ whose core is $\beta$ in Fig. 3, we obtain an unknotted surface. Note that this 1-handle corresponds to $y^{-1}x \in \pi T_{2,3} \subset \pi \Sigma(J)$. More generally, we have:

**Lemma 5.** For any 2-knot $J$, $u(\Sigma(J)) = 1$.

Now we consider $\Sigma(kT_{2,5}^*)$, the satellite knot whose companion is the composition of $k$ copies of the spun 2-knot of the $(2,5)$-torus knot. Let $K(k, l)$ be the composition of $\Sigma(kT_{2,5}^*)$ and the composition of $l$ copies of the spun trefoil: $K(k, l) = \Sigma(kT_{2,5}^*) \# l T_{2,3}^*$, $k, l \geq 0$. Then we have:
**Proposition 8.**

\[ u(K(k, l)) = l + 1, \quad f(K(k, l)) = 2k + l + 1. \]

**Proof.** Since \( u(\Sigma(kT_{2,5}^*)) = u(T_{2,3}^*) = 1 \), we have \( u(K(k, l)) \leq l + 1 \) by (5). On the other hand, the Alexander invariant of \( K(k, l) \) is the sum of the \( l + 1 \) copies of \( \Lambda/(t^2 - t + 1) \), and so \( e(K(k, l)) = l + 1 \). Therefore, from (3) and (4), we have \( u(K(k, l)) = l + 1 \).

Next we consider the fusion number. By Lemma 5 and (6), we have \( f(K(k, l)) \leq 2k + l + 1 \). Using the presentation

\[ \langle a, b_1, b_2, \ldots, b_k | ab_iab_i = b_iab_iab_i \ (1 \leq i \leq k) \rangle \]
of $\pi(kT^*_2,5)$, we have a presentation of $\pi\Sigma(kT^*_2,5)$:

$$\langle x, y, a, b_1, b_2, \ldots, b_k \mid xyx = yxy, a = y^{-1}x, \quad ab_ia = b_iaab_i \ (1 \leq i \leq k) \rangle.$$

Thus $\pi K(k,l)$ has a presentation

$$\langle x, y, z_1, z_2, \ldots, z_l, a, b_1, b_2, \ldots, b_k \mid xyx = yxy, \quad xz_jx = z_jxz_j \ (1 \leq j \leq l), a = y^{-1}x, \quad ab_ia = b_iaab_i \ (1 \leq i \leq k) \rangle.$$

There is a homomorphism from $\pi K(k,l)$ to $\mathfrak{S}_{2k+l+3}$, defined by

$$x \mapsto (12), \quad y \mapsto (23), \quad z_j \mapsto (2, 2k + j + 3) \ (1 \leq j \leq l), \quad a \mapsto (123), \quad b_i \mapsto (3, 2i + 2, 2i + 3) \ (1 \leq i \leq k).$$

Then we have $b_jy_i^{-1} \mapsto (2, 2i + 3)$ and $b_i^{-1}yb_i \mapsto (2, 2i + 4)$, and so this homomorphism is surjective. Since we need at least $2k + l + 2$ transpositions to generate $\mathfrak{S}_{2k+l+3}$, $f(K(k,l)) \geq 2k + l + 1$. The proof is complete.

Clearly many more examples can be constructed using satellite knots.

**Remark.** The results of this paper may be generalized to higher dimensions.

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