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ANSWER TO A QUESTION BY NAKAMURA, NAKANISHI, AND SATOH INVOLVING CROSSING NUMBERS OF KNOTS

JUN GE, XIAN’AN JIN, LOUIS H. KAUFFMAN, PEDRO LOPES and LIANZHU ZHANG
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Abstract
In this paper we give a positive answer to a question raised by Nakamura, Nakanishi, and Satoh concerning an inequality involving crossing numbers of knots. We show it is an equality only for the trefoil and for the figure-eight knots.

1. Introduction

A Fox $m$-coloring [4] is an assignment of elements from $\{0, 1, \ldots, m - 1\}$ to the arcs of a link diagram such that at each crossing twice the integer assigned to the over-arc equals to the sum of the integers assigned to the two under-arcs mod $m$. For each link diagram and each modulus $m > 1$, there are always $m$ trivial colorings, namely by assigning the same integer mod $m$ to every arc of the diagram. A coloring with at least two distinct colors (i.e., two distinct integers mod $m$ assigned to two arcs) is called a non-trivial coloring. It is easy to check that if one diagram of a link has a non-trivial $m$-coloring, then each diagram of that link has a non-trivial $m$-coloring. A link is called $m$-colorable if it admits a diagram with non-trivial $m$-colorings. The following well-known theorem (see Exercise 8, page 133 of [3]) presents a criterion for checking if a given link is $m$-colorable.

Theorem 1. [3] A link $L$ is $m$-colorable if and only if the determinant of $L$ $(\text{det} L)$ and $m$ are not relatively prime.

For the proof of Theorem 1, for example, refer to [6].

The following definition was introduced by Harary and Kauffman in [5].

Definition 1. Given an integer $m$ greater than 1. Let $L$ be a link admitting non-trivial $m$-colorings. Let $D$ be a diagram of $L$, and let $n_{m,D}$ be the minimum number of colors mod $m$ it takes to construct a non-trivial $m$-coloring on $D$. Set

$$\text{mincol}_m L = \min\{n_{m,D} \mid D \text{ is a diagram of } L\}. $$

We call $\text{mincol}_m L$ the minimum number of colors of $L$, mod $m$.

We call any non-trivial $m$-coloring of $L$ using $\text{mincol}_m L$ colors a minimal $m$-coloring of $L$.

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Nakamura, Nakanishi, and Satoh proved the following theorem in [7].

**Theorem 2.** Let \( p \) be an odd prime. Any \( p \)-colorable knot \( K \) satisfies

\[
\text{mincol}_p(K) \geq \lfloor \log_2 p \rfloor + 2
\]

where \( \lfloor x \rfloor \) is the largest integer less than or equal to \( x \).

Let \( c(K) \) denote the crossing number of \( K \). Since \( c(K) \geq \text{mincol}_p(K) \), any \( p \)-colorable knot \( K \) satisfies \( c(K) \geq \lfloor \log_2 p \rfloor + 2 \). In Remark 3.3 (iii) on page 96 of [7], Nakamura, Nakanishi, and Satoh ask if the equality only holds for the trefoil knot \((p = 3)\) and the figure-eight knot \((p = 5)\). We give a positive answer to this question for classical knots.

### 2. Answering the Question

We recall that Theorem 2 which states that \( \text{mincol}_p(K) \geq \lfloor \log_2 p \rfloor + 2 \) is proved in Nakamura et al. [7]. Since the crossing number of knot \( K, c(K) \), satisfies \( c(K) \geq \text{mincol}_p(K) \), for any \( p \)-colorable knot \( K \), these authors wonder if the equality \( c(K) = \lfloor \log_2 p \rfloor + 2 \) only holds for the trefoil and the figure-eight knots, see (iii) in Remark 3.3 on page 96 of [7]. Here we settle this matter with Theorem 3.

**Theorem 3.** Let \( p \) be an odd prime. Let \( K \) be a \( p \)-colorable classical knot. Then the equality in \( c(K) \geq \lfloor \log_2 p \rfloor + 2 \) only holds for the trefoil knot \((p = 3)\) and the figure-eight knot \((p = 5)\).

Let \( D \) be a link diagram. Let

\[
d^\infty_n := \max\{\det(D) \mid D \text{ is a link diagram of } n \text{ crossings}\}.
\]

In [8], Stoimenow showed

\[
d^\infty_n \leq d^\infty_{n-1} + d^\infty_{n-2} + d^\infty_{n-3} \quad (n > 2),
\]

and then proved the following theorem.

**Theorem 4.** [8] Let \( \delta \approx 1.83929 \) be the real positive root of \( x^3 - x^2 - x - 1 = 0 \). There exists a constant \( C > 0 \) such that for any link diagram \( D \) of \( c(D) \) crossings

\[
\det(D) \leq C \cdot \delta^{c(D)}.
\]

We now prove that \( C = 2/\delta^2 \approx 0.59120 \) is always valid for non-trivial link diagrams.

**Theorem 5.** Let \( \delta \approx 1.83929 \) be the real positive root of \( x^3 - x^2 - x - 1 = 0 \). Then for any non-trivial link diagram \( D \) of \( c(D) \) crossings,

\[
\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}.
\]

Proof. We prove it by induction.

Since \( d^\infty_1 = 1, d^\infty_2 = 2, d^\infty_3 = 3 \), it is easy to check that \( \det(D) \leq \frac{2}{\delta} \cdot \delta^{c(D)} \) holds for any diagram \( D \) with \( 1 \leq c(D) \leq 3 \).

For any given integer \( n \geq 3 \), suppose that for any link diagram \( D \) with \( 1 \leq c(D) \leq n \), \( \det(D) \leq \frac{2}{\delta} \cdot \delta^{c(D)} \) holds. Then by the definition of \( d^\infty_k \), we have \( d^\infty_k \leq \frac{2}{\delta^2} \cdot \delta^k \) for each \( 1 \leq k \leq n \).
So for any link diagram \( D' \) with \( n + 1 \) crossings, we have
\[
\det(D') \leq d_{n+1}^\infty \\
\leq d_n^\infty + d_{n-1}^\infty + d_{n-2}^\infty \\
\leq \frac{2}{\delta^2} \cdot \delta^{n-2} \cdot (\delta^2 + \delta + 1) \\
= \frac{2}{\delta^2} \cdot \delta^{n+1}.
\]
□

Proof of Theorem 3. The unknot is not \( p \)-colorable for any prime \( p \), so we only need to consider non-trivial knots.

Let \( \tilde{D} \) be a minimal diagram of \( K \). Since \( K \) is a \( p \)-colorable knot, we have \( p \mid \det(K) \) and \( \det K > 0 \). By Theorem 4,
\[
\log_2 p \leq \log_2 \det(K) = \log_2 \det(\tilde{D}) \leq c(\tilde{D}) \log_2 \delta + \log \frac{2}{\delta^2} < 0.87915 \cdot c(K) - 0.5256.
\]
It is easy to see, for \( c(K) \geq 13, \)
\[
c(K) > 0.87915 \cdot c(K) - 0.5256 + 2 > \log_2 p + 2 > [\log_2 p] + 2.
\]

Table 1 shows the numerical results of \( d_n^\infty \) and \( [\log_2 d_n^\infty] + 2 \) for \( 3 \leq n \leq 12 \). The first 9 values of \( d_n^\infty (n \geq 3) \) were obtained by using KnotInfo [1] and LinkInfo [2] and the last value is estimated by formula (2).

<table>
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<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
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<tr>
<td>( d_n^\infty )</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>130</td>
<td>\leq 429</td>
</tr>
<tr>
<td>([\log_2 d_n^\infty] + 2)</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>\leq 10</td>
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Hence, for any knot \( K \) with crossing number between 7 and 16, we obtain
\[
c(K) > [\log_2 d_n^\infty] + 2 \geq [\log_2 \det(K)] + 2 \geq [\log_2 p] + 2.
\]

For any knot \( K \) with crossing number 5 or 6, it is easy to check that \( c(K) > [\log_2 p] + 2 \).

The proof is complete. □

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