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ANSWER TO A QUESTION BY NAKAMURA, NAKANISHI, AND SATOH INVOLVING CROSSING NUMBERS OF KNOTS

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Abstract

In this paper we give a positive answer to a question raised by Nakamura, Nakanishi, and Satoh concerning an inequality involving crossing numbers of knots. We show it is an equality only for the trefoil and for the figure-eight knots.

1. Introduction

A Fox m-coloring [4] is an assignment of elements from $\{0, 1, \ldots, m-1\}$ to the arcs of a link diagram such that at each crossing twice the integer assigned to the over-arc equals to the sum of the integers assigned to the two under-arcs mod m. For each link diagram and each modulus $m > 1$, there are always m trivial colorings, namely by assigning the same integer mod m to every arc of the diagram. A coloring with at least two distinct colors (i.e., two distinct integers mod m assigned to two arcs) is called a non-trivial coloring. It is easy to check that if one diagram of a link has a non-trivial m-coloring, then each diagram of that link has a non-trivial m-coloring. A link is called m-colorable if it admits a diagram with non-trivial m-colorings. The following well-known theorem (see Exercise 8, page 133 of [3]) presents a criterion for checking if a given link is m-colorable.

Theorem 1. [3] *A link L is m-colorable if and only if the determinant of L ($\det L$) and m are not relatively prime.*

For the proof of Theorem 1, for example, refer to [6].

The following definition was introduced by Harary and Kauffman in [5].

Definition 1. Given an integer m greater than 1. Let L be a link admitting non-trivial m-colorings. Let D be a diagram of L, and let $n_{m,D}$ be the minimum number of colors mod m it takes to construct a non-trivial m-coloring on D. Set

$$\mathrm{mincol}_m L \doteq \min\{n_{m,D} \mid D \text{ is a diagram of } L\}.$$

We call $\mathrm{mincol}_m L$ the minimum number of colors of L, mod m.

We call any non-trivial m-coloring of L using $\mathrm{mincol}_m L$ colors a minimal m-coloring of L.

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Nakamura, Nakanishi, and Satoh proved the following theorem in [7].

Theorem 2. *Let p be an odd prime. Any p-colorable knot K satisfies*

$$mincol_p(K) \geq \lfloor \log_2 p \rfloor + 2 \tag{1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Let $c(K)$ denote the crossing number of K. Since $c(K) \geq \mathrm{mincol}_p(K)$, any p-colorable knot K satisfies $c(K) \geq \lfloor \log_2 p \rfloor + 2$. In Remark 3.3 (iii) on page 96 of [7], Nakamura, Nakanishi, and Satoh ask if the equality only holds for the trefoil knot ($p = 3$) and the figure-eight knot ($p = 5$). We give a positive answer to this question for classical knots.

2. Answering the Question

We recall that Theorem 2 which states that $\mathrm{mincol}_p(K) \geq \lfloor \log_2 p \rfloor + 2$ is proved in Nakamura et al. [7]. Since the crossing number of knot K, $c(K)$, satisfies $c(K) \geq \mathrm{mincol}_p(K)$, for any p-colorable knot K, these authors wonder if the equality $c(K) = \lfloor \log_2 p \rfloor + 2$ only holds for the trefoil and the figure-eight knots, see (iii) in Remark 3.3 on page 96 of [7]. Here we settle this matter with Theorem 3.

Theorem 3. *Let p be an odd prime. Let K be a p-colorable classical knot. Then the equality in $c(K) \geq \lfloor \log_2 p \rfloor + 2$ only holds for the trefoil knot ($p = 3$) and the figure-eight knot ($p = 5$).*

Let D be a link diagram. Let

$$d_n^\infty := \max\{\det(D) \mid D \text{ is a link diagram of } n \text{ crossings}\}.$$

In [8], Stoimenow showed

$$d_n^\infty \leq d_{n-1}^\infty + d_{n-2}^\infty + d_{n-3}^\infty \quad (n > 2), \tag{2}$$

and then proved the following theorem.

Theorem 4. [8] *Let $\delta \approx 1.83929$ be the real positive root of $x^3 - x^2 - x - 1 = 0$. There exists a constant $C > 0$ such that for any link diagram D of $c(D)$ crossings*

$$\det(D) \leq C \cdot \delta^{c(D)}.$$

We now prove that $C = 2/\delta^2 \approx 0.59120$ is always valid for non-trivial link diagrams.

Theorem 5. *Let $\delta \approx 1.83929$ be the real positive root of $x^3 - x^2 - x - 1 = 0$. Then for any non-trivial link diagram D of $c(D)$ crossings,*

$$\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}.$$

Proof. We prove it by induction.

Since $d_1^\infty = 1$, $d_2^\infty = 2$, $d_3^\infty = 3$, it is easy to check that $\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}$ holds for any diagram D with $1 \leq c(D) \leq 3$.

For any given integer $n \geq 3$, suppose that for any link diagram D with $1 \leq c(D) \leq n$, $\det(D) \leq \frac{2}{\delta^2} \cdot \delta^{c(D)}$ holds. Then by the definition of d_k^∞, we have $d_k^\infty \leq \frac{2}{\delta^2} \cdot \delta^k$ for each $1 \leq k \leq n$.

So for any link diagram D' with $n+1$ crossings, we have

$$\begin{aligned} \det(D') &\leq d_{n+1}^{\infty} \\ &\leq d_n^{\infty} + d_{n-1}^{\infty} + d_{n-2}^{\infty} \\ &\leq \frac{2}{\delta^2} \cdot \delta^{n-2} \cdot (\delta^2 + \delta + 1) \\ &= \frac{2}{\delta^2} \cdot \delta^{n+1}. \end{aligned}$$

□

Proof of Theorem 3. The unknot is not p-colorable for any prime p, so we only need to consider non-trivial knots.

Let $\tilde{D}$ be a minimal diagram of K. Since K is a p-colorable knot, we have $p \mid \det(K)$ and $\det K > 0$. By Theorem 4,

$$\log_2 p \leq \log_2 \det(K) = \log_2 \det(\tilde{D}) \leq c(\tilde{D}) \log_2 \delta + \log \frac{2}{\delta^2} < 0.87915 \cdot c(K) - 0.5256.$$

It is easy to see, for $c(K) \geq 13$,

$$c(K) > 0.87915 \cdot c(K) - 0.5256 + 2 > \log_2 p + 2 > \lfloor \log_2 p \rfloor + 2.$$

Table 1 shows the numerical results of d_n^{∞} and $\lfloor \log_2 d_n^{\infty} \rfloor + 2$ for $3 \leq n \leq 12$. The first 9 values of d_n^{∞} ($n \geq 3$) were obtained by using KnotInfo [1] and LinkInfo [2] and the last value is estimated by formula (2).

Table 1. d_n^{∞} and $\lfloor \log_2 d_n^{\infty} \rfloor + 2$ for $3 \leq n \leq 16$.

n	3	4	5	6	7	8	9	10	11	12
d_n^{∞}	3	5	8	16	24	45	75	130	224	≤ 429
$\lfloor \log_2 d_n^{\infty} \rfloor + 2$	3	4	5	6	6	7	8	9	9	≤ 10

Hence, for any knot K with crossing number between 7 and 16, we obtain

$$c(K) > \lfloor \log_2 d_n^{\infty} \rfloor + 2 \geq \lfloor \log_2 \det(K) \rfloor + 2 \geq \lfloor \log_2 p \rfloor + 2.$$

For any knot K with crossing number 5 or 6, it is easy to check that $c(K) > \lfloor \log_2 p \rfloor + 2$. The proof is complete. □

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