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FRACTIONAL SPDES DRIVEN BY SPATIALLY CORRELATED NOISE: EXISTENCE OF THE SOLUTION AND SMOOTHNESS OF ITS DENSITY

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Abstract

In this paper we study a class of stochastic partial differential equations in the whole space $\mathbb{R}^d$, with arbitrary dimension $d \geq 1$, driven by a Gaussian noise white in time and correlated in space. The differential operator is a fractional derivative operator. We show the existence, uniqueness and Hölder's regularity of the solution. Then by means of Malliavin calculus, we prove that the law of the solution has a smooth density with respect to the Lebesgue measure.

1. Introduction and general framework

In this paper we consider the following stochastic partial differential equation (SPDE for abbreviation) given by

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \mathcal{D}_\delta^\alpha u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{F}(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $(t,x) \in [0,T] \times \mathbb{R}^d$, $d \geq 1$, $\alpha = (\alpha_1, \dots, \alpha_d)$, $\delta = (\delta_1, \dots, \delta_d)$ and $\mathcal{D}_\delta^\alpha$ denotes a non-local fractional differential operator defined by

$$\mathcal{D}_\delta^\alpha = \sum_{i=1}^d D_{\delta_i}^{\alpha_i},$$

where $D_{\delta_i}^{\alpha_i}$ denotes the fractional differential derivative w.r.t. the i-th coordinate defined

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via its Fourier transform $\mathcal{F}$ by

$$\mathcal{F}(D_\delta^\alpha \varphi)(\xi) = -|\xi|^\alpha \exp\left(-\imath \delta \frac{\pi}{2} \operatorname{sgn}(\xi)\right)\mathcal{F}(\varphi)(\xi),$$

where $\imath^2 + 1 = 0$. Along this paper, we will assume that $\alpha_i \in]0, 2] \setminus \{1\}$ and $|\delta_i| \le \min\{\alpha_i, 2 - \alpha_i\}$, $i = 1, \dots, d$. The noise $F(t, x)$ is a martingale measure (in the sense given by Walsh in [30]) to be defined with more details in the sequel. The coefficients b and $\sigma\colon \mathbb{R} \to \mathbb{R}$ are given functions. We shall refer to Eq. (1.1) as $Eq_\delta^\alpha(d, b, \sigma)$. Here, we are interested in solutions which are real valued stochastic processes. Firstly, we establish sufficient conditions ensuring the existence, the uniqueness and the Hölder regularity of solutions $u(t, x)$ whenever they exist. Secondly, we study the existence and smoothness of the density of the law of the solution $u(t, x)$ for fixed $t > 0$ and $x \in \mathbb{R}^d$. Our results of existence, uniqueness and regularity of the solution extend those obtained in [10] for $Eq_\delta^\alpha(1, b, \sigma)$ and [1] for $Eq_0^\alpha(1, b, \sigma)$ both driven by a space-time white noise W and $1 < \alpha < 2$. However in their framework, the regularity of the noise rises up $\alpha > 1$ as a necessary condition for the existence of the stochastic integral as an $L^2(\Omega)$ random variable. This problem does not appear in our case ($\alpha \in]0, 2] \setminus \{1\}$) because the noise F is smoother than W. The equation $Eq_\delta^\alpha(d, b, \sigma)$ recovers for instance the stochastic heat equation in spatial dimension $d \ge 1$ studied by many authors among others [6], [25]. With the notations adopted above it corresponds to $Eq_0^2(d, b, \sigma)$. Indeed, when $\delta_i = 0$ and $\alpha_i = 2$ for $i = 1, \dots, d$ the operator $\mathcal{D}_0^2$ coincides with the classical Laplacian operator in $\mathbb{R}^d$. Various physical phenomena involving diffusion and interaction of particles can be described by the equations $Eq_\delta^\alpha(d, b, \sigma)$ when suitable assumptions are made on the coefficient b and σ. The non-local property in this equation is due to the presence of $\mathcal{D}_\delta^\alpha$ and the non-linearity comes from the general form of b and σ. These equations can also be interpreted as random perturbation of deterministic equations $Eq_\delta^\alpha(d, b, 0)$ by multiplicative noise of the form $\sigma(u(t, x))\dot{F}(t, x)$. In probabilistic terms, replacing the Laplacian by its fractional power (which is an integro-differential operator) leads to interesting and largely open questions of extensions of results for Brownian motion driven stochastic equations to those driven by Lévy stable processes. In the physical literature, such fractal anomalous diffusions have been recently enthusiastically embraced by a slew of investigators in the context of hydrodynamics, acoustics, trapping effects in surface diffusion, statistical mechanics, relaxation phenomena, and biology (see e.g. [28], [29], [32], [33], [23], [31]). A probabilistic approach to the equation $Eq_0^\alpha(d, b, 0)$ is made by means of the Feynman–Kac formula (see [3], [4], [12]). Solutions to other particular fractional differential equations are given as functionals of stable subordinators. This representation provides explicit form to the density of the 3/2-stable law and to the density of escaping island vicinity in vortex medium, see [9]. In other words the laws of stable Lévy processes or stable subordinators satisfies fractional equations like $Eq_\delta^\alpha(d, b, 0)$.

The case where $0 < \alpha < 1$ was motivated by the work [5] where he attempts to use the $2/3$-order fractional Laplacian modeling of enhanced diffusing movements of random turbulent particle resulting from non-linear inertial interactions. A combined effect of the inertial interactions and the molecule Brownian diffusivities is found to be the bi-fractal mechanism behind multifractal scaling in the inertial range of scales of moderate Reynolds number turbulence. Accordingly, a stochastic equation is proposed to describe turbulence intermittency. The $2/3$-order fractional Laplacian representation is also used to construct a fractional Reynolds equation for non-linear interactions of fluctuating velocity components, underlying turbulence space-time fractal structures of Levy $2/3$-stable distribution. The new perspective of his study is that the fractional calculus is an effective approach modeling of chaotic fractal phenomena induced by non-linear interactions.

In the literature, various fractional differential operators are defined (see [11], [18], [21], [24] and [7]). The results of this paper apply to several of them, such as fractional Laplacian, Nishimoto operator and the non-self adjoint fractional operator introduced in [13] and used in [14] to the study of stochastic partial differential equation.

Basic notations, definitions and preliminary results of the operator $\mathcal{D}_\delta^\alpha$ and the noise F will be presented in the following subsequent two subsections. Section 2 is devoted to the existence and uniqueness and Section 3 deals with the Hölder regularity result. Section 4 contains the existence and smoothness of the density of the law of the solution. In appendix we prove some technical results which will be used in the proofs.

The value of the constants along this article may change from line to line and some of the standing parameters are not always indicated.

1.1. The operator D_δ^α. In one space dimension, the operator D_δ^α is a closed, densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. It is self adjoint only when $\delta = 0$ and in this case, it coincides with the fractional power of the Laplacian. According to [13], D_δ^α can be represented for $1 < \alpha < 2$, by

$$D_\delta^\alpha \varphi(x) = \int_{-\infty}^{+\infty} \frac{\varphi(x+y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} (\kappa_-^\delta \mathbf{1}_{(-\infty,0)}(y) + \kappa_+^\delta \mathbf{1}_{(0,+\infty)}(y))\, dy$$

and for $0 < \alpha < 1$, by

$$D_\delta^\alpha \varphi(x) = \int_{-\infty}^{+\infty} \frac{\varphi(x+y) - \varphi(x)}{|y|^{1+\alpha}} (\kappa_-^\delta \mathbf{1}_{(-\infty,0)}(y) + \kappa_+^\delta \mathbf{1}_{(0,+\infty)}(y))\, dy$$

where κ_-^δ and κ_+^δ are two non-negative constants satisfying $\kappa_-^\delta + \kappa_+^\delta > 0$ and φ is a smooth function for which the integral exists, and φ' is its derivative. This representation identifies it as the infinitesimal generator for a non-symmetric α-stable Lévy process.

Let $G_{\alpha,\delta}(t,x)$ denote the fundamental solution of the equation $Eq_\delta^\alpha(1,0,0)$ that is, the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = D_\delta^\alpha u(t,x), \\ u(0,x) = \delta_0(x), \qquad\qquad t > 0,\ x \in \mathbb{R}, \end{cases}$$

where δ_0 is the Dirac distribution. Using Fourier's calculus we obtain

$$G_{\alpha,\delta}(t,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-\imath zx - t|z|^\alpha \exp\left(-\imath\delta\frac{\pi}{2}\operatorname{sgn}(z)\right)\right) dz.$$

The relevant parameters, α, called the index of *stability* and δ (related to the asymmetry) improperly referred to as the *skewness* are real numbers satisfying $\alpha \in]0,2]$ and $|\delta| \le \min(\alpha, 2-\alpha)$.

The function $G_{\alpha,\delta}(t,\cdot)$ has the following properties. For the proof, one can see e.g. [10].

Lemma 1.1. *For $\alpha \in (0,2] \setminus \{1\}$ such that $|\delta| \le \min(\alpha, 2-\alpha)$, we have the following*:

(i) *The function $G_{\alpha,\delta}(t,\cdot)$ is the density of a Lévy α-stable process in time t.*

(ii) *The function $G_{\alpha,\delta}(t,x)$ is not in general symmetric relatively to x.*

(iii) *Semigroup property*: *$G_{\alpha,\delta}(t,x)$ satisfies the Chapman Kolmogorov equation, i.e. for $0 < s < t$*

$$G_{\alpha,\delta}(t+s,x) = \int_{-\infty}^{+\infty} G_{\alpha,\delta}(t,y) G_{\alpha,\delta}(s,x-y)\,dy.$$

(iv) *Scaling property*: $G_{\alpha,\delta}(t,x) = t^{-1/\alpha} G_{\alpha,\delta}(1, t^{-1/\alpha}x)$.

(v) *There exists a constant c_α such that* $0 \le G_{\alpha,\delta}(1,x) \le c_\alpha/(1+|x|^{1+\alpha})$, *for all $x \in \mathbb{R}$.*

For $d \ge 1$ and any multi index $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\delta = (\delta_1, \dots, \delta_d)$, let $\mathbf{G}_{\alpha,\delta}(t,x)$ be the Green function of the deterministic equation $Eq_\delta^\alpha(d,0,0)$. Clearly

$$\begin{aligned} \mathbf{G}_{\alpha,\delta}(t,x) &= \prod_{i=1}^d G_{\alpha_i,\delta_i}(t,x_i) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(-\imath\langle\xi, x\rangle - t\sum_{i=1}^d |\xi_i|^{\alpha_i} \exp\left(-\imath\delta_i\frac{\pi}{2}\operatorname{sgn}(\xi_i)\right)\right) d\xi, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product in $\mathbb{R}^d$.

1.2. The driving noise F. Let $\mathcal{S}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions. On a complete probability space $(\Omega, \mathcal{G}, P)$, the noise $F = \{F(\varphi), \varphi \in \mathcal{S}(\mathbb{R}^{d+1})\}$ is assumed to be an $L^2(\Omega, \mathcal{G}, P)$-valued Gaussian process with mean zero and covariance functional given by

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)(\varphi(s, \cdot) * \tilde{\psi}(s, \cdot))(x), \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^{d+1}),$$

where $\tilde{\psi}(s, x) = \psi(s, -x)$ and Γ is a non-negative and non-negative definite tempered measure, therefore symmetric. Let μ denote the spectral measure of Γ, which is also a on trivial tempered measure (see [27]) that is $\mu = \mathcal{F}^{-1}(\Gamma)$ and this gives

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(s, \cdot)(\xi)\overline{\mathcal{F}\psi(s, \cdot)}(\xi), \tag{1.2}$$

where $\bar{z}$ is the complex conjugate of z. Following the same approach as in [6], the Gaussian process F can be extended to a worthy martingale measure $M = \{M(t, A) := F([0, t] \times A) \colon t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ which shall acts as integrator, in the sense of Walsh [30], where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$. Let $\mathcal{G}_t$ be the completion of the σ-field generated by the random variables $\{M(s, A), \ 0 \leq s \leq t, \ A \in \mathcal{B}_b(\mathbb{R}^d)\}$. The properties of F ensure that the process $M = \{M(t, A), \ t \geq 0, \ A \in \mathcal{B}_b(\mathbb{R}^d)\}$, is a martingale with respect to the filtration $\{\mathcal{G}_t \colon t \geq 0\}$.

Then one can give a rigorous meaning to the solution of equation $Eq_\delta^\alpha(d, b, \sigma)$, by means of a jointly measurable and $\mathcal{G}_t$-adapted process $\{u(t, x) \colon (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfying, for each $t \geq 0$, a.s. for almost $x \in \mathbb{R}^d$, the following evolution equation:

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x - y)u_0(y)\, dy \\ &\quad + \int_0^t ds \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t - s, x - y)b(u(s, y))\, dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t - s, x - y)\sigma(u(s, y))\, M(ds, dy). \end{aligned} \tag{1.3}$$

Throughout this paper we adopt the following definition.

DEFINITION 1.1. A stochastic process u defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}^d$, which is jointly measurable and $\mathcal{G}_t$-adapted, is said to be a solution to the fractional SPDE $Eq_\delta^\alpha(d, b, \sigma)$, if it is an $\mathbb{R}$-valued field which satisfies (1.3) and

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} E|u(t, x)|^p < +\infty, \quad \text{for some} \quad p \geq 2.$$

Let us formulate our assumption concerning the fractional differential operator D_δ^α and the correlation of the noise M.

If we take $\sigma \equiv 1$ and use the formula (1.2) the stochastic integral

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(T-s, x-y)\, M(ds, dy)$$

appearing in (1.3) is well defined if and only if

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi) |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, \cdot)(\xi)|^2 < +\infty.$$

Indeed

$$\begin{aligned} & E\left|\int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)\, M(ds, dy)\right|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dz)(\mathbf{G}_{\alpha,\delta}(t-s, \cdot) * \widetilde{\mathbf{G}_{\alpha,\delta}}(t-s, \cdot))(x-z) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\mathbf{G}_{\alpha,\delta}(t-s, \cdot)(\xi)|^2, \quad (\mathcal{F}\mu = \Gamma). \end{aligned}$$

For a given multi index $\alpha = (\alpha_1, \dots, \alpha_d)$ such that $0 < \alpha_i \leq 2$, $i = 1, \dots, d$ and any $\xi \in \mathbb{R}^d$ we use the notation

$$S_\alpha(\xi) = \sum_{i=1}^d |\xi_i|^{\alpha_i}.$$

The following lemma gives sufficient condition for the existence of stochastic integrals w.r.t. the martingale measure M.

Lemma 1.2. *There exist two positive constants c_1 and c_2 such that*

$$c_1 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+S_\alpha(\xi)} \leq \int_0^T \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, \cdot)(\xi)|^2\, ds \leq c_2 \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+S_\alpha(\xi)}. \tag{1.4}$$

Proof. By the definition of $\mathbf{G}_{\alpha,\delta}$ we have

$$\mathcal{F}\mathbf{G}_{\alpha,\delta}(t, \cdot)(\xi) = \exp\left(-t \sum_{i=1}^d |\xi_i|^{\alpha_i} \exp\left(-\imath \delta_i \frac{\pi}{2} \operatorname{sgn}(\xi_i)\right)\right).$$

Therefore,

$$\int_0^t |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, \cdot)(\xi)|^2\, ds = \Phi\left(2t \sum_{i=1}^d |\xi_i|^{\alpha_i} \cos\left(\delta_i \frac{\pi}{2}\right)\right), \tag{1.5}$$

where $\Phi(x) = t(1 - e^{-x})/x$. Since Φ is a decreasing function, we get

$$\Phi(2tS_\alpha(\xi)) \le \Phi\left(2t \sum_{i=1}^{d} |\xi_i|^{\alpha_i} \cos\left(\delta_i \frac{\pi}{2}\right)\right) \le \Phi(2t\kappa S_\alpha(\xi)) \tag{1.6}$$

with $\kappa = \min_{1 \le i \le d}(\cos(\delta_i \pi/2)) > 0$. Indeed, the conditions $\alpha_i \in]0, 2]\backslash\{1\}$ and $|\delta_i| \le \min\{\alpha_i, 2 - \alpha_i\}$ imply that $|\delta_i| < 1$, and consequently $-\pi/2 < \delta_i \pi/2 < \pi/2$. The mean value theorem applied to the function $(1 + x)(1 - e^{-x})$ yields

$$\frac{t}{1 + x} \le \Phi(x) \le \frac{2t}{1 + x}$$

for all $x > 0$ and $t \in [0, T]$. This together with (1.5) and (1.6) imply that for all $t \in [0, T]$

$$\frac{t}{1 + 2tS_\alpha(\xi)} \le \int_0^t |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, \cdot)(\xi)|^2 \, ds \le \frac{2t}{1 + 2t\kappa S_\alpha(\xi)}. \tag{1.7}$$

The conclusion follows by taking the integral over $\mathbb{R}^d$ with respect to the measure μ. $\square$

REMARK 1.1. The upper and lower bounds in (1.4) do not depend on the parameter δ. When $\alpha_i = 2$ for all $i = 1, \dots, d$ then $S_2(\xi) = \sum_{i=1}^d |\xi_i|^2 =: |\xi|^2$ and the bounds in (1.4) are the same ones which appeared in [6] (see also [25]), that is

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < +\infty.$$

Our main assumption on the noise relies an the integrability condition w.r.t. the spectral measure μ.

ASSUMPTION $(\mathbf{H}_\eta^\alpha)$.

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + S_\alpha(\xi))^\eta} < +\infty \quad \text{where} \quad \eta \in]0, 1].$$

This condition states that there are not too many high frequencies in the noise, which turns out to be reformulated into a condition on the integrability of Γ (measure characterizing the covariance of the noise), near the origin in the case where Γ is a non-negative measure on $\mathbb{R}^d$. The condition $(\mathbf{H}_0^\alpha)$ means that μ is a finite measure, which is equivalent to say that Γ is a uniformly continuous and bounded function on $\mathbb{R}^d$. For this reason we do not consider this case in this paper. When μ is the Lebesgue measure on $\mathbb{R}^d$ which is the spectral measure of the white noise on $\mathbb{R}^d$ that is the noise corresponding to $\Gamma = \delta_0$, the condition $(\mathbf{H}_\eta^\alpha)$ is equivalent to $\int_0^{+\infty} r^{-1+\sum_{i=1}^d 1/\alpha_i}(1 + r)^{-\eta} \, dr < +\infty$ which is finite in and only if $\eta > \sum_{i=1}^d (1/\alpha_i)$.

1.2.1. Some examples. Let us now, consider a class of covariances measures Γ for which the condition $(\mathbf{H}_\eta^2)$ is satisfied. In this part we assume $\alpha_i = 2$, $i = 1, \dots, d$ and use $|x|$ as the Euclidian norm of x. These are considered by many authors, see e.g. [8] or [17].

1. Riesz kernels. Let $\Gamma(dx) = |x|^{-\gamma}\, dx$, $\gamma \in\,]0, d[$. It is known that the spectral measure is $\mu(dx) = c|x|^{\gamma-d}\, dx$. Then $(\mathbf{H}_\eta^2)$ is satisfied if and only if $\gamma \in\,]0, 2\eta \wedge d[$.
2. Assume that the spectral measure μ is either finite or absolutely continuous with respect to Lebesgue measure and $\mu(dx)/dx = \rho(x) \in L^p(\mathbb{R}^d)$ for some $p \geq 1$. For μ finite $(\mathbf{H}_\eta^2)$ holds for any $\gamma \in\,]0, 1[$. In the second case, $(\mathbf{H}_\eta^2)$ holds if $d(p-1)/2p < \eta < 1$. That is if $p \in [1, d/(d-2) \vee 1]$.
3. Brownian free field. Let $J(\varphi, \psi) = \langle(-\Delta + m^2)^{-1}\varphi, \psi\rangle$, where Δ is the Laplace operator in $\mathbb{R}^d$ and $m > 0$. In this case $\mu(dx) = (2\pi)^{-d/2}(|x|^2 + m^2)^{-1}\, dx$. Therefore, $(\mathbf{H}_\eta^2)$ holds if and only if $\eta \in\,](d-2)^+/2 \wedge 1, 1[$. That is, for $d = 1, 2$, $(\mathbf{H}_\eta^2)$ holds for any $\eta \in (0, 1)$, for $d = 3$, $(\mathbf{H}_\eta^2)$ holds for any $\eta \in\,]1/2, 1[$; for $d \geq 4$, $(\mathbf{H}_\eta^2)$ does not hold.
4. Bessel kernel. Assume $\beta > 0$ and let Γ be the Bessel kernel of order β. That is $\Gamma(dx) = B_\beta(x)\, dx$ where $B_\beta(x) = \nu_{\beta,d} \int_0^{+\infty} t^{-1+(\beta-d)/2} e^{-t-|x|^2/4t}\, dt$, $x \in \mathbb{R}^d$ and $\nu_{\beta,d}$ is a constant. The spectral measure μ of Γ is given by $\mathcal{F}(B_\beta)(\xi) = (1 + |\xi|^2)^{-\beta/2}$. Therefore $(\mathbf{H}_\eta^2)$ is satisfied if and only if

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-\eta-\beta/2}\, d\xi = c_d \int_0^{+\infty} (1+t)^{-\eta-\beta/2} t^{-1+d/2}\, dt$$

which is convergent for $d < 2\eta + \beta$. Hence $(\mathbf{H}_\eta^2)$ holds for $\eta \in\,](d-\beta)^+/2, 1[$.

2. Existence and uniqueness of the solution

We shall also need a hypothesis on the initial condition u_0:
(H.1) u_0 is an $\mathcal{G}_0$-measurable random variable such that $\sup_{x\in\mathbb{R}^d} E|u_0(x)|^{p_0} < +\infty$ for some p_0 large enough in $[2, +\infty[$.

The main result of this section is the following:

Theorem 2.1. *Suppose that b and σ are Lipschitz functions and assume that* **(H.1)** *and* $(\mathbf{H}_1^\alpha)$ *are satisfied. Then, there exists a unique solution $u(t, x)$ to* (1.3) *such that*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E|u(t, x)|^p < +\infty, \quad \textit{for any} \quad T > 0 \quad \textit{and} \quad p \in [2, p_0]. \tag{2.8}$$

REMARK 2.1. Our results improve those of [10] and [1] to d-dimensional case and extend the result in [6] to the fractional setting, both with α_i in $]0, 2] \setminus \{1\}$ for $i = 1, \dots, d$.

Proof of Theorem 2.1. The proof of the existence is done by the Picard iteration scheme. That is, we define recursively for all $n \geq 1$.

$$\begin{cases} u^0(t,x) = \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x-y) u_0(y)\, dy, \\ u^n(t,x) = u^0(t,x) + \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)\sigma(u^{n-1}(s,y))\, M(ds,dy) \\ \qquad + \int_0^t ds \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y) b(u^{n-1}(s,y))\, dy. \end{cases} \tag{2.9}$$

Fix $p \in [2, p_0]$. To prove the theorem we only need the following facts:
(i) For $n \geq 0$, $u^n = \{u^n(t,x) \colon (t,x) \in [0,T] \times \mathbb{R}^d\}$ is predictable process which satisfies:

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E|u^n(t,x)|^2 < +\infty. \tag{P_n}$$

This proves that the sequence u^n is well defined.
(ii) For $T > 0$ and any $p \in [2, p_0]$, $\sup_{n\geq 0} \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E|u^n(t,x)|^p < +\infty$.
(iii) Set $v_n(t) = \sup_{x\in\mathbb{R}^d} E|u^{n+1}(t,x) - u^n(t,x)|^p$, $n \geq 0$. Then

$$v_n(t) \leq c \int_0^t v_{n-1}(s)(J(t-s)+1)\, ds, \tag{2.10}$$

where

$$J(t) = \int_{\mathbb{R}^d} |\mathcal{F}\mathbf{G}_{\alpha,\delta}(t, \cdot)(\xi)|^2\, \mu(d\xi).$$

From this we conclude that the sequence $u^n(t,x)$ converges in $L^p(\Omega)$.

Statement (i) is checked by induction on n as a consequence of the hypotheses (**H.1**), ($\mathbf{H}_1^\alpha$) and the properties on the coefficients. Indeed, Hölder's inequality applied to the probability measure $\mathbf{G}_{\alpha,\delta}(t, x-y)\, dy$ gives

$$\begin{aligned} E|u^0(t,x)|^2 &\leq E \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x-y)|u_0(y)|^2\, dy \\ &\leq \sup_{y\in\mathbb{R}^d} E|u_0(y)|^2 \leq \sup_{y\in\mathbb{R}^d} E|u_0(y)|^{p_0} < +\infty. \end{aligned}$$

This proves (P_0). Now, we assume that the property (P_l) is true for any integer $l \leq n-1$ ($n \geq 2$). We want to check (P_n). Using Burkholder's inequality, the Lipschitz condition on σ, the induction hypothesis and ($\mathbf{H}_1^\alpha$), for the stochastic integral, and by the Hölder's inequality, the Lipschitz condition on b, the induction hypothesis and ($\mathbf{H}_1^\alpha$), for the third term in (2.9), we get

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E|u^n(t,x)|^2 \leq c \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} E(1 + |u^{n-1}(t,x)|^2).$$

Hence, we deduce (P_n). It is easy to check that u^n is predictable.

Statement (ii): Fix $p \in [2, p_0]$. We first prove that for any $n \geq 1$, $t \in [0, T]$

$$\sup_{x\in\mathbb{R}^d} E|u^n(t,x)|^p \leq c_1 + c_2 \int_0^t \sup_{x\in\mathbb{R}^d} E|u^{n-1}(s,x)|^p (J(t-s)+1)\,ds. \tag{2.11}$$

The arguments are not very far from those used in the proof of (i). Indeed, we have

$$E|u^n(t,x)|^p \leq c(C_0(t,x) + A_n(t,x) + B_n(t,x)) \tag{2.12}$$

with

$$\begin{aligned}
C_0(t,x) &= E|u^0(t,x)|^p,\\
A_n(t,x) &= E\left|\int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)\sigma(u^{n-1}(s,y))\,M(ds,dy)\right|^p,\\
B_n(t,x) &= E\left|\int_0^t ds \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y) b(u^{n-1}(s,y))\,dy\right|^p.
\end{aligned}$$

Jensen's inequality and assumption (**H.1**) yield

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} C_0(t,x) \leq \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d} (E|u^0(t,x)|^{p_0})^{p/p_0} < +\infty. \tag{2.13}$$

Burkholder's inequality and the linear growth condition of σ lead to

$$\sup_{x\in\mathbb{R}^d} A_n(t,x) \leq c \int_0^t \left(1 + \sup_{y\in\mathbb{R}^d} E|u^{n-1}(s,y)|^p\right) J(t-s)\,ds. \tag{2.14}$$

Moreover, Hölder's inequality, the linear growth condition of b and (i) of Lemma 1.1 imply

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} B_n(t,x) &\leq c \int_0^t ds \left(1 + \sup_{y\in\mathbb{R}^d} E|u^{n-1}(s,y)|^p\right) \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s,y)\,dy\\
&= c \int_0^t \left(1 + \sup_{y\in\mathbb{R}^d} E|u^{n-1}(s,y)|^p\right) ds.
\end{aligned} \tag{2.15}$$

Plugging the estimates (2.13) to (2.15) into (2.12) yields (2.11). Finally, the conclusion of part (ii) follows by applying Lemma 5.1 quoted in the appendix below to the situation: $f_n(t) = \sup_{x\in\mathbb{R}^d} E|u^n(t,x)|^p$, $k_1 = c_1$, $k_2 = 0$, $g(s) = c_2(J(s)+1)$, with c_1, c_2 given in (2.11).

Statement (iii): The same arguments as above imply that

$$\begin{aligned}
&E|u^{n+1}(t,x) - u^n(t,x)|^p\\
&\leq c \int_0^t \left(\sup_{y\in\mathbb{R}^d} E|u^n(s,y) - u^{n-1}(s,y)|^p\right)(J(t-s)+1)\,ds.
\end{aligned}$$

This yields (2.10). We finish the proof by applying Lemma 5.1 in the particular case: $f_n(t) = v_n(t)$, $k_1 = k_2 = 0$, $g(s) = c(J(s) + 1)$, with c given in (2.10). Notice that the results proved in part (ii) show that $v := \sup_{0\leq s\leq T} f_0(s)$ is finite. Hence the series $\sum_{n\geq 0}(v_n(t))^{1/p}$ converges for any $p \in [2, p_0]$. We then conclude that $\{u_n(t, x)\colon (t, x) \in [0, T]\times\mathbb{R}^d\}$ converges uniformly in $L^p(\Omega)$ to a limit $u = \{u(t, x)\colon (t, x) \in [0, T]\times\mathbb{R}^d\}$. It is not difficult to check that u satisfies conditions of Definition 1.1 and therefore the existence is completely proved. □

Let us now give a short proof for the uniqueness. Let u_1 and u_2 be two solutions to (1.3)

$$F(t, x) := E|u_1(t, x) - u_2(t, x)|^2$$

and $H(t) = \sup_{x\in\mathbb{R}^d} F(t, x)$. Then the isometry property for stochastic integrals, Hölder's inequality and the Lipschitz condition on b and σ yield

$$H(t) \leq c\int_0^t H(s)(J(t-s) + 1)\, ds.$$

By iterating this formula and using Lemma 5.1 we deduce that $H \equiv 0$, hence, $u_1(t, x) = u_2(t, x)$ t, x a.e. □

3. Path regularity of the solution

In this section we analyze the path Hölder regularity of $u(t, x)$. The next theorem extends and improves similar results known for stochastic heat equation, corresponding to the case $\alpha_i = 2$, $\delta_i = 0$, $i = 1, \dots, d$, (see for instance [26]). Contrary to the factorization method usually used in high dimension (see e.g. [25] and [8]) we use a direct method to prove our regularity results; in which the Fourier transform and the representation of the Green kernel play a crucial role.

Let $\gamma = (\gamma_1, \gamma_2)$ such that $\gamma_1, \gamma_2 > 0$ and let K be a compact set of $\mathbb{R}^d$. We denote by $\mathcal{C}^\gamma([0, T]\times K; \mathbb{R})$ the set of γ-Hölder continuous functions equipped with the norm defined by:

$$\|f\|_{\gamma,T,K} = \sup_{(t,x)\in[0,T]\times K} |f(t, x)| + \sup_{s\neq t\in[0,T]}\ \sup_{x\neq y\in K} \frac{|f(t, x) - f(s, y)|}{|t-s|^{\gamma_1} + |x-y|^{\gamma_2}}.$$

(**H.2**) There exists $\rho \in\,]0, 1[$ such that for all $z \in K$ (compact subset of $\mathbb{R}^d$)

$$\sup_{y\in\mathbb{R}^d} E|u_0(y+z) - u_0(y)|^{p_0} \leq c|z|^{\rho p_0}$$

for some p_0 large enough in $[2, \alpha_0/\rho[$, where $\alpha_0 = \min_{1\leq i\leq d}\{\alpha_i\}$.

The main result of this section is

Theorem 3.1. *Suppose that b and σ are Lipschitz functions. Assume that* (**H.1**), (**H.2**) *and* ($\mathbf{H}_\eta^\alpha$) *hold for $\eta \in \,]0, 1[$ and let u be a solution to equation* (1.3). *Then u belongs a.s. to the Hölder space $\mathcal{C}^\gamma([0, T] \times K; \mathbb{R})$ for $0 < \gamma_1 < \min\{\sum_{i=1}^d (\rho/\alpha_i), (1 - \eta)/2\}$, $0 < \gamma_2 < \min\{\rho, \alpha_0(1 - \eta)/2, 1/2\}$ and K compact subset of $\mathbb{R}^d$. Moreover $E\|u\|_{\gamma,T,K}^p < +\infty$ for any $p \in [2, p_0]$.*

In what follows, we establish some technical and useful results that will be needed in the proof of the regularity of the solution.

Let u_0 be a given random fields satisfying (**H.1**). Set

$$u^0(t, x) = \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x - y) u_0(y)\, dy.$$

Proposition 3.1. *Suppose that u_0 satisfies* (**H.2**). *Then we have the following*:

(i) *For each $x \in \mathbb{R}^d$ a.s. the mapping $t \mapsto u^0(t, x)$ is γ_1-Hölder continuous for $0 < \gamma_1 < \min\{\sum_{i=1}^d (\rho/\alpha_i), 1\}$.*

(ii) *For each $t \in [0, T]$ a.s. the mapping $x \mapsto u^0(t, x)$ is γ_2-Hölder continuous for $0 < \gamma_2 < \rho$.*

Proof. *Proof of* (i). Using the semigroup property of the Green kernel $\mathbf{G}_{\alpha,\delta}$,

$$\mathbf{G}_{\alpha,\delta}(t + h, x - y) = \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x - y - z) \mathbf{G}_{\alpha,\delta}(h, z)\, dz,$$

we have

$$\begin{aligned} &u^0(t + h, x) - u^0(t, x) \\ &= \int_{\mathbb{R}^d} [\mathbf{G}_{\alpha,\delta}(t + h, x - y) - \mathbf{G}_{\alpha,\delta}(t, x - y)] u_0(y)\, dy \\ &= \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(h, z) \left(\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x - y)[u_0(y - z) - u_0(y)]\, dy \right) dz. \end{aligned}$$

By concavity of the mapping $x \mapsto x^{p/p_0}$ (since $p \le p_0$), the Hölder's inequality, the assumption (**H.2**) and Fubini's theorem we obtain for $p \in [2, p_0]$

$$\begin{aligned} E|u^0(t + h, x) - u^0(t, x)|^p &\le \left(\int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} E|u_0(y - z) - u_0(y)|^{p_0} \mathbf{G}_{\alpha,\delta}(h, z)\, dz \right)^{p/p_0} \\ &\le c \left(\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(h, z) |z|^{\rho p_0}\, dz \right)^{p/p_0}. \end{aligned}$$

Now, due to the scaling property of $\mathbf{G}_{\alpha,\delta}$

$$\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(h, z)|z|^{\rho p_0}\, dz = \int_{\mathbb{R}^d} h^{-\sum_{i=1}^d 1/\alpha_i} \mathbf{G}_{\alpha,\delta}\big(1, h^{-\sum_{i=1}^d 1/\alpha_i} z\big)|z|^{\rho p_0}\, dz$$
$$= h^{\sum_{i=1}^d \rho p_0/\alpha_i} \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(1, y)|y|^{\rho p_0}\, dy.$$

Using property (v) of Lemma 1.1 we obtain that

$$\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(1, y)|y|^{\rho p_0}\, dy \le c \sum_{i=1}^d \int_{\mathbb{R}} \mathbf{G}_{\alpha,\delta}(1, y_i)|y_i|^{\rho p_0}\, dy$$
$$\le c \sum_{i=1}^d \int_{\mathbb{R}} \frac{|y|^{\rho p_0}}{1 + |y|^{1+\alpha_i}}\, dy \le c \int_0^{+\infty} \frac{x^{\rho p_0}}{1 + x^{1+\alpha_0}}\, dx.$$

The last integral is convergent for $\rho p_0 < \alpha_0$. Therefore we have proved the assertion (i).

Proof of (ii). A change of variable gives immediately

$$u^0(t, x+z) - u^0(t, x) = \int_{\mathbb{R}^d} [\mathbf{G}_{\alpha,\delta}(t, x+z-y) - \mathbf{G}_{\alpha,\delta}(t, x-y)]u_0(y)\, dy$$
$$= \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x-y)[u_0(y+z) - u_0(y)]\, dy.$$

Applying again concavity of $x \mapsto x^{p/p_0}$, the Hölder's inequality, the assumption (**H.2**) and Fubini's theorem, we obtain for all $p \in [2, p_0]$

$$E|u^0(t, x+z) - u^0(t, x)|^p$$
$$\le c\bigg(\int_{\mathbb{R}^d} \sup_{y\in\mathbb{R}^d} E(|u_0(y+z) - u_0(y)|^{p_0})\mathbf{G}_{\alpha,\delta}(t, x-y)\, dy\bigg)^{p/p_0}$$
$$\le c\bigg(\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t, x-y)|z|^{\rho p_0}\, dy\bigg)^{p/p_0} = c|z|^{\rho p}.$$

We complete the proof of the lemma by Kolmogorov's criterion. □

The next proposition studies the Hölder regularity of stochastic integrals with respect to the martingale measure M. For a given predictable random field V we set

$$U(t, x) = \int_0^t \int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)V(s, y)\, M(ds, dy).$$

Proposition 3.2. *Assume that* $\sup_{0\le t\le T} \sup_{x\in\mathbb{R}^d} E(|V(t, x)|^p)$ *is finite for some* p *large enough. Then under* $(\mathbf{H}^{\alpha}_{\eta})$ *we have the following*:

(i) *For each* $x \in \mathbb{R}^d$ *a.s. the mapping* $t \mapsto U(t,x)$ *is* γ_1*-Hölder continuous for* $0 < \gamma_1 < (1-\eta)/2$.
(ii) *For each* $t \in [0,T]$ *a.s. the mapping* $x \mapsto U(t,x)$ *is* γ_2*-Hölder continuous for* $0 < \gamma_2 < \min\{\alpha_0(1-\eta)/2, 1/2\}$.

Proof. *Proof of* (i). We have

$$\begin{aligned}
&U(t+h,x) - U(t,x)\\
&= \int_0^{t+h}\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t+h-s,x-y)V(s,y)\,M(ds,dy)\\
&\quad - \int_0^{t}\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s,x-y)V(s,y)\,M(ds,dy)\\
&= \int_0^{t}\int_{\mathbb{R}^d} [\mathbf{G}_{\alpha,\delta}(t+h-s,x-y) - \mathbf{G}_{\alpha,\delta}(t-s,x-y)]V(s,y)\,M(ds,dy)\\
&\quad + \int_t^{t+h}\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t+h-s,x-y)V(s,y)\,M(ds,dy).
\end{aligned}$$

For $p \geq 2$, the Burkholder inequality yields

$$\begin{aligned}
&E|U(t+h,x) - U(t,x)|^p\\
&\leq c \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E|V(s,y)|^p[(I_1(h,x))^{p/2} + (I_2(h,x))^{p/2}],
\end{aligned}$$

where

$$I_1(h,x) = \int_0^T ds \int_{\mathbb{R}^d} |\mathcal{F}[\mathbf{G}_{\alpha,\delta}(h+s,x-\cdot\,) - \mathbf{G}_{\alpha,\delta}(s,x-\cdot\,)](\xi)|^2\,\mu(d\xi)$$

and

$$I_2(h,x) = \int_0^h ds \int_{\mathbb{R}^d} |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s,x-\cdot\,)(\xi)|^2\,\mu(d\xi).$$

Estimation of $I_1(h,x)$. Set

$$\psi_{\alpha,\xi}(t) := \exp\left\{-t\sum_{i=1}^d |\xi_i|^{\alpha_i} \exp\left(-\imath\delta_i \frac{\pi}{2}\operatorname{sgn}(\xi_i)\right)\right\}. \tag{3.16}$$

By the definition of the Green function $\mathbf{G}_{\alpha,\delta}$ we can write

$$\begin{aligned}
I_1(h,x) &= \int_0^T dr \int_{\mathbb{R}^d} |\psi_{\alpha,\xi}(r)\psi_{\alpha,\xi}(h) - \psi_{\alpha,\xi}(r)|^2\,\mu(d\xi)\\
&= \int_0^T dr \int_{\mathbb{R}^d} |\psi_{\alpha,\xi}(r)|^2|\psi_{\alpha,\xi}(h) - 1|^2\,\mu(d\xi).
\end{aligned}$$

From the proof of Lemma 1.2 we know that $|\psi_{\alpha,\xi}(r)|^2 \le \exp(-2r\kappa S_\alpha(\xi))$. The mean value theorem to the function $\psi_{\alpha,\xi}(\,\cdot\,)$ between 0 and h implies that

$$|\psi_{\alpha,\xi}(h) - 1| \le h S_\alpha(\xi).$$

Moreover, for any $\beta \in\,]0,1[$, we have $|\psi_{\alpha,\xi}(h) - 1| \le 2^{1-\beta}|\psi_{\alpha,\xi}(h) - 1|^\beta$. Therefore

$$I_1(h,x) \le c_\beta h^{2\beta} \int_0^T dr \int_{\mathbb{R}^d} (S_\alpha(\xi))^{2\beta} \exp(-2\kappa r S_\alpha(\xi))\, \mu(d\xi).$$

Hence, under $(\mathbf{H}^\alpha_\eta)$ Lemma 5.2 implies that the right hand side of the above inequality is finite for any β in $]0,(1-\eta)/2[$. *Estimation of* $I_2(h,x)$. Fubini's theorem and a change of variable lead to

$$\begin{aligned} I_2(h,x) &= \int_{\mathbb{R}^d} \left[\int_0^h ds |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, x - \,\cdot\,)(\xi)|^2\right] \mu(d\xi) \\ &= \int_{\mathbb{R}^d} \left[\int_0^h |\psi_{\alpha,\xi}(s)|^2\, ds\right] \mu(d\xi) \\ &\le \int_{\mathbb{R}^d} \left[\int_0^h \exp(-2\kappa r S_\alpha(\xi))\, dr\right] \mu(d\xi). \end{aligned}$$

By Lemma 5.3 the last term is bounded by $c(h + h^\beta)$ for any $\beta \in\,]0, 1-\eta[$.

Proof of (ii). Let $x \in \mathbb{R}^d$ and z belongs to a compact subset K of $\mathbb{R}^d$

$$\begin{aligned} &U(t,x+z) - U(t,x) \\ &\quad = \int_0^t \int_{\mathbb{R}^d} [\mathbf{G}_{\alpha,\delta}(t-s, x+z-y) - \mathbf{G}_{\alpha,\delta}(t-s, x-y)] V(s,y)\, M(ds,dy). \end{aligned}$$

For $p \ge 2$, Burkholder's inequality yields

$$E|U(t,x+z) - U(t,x)|^p \le c \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E|V(s,y)|^p (J(x,z))^{p/2},$$

where

$$J(x,z) = \int_0^T ds \int_{\mathbb{R}^d} |\mathcal{F}[\mathbf{G}_{\alpha,\delta}(s, x+z-\,\cdot\,) - \mathbf{G}_{\alpha,\delta}(s, x-\,\cdot\,)](\xi)|^2\, \mu(d\xi).$$

The property $\mathcal{F}(f)(\xi + a) = \mathcal{F}(e^{-2i\pi\langle a,\cdot\rangle} f)(\xi)$ together with Lemma 5.4 imply

$$\begin{aligned} J(x,z) &= \int_0^T ds \int_{\mathbb{R}^d} |\psi_{\alpha,\xi}(s)[\exp(2\imath\pi\langle \xi, z\rangle) - 1]|^2\, \mu(d\xi) \\ &\le 4 \int_{\mathbb{R}^d} \int_0^T dr \exp(-2\kappa r S_\alpha(\xi))[1 - \cos(2\pi\langle \xi, z\rangle)]\, \mu(d\xi) \\ &\le c(|z| + |z|^{2\beta}), \end{aligned}$$

for all $0 < \beta < \min\{(1-\eta)\alpha_0/2, 1/2\}$. □

Proof of Theorem 3.1. Let u be a solution to equation (1.3). For any $x \in \mathbb{R}^d$, $z \in K$ and $t \in [0, T]$, and for $p \in [2, p_0]$,

$$\begin{aligned}
&E|u(t+h, x+z) - u(t, x)|^p \\
&\le cE|u^0(t+h, x+z) - u^0(t, x)|^p \\
&\quad + cE\biggl(\biggl|\int_0^{t+h}\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t+h-s, x+z-y)\sigma(u(s, y))\, M(ds, dy) \\
&\qquad\qquad - \int_0^t\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)\sigma(u(s, y))\, M(ds, dy)\biggr|\biggr)^p \\
&\quad + cE\biggl(\biggl|\int_0^{t+h}\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t+h-s, x+z-y)b(u(s, y))\, ds\, dy \\
&\qquad\qquad - \int_0^t\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)b(u(s, y))\, ds\, dy\biggr|\biggr)^p \\
&= A_1 + A_2 + A_3.
\end{aligned}$$

The terms A_1, A_2 are estimated by Propositions 3.1 and 3.2. More precisely we get

$$A_1 + A_2 \le c\bigl(h^{\sum_{i=1}^d \rho p/\alpha_i} + h^{\beta_1 p} + |z|^{\rho p} + |z|^{\beta_2 p}\bigr)$$

for $\beta_1 < (1-\eta)/2$ and $\beta_2 < \min(\rho, \alpha_0(1-\eta)/2, 1/2)$. Let us now give the estimation of A_3. After a change of variable $A_3 = |B|^p$, where

$$\begin{aligned}
B = &\int_0^h\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(h+t-s, x+z-y)b(u(s, y))\, ds\, dy \\
&+ \int_0^t\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x-y)[b(u(s+h, y+z)) - b(u(s, y))]\, ds\, dy.
\end{aligned}$$

By Hölder inequality, Lipschitz and linear growth condition of b, we deduce

$$\begin{aligned}
A_3 \le &\int_0^h\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t+h-s, x+z-y)E|b(u(s, y))|^p\, ds\, dy \\
&+ \int_0^t\int_{\mathbb{R}^d} \mathbf{G}_{\alpha,\delta}(t-s, x+z-y)E|b(u(s+h, y+z)) - b(u(s, y))|^p\, ds\, dy \\
\le &\; c_1 \cdot h + c_2 \int_0^t \sup_{y\in\mathbb{R}^d} E|u(s+h, y+z) - u(s, y)|^p\, ds,
\end{aligned}$$

where we have used assertion (i) of Lemma 1.1.

Put $\varphi(s, h, z) = \sup_{y\in\mathbb{R}^d} E|u(s+h, y+z) - u(s, y)|^p$. Hence,

$$\varphi(t, h, z) \le c_3\bigl(h + h^{\sum_{i=1}^d \rho p/\alpha_i} + h^{\beta_1 p} + |z|^{\rho p} + |z|^{\beta_2 p}\bigr) + c_2 \int_0^t \varphi(s, h, z)\, ds.$$

Therefore by the Gronwall's lemma

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} E|u(t+h,x+z)-u(t,x)|^p$$
$$\le c\big(h+h^{\sum_{i=1}^d \rho p/\alpha_i}+h^{\beta_1 p}+|z|^{\rho p}+|z|^{\beta_2 p}\big).$$

The conclusion of Theorem 3.1 is a consequence of the Kolmogorov continuity criterium. □

REMARK 3.1. Note that our results of regularity of the solution generalizes those obtained in [26].

4. Smoothness of the law

We prove that, under non-degeneracy condition on the diffusion coefficient σ, the law of $u(t,x)$ (solution of $Eq_\delta^\alpha(d,b,\sigma)$), has a smooth density for fixed $t>0$, $x\in\mathbb{R}^d$. Since the noise F has a space correlation, the setting of the corresponding stochastic calculus of variations is that used in [17] (see also [16]). Let $\mathcal{E}$ be the inner-product space consisting of functions $\varphi\colon \mathbb{R}^d\to\mathbb{R}$ such that

$$\int_{\mathbb{R}^d}\Gamma(dx)(\varphi*\tilde{\varphi})(x)<+\infty,$$

where $\tilde{\varphi}(x)=\varphi(-x)$. This space is endowed with the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{E}}=\int_{\mathbb{R}^d}\Gamma(dx)(\varphi*\tilde{\psi})(x)=\int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d}dy\,\varphi(x-y)\psi(-y).$$

Let $\mathcal{H}$ be the completion of $\mathcal{E}$ with respect the norm derived from $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. Set $\mathcal{H}_T=L^2([0,T];\mathcal{H})$, this space is a real separable Hilbert space isomorphic to the reproducing kernel Hilbert space of the centred Gaussian noise F which can be identified with a Gaussian process $\{W(h)\colon h\in\mathcal{H}_T\}$ as follows. For any complete orthonormal system $\{e_k\colon k\in\mathbb{N}\}\subset\mathcal{E}$ of the Hilbert space $\mathcal{H}$, define

$$W_k(t)=\int_0^t\int_{\mathbb{R}^d}e_k(x)\,F(ds,dx),\quad k\in\mathbb{N},\ t\in[0,T],$$

where the integral must be understood in Dalang's sense. The process $\{W_k(t)\colon t\in[0,T], k\in\mathbb{N}\}$ is a sequence of independent standard Brownian motions, such that for any predictable process X

$$\int_0^T\int_{\mathbb{R}^d}X(s,x)\,F(ds,dx)=\sum_{k=0}^{\infty}\int_0^T\langle X(s,\cdot),e_k(\cdot)\rangle_{\mathcal{H}}\,dW_k(s).$$

In particular, for any $\varphi \in \mathcal{S}(\mathbb{R}^{d+1})$

$$F(\varphi) = \sum_{k=0}^{\infty} \int_0^T \langle \varphi(s, \cdot), e_k(\cdot) \rangle_{\mathcal{H}} \, dW_k(s).$$

For $h \in \mathcal{H}_T$, set

$$W(h) = \sum_{k=1}^{\infty} \int_0^T \langle h(s), e_k \rangle_{\mathcal{H}} \, dW_k(s).$$

Thus, we can use the differential Malliavin calculus based on $\{W(h) \colon h \in \mathcal{H}_T\}$ (for more details see [19]). Recall that the Sobolev spaces $\mathbb{D}^{k,p}$ are defined by means of iterations of the derivative operator D. For a random variable X, $D^k X$ defines a $\mathcal{H}_T^{\otimes k}$-valued random variable if it exists. For $h \in \mathcal{H}_T$, set $D_h X = \langle DX, h \rangle_{\mathcal{H}_T}$ and for $r \in [0, T]$, $D_r X$ defines an element of $\mathcal{H}$, which is denoted by $D_{r,\cdot} X$. Then for any $h \in \mathcal{H}_T$

$$D_h X = \int_0^T \langle D_{r,\cdot} X, h(r) \rangle_{\mathcal{H}} \, dr.$$

We write $D_{r,\varphi} X = \langle D_{r,\cdot} X, \varphi \rangle_{\mathcal{H}}$ for $\varphi \in \mathcal{H}$.

The main result in this section is

Theorem 4.1. *Fix $t > 0$ and $x \in \mathbb{R}^d$. Assume that* $(\mathbf{H}_1^\alpha)$ *is satisfied, the coefficients σ and b are $\mathcal{C}^\infty$ with bounded derivatives of any order and in addition there exists $a > 0$ such that $|\sigma(z)| \geq a$ for any $z \in \mathbb{R}$. Then, the law of $u(t, x)$, solution of* (1.3)*, has a density which is infinitely differentiable.*

REMARK 4.1. If $d = 1$ and F is a space-time white noise, we obtain existence and smoothness of the density of the law for solution to equations studied in [10] and [1]. In the case where $\alpha_i = 2$, $\delta_i = 0$, $i = 1, \dots, d$, we obtain the result obtained in [16].

Classical results on the existence and smoothness of the density, using the approach of Malliavin calculus, are based on the next proposition.

Proposition 4.1 ([19]). *Let $F = (F^1, \dots, F^m)$, $m \geq 1$, be a random vector satisfying the following conditions*:

(i) *F^i belongs to $\mathbb{D}^\infty = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}$ for all $i = 1, \dots, m$.*

(ii) *The Malliavin matrix $\gamma_F = (\langle F^i, F^j \rangle_{\mathcal{H}})_{1 \leq i,j \leq m}$ satisfies*

$$(\det \gamma_F)^{-1} \in \bigcap_{p \geq 1} L^p(\Omega).$$

Then, F has an infinitely differentiable density with respect to the Lebesgue measure on $\mathbb{R}^m$.

According to Theorem 6.2 of [20], the proof of Theorem 4.1 is a consequence of the following lemma.

Lemma 4.1. *Under the assumption* $(\mathbf{H}_1^\alpha)$, *there exists a positive constant c_1 such that for any $\rho \in [0, 1/2]$,*

$$\int_0^\rho ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, \cdot)(\xi)|^2 \geq c_1\rho^{\theta_1}$$

for any $\theta_1 \geq 1$.

Proof. Using the lower bound of (1.7), we have

$$\begin{aligned}\int_0^\rho ds \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, \cdot)(\xi)|^2 &\geq \int_{\mathbb{R}^d} \frac{\rho\mu(d\xi)}{1+2\rho S_\alpha(\xi)} \\ &\geq \rho \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+S_\alpha(\xi)} \geq c_1\rho^{\theta_1}\end{aligned}$$

for any $\theta_1 \geq 1$. □

5. Appendix

In this last section, we collect some of the technical results which have been used the in previous sections.

We state below a version of Gronwall's Lemma that plays a crucial role in the proof of the existence and uniqueness results.

Lemma 5.1 ([6], Lemma 15). *Let g: $[0, T] \to \mathbb{R}_+$ be a non-negative function such that $\int_0^T g(s)\,ds < +\infty$. Then, there is a sequence $\{a_n,\ n \in \mathbb{N}\}$ of non-negative real numbers such that for all $p \geq 1$, $\sum_{n=1}^{+\infty}(a_n)^{1/p} < +\infty$, and having the following property: Let $(f_n,\ n \in \mathbb{N})$ be a sequence of non-negative functions on $[0, T]$, k_1, k_2 be non-negative numbers such that for $0 \leq t \leq T$,*

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s))g(t-s)\,ds.$$

If $\sup_{0\leq s\leq T} f_0(s) = c$, then for $n \geq 1$,

$$f_n(t) \leq k_1 + (k_1 + k_2)\sum_{i=1}^{n-1} a_i + (k_2 + c)a_n.$$

In particular, $\sup_{n\geq 0}\sup_{0\leq t\leq T} f_n(t) < +\infty$. *If* $k_1 = k_2 = 0$, *then* $\sum_{n\geq 0}(f_n(t))^{1/p}$ *converges uniformly on* $[0, T]$.

Lemma 5.2. *Let* η *and* β *be in* $(0, 1)$. *If* $(\mathbf{H}_\eta^\alpha)$ *holds then*

$$\int_0^T dr \int_{\mathbb{R}^d} \exp(-2r\kappa S_\alpha(\xi))(S_\alpha(\xi))^{2\beta}\mu(d\xi)$$

is finite for all β *in* $]0, (1-\eta)/2[$.

REMARK 5.1. Since the spectral measure μ is non-trivial positive tempered measure, we can ensure that there exist positive constants c_1, c_2 and k such that

$$c_1 < \int_{|\xi|<k} \mu(d\xi) < c_2.$$

Proof. Let $\beta \in]0, 1[$, and set

$$\begin{aligned} I &:= \int_0^T dr \int_{\mathbb{R}^d} \exp(-2\kappa r S_\alpha(\xi))(S_\alpha(\xi))^{2\beta}\mu(d\xi) \\ &= I_1 + I_2 \end{aligned}$$

where

$$I_1 := \int_0^T dr \int_{\{S_\alpha(\xi)\leq 1\}} \exp(-2\kappa r S_\alpha(\xi))(S_\alpha(\xi))^{2\beta}\mu(d\xi)$$

and

$$\begin{aligned} I_2 &:= \int_{\{S_\alpha(\xi)\geq 1\}} \left[\int_0^T ds \exp(-2\kappa r S_\alpha(\xi))\right](S_\alpha(\xi))^{2\beta}\mu(d\xi) \\ &= \int_{\{S_\alpha(\xi)\geq 1\}} \left[\frac{1-\exp(-2T\kappa S_\alpha(\xi))}{2\kappa}\right](S_\alpha(\xi))^{2\beta-1}\mu(d\xi). \end{aligned}$$

Clearly I_1 is finite. And the term I_2 is bounded by

$$\begin{aligned} \frac{1}{\kappa}\int_{\{S_\alpha(\xi)\geq 1\}} (S_\alpha(\xi))^{2\beta-1}\mu(d\xi) &\leq \frac{2^\eta}{\kappa}\int_{\{S_\alpha(\xi)\geq 1\}} (S_\alpha(\xi))^{2\beta-1+\eta}\frac{\mu(d\xi)}{(1+S_\alpha(\xi))^\eta} \\ &\leq \frac{2^\eta}{\kappa}\sup_{\{S_\alpha(\xi)\geq 1\}} (S_\alpha(\xi))^{2\beta-1+\eta}\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+S_\alpha(\xi))^\eta}. \end{aligned}$$

Now, choose β in $]0, (1-\eta)/2[$ and by the hypothesis $(\mathbf{H}_\eta^\alpha)$ the last term is finite. $\square$

Lemma 5.3. *If* $(\mathbf{H}_\eta^\alpha)$ *holds, then for any* $h \in [0, T]$

$$\int_0^h ds \int_{\mathbb{R}^d} |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, x - \cdot)(\xi)|^2 \mu(d\xi) \le c(h + h^\beta)$$

for any $\beta \in]0, 1 - \eta[$.

Proof. Let h be in $[0, T]$ and set

$$I_2(h) = \int_0^h ds \int_{\mathbb{R}^d} |\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, x - \cdot)(\xi)|^2 \mu(d\xi).$$

By the definition of $\mathbf{G}_{\alpha,\delta}$ we recall that $\mathcal{F}\mathbf{G}_{\alpha,\delta}(s, x - \cdot)(\xi) = \psi_{\alpha,\xi}(s)$ where $\psi_{\alpha,\xi}(s)$ is given by (3.16). Using Fubini's theorem we can write

$$\begin{aligned} I_2(h) &= \int_{\mathbb{R}^d} \left[\int_0^h |\psi_{\alpha,\xi}(s)|^2 \, ds \right] \mu(d\xi) \\ &\le \int_{\mathbb{R}^d} \left[\int_0^h \exp(-2\kappa r S_\alpha(\xi)) \, dr \right] \mu(d\xi) \\ &= I_{2.1}(h) + I_{2.2}(h), \end{aligned}$$

where

$$I_{2.1}(h) := \int_{\{S_\alpha(\xi) \le 1\}} \left[\int_0^h \exp(-2\kappa r S_\alpha(\xi)) \, dr \right] \mu(d\xi)$$

and

$$\begin{aligned} I_{2.2}(h) &:= \int_{\{S_\alpha(\xi) \ge 1\}} \left[\int_0^h \exp(-2\kappa r S_\alpha(\xi)) \, dr \right] \mu(d\xi) \\ &= \int_{\{S_\alpha(\xi) \ge 1\}} \left[\frac{1 - \exp(-2\kappa h S_\alpha(\xi))}{2\kappa S_\alpha(\xi)} \right] \mu(d\xi). \end{aligned}$$

It is easy to see that $I_{2.1}(h)$ is bounded by $c \cdot h$. Moreover for each $\beta \in]0, 1[$ we have

$$|1 - \exp(-2\kappa h S_\alpha(\xi))| \le 2^{1-\beta} |1 - \exp(-2\kappa h S_\alpha(\xi))|^\beta.$$

Using the mean value theorem we get

$$I_{2.2}(h) \le c_\beta h^\beta \int_{\{S_\alpha(\xi) \ge 1\}} (S_\alpha(\xi))^{\beta - 1} \mu(d\xi).$$

Choosing $\beta \in]0, 1 - \eta[$ and using hypothesis $(\mathbf{H}_\eta^\alpha)$, we show easily that

$$\int_{\{S_\alpha(\xi) \ge 1\}} (S_\alpha(\xi))^{\beta - 1} \mu(d\xi) < +\infty. \qquad \square$$

Lemma 5.4. *Let z belongs to a compact subset K of $\mathbb{R}^d$. Under* $(\mathbf{H}_\eta^\alpha)$ *we have*

$$\int_{\mathbb{R}^d}\int_0^T dr\,\exp(-2\kappa r S_\alpha(\xi))[1-\cos(2\pi\langle\xi,z\rangle)]\,\mu(d\xi)\le c(|z|+|z|^{2\beta})$$

for any $0<\beta<\min\{1/2,\alpha_0(1-\eta)/2\}$.

Proof. Let us put

$$\begin{aligned}J(z)&=\int_{\mathbb{R}^d}\left(\int_0^T dr\,\exp(-2\kappa r S_\alpha(\xi))\right)[1-\cos(2\pi\langle\xi,z\rangle)]\,\mu(d\xi)\\&=J_1(z)+J_2(z),\end{aligned}$$

where

$$\begin{aligned}J_1(z)&:=\int_0^T dr\int_{\{S_\alpha(\xi)\le 1\}}\exp(-2\kappa r S_\alpha(\xi))[1-\cos(2\pi\langle\xi,z\rangle)]\,\mu(d\xi)\\&\le\int_{\{S_\alpha(\xi)\le 1\}}[1-\cos(2\pi\langle\xi,z\rangle)]\,\mu(d\xi)\end{aligned}$$

and

$$\begin{aligned}J_2(z)&:=\int_{\{S_\alpha(\xi)>1\}}\frac{1-\exp(-2\kappa T S_\alpha(\xi))}{2\kappa S_\alpha(\xi)}[1-\cos(2\pi\langle\xi,z\rangle)]\,\mu(d\xi)\\&\le\int_{\{S_\alpha(\xi)>1\}}\frac{[1-\cos(2\pi\langle\xi,z\rangle)]}{2\kappa S_\alpha(\xi)}\,\mu(d\xi)\\&\le\frac{2^{1-2\beta}}{\kappa}\int_{\{S_\alpha(\xi)>1\}}\frac{|1-\cos(2\pi\langle\xi,z\rangle)|^{2\beta}}{S_\alpha(\xi)}\,\mu(d\xi)\end{aligned}$$

for any $\beta\in\,]0,1/2[$.

By the mean value theorem and Remark 5.1

$$J_1(z)\le c\sum_{i=1}^d|z_i|\le c\cdot d\cdot|z|$$

and

$$J_2(z)\le c_{\beta,\kappa}\int_{\{S_\alpha(\xi)>1\}}\frac{\left(\sum_{i=1}^d|z_i|\,|\xi_i|\right)^{2\beta}}{S_\alpha(\xi)}\,\mu(d\xi).$$

Set $\alpha_0=\min_{1\le i\le d}\{\alpha_i\}$. By the Hölder inequality

$$\sum_{i=1}^d|z_i|\,|\xi_i|\le\left(\sum_{i=1}^d|z_i|^\lambda\right)^{1/\lambda}\left(\sum_{i=1}^d|\xi_i|^\gamma\right)^{1/\gamma},\tag{5.17}$$

where γ and λ are positive real numbers such that $\gamma \geq \max\{\alpha_0, 1\}$ and $1/\gamma + 1/\lambda = 1$. Taking in account the equivalence of the norms on $\mathbb{R}^d$, there exists a constant c such that

$$\left(\sum_{i=1}^{d} |z_i|^{\lambda}\right)^{1/\lambda} \leq c|z|. \tag{5.18}$$

On the other hand, using the fact that for any positive real numbers $a_1, a_2, \dots, a_d$, and $p \geq 1$, $\sum_{i=1}^{d} a_i^p \leq \left(\sum_{i=1}^{d} a_i\right)^p$, we can write

$$\begin{aligned} \left(\sum_{i=1}^{d} |\xi_i|^{\gamma}\right)^{1/\gamma} &= \left(\sum_{i=1}^{d} (|\xi_i|^{\alpha_0})^{\gamma/\alpha_0}\right)^{1/\gamma} \\ &\leq \left(\sum_{i=1}^{d} |\xi_i|^{\alpha_0}\right)^{1/\alpha_0} \\ &\leq \left(\sum_{i=1}^{d} |\xi_i|^{\alpha_i}\right)^{1/\alpha_0}. \end{aligned} \tag{5.19}$$

Combining (5.17), (5.18) and (5.19), we obtain

$$J_2(z) \leq c_{\beta,\kappa} |z|^{2\beta} \int_{\{S_\alpha(\xi) > 1\}} (S_\alpha(\xi))^{2\beta/\alpha_0 - 1} \, \mu(d\xi)$$

which is finite according to $(\mathbf{H}_\eta^\alpha)$ for all $0 < \beta < \alpha_0(1 - \eta)/2$. □

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