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KNOTTED KLEIN BOTTLES WITH ONLY DOUBLE POINTS

AKIKO SHIMA

(Received September 27, 1999)

1. Introduction

If an embedded 2-sphere in 4-space \( \mathbb{R}^4 \) has the singular set of the projection in 3-space \( \mathbb{R}^3 \) consisting of double points, then the 2-sphere is ambient isotopic to a ribbon 2-sphere (see [19]). Similarly, if an embedded torus in \( \mathbb{R}^4 \) has the singular set of the projection in \( \mathbb{R}^3 \) consisting only of double points, then the torus is ambient isotopic to either a ribbon torus or a torus obtained from a symmetry-spun torus by \( m \)-fusion (see [15]). In this paper we will show a similar theorem for an embedded Klein bottle in \( \mathbb{R}^4 \). The following is the main results in this paper.

**Theorem 1.1.** Let \( F \) be an embedded Klein bottle in \( \mathbb{R}^4 \). If the singular set \( \Gamma^*(F) \) of the projection of \( F \) in \( \mathbb{R}^3 \) consists only of double points, then \( F \) is ambient isotopic to either a ribbon Klein bottle or a Klein bottle obtained from a spun Klein bottle by \( m \)-fusion.

**Corollary 1.2.** Let \( F \) be an embedded Klein bottle in \( \mathbb{R}^4 \). Suppose that the singular set \( \Gamma^*(F) \) of the projection of \( F \) in \( \mathbb{R}^3 \) consists of double points, and every component of the singular set \( \Gamma(F) \) on \( F \) is not homotopic to zero. If the fundamental group of the complement of \( F \) is isomorphic to \( \mathbb{Z}_2 \), then \( F \) is trivial, i.e., \( F \) bounds a solid Klein bottle in \( \mathbb{R}^4 \).

Let \( F \) be an oriented closed surface in \( \mathbb{R}^4 \). Is \( F \) trivial if the fundamental group of the complement of \( F \) is isomorphic to \( \mathbb{Z} \)? In the topological category, the question is affirmatively soloved when it is a 2-sphere (see [3]). In the PL or smooth category, this is an open question, it is affirmatively soloved when \( F \) is one of the following:

(i) \( F \) is a 1-fusion ribbon 2-knot ([8]).
(ii) \( F \) is a 2-sphere with four critical points ([11]).
(iii) \( F \) is a symmetry-spun torus ([17]).
(iv) \( F \) is a torus whose singular set on the torus consists only of disjoint simple closed curves with non-homotopic to zero in the torus ([15]).

All homology groups are taken with coefficients in \( \mathbb{Z} \), and all submanifolds are

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assumed to be locally flat, throughout in this paper. We will work in the PL category, throughout in this paper. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. Moreover, we regard 3-space $\mathbb{R}^3$ as the subset $\mathbb{R}^3 \times \{0\}$ of $\mathbb{R}^4$.

The paper is organized as follows. In Section 2, we define a ribbon surface, and a Klein bottle obtained from a spun Klein bottle by $m$-fusion. In Section 3, we study certain types of 2-complexes in $\mathbb{R}^3$. In Section 4, we define diagrams for embedded surfaces. In Section 5, we consider spun Klein bottles in $\mathbb{R}^4$. In Section 6, we prove the main theorem.

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2. Preliminaries and definitions

In this section, we define an $m$-fusion, a ribbon surface, and a spun Klein bottle.

Let $F$ be a closed surface. A map $f$ from $F$ to $\mathbb{R}^3$ is a generic map if for at every point $x$ of $F$, there exists a regular neighborhood $N$ of $f(x)$ in $\mathbb{R}^3$ such that $(N, f(F) \cap N)$ is homeomorphic to $(B^3, Z_1), (B^3, Z_1 \cup Z_2), (B^3, Z_1 \cup Z_2 \cup Z_3)$ or $(B^3, \text{the cone on a figure 8})$, where $B^3$ is the unit 3-ball in $\mathbb{R}^3$, $Z_i$ is the intersection of $B^3$ and $x_j, x_k$-plane ($\{1, 2, 3\} = \{i, j, k\}$). If $(N, f(F) \cap N)$ is homeomorphic to $(B^3, \text{the cone on a figure 8})$, then the point $f(x)$ is called a branch point. The point is also known as “Whitney’s umbrella” or “a pinch point”. A point $x \in f(F)$ is called a double point if $f^{-1}(x)$ consists of two points, and a triple point if $f^{-1}(x)$ consists of three points.

Let $F$ be an embedded surface in $\mathbb{R}^4$, and let $p$ be the projection defined by $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$. If $p|F$ is a generic map, then we associate the subset $F^* = p(F)$, and we denote by $\Gamma^*(F)$ the set of all double points, triple points and branch points. And put $\Gamma(F) = p^{-1}(\Gamma^*(F)) \cap F$. In this paper we assume that $p|F$ is a generic map.

An oriented closed surface in $\mathbb{R}^4$ is said to be trivial if it is the boundary of the disjoint union of handlebodies in $\mathbb{R}^4$. Note that the boundary of a handlebody is unique up to ambient isotopies of $\mathbb{R}^4$ (see [5]). An embedded Klein bottle in $\mathbb{R}^4$ is said to be trivial if it is the boundary of a solid Klein bottle in $\mathbb{R}^4$. Here the solid Klein bottle is homomorphic to the 3-manifold by attaching $B^2 \times \{0\}$ and $B^2 \times \{1\}$ from $B^2 \times [0, 1]$ via the map $q(x, 0) = (-x, 1)$, where $B^2$ is the unit 2-ball. In other word, the trivial Klein bottle is ambient isotopic to the surface with projection in $\mathbb{R}^3$ as illustrated in Fig. 1.

Let $G$ be an embedded closed surface in $\mathbb{R}^4$, $I = [0, 1]$, $B^2$ the unit 2-ball. An embedded surface $F$ in $\mathbb{R}^4$ is a surface obtained from $G$ by $m$-fusion if there exists a collection of embeddings $h_i : B^2 \times I \rightarrow \mathbb{R}^4, i = 1, 2, \ldots, m$, satisfying the following three conditions:

(i) The images of any two maps $h_i, h_j$ are disjoint for any distinct $i, j$. 
 iii) \( F = (G \setminus \bigcup_{i=1}^{m} (h_i(B^2 \times \partial I))) \cup \bigcup_{i=1}^{m} h_i(\partial B^2 \times I) \).

An embedded surface in \( \mathbb{R}^4 \) is a **ribbon surface** if it is obtained from a trivial 2-spheres by \( m \)-fusion.

Next, we define a spun Klein bottle in \( \mathbb{R}^4 \). For \( \theta \in [0, 2\pi] \), let \( \mathbb{R}^3_\theta = \{(x, y \cos \theta, y \sin \theta, z) \mid y \geq 0\} \), and

\[
B_0 = \{(x, y, 0, z) \mid x^2 + (y - 2)^2 + z^2 \leq 1\}.
\]

Then \( B_0 \) is the 3-ball in \( \mathbb{R}^3_\theta \), and the union of \( \mathbb{R}^3_\theta \) for all \( \theta \in [0, 2\pi] \) is \( \mathbb{R}^4 \). Let \( r_\theta : B_0 \to B_0 \) be the \( \theta \)-rotation map through the axis \( (0, 2, 0) \times [-1, 1] \) for \( \theta \in [0, 2\pi] \). An embedded Klein bottle \( F \) in \( \mathbb{R}^4 \) is called a **spun Klein bottle** if there exist an integer \( a \) and a knot \( K \) in the 3-ball \( B_0 \) as shown in Fig. 2 (1) such that

(i) \( K \) intersects two points to the axis \( (0, 2, 0) \times [-1, 1] \),
(ii) \( r_\theta(K) = K \), and
(iii) \( F = \{(x, y \cos \theta, y \sin \theta, z) \mid (x, y, 0, z) \in r_{(a+1/2)}(K), \theta \in [0, 2\pi]\} \).

We denote it by \( K^{\theta}(K) \). In particular, if \( K \) is a connected sum \( L\#(-L) \) of a knot \( L \) as shown in Fig. 2 (2), then \( K^{\theta}(K) \) is called a **simple** spun Klein bottle, where \(-L\) is the knot with the reverse orientation of \( L \). The symbol \( L \) in Fig. 2 (2) is the 1-string tangle so that the tangle sum of \( L \) and the trivial tangle is the knot \( L \). In particular, a Klein bottle obtained from a split union of a trivial 2-spheres and a spun Klein bottle by \( m \)-fusion is simply called a **Klein bottle obtained from a spun Klein bottle by \( m \)-fusion**.

**Remark 2.1.** (1) Let \( K^{\theta}(L\#(-L)) \) be a simple spun Klein bottle. Then, the fundamental group of the complement of \( K^{\theta}(L\#(-L)) \) is isomorphic to \( \pi_1(S^3 \setminus L)/\langle m^2 = 1 \rangle \) where \( m \) is a meridian curve of \( L \) (see [18]).
(2) The Klein bottle \( K^{\theta}(K) \) is ambient isotopic to \( K^{\theta \pm 2}(K) \) (cf. [17]).
Fig. 2. The center of each figure is z-axis.

Fig. 3.

3. 2-complexes in \( \mathbb{R}^3 \)

3.1. Embedded Klein bottles in \( \mathbb{R}^4 \). Let \( F \) be an embedded Klein bottle in \( \mathbb{R}^4 \) such that \( p|F \) is a generic map. In this section, we assume that \( \Gamma^*(F) \) consists only of double points. First, we consider the singular set \( \Gamma(F) \) on \( F \). Let \( c_1 = 0 \times I \), \( c_2 = (1/2) \times I \), \( c_i = i/(2n+1) \times I \cup (2n+1 - i)/(2n+1) \times I \), and \( d_j = I \times j/(2n) \) where \( i = 3, \ldots, 2n \) and \( j = 1, 2, \ldots, 2n - 1 \). Let \( \Gamma_1 = c_1 \cup c_2 \cup \cdots \cup c_n / \sim \), \( \Gamma_2 = d_1 \cup d_2 \cup \cdots \cup d_{2n-1} / \sim \) where \( \sim \) is the relation on \( I \times I \) with \( (0,t) \sim (1,t) \) and \( (t,0) \sim (1-t,1) \) for all \( t \in I \). Then each of \( \Gamma_1 \) and \( \Gamma_2 \) is a union of disjoint simple closed curves on a Klein bottle (see Fig. 3). Note that \( \Gamma_2 \) consists of an odd number of disjoint simple closed curves.

Lemma 3.1 ([16, Lemma 1.4]). Let \( F \) be a Klein bottle in \( \mathbb{R}^4 \) such that \( \Gamma^*(F) \) consists only of double points. Let \( \Gamma \) be the union of the components of \( \Gamma(F) \) each of
which is not homotopic to zero in \( F \). Then the pair \((\Gamma, F)\) is homeomorphic to \((\Gamma_1, F)\) or \((\Gamma_2, F)\).

### 3.2. Certain types of 2-complexes in \( \mathbb{R}^3 \)

In this subsection, we define certain types of 2-complexes in \( \mathbb{R}^3 \). For \( \theta \in [0, 2\pi] \), let \( \mathbb{R}_\theta^2 = \{(x, y \cos \theta, y \sin \theta) \mid y \geq 0\} \), and

\[
\overline{B}_0 = \{(x, y, 0) \mid x^2 + (y - 2)^2 \leq 1\}.
\]

Then \( \overline{B}_0 \) is the 2-ball in \( \mathbb{R}_\theta^2 \), and the union of \( \mathbb{R}_\theta^2 \) for all \( \theta \in [0, 2\pi] \) is \( \mathbb{R}^3 \). Let \( \phi_\theta : \overline{B}_0 \to \overline{B}_0 \) be the \( \theta \)-rotation map through the point \((0, 2, 0)\) for \( \theta \in [0, 2\pi] \). Let \( \alpha \) be a 1-complex in \( \overline{B}_0 \) such that each vertex is a vertex of degree four or three. A 2-complex \( K \) in \( \mathbb{R}^3 \) is called a 2-complex obtained from \( \alpha \) if there exist integers \( b, c \) with \( c \neq 0 \) such that

(i) If \( \alpha \) intersects the point \((0, 2, 0)\), then the point \((0, 2, 0)\) is the vertex of degree four and \( c = 2 \).

(ii) \( \phi_{2\pi/c}(\alpha) = \alpha \), and

(iii) \( K = \{(x, y \cos \theta, y \sin \theta) \mid (x, y, 0) \in \phi_{b/c}(\alpha), \theta \in [0, 2\pi]\} \).

We denote the 2-complex \( K \) by \( \alpha(b, c) \), and the above 1-complex \( \alpha \) is called a \( c \)-symmetric 1-complex.

**Example 3.2.**

(i) Let \( \alpha_1 \) be the 2-symmetric 1-complex in \( \overline{B}_0 \) as shown in Fig. 4 (1) such that the vertex of \( \alpha_1 \) is the point \((0, 2, 0)\). Then if \( b \) is an odd integer (resp. even integer), then the 2-complex \( \alpha_1(b, 2) \) is an immersed Klein bottle (resp. torus) in \( \mathbb{R}^3 \).

(ii) Let \( c \) be an integer with \( c \neq 0 \), and \( \alpha_i \) the \( c \)-symmetric 1-complex in \( \overline{B}_0 \) as shown in Fig. 4 (i) such that \( \alpha_i \) does not intersect the point \((0, 2, 0)\) for \( i = 2, 3 \). Then \( c \) is the number of vertices of \( \alpha_i \), and then the 2-complex \( \alpha_i(b, c) \) is immersed tori for any integer \( b \).
Lemma 3.3. Let \( \alpha \) be a \( c \)-symmetric 1-complex, and \( \alpha(b, c) \) a 2-complex in \( \mathbb{R}^3 \) obtained from \( \alpha \).

1. Let \( C \) be a component of \( S(\alpha(b, c)) \). Then, a regular neighborhood of \( C \) in \( \alpha(b, c) \) is two immersed annuli, two immersed Möbius bands. Moreover, there is at most one regular neighborhood consisting of two immersed Möbius bands.

2. Removing \( S(\alpha(b, c)) \), we obtain open annuli. Here \( S(\alpha(b, c)) \) is the set of all points whose neighborhood in \( \alpha(b, c) \) is the intersection of two sheets or \( Y \times [0, 1] \), where \( Y \) is the cone on three points.

Proof. (1) If \( c = 2 \), if \( b \) is odd, and if \( \alpha \) intersects the point \( (0, 2, 0) \) in \( \overline{B}_0 \), then we have the component with \( (0, 2, 0) \) in \( S(\alpha(b, c)) \) whose regular neighborhood in \( \alpha(b, c) \) consists of two immersed Möbius bands. Conversely, such a component can be obtained only as above, which yields the result.

(2) From the condition (ii) of the definition of symmetric 1-complexes, we can show (2).

From Lemma 3.3, we have the following remark:

Remark 3.4. (1) Let \( b, c \) be integers with \( c \neq 0 \), and \( \alpha \) a \( c \)-symmetric 1-complex in \( \overline{B}_0 \). If \( \alpha(b, c) \) is an immersed Klein bottle, then \( b \) is odd, \( c = 2 \) and there exists a knot \( K \) in \( B_0 \) with \( (K^{b+1/2})_\ast = \alpha(b, 2) \).

(2) Let \( K \) be a knot in \( B_0 \) satisfying (i) and (ii) in the definition of spun Klein bottles. Then for any integer \( a \), the projection \( (K^a(K))_\ast \) in \( \mathbb{R}^3 \) is the 2-complex obtained from \( p(K) \), i.e., \( (K^a(K))_\ast = p(K)(2a + 1, 2) \).

Definition 3.5. Let \( \alpha_1 \) be the 2-symmetric 1-complex as shown in Fig. 4 (1) with \( \alpha_1 \subset \overline{B}_0 \). Then there exist two 2-balls \( D_1, D_2 \) in \( \overline{B}_0 \) such that \( D_1 \cap D_2 \) is the point \( (0, 2, 0) \) and \( \alpha_1 = \partial D_1 \cup \partial D_2 \). For an integer \( b \), the 3-complex \( X_b \) is defined by

\[
X_b = \{(x, y \cos \theta, y \sin \theta) \mid (x, y, 0) \in \overline{B}((2b+1/2)\partial(D_1 \cup D_2), \theta \in [0, 2\pi])\}.
\]

Note that the closure of one component of \( \mathbb{R}^3 \setminus \{\alpha_1(b, 2)\} \) is \( X_b \). Let \( S^1 \) be the unit 1-sphere. Then, the 1-sphere \( S^1 \) is identified with \([0, 2\pi]/0 \sim 2\pi \). We have a natural embedding \( \psi \) of the solid torus \( \overline{B}_0 \times S^1 \) in \( \mathbb{R}^3 \) defined by \( \psi(x, y, \theta) = (x, y \cos \theta, y \sin \theta) \). Let \( g: \overline{B}_0 \times S^1 \to \mathbb{R}^3 \) be an embedding. Then \( g(\psi^{-1}(X_b)) \) is also a 3-complex in \( \mathbb{R}^3 \). We call it a coiled solid torus. Let \( \alpha \) be a \( c \)-symmetric 1-complex. Then we also call \( g(\psi^{-1}(\alpha(b, c))) \) a 2-complex obtained from \( \alpha \) for any integer \( b \).

Let \( F \) be an embedded surface in \( \mathbb{R}^4 \) such that

\( K0 \) \( F \) is the disjoint union of one Klein bottle and tori, or the disjoint union of tori,

\( K1 \) \( \Gamma^*(F) \) consists only of double points, and

\( K2 \) each component of \( \Gamma(F) \) is not homotopic to zero in \( F \), and \( F^* \) is connected.

From Lemma 3.1, we have the following lemma.
Lemma 3.6.  Let $F$ be as above. Then we have the following.

(1) $F^* \setminus \Gamma^*(F)$ consists of open annuli.

(2) Let $C$ be a component of $\Gamma^*(F)$, and $N(C)$ a regular neighborhood of $C$ in $\mathbb{R}^3$. Then $N(C) \cap F^*$ consists of two immersed annuli or two immersed Möbius bands.

A curve $C$ is an $A$-curve if $N(C) \cap F^*$ is two immersed annuli, and is an $M$-curve if $N(C) \cap F^*$ is two immersed Möbius bands.

In the case of classical knots, any knot diagram in $\mathbb{R}^2$ can be considered in the 2-sphere. Because, by ambient isotopies the bounded region of $\mathbb{R}^3 \setminus \{\text{a knot projection}\}$ can be changed. Similarly, without loss of generality we may consider that the projection of knotted surfaces is in the 3-sphere $S^3$. Here, we consider the 3-sphere $S^3$ as a one point compactification of $\mathbb{R}^3$. We discuss about a 2-complex which is the projection into $\mathbb{R}^3$ of an embedded surface in $\mathbb{R}^4$ satisfying (K0), (K1) and (K2). Note that the above 2-complex is called a 2-complex consisting of annuli in [14]. From now on, we assume that such a projection is in the 3-sphere $S^3$ in this section.

Lemma 3.7 ([16, Lemma 2.1]). Let $F$ be an embedded Klein bottle in $\mathbb{R}^4$ such that $\Gamma^*(F)$ consists only of one simple closed curve, and each component of $\Gamma(F)$ has a Möbius band neighborhood. Then there exists an odd integer $b$ and an embedding $g: \overline{B_0} \times S^1 \to S^3$ such that $F^*$ can be moved by an ambient isotopy of $S^3$, where $\alpha_1$ is the 2-symmetric 1-complex as shown in Fig. 4 (1).

3.3. Good solid tori sequences. Let $F$ be an embedded surface in $\mathbb{R}^4$ satisfying the conditions (K0), (K1) and (K2). Then $\Gamma^*(F)$ consists only of A-curves and at most one M-curve. Let $V_1, V_2, \ldots, V_k$ be solid tori in $S^3$, and $\mathcal{V} = \{V_1, V_2, \ldots, V_k\}$. We say that $\mathcal{V}$ is a solid tori sequence for $F^*$ if $\mathcal{V}$ satisfies the following two conditions:

(i) $\partial V_i \subset F^*$ for all $i$.

(ii) If $i \neq j$, then $V_i \cap V_j = \partial V_i \cap \partial V_j$ is one simple closed curve, an annulus or empty. Let $X$ be a coiled solid torus, and $\mathcal{V}$ as above. We say that $\mathcal{V} \cup \{X\}$ is an almost solid torus sequence for $F^*$ if $\mathcal{V} \cup \{X\}$ satisfies the above conditions (i), (ii), and (iii) the intersection of $X$ and $S^3 \setminus X$ is contained in $F^*$, and

(iv) $X \cap V_i$ is one simple closed curve, an annulus or empty for all $i$.

Example 3.8. Let $\alpha_3$ be a $c$-symmetric 1-complex as shown in Fig. 4 (3), and let $D_1, D_2$ be 2-balls in $\overline{B_0}$ such that $D_1 \subset D_2$ and $\alpha_3 = \partial D_1 \cup \partial D_2$. For an integer $b$ with $(b, c) = 1$, let $W_i = \{(x, y \cos \theta, y \sin \theta) \mid (x, y, 0) \in \overline{B}_b/c\theta(D_i), \theta \in [0, 2\pi]\}$. Then $W_1$, $W_2$ are the solid tori in $S^3$ with $W_1 \subset W_2$ and $\partial W_1 \cup \partial W_2 = \alpha_3(b, c)$. We see that $\{W_2\}$ is a solid torus sequence for the 2-complex $\alpha_3(b, c)$. Let $V_2 = S^3 \setminus \overline{W_2}$. Then $V_2$ is a solid torus, $\partial W_1 \cup \partial V_2 = \alpha_3(b, c)$, and $W_1 \cap V_2 = \partial W_1 \cap \partial V_2$ is one simple closed curve, say $L$. The set $\{W_1, V_2\}$ is a solid torus sequence for $\alpha_3(a, b)$. Let $N$ be a
Fig. 5. (1) the 2-complex $K_1$ (2) the 2-complex $K_2$.

regular neighborhood of $L$ in $S^3$. Note that if $L$ is not a trivial knot, then $W_1 \cup V_2 \cup N$ is not a solid torus. Because, $W_1 \cup V_2 \cup N$ is homeomorphic to the complement of an open regular neighborhood of $L$.

Let $F$ be an embedded surface in $\mathbb{R}^4$ satisfying (K0), (K1) and (K2). Let $\mathcal{V} = \{V_1, V_2, \ldots, V_k\}$ be a solid tori sequence for the 2-complex $F^*$. Let $c_i$ be a component of $\Gamma^*(F)$ with $c_i \subset \partial V_i$. Let $n$ be the minimal number of intersection points of $c_i$ and a meridian disk of the solid torus $V_i$. For the solid torus $V_i$ we define $n(V_i)$ as follows:

$$n(V_i) = \begin{cases} 
    n & \text{if } n \geq 1, \\
    0 & \text{if } n = 0, V_i \text{ is non-standard}, \\
    \infty & \text{if } n = 0, V_i \text{ is standard}.
\end{cases}$$

Here, a standard solid torus means a regular neighborhood of a trivial knot in $S^3$. We would like to distinguish standard and non-standard solid tori. Let $T_1, T_2, T$ be tori in $S^3$ such that

- $T$ bounds a standard solid torus $V$,
- $T_1, T_2 \subset V$,
- $T_i \cap T$ is a simple closed curve for $i = 1, 2$,
- $T_1$ bounds the complement of an open regular neighborhood of a trefoil knot in $V$, and
- $T_2$ bounds a solid torus $V_2$ in $V$ so that $V_2$ has a 2-ball $D$ in $V$ with $D \cap V_2 = \partial D$.

See Fig. 5. For the torus $T_1$, there exists a solid torus $V_1$ with $\partial V_1 = T_1$. Let $K_i = T_i \cup T$ for $i = 1, 2$. Then $\{V\}$ is a solid tori sequence for $K_i$ with $K_i \subset V$ and $n(V) = 0$, and $\{V_3\}$ is a solid torus sequence for $K_1$ with $n(V_3) = \infty$. However, $K_1$ is not a 2-complex $\alpha(D_3, c)$ obtained from any symmetric 1-complex $\alpha$. If an embedded torus in $\mathbb{R}^4$ has such a projection $K_1$ into $S^3$, then by an ambient isotopy of $\mathbb{R}^4$ we can assume that its projection in $S^3$ is $K_2$. Let $W = S^3 \setminus V$. Note that $K_2$ has a solid tori sequence.
\( \mathcal{W} = \{ V_2, W \} \) with \( K_2 \subset \cup \mathcal{W} \), \( n(V_2) = 1 \) and \( n(W) = 1 \). By Proposition 3.18, we see that \( K_2 \) is a 2-complex obtained from some symmetric 1-complex. In this paper we discuss about immersed Klein bottles. It is not important a solid torus \( V \) with \( n(V) = 0 \) or \( n(V) = \infty \).

We construct the graph \( G(\mathcal{W}) \) obtained by a solid tori sequence \( \mathcal{W} \) as follows. The vertices are in one to one correspondence with the solid tori \( \{ V_i \} \), and the edges are in one to one correspondence with the set \( \{ V_i \cap V_j \neq \emptyset \} \). If \( V_i \cap V_j \neq \emptyset \), then we connect the vertices \( u(V_i) \) and \( u(V_j) \) by the edge \( e_{ij} \).

**Definition 3.9.** Let \( F \) be an embedded surface in \( \mathbb{R}^4 \) satisfying (K0), (K1) and (K2), and \( \mathcal{W} = \{ V_1, V_2, \ldots, V_k \} \) a solid tori sequence for the 2-complex \( F^* \). A solid tori sequence \( \mathcal{W} \) is *good*, if \( \mathcal{W} \) satisfies the following four conditions:

(i) \( G(\mathcal{W}) \) is a connected tree.

(ii) If \( B \) is an annulus with \( B \subset F^* \) and if \( (\cup \mathcal{W}) \cap B = \partial B \), then \( \partial B \subset \partial V_i \) for some \( i \). Namely, for any annulus \( B \) in \( F^* \) with \( \partial B \cap (\cup \mathcal{W}) = \partial B \), the boundary of \( B \) is not contained in different two solid tori.

(iii) There exists a vertex \( u(V_i) \) of \( G(\mathcal{W}) \) such that if \( V_i \neq V_1 \) then \( n(V_i) = 1 \).

(iv) If \( i \neq j \), then \( V_i \cap V_j \) is either one simple closed curve or empty. The vertex \( u(V_i) \) is called the *special vertex*.

**Example 3.10.** We give not good solid tori sequences as follows. Let \( M \) be the 1-complex in \( \overline{B}_0 \) as shown in Fig. 6, and let \( D_1, D_2, D_3, D_4 \) be the closures of the bounded components of \( \overline{B}_0 \setminus M \) as shown in Fig. 6. We naturally embed the 2-complex \( M \times S^1 \subset \overline{B}_0 \times S^1 \) in \( S^4 \) via \( \psi \).

(i) The solid tori sequence \( \mathcal{W}_1 = \{ D_1 \times S^1, D_2 \times S^1, D_3 \times S^1 \} \) is not a good solid tori for \( M \times S^1 \), because \( G(\mathcal{W}_1) \) is a circle.

(ii) Let \( A \) be the closure of a component of \( M \setminus D_1 \cup D_2 \). Then \( A \) is an arc in \( \partial D_3 \). The solid tori sequence \( \mathcal{W}_2 = \{ D_1 \times S^1, D_2 \times S^1 \} \) is not a good solid tori sequence for \( M \times S^1 \), because there exists the annulus \( A \times S^1 \) with \( (\partial A \times S^1) \cap (\partial D_i \times S^1) \neq \emptyset \) for \( i = 1, 2 \).

(iii) Let \( L, \alpha_3(b, c), W_1, W_2 \) be as in Example 3.8. Suppose that \( b, c \) are integers with \( b > 1 \) and \( c > 1 \). Then the knot \( L \) wraps \( b \) times in the longitudinal direction of \( W_1 \), and then \( L \) wraps \( c \) times in the longitudinal direction of \( V_2 \). Moreover, \( n(W_1) = b \)
and \( n(V_2) = c \). Since \( b > 1 \) and \( c > 1 \), \( \{W_1, V_2\} \) is not a good solid tori sequence for \( \alpha_3(b, c) \).

However, there exist good solid tori sequences \( \mathfrak{V} \) and \( \mathfrak{W} \) for \( M \times S^1 \) and \( \alpha_3(b, c) \), respectively, such that \( \alpha_3(b, c) \subset (\cup \mathfrak{V}) \) and \( M \times S^1 \subset (\cup \mathfrak{W}) \). In the case of \( M \times S^1 \), let \( D = D_1 \cup D_2 \cup D_3 \cup D_4 \), then \( \mathfrak{W} = \{ D \times S^1 \} \) is a desired solid tori sequence. In the case of \( \alpha_3(b, c) \), since \( V_2 \) is a standard solid torus, \( W = S^3 \setminus V_2 \) is a solid torus with \( W_1 \subset W \). Hence, \( \mathfrak{W} = \{ W \} \) is a desired solid tori sequence.

For a coiled solid torus \( X \), we define \( n(X) = 2 \). For an almost solid tori sequence \( \mathfrak{V} \), we construct the graph \( G(\mathfrak{V}) \) in a similar way as above.

**Definition 3.11.** Let \( F \) be an embedded surface in \( \mathbb{R}^4 \) satisfying (K0), (K1) and (K2). Let \( X \) be a coiled solid torus, and \( \mathfrak{V} = \{ X, V_1, V_2, \ldots, V_k \} \) an almost solid tori sequence for \( F^* \). An almost solid tori sequence \( \mathfrak{V} \) is **good**, if \( \mathfrak{V} \) satisfies the following four conditions:

(i) \( G(\mathfrak{V}) \) is a connected tree.
(ii) If \( B \) is an annulus with \( B \subset F^* \) and if \( (\cup \mathfrak{V}) \cap B = \partial B \), then \( \partial B \subset \partial V_i \) for some \( i \) or \( \partial B \subset X \cap S^3 \setminus X \).
(iii) \( n(V_i) = 1 \) for all solid tori \( V_i \).
(iv) If \( i \neq j \), then \( V_i \cap V_j \) and \( X \cap V_i \) are one simple closed curve or empty.

The vertex \( n(X) \) is called the **special vertex**.

Let \( \mathfrak{V} = \{ V_1, \ldots, V_k \} \) be a (almost) solid tori sequence. If \( V_i \cap V_j \) is a simple closed curve, let \( \mathcal{N}_{ij} \) be a regular neighborhood of \( V_i \cap V_j \) in \( S^3 \). If \( V_i \cap V_j = \emptyset \), let \( \mathcal{N}_{ij} = \emptyset \). If \( V_i \cap V_j \) is an annulus, let \( \mathcal{N}_{ij} = V_i \cap V_j \). Then we say that \( (\cup \mathfrak{V}) \cup (\cup \mathcal{N}_{ij}) \) is a shape of \( \mathfrak{V} \).

**Lemma 3.12** ([15, Lemma 3.4]). Let \( \{ V_1, V_2 \} \) be a solid tori sequence. Let \( V \) be a shape of \( \mathfrak{V} \).

1. If \( V \) is a solid torus, then \( n(V_1) = 1 \) or \( n(V_2) = 1 \).
2. If \( V \) is not a solid torus, then \( n(V_1) > 1 \), \( n(V_2) > 1 \), and \( V_1, V_2 \) are standard solid tori in \( S^3 \).

Here a standard solid torus means a regular neighborhood of a trivial knot in \( S^3 \).

**Lemma 3.13.** Let \( \{ V_1 \}, \{ V_2 \} \) be solid tori sequences such that \( V_2 \subset V_1 \), and \( \partial V_1 \cap \partial V_2 \) is one simple closed curve or an annulus. If \( n(V_2) \) is not equal to 0, 1, and \( \infty \), then \( \partial V_1 \cup \partial V_2 \) can be moved a 2-complex obtained from one of Fig. 7 (1), (3) by an ambient isotopy of \( S^3 \). Hence \( V_1 \) can be moved to \( V_2 \) by an ambient isotopy of \( S^3 \).

Proof. In the case that \( \partial V_1 \cap \partial V_2 \) is an annulus, by [12, Lemma 2.1] the annulus \( B = \text{Int} V_1 \cap \partial V_2 \) is parallel to a boundary annulus in \( \partial V_2 \). The annulus \( B \) is decomp...
posed $V_1$ into two solid tori $V_2$ and $\overline{V_2 \setminus V_1}$. Note that cutting a meridian disk of $V_2$, then we have Fig. 7 (1) which is the intersection of the meridian disk and $\partial V_2$. Since $n(V_2) \neq 0, 1, \infty$, $V_1$ can be moved to $V_2$ by an ambient isotopy of $S^3$.

In the case that $\partial V_1 \cap \partial V_2$ is one simple closed curve $C$, let $N$ be a regular neighborhood of $C$ in $V_1$. Let $K = (\partial V_1 \cup \partial V_2) \setminus N \cup \text{Int} V_1 \cap \partial N$. Then, the solid tori sequence $\{V_1 \setminus N, V_2 \setminus N\}$ for $K$ satisfies the above condition. Cutting a meridian disk of $V_1$, then we have Fig. 7 (2) or (3) which is the intersection of the meridian disk and $\partial V_2$. If $\partial V_2 \cap \partial V_1$ is a longitude curve of $V_2$, i.e., $n(V_2) = 1$, then we see Fig. 7 (2). We have that $V_2$ can be moved to $V_1$ by an ambient isotopy of $S^3$ if and only if we see Fig. 7 (3). Since $n(V_2) \neq 0, 1, \infty$, $V_1$ can be moved to $V_2$ by an ambient isotopy of $S^3$. □

Remark 3.14. Let $F$ be an embedded Klein bottle in $\mathbb{R}^4$ satisfying (K1) and (K2). Let $\mathcal{V}$ be a good almost solid tori sequence for $F^*$, $C$ the M-curve in the coiled solid torus $X$. Let $N$ be a regular neighborhood of $C$ in $S^3$, $X' = X \cup N$, $K = (F^* \setminus N) \cup (\partial N \cap \partial X')$. Then $X'$ is a solid torus, $\mathcal{V}' = \{X'\} \cup (\mathcal{V} \setminus \{X\})$ is a good solid tori sequence for $K$ with $n(X') = 2$.

Lemma 3.15. Let $F$ be an embedded surface satisfying (K0), (K1) and (K2). Let $\mathcal{V}$ be a good (almost) solid tori sequence for $F^*$ such that $n(V_1) = 2$, where $n(V_1)$ is the special vertex. Let $C$ be an A-curve in $\cup \mathcal{V}$, and $V$ a shape of $\mathcal{V}$. Then $V$ is a coiled solid torus if $\mathcal{V}$ is almost, and $V$ is a solid torus otherwise. Moreover, $|C| = \pm 2 \in H_1(V)$.

Proof. In the case of a solid tori sequence, we showed in [14, Lemma 7.5]. So, we may assume that $\mathcal{V}$ is almost. By Remark 3.14, $\mathcal{V}$ can be changed a solid tori sequence. Given that $N$ is a regular neighborhood of the M-curve in $S^3$, we have $V \cup N$ is a solid torus, and $|C| = \pm 2 \in H_1(V \cup N) \cong H_1(V)$. This and Lemma 3.13 imply...
that $V$ can be moved to the coiled solid torus $X$ in $\mathcal{V}$ by an ambient isotopy of $S^3$.

Lemma 3.16. Let $F$ be an embedded surface satisfying (K0), (K1) and (K2). Let $\mathcal{V}$ be a good (almost) solid tori sequence for $F^*$ with $\cup \mathcal{V} \not\ni F^*$, and $n(V_1) = 2$, where $\nu(V_1)$ is the special vertex. Then, there exists a solid torus $V$ such that $\partial V \subset F^*$ and $\partial V \cap (\cup \mathcal{V})$ is a simple closed curve or an annulus, $n(V) = 2$ if $V$ contains $V_1$, and $n(V) = 1$ otherwise. Moreover, if the M-curve is a trivial knot in $S^3$, then there exists a coiled solid torus $X$ with $X \cap (\mathcal{V} \setminus \mathcal{X}) \subset F^*$ such that $X$ can be moved to the 3-complex $X_b$ for some integer $b$ of an ambient isotopy of $S^3$, where $X_b$ is the set in Definition 3.5. In particular, if $b = 0$ or $-1$, then we can take a solid torus $V$ with $n(V) = 1$.

Proof. By Remark 3.14, it suffices to prove for a solid tori sequence. Let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be a good solid tori sequence. Since $\cup \mathcal{V} \not\ni F^*$, by the definition of good, there exists a torus or an annulus, $B$, in $F^*$ such that

$$B \cap (\cup \mathcal{V}) = \begin{cases} \text{one simple closed curve,} & \text{if } B \text{ is a torus,} \\ B \cap \partial V_i = \partial B, & \text{if } B \text{ is an annulus.} \end{cases}$$

By the solid torus theorem in [10], there exists a solid torus $V$ with $B \subset \partial V \subset F^*$. We see that $\partial V \cap (\cup \mathcal{V})$ is a simple closed curve or an annulus. Let $C$ be a component of $\Gamma^*(F)$ in $\partial V \cap (\cup \mathcal{V})$.

Case 1. $V$ contains $V_1$.

Let $\mathcal{V}' = \{V_i \in \mathcal{V} \mid V_i \subset V\}$. Then $\mathcal{V}'$ is a good solid tori sequence for $F^*$. By Lemma 3.15, a shape $V'$ of $\mathcal{V}'$ is a solid torus and $V_1 \subset V'$. By $[C] = \pm 2 \in H_1(V')$ and Lemma 3.13, we can show that $V'$ can be moved to $V$ by an ambient isotopy of $S^3$. This implies $n(V) = 2$.

Case 2. $V$ does not contain $V_1$.

Let $\mathcal{V}' = \{V_i \in \mathcal{V} \mid V_i \not\subset V\}$, then $\mathcal{V}'$ is a good solid tori sequence for $F^*$. By Lemma 3.15, a shape $V'$ of $\mathcal{V}'$ is a solid torus. Since $V' \cap V = \partial V' \cap \partial V$ is a simple closed curve or an annulus, by Lemma 3.12, $n(V) = 1$ or $V$ is standard. If $V$ is standard, then this case can be proved in a similar way to Case 1 by replacing $V$ by $S^3 \setminus V$. If $n(V) = 1$, then there is nothing to do.

Moreover, we assume that $\mathcal{V}'$ is a good almost solid tori sequence and the M-curve is a trivial knot in $S^3$. Then there exists a 2-complex $K \subset F^*$ such that $K$ is a projection of an embedded Klein bottle satisfying (K1) and (K2). $K$ contains only one M-curve and no A-curve. By Lemma 3.7, there exists a coiled solid torus $X$. Since the M-curve is a trivial knot, we can easily prove that $X$ can be moved to the 3-complex $X_b$ for some $b$ of an ambient isotopy of $S^3$. Suppose that $b = 0$ or $-1$. In the case of $n(V) = 1$, there is nothing to do. Suppose $n(V) = 2$. Let $\mathcal{V}' = \{V_i \in \mathcal{V} \mid V_i \subset V\}$ and let $V'$ be a shape of $\mathcal{V}'$. Then $V' \subset V$, $\partial V' \cap \partial V$ is an
annulus or a simple closed curve, and \( V' \) is the coiled solid torus by Lemma 3.13. So we may assume \( V' = X_b \). Since the M-curve is a trivial knot, \( V \) is a standard solid torus. Let \( W = S^3 \setminus \bar{V} \). Since \( b = 0 \) or \(-1\), a simple closed curve of \( \partial V' \cap \partial V \) is homologous to \( \pm 2l \pm m \in H_1(\partial V) \), where \( m \) is a meridian curve of \( V \), \( l \) is a preferred longitude of \( V \). This implies \( n(W) = 1 \), and \( W \) is a desired solid torus.

**Proposition 3.17.** Let \( F \) be an embedded surface satisfying (K0), (K1) and (K2). Then there exists a good (almost) solid tori sequence \( \mathcal{W} \) for \( F^* \) with \( \cup \mathcal{W} \supseteq F^* \). Moreover, suppose that the M-curve in \( F^* \) is a trivial knot in \( S^3 \), and suppose that there exists a good almost solid tori sequence \( \{X\} \) for \( F^* \) such that \( X \) can be moved to the 3-complexes \( X_0 \) or \( X_{-1} \) of an ambient isotopy of \( S^3 \). Then we can take that \( \mathcal{W} \) is almost.

Proof. We only prove for the case that \( F^* \) contains an M-curve. There exists a good almost solid tori sequence \( \{X\} \) for \( F^* \) such that \( X \) is maximal, i.e., \( X \) is not contained in another coiled solid torus. We prove by induction on the number of the components of \( F^* \setminus \Gamma^*(F) \) in a good (almost) solid tori sequence. Let \( \mathcal{W} \) be a good (almost) solid tori sequence for \( F^* \). If \( \cup \mathcal{W} \not\supset F^* \), then by Lemma 3.16 there exists a solid torus \( V \) satisfying the condition in Lemma 3.16. By Lemma 3.16, there exists only one solid torus \( V_j \in \mathcal{W} \) such that \( (\cup \mathcal{W}) \cap \partial V = V_j \cap \partial V \) is an annulus or a simple closed curve. Let \( \hat{V} = V \cup V_j \) if \( V_j \cap \partial V \) is an annulus, let \( \hat{V} = V \) otherwise. Since \( n(V_j) = 1 \), \( \hat{V} \) is a solid torus. We have a good solid tori sequence \( \mathcal{W} = \{V_i \in \mathcal{W} \mid V_i \not\subseteq \hat{V}\} \cup \{V\} \) for \( F^* \) with \( \cup \mathcal{W} \subseteq \cup \mathcal{W} \). In particular, if the M-curve is trivial, and if the coiled solid torus \( X \in \mathcal{W} \) can be moved to the 3-complexes \( X_0 \) or \( X_{-1} \) of an ambient isotopy of \( S^3 \), by Lemma 3.16, then \( n(V) = 1 \), and \( \mathcal{W} \) contains the coiled solid torus \( X \). Inductively, this completes the proof of Proposition 3.17.

**Proposition 3.18.** Let \( F \) be an embedded surface satisfying (K0), (K1) and (K2). Let \( \mathcal{W} \) be a good (almost) solid tori sequence for \( F^* \) with \( \cup \mathcal{W} \supseteq F^* \), \( n(V_i) \neq 0 \) and \( n(V_i) \neq \infty \), where \( v(V_i) \) is the special vertex. Then \( F^* \) can be moved to a 2-complex obtained by a \( c \)-symmetric 1-complex by an ambient isotopy of \( S^3 \), where \( b = n(V_i) \), \( (b, c) = 1 \). In particular, if \( \mathcal{W} \) is almost, then \( b = 2 \).

Proof. In the case that \( \Gamma^*(F) \) consists only \( A \)-curves, we showed in [14, Proposition 7.15].

Assume that \( \Gamma^*(F) \) contains one \( M \)-curve. Let \( C \) be the M-curve, \( N \) a regular neighborhood of \( C \) in \( S^3 \), \( V = V_i \cup N \) and \( K = (F^* \setminus N) \cup (\partial N \cap \partial V) \). By Remark 3.14, \( \mathcal{W} \setminus \{V_i\} \cup \{V\} \) is a good solid tori sequence for \( K \). Since it is true for the case of only \( A \)-curves, we see that \( K \) is a 2-complex obtained from some symmetric 1-complex. Hence, \( F^* \) is also a 2-complex obtained from some symmetric 1-complex.
4. Spun Klein bottles

Proposition 4.1. Let $F$ be an embedded Klein bottle in $\mathbb{R}^4$ such that $\Gamma^*(F)$ consists only of double points, and each component of $\Gamma(F)$ is not homotopic to zero in $\pi_1(F)$. Then $F^*$ is the projection into $\mathbb{R}^3$ of a spun Klein bottle in $\mathbb{R}^4$. In particular, $F$ is ambient isotopic to a simple spun Klein bottle in $\mathbb{R}^4$.

Proof. By [16, Remark 1.5], the number of components of $\Gamma(F)$ is even. Hence, by Lemma 3.1, $(\Gamma(F), F)$ is homeomorphic to $(\Gamma_1, F)$. We see that $\Gamma^*(F)$ consists only of A-curves and one M-curve. By Proposition 3.17, there exists a good (almost) solid tori sequence $\mathcal{V}$ for $F^*$ with $F^* \subset \cup \mathcal{V}$. By Proposition 3.18 and Remark 3.4, there exists a spun Klein bottle $K\mathcal{F}(K)$ in $\mathbb{R}^4$ such that $F$ is ambient isotopic to $K\mathcal{F}(K)$.

If $a \neq 0$ and $a \neq -1$, by Remark 2.1 (2), then we may assume that $a = 0$ or $-1$, and the M-curve of $F^*$ is a trivial knot. Applying Proposition 3.18 again, we obtain a good almost solid tori sequence $\mathcal{V}$ for $F^*$ with $F^* \subset \cup \mathcal{V}$. Hence $F$ is simple. \hfill $\square$

5. Diagrams for embedded surfaces

For an embedded surface, we define a ‘diagram’ in $\mathbb{R}^3$. In classical knots, it is convenient to represent by a diagram, i.e., an immersed closed curve in the plane that has crossing information indicated at its double points. A ‘diagram’ for an embedded surface is like a diagram of classical knots.

Let $\varphi : F \to \mathbb{R}^3$ be an immersion of a closed surface $F$ (possibly disconnected, non-orientable) such that the singular set of $\varphi$ has only transverse double points; each component of its is a circle. Such a circle is called a crossing circle. A diagram $D$ is an immersion of a union of 2-spheres and a Klein bottle with a mark at each crossing circle satisfying the two conditions:

(i) For any crossing circle $C$, let $N$ be a regular neighborhood of $C$ in $\mathbb{R}^3$. Then $N \cap \text{Im } D$ consists of two annuli or two Möbius bands, say $A_1, A_2$.
(ii) One of $A_1, A_2$ is marked either by ‘a’ (for ‘above’) or by ‘b’ (for ‘below’).

We define that there is a mark ‘a’ on $A_i$ if and only if there is a mark ‘b’ on $A_j$ (i ≠ j).

We usually place a mark ‘a’ or ‘b’ on only one $A_i$. A surface $A_i$ with mark ‘a’ (resp. ‘b’) is called an a-tube (resp. a b-tube). We define the associated embedded surface $L_D$ of a diagram $D$ by the following properties.

(i) $p(L_D) = \text{Im } D$, where $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the projection onto $\mathbb{R}^3$.
(ii) $L_D \cap (\mathbb{R}^3 \times \{0\}) = (\text{Im } D \setminus \text{Int}(a-tubes in } D)) \times \{0\}$, and $L_D \subset \mathbb{R}^3 \times \{0, \infty\}$.

These conditions determine an embedded surface up to ambient isotopy.

5.1. 1-handles for diagrams. In this subsection, we define a 1-handle for a diagram.

Let \( D \) be a diagram. Let \( h_i: B^2 \times I \to \mathbb{R}^3, i = 1, 2, \ldots, m, \) be a collection of embeddings with mutually disjoint images such that

\[
h_i(B^2 \times I) \cap \text{Im } D = h_i(B^2 \times \{0, t_1, \ldots, t_k, 1\})
\]

for some \( t_1, t_2, \ldots, t_k \) with \( 0 < t_1 < t_2 < \cdots < t_k < 1 \), where \( B^2 \) is a 2-ball and \( I = [0, 1] \). Define the immersed surface \( D + \sum_{i=1}^{m} h_i \) to be

\[
\left( \text{Im } D \setminus \bigcup_{i=1}^{m} h_i(B^2 \times \partial I) \right) \cup \left( \bigcup_{i=1}^{m} h_i(\partial B^2 \times I) \right).
\]

We call the embedding \( h_i \) a 1-handle on the diagram \( D \), and the diagram with \( D + \sum_{i=1}^{m} h_i \) a diagram obtained from \( D \) by attaching 1-handles. For a 1-handle \( h_i \), we call the disks \( h_i(B^2 \times 0) \) and \( h_i(B^2 \times 1) \), attaching disks, the disk \( h_i(B^2 \times t) \), \( 0 < t < 1 \), a cocore of \( h_i \), and the arc \( h_i(x \times I), x \in \text{Int } B^2 \), a core; see [7, Fig. 1].

5.2. Local moves. Local moves between diagrams are defined in [7]. They do not change the ambient isotopy classes of associated embedded surfaces of diagrams. Now, we define three of them.

\((\Omega 1)\) Moving a 1-handle through a sheet as shown in Fig. 8, where \( c_1, c_2 \in \{a, b\} \) and

\[
c_3 = c_4 = \begin{cases} c_1 & \text{if } c_1 = c_2, \\ c_2 & \text{either } a \text{ or } b & \text{if } c_1 \neq c_2. \end{cases}
\]

This move adds two crossing circles. (cf. Fig. 4 in [19])

\((\Omega 2)\) Sliding a 1-handle through a sheet as shown in Fig. 9, where \( c_1 = c_2 \in \{a, b\} \). This move adds one crossing circle.
Pulling out a 2-sphere with 1-handles across a sheet as shown in Fig. 10, where $S$ is a 2-sphere bounding a 3-ball $B$, and $h_i$, $1 \leq i \leq p + q + r$, are 1-handles such that

(i) $h_1, \ldots, h_p$ are passing through $S$,
(ii) $h_{p+1}, \ldots, h_{p+q+r}$ are attached on $S$ whose one attaching disks are in $S$,
(iii) the pair $(B, B \cap (\bigcup_{i=1}^{p+q} \alpha_i))$, where $\alpha_i$ is a core of $h_i$, is a trivial tangle, meaning that it is homeomorphic to the pair $(D^2, \{x_1, \ldots, x_{p+q}\}) \times [0, 1]$, where $x_i$ are interior points of the 2-ball $D^2$, and
(iv) \( c_i, c_j, d \in \{a, b\} \), where \( 1 \leq i \leq p + q, 1 \leq j \leq p \).

The following move is a generalization of the move \((6\omega)\). 

\( (6\omega) \) Pulling out a 2-sphere \( S \) across a sheet as shown in Fig. 11, where \( S \) is bounding a 3-ball \( B \), and \( h_i, 1 \leq i \leq p + q \), are 1-handles such that

(i) \( h_1, \ldots, h_{p+q} \) are passing through \( S \) or are attached on \( S \), and

(ii) \( d \in \{a, b\} \).

(cf. Lemma 4.6 in [19])

A diagram \( D \) is with good position, if it is obtained by attaching 1-handles from 2-spheres \( S_1, \ldots, S_m \) and an immersed Klein bottle \( K \) in \( \mathbb{R}^3 \) such that

(i) \( K \) is the projection of an embedded Klein bottle in \( \mathbb{R}^3 \) satisfying (K1) and (K2), and

(ii) there exist disjoint 3-balls \( B_1, \ldots, B_{m+1} \) in \( \mathbb{R}^3 \) with \( S_i \subset \text{Int} \; B_i \) and \( K \subset \text{Int} \; B_{m+1} \).

Observe that an associated surface as above is a Klein bottle obtained from a spun Klein bottle by \( m \)-fusion. Also, a diagram obtained by attaching 1-handles from only 2-spheres \( S_1, \ldots, S_m \) is called a diagram with good position. Observe that an associated surface of its diagram is a ribbon surface.

**Proposition 5.1.** Any diagram can be transformed into a diagram with good position by a sequence of moves \((\Omega_1), (\Omega_2)\) and \((6\omega)\).

Proof. First of all, we show that any diagram can be transformed into a diagram by attaching 1-handles from disjoint 2-spheres in \( \mathbb{R}^3 \), or a diagram by attaching 1-handles from disjoint 2-spheres and the projection of a spun Klein bottle. Let \( D \) be a diagram obtained from a diagram \( D_0 \) by attaching 1-handles \( h_1, \ldots, h_m \), where \( D_0 \) is the image of an immersion of a surface \( F \). Let \( R(D_0) \) be the components in the
singular set of \(D_0\) in \(\mathbb{R}^3\) such that one of the preimage bounds a disk in \(F\). We use induction on the number of the components in \(R(D_0)\), say \(n\).

In case of \(n = 0\), i.e., \(R(D_0) = \emptyset\), by Proposition 4.1, \(D_0\) is disjoint 2-spheres in \(\mathbb{R}^3\), or disjoint 2-spheres and the projection of a spun Klein bottle in \(\mathbb{R}^3\). This implies the desired result.

Assume it is true for less than \(n\), and the number of the components in \(R(D_0)\) is \(n\). Choose the disk \(E\) in \(D_0\) such that \(\partial E\) is a component of \(R(D_0)\), and \(E\) is a non-singular disk in \(\mathbb{R}^3\). If \(E\) intersects a cocore of a 1-handle, perform the 1-handle by the move \((\Omega 1)\) in Fig. 8. See the first move in Fig. 12. By the move \((\Omega 1)\), two crossing circles appear, but the number of the components in \(R(D_0)\) does not change. If \(E\) intersects an attaching disk of a 1-handle, then perform the 1-handle by the move \((\Omega 2)\) in Fig. 9. See the second move in Fig. 12. Similarly, we see that the number of the components in \(R(D_0)\) does not change. Hence, we may assume that \(E\) does not intersect 1-handles. A regular neighborhood of \(E\) in \(\mathbb{R}^3\) consists of an annulus \(A\) and a disk \(E'\) containing \(E\). By replacing the annulus \(A\) with two disks, each of which is parallel to \(E\). Then we obtain a diagram \(D_1\) such that \(D_0\) is obtained from \(D_1\) by attaching a 1-handle \(h\) such that \(h(\partial B^2 \times I) = A\). Thus, \(D\) is obtained from \(D_1\) by attaching 1-handles \(h_1, \ldots, h_m, h\). The number of the components of \(R(D_1)\) is less than that of \(R(D_0)\), which yields the result.

Next, we consider a diagram obtained by attaching 1-handles \(h_1, \ldots, h_n\) on 2-spheres \(S_1, \ldots, S_m\) and immersed Klein bottle \(K\) such that \(K\) is a 2-complex consisting of annuli. If the 2-spheres and \(K\) are contained in the interior of disjoint 3-balls, respectively, then the diagram is a desired diagram. Otherwise, take a 2-sphere, say \(S_1\), such that \(S_1\) does not contain any other 2-sphere in \(\mathbb{R}^3\). Let \(B, B_i\) be 3-balls in \(\mathbb{R}^3\) such that the interior of \(B\) contains \(K\), \(\partial B \cap S_i = \emptyset\) for all \(i\), and \(\partial B_i = S_i\). If \(B_i\) does not contain \(K\), by a sequence of the move \((\Omega 6)'\), then we pull out \(S_i\) from the 2-sphere that contains \(S_i\). If not, by a sequence of the move \((\Omega 6)'\), then we pull \(K\), and then we pull out \(S_i\) from the 2-sphere that contains \(S_i\). Inductively, we have a diagram with good position. Similarly, we can prove for the case of a diagram obtained by attaching 1-handles on 2-spheres.

The technique in Proposition 5.1 was used in [7] and [19].

6. Proof of the main theorem

From Proposition 5.1, we have:

**Theorem 6.1** (Theorem 1.1). Let \(F\) be an embedded Klein bottle in \(\mathbb{R}^4\). If \(\Gamma^*(F)\) consists of double points, then \(F\) is ambient isotopic to either a ribbon Klein bottle, or a Klein bottle obtained from a spun Klein bottle by \(m\)-fusion.
Fig. 12. A transformation for the case that $E$ intersects one cocore and one attaching disk.

**Lemma 6.2.** Let $L$ be a knot in $S^3$. If $\pi_1(S^3 \setminus L)/\langle m^2 = 1 \rangle$ is isomorphic to $\mathbb{Z}_2$, then $L$ is trivial.

Proof. Let $N$ be a regular neighborhood of $L$ in $S^3$, $E = S^3 \setminus N$, $E_2$ the 2-fold cover, $X_2$ the 2-fold branch cover. Then we obtain the following exact sequences:

$$
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(E_2) & \rightarrow & \pi_1(E) & \rightarrow & \mathbb{Z}_2 & \rightarrow & 1 \\
\downarrow_{m=1} & & \downarrow_{m^2=m^2} & & \downarrow_{m^2=1} & & \downarrow_{\cong} & & \downarrow & & \downarrow & & \downarrow_{\cong} & & 1 \\
1 & \rightarrow & \pi_1(X_2) & \rightarrow & \pi_1(E)/\langle m^2 = 1 \rangle & \rightarrow & \mathbb{Z}_2 & \rightarrow & 1
\end{array}
$$

where $m$ is a meridian curve of $L$. By the above diagram, we have $\pi_1(X_2) \cong 1$. By the Smith Conjecture [9], if $\pi_1(X_2) \cong 1$, then the branch set of $X_2$ is a trivial knot. And we can show that $L$ is trivial. \qed

**Corollary 6.3** (Corollary 1.2). Let $F$ be an embedded Klein bottle in $\mathbb{R}^4$. Suppose that $\Gamma^*(F)$ consists only of double points, and all components of the singular set $\Gamma(F)$ are not homotopic to zero in $\pi_1(F)$. If $\pi_1(\mathbb{R}^4 \setminus F)$ is isomorphic to $\mathbb{Z}_2$, then $F$
is trivial.

Proof. By assumption, $F^*$ consists only of A-curves and one M-curve. By Proposition 4.1, $F$ is ambient isotopic to a simple spun Klein bottle $K^0(L\#(-L))$. By Lemma 6.2 and Remark 2.1 (2), if the fundamental group of the complement of $K^0(L\#(-L))$ is isomorphic to $\mathbb{Z}_2$, then the knot $L$ is trivial in $S^3$. Hence $K^0(L\#(-L))$ is ambient isotopic to a Klein bottle $F'$ such that $\Gamma^*(F')$ consists only of one simple closed curve. Hence $F'$ is a boundary of a solid Klein bottle in $\mathbb{R}^4$. Therefore $F$ is trivial.

6.1. Example of a non-ribbon surface. In [12], [13], and [14], we classified for an embedded torus $T$ whose singular set $\Gamma^*(T)$ consists of at most three disjoint simple closed curves. The twist spun torus of the trefoil knot has the projection into $\mathbb{R}^3$ with the singular set consisting of three disjoint simple closed curves. This example is given in [1] or [14].

Proposition 6.4. The twist spun torus $F$ is not a ribbon surface.

Proof. Suppose that $F$ is a ribbon surface. Let $N$ be a regular neighborhood of $F$ in $\mathbb{R}^4$. Boyle [1] defined the $\mathbb{Z}_2$-invariant $q$ for a curve $c$ in $\partial N$ which is homologous to zero in $\mathbb{R}^4 \setminus \overline{N}$, this is modulo 2 to the intersection number of a surface with boundary $c$ in $\mathbb{R}^4 \setminus \overline{N}$. Then, there exists a unique simple closed curve $C$ on the boundary of $N$ such that $C$ is homotopic to zero in $\mathbb{R}^4 \setminus \overline{N}$. We see that $q(C) = 1$. However, a ribbon torus has a curve $C'$ on $\partial N$ such that $C'$ is homotopic to zero in $\mathbb{R}^4 \setminus \overline{N}$, and $q(C') = 0$. This is a contradiction. Hence, $F$ is not a ribbon surface.

Question 6.5. For a trefoil knot $L$, is the spun Klein bottle $K^0(L\#(-L))$ a non-ribbon surface?

References

NOTED KLEIN BOTTLES WITH DOUBLE POINTS


