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## KNOTTED KLEIN BOTTLES WITH ONLY DOUBLE POINTS

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### 1. Introduction

If an embedded 2-sphere in 4-space  $\mathbf{R}^4$  has the singular set of the projection in 3-space  $\mathbf{R}^3$  consisting of double points, then the 2-sphere is ambient isotopic to a ribbon 2-sphere (see [19]). Similarly, if an embedded torus in  $\mathbf{R}^4$  has the singular set of the projection in  $\mathbf{R}^3$  consisting only of double points, then the torus is ambient isotopic to either a ribbon torus or a torus obtained from a symmetry-spun torus by  $m$ -fusion (see [15]). In this paper we will show a similar theorem for an embedded Klein bottle in  $\mathbf{R}^4$ . The following is the main results in this paper.

**Theorem 1.1.** *Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$ . If the singular set  $\Gamma^*(F)$  of the projection of  $F$  in  $\mathbf{R}^3$  consists only of double points, then  $F$  is ambient isotopic to either a ribbon Klein bottle or a Klein bottle obtained from a spun Klein bottle by  $m$ -fusion.*

**Corollary 1.2.** *Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$ . Suppose that the singular set  $\Gamma^*(F)$  of the projection of  $F$  in  $\mathbf{R}^3$  consists of double points, and every component of the singular set  $\Gamma(F)$  on  $F$  is not homotopic to zero. If the fundamental group of the complement of  $F$  is isomorphic to  $\mathbf{Z}_2$ , then  $F$  is trivial, i.e.,  $F$  bounds a solid Klein bottle in  $\mathbf{R}^4$ .*

Let  $F$  be an oriented closed surface in  $\mathbf{R}^4$ . Is  $F$  trivial if the fundamental group of the complement of  $F$  is isomorphic to  $\mathbf{Z}$ ? In the topological category, the question is affirmatively soloved when it is a 2-sphere (see [3]). In the PL or smooth category, this is an open question, it is affirmatively soloved when  $F$  is one of the following:

- (i)  $F$  is a 1-fusion ribbon 2-knot ([8]).
- (ii)  $F$  is a 2-sphere with four critical points ([11]).
- (iii)  $F$  is a symmetry-spun torus ([17]).
- (iv)  $F$  is a torus whose singular set on the torus consists only of disjoint simple closed curves with non-homotopic to zero in the torus ([15]).

All homology groups are taken with coefficients in  $\mathbf{Z}$ , and all submanifolds are

assumed to be locally flat, throughout in this paper. We will work in the PL category, throughout in this paper. Let  $\mathbf{R}^n$  be the  $n$ -dimensional Euclidean space. Moreover, we regard 3-space  $\mathbf{R}^3$  as the subset  $\mathbf{R}^3 \times \{0\}$  of  $\mathbf{R}^4$ .

The paper is organized as follows. In Section 2, we define a ribbon surface, and a Klein bottle obtained from a spun Klein bottle by  $m$ -fusion. In Section 3, we study certain types of 2-complexes in  $\mathbf{R}^3$ . In Section 4, we define diagrams for embedded surfaces. In Section 5, we consider spun Klein bottles in  $\mathbf{R}^4$ . In Section 6, we prove the main theorem.

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## 2. Preliminaries and definitions

In this section, we define an  $m$ -fusion, a ribbon surface, and a spun Klein bottle.

Let  $F$  be a closed surface. A map  $f$  from  $F$  to  $\mathbf{R}^3$  is a *generic map* if for at every point  $x$  of  $F$ , there exists a regular neighborhood  $N$  of  $f(x)$  in  $\mathbf{R}^3$  such that  $(N, f(F) \cap N)$  is homeomorphic to  $(B^3, Z_1)$ ,  $(B^3, Z_1 \cup Z_2)$ ,  $(B^3, Z_1 \cup Z_2 \cup Z_3)$  or  $(B^3, \text{the cone on a figure 8})$ , where  $B^3$  is the unit 3-ball in  $\mathbf{R}^3$ ,  $Z_i$  is the intersection of  $B^3$  and  $x_j x_k$ -plane ( $\{1, 2, 3\} = \{i, j, k\}$ ). If  $(N, f(F) \cap N)$  is homeomorphic to  $(B^3, \text{the cone on a figure 8})$ , then the point  $f(x)$  is called a *branch point*. The point is also known as “Whitney’s umbrella” or “a pinch point”. A point  $x \in f(F)$  is called a *double point* if  $f^{-1}(x)$  consists of two points, and a *triple point* if  $f^{-1}(x)$  consists of three points.

Let  $F$  be an embedded surface in  $\mathbf{R}^4$ , and let  $p$  be the projection defined by  $p(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3)$ . If  $p|_F$  is a generic map, then we associate the subset  $F^* = p(F)$ , and we denote by  $\Gamma^*(F)$  the set of all double points, triple points and branch points. And put  $\Gamma(F) = p^{-1}(\Gamma^*(F)) \cap F$ . In this paper we assume that  $p|_F$  is a generic map.

An oriented closed surface in  $\mathbf{R}^4$  is said to be *trivial* if it is the boundary of the disjoint union of handlebodies in  $\mathbf{R}^4$ . Note that the boundary of a handlebody is unique up to ambient isotopies of  $\mathbf{R}^4$  (see [5]). An embedded Klein bottle in  $\mathbf{R}^4$  is said to be *trivial* if it is the boundary of a solid Klein bottle in  $\mathbf{R}^4$ . Here the solid Klein bottle is homeomorphic to the 3-manifold by attaching  $B^2 \times \{0\}$  and  $B^2 \times \{1\}$  from  $B^2 \times [0, 1]$  via the map  $q(x, 0) = (-x, 1)$ , where  $B^2$  is the unit 2-ball. In other word, the trivial Klein bottle is ambient isotopic to the surface with projection in  $\mathbf{R}^3$  as illustrated in Fig. 1.

Let  $G$  be an embedded closed surface in  $\mathbf{R}^4$ ,  $I = [0, 1]$ ,  $B^2$  the unit 2-ball. An embedded surface  $F$  in  $\mathbf{R}^4$  is a *surface obtained from  $G$  by  $m$ -fusion* if there exists a collection of embeddings  $h_i: B^2 \times I \rightarrow \mathbf{R}^4$ ,  $i = 1, 2, \dots, m$ , satisfying the following three conditions:

- (i) The images of any two maps  $h_i, h_j$  are disjoint for any distinct  $i, j$ .

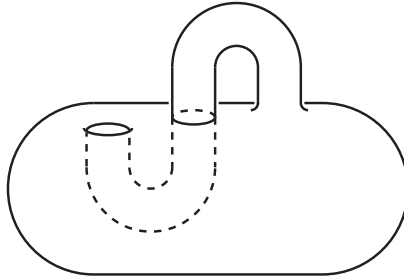


Fig. 1.

- (ii)  $h_i(B^2 \times I) \cap G = h_i(B^2 \times \partial I)$  for all  $i$ .
- (iii)  $F = (G \setminus \bigcup_{i=1}^m (h_i(B^2 \times \partial I))) \cup (\bigcup_{i=1}^m h_i(\partial B^2 \times I))$ .

An embedded surface in  $\mathbf{R}^4$  is a *ribbon surface* if it is obtained from a trivial 2-spheres by  $m$ -fusion.

Next, we define a spun Klein bottle in  $\mathbf{R}^4$ . For  $\theta \in [0, 2\pi]$ , let  $\mathbf{R}_\theta^3 = \{(x, y \cos \theta, y \sin \theta, z) \mid y \geq 0\}$ , and

$$B_0 = \{(x, y, 0, z) \mid x^2 + (y - 2)^2 + z^2 \leq 1\}.$$

Then  $B_0$  is the 3-ball in  $\mathbf{R}_0^3$ , and the union of  $\mathbf{R}_\theta^3$  for all  $\theta \in [0, 2\pi]$  is  $\mathbf{R}^4$ . Let  $r_\theta: B_0 \rightarrow B_0$  be the  $\theta$ -rotation map through the axis  $(0, 2, 0) \times [-1, 1]$  for  $\theta \in [0, 2\pi]$ .

An embedded Klein bottle  $F$  in  $\mathbf{R}^4$  is called a *spun Klein bottle* if there exist an integer  $a$  and a knot  $K$  in the 3-ball  $B_0$  as shown in Fig. 2 (1) such that

- (i)  $K$  intersects two points to the axis  $(0, 2, 0) \times [-1, 1]$ ,
- (ii)  $r_\pi(K) = K$ , and
- (iii)  $F = \{(x, y \cos \theta, y \sin \theta, z) \mid (x, y, 0, z) \in r_{(a+(1/2)\theta)}(K), \theta \in [0, 2\pi]\}$ .

We denote it by  $Kl^a(K)$ . In particular, if  $K$  is a connected sum  $L\#(-L)$  of a knot  $L$  as shown in Fig. 2 (2), then  $Kl^a(K)$  is called a *simple spun Klein bottle*, where  $-L$  is the knot with the reverse orientation of  $L$ . The symbol  $L$  in Fig. 2 (2) is the 1-string tangle so that the tangle sum of  $L$  and the trivial tangle is the knot  $L$ . In particular, a Klein bottle obtained from a split union of a trivial 2-spheres and a spun Klein bottle by  $m$ -fusion is simply called a *Klein bottle obtained from a spun Klein bottle by  $m$ -fusion*.

REMARK 2.1. (1) Let  $Kl^a(L\#(-L))$  be a simple spun Klein bottle. Then, the fundamental group of the complement of  $Kl^a(L\#(-L))$  is isomorphic to  $\pi_1(S^3 \setminus L)/\langle m^2 = 1 \rangle$  where  $m$  is a meridian curve of  $L$  (see [18]).

(2) The Klein bottle  $Kl^a(K)$  is ambient isotopic to  $Kl^{a\pm 2}(K)$  (cf. [17]).

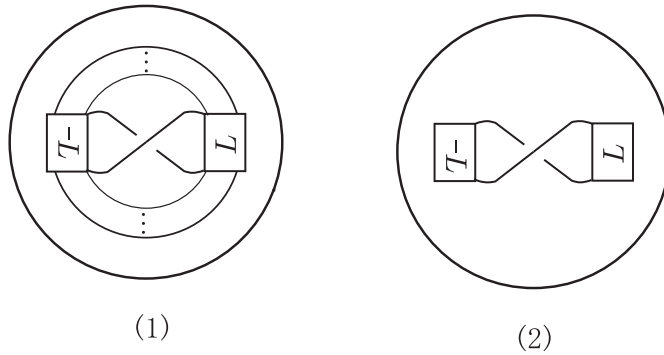


Fig. 2. The center of each figure is z-axis.

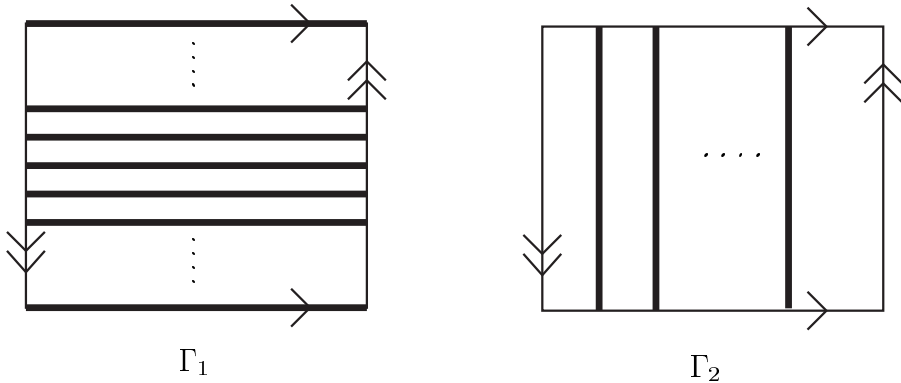


Fig. 3.

### 3. 2-complexes in $\mathbb{R}^3$

**3.1. Embedded Klein bottles in  $\mathbb{R}^4$ .** Let  $F$  be an embedded Klein bottle in  $\mathbb{R}^4$  such that  $p|_F$  is a generic map. In this section, we assume that  $\Gamma^*(F)$  consists only of double points. First, we consider the singular set  $\Gamma(F)$  on  $F$ . Let  $c_1 = 0 \times I$ ,  $c_2 = (1/2) \times I$ ,  $c_i = i/(2n+1) \times I \cup (2n+1-i)/(2n+1) \times I$ , and  $d_j = I \times j/(2n)$  where  $i = 3, \dots, 2n$  and  $j = 1, 2, \dots, 2n-1$ . Let  $\Gamma_1 = c_1 \cup c_2 \cup \dots \cup c_n / \sim$ ,  $\Gamma_2 = d_1 \cup d_2 \cup \dots \cup d_{2n-1} / \sim$  where  $\sim$  is the relation on  $I \times I$  with  $(0, t) \sim (1, t)$  and  $(t, 0) \sim (1-t, 1)$  for all  $t \in I$ . Then each of  $\Gamma_1$  and  $\Gamma_2$  is a union of disjoint simple closed curves on a Klein bottle (see Fig. 3). Note that  $\Gamma_2$  consists of an *odd* number of disjoint simple closed curves.

**Lemma 3.1** ([16, Lemma 1.4]). *Let  $F$  be a Klein bottle in  $\mathbb{R}^4$  such that  $\Gamma^*(F)$  consists only of double points. Let  $\Gamma$  be the union of the components of  $\Gamma(F)$  each of*

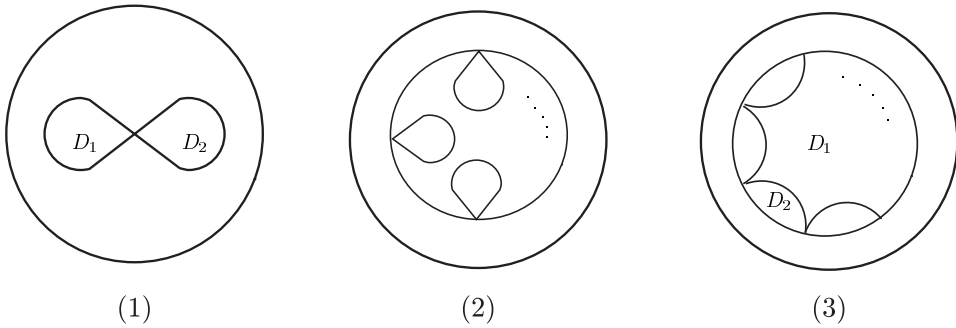


Fig. 4.

which is not homotopic to zero in  $F$ . Then the pair  $(\Gamma, F)$  is homeomorphic to  $(\Gamma_1, F)$  or  $(\Gamma_2, F)$ .

**3.2. Certain types of 2-complexes in  $\mathbf{R}^3$ .** In this subsection, we define certain types of 2-complexes in  $\mathbf{R}^3$ . For  $\theta \in [0, 2\pi]$ , let  $\mathbf{R}_\theta^2 = \{(x, y \cos \theta, y \sin \theta) \mid y \geq 0\}$ , and

$$\overline{B}_0 = \{(x, y, 0) \mid x^2 + (y - 2)^2 \leq 1\}.$$

Then  $\overline{B}_0$  is the 2-ball in  $\mathbf{R}_0^2$ , and the union of  $\mathbf{R}_\theta^2$  for all  $\theta \in [0, 2\pi]$  is  $\mathbf{R}^3$ . Let  $\overline{r}_\theta: \overline{B}_0 \rightarrow \overline{B}_0$  be the  $\theta$ -rotation map through the point  $(0, 2, 0)$  for  $\theta \in [0, 2\pi]$ . Let  $\alpha$  be a 1-complex in  $\overline{B}_0$  such that each vertex is a vertex of degree four or three. A 2-complex  $K$  in  $\mathbf{R}^3$  is called a 2-complex obtained from  $\alpha$  if there exist integers  $b, c$  with  $c \neq 0$  such that

- (i) If  $\alpha$  intersects the point  $(0, 2, 0)$ , then the point  $(0, 2, 0)$  is the vertex of degree four and  $c = 2$ .
- (ii)  $\overline{r}_{2\pi/c}(\alpha) = \alpha$ , and
- (iii)  $K = \{(x, y \cos \theta, y \sin \theta) \mid (x, y, 0) \in \overline{r}_{(b/c)\theta}(\alpha), \theta \in [0, 2\pi]\}$ ,

We denote the 2-complex  $K$  by  $\alpha(b, c)$ , and the above 1-complex  $\alpha$  is called a  $c$ -symmetric 1-complex.

EXAMPLE 3.2. (i) Let  $\alpha_1$  be the 2-symmetric 1-complex in  $\overline{B}_0$  as shown in Fig. 4 (1) such that the vertex of  $\alpha_1$  is the point  $(0, 2, 0)$ . Then if  $b$  is an odd integer (resp. even integer), then the 2-complex  $\alpha_1(b, 2)$  is an immersed Klein bottle (resp. torus) in  $\mathbf{R}^3$ .

(ii) Let  $c$  be an integer with  $c \neq 0$ , and  $\alpha_i$  the  $c$ -symmetric 1-complex in  $\overline{B}_0$  as shown in Fig. 4 (i) such that  $\alpha_i$  does not intersect the point  $(0, 2, 0)$  for  $i = 2, 3$ . Then  $c$  is the number of vertices of  $\alpha_i$ , and then the 2-complex  $\alpha_i(b, c)$  is immersed tori for any integer  $b$ .

**Lemma 3.3.** *Let  $\alpha$  be a  $c$ -symmetric 1-complex, and  $\alpha(b, c)$  a 2-complex in  $\mathbf{R}^3$  obtained from  $\alpha$ .*

(1) *Let  $C$  be a component of  $S(\alpha(b, c))$ . Then, a regular neighborhood of  $C$  in  $\alpha(b, c)$  is two immersed annuli, two immersed Möbius bands. Moreover, there is at most one regular neighborhood consisting of two immersed Möbius bands.*

(2) *Removing  $S(\alpha(b, c))$ , we obtain open annuli.*

*Here  $S(\alpha(b, c))$  is the set of all point whose neighborhood in  $\alpha(b, c)$  is the intersection of two sheets or  $Y \times [0, 1]$ , where  $Y$  is the cone on three points.*

*Proof.* (1) If  $c = 2$ , if  $b$  is odd, and if  $\alpha$  intersects the point  $(0, 2, 0)$  in  $\overline{B}_0$ , then we have the component with  $(0, 2, 0)$  in  $S(\alpha(b, c))$  whose regular neighborhood in  $\alpha(b, c)$  consists of two immersed Möbius bands. Conversely, such a component can be obtained only as above, which yields the result.

(2) From the condition (ii) of the definition of symmetric 1-complexes, we can show (2). □

From Lemma 3.3, we have the following remark:

**REMARK 3.4.** (1) Let  $b, c$  be integers with  $c \neq 0$ , and  $\alpha$  a  $c$ -symmetric 1-complex in  $\overline{B}_0$ . If  $\alpha(b, c)$  is an immersed Klein bottle, then  $b$  is odd,  $c = 2$  and there exists a knot  $K$  in  $B_0$  with  $(Kl^{(b-1)/2}(K))^* = \alpha(b, 2)$ .

(2) Let  $K$  be a knot in  $B_0$  satisfying (i) and (ii) in the definition of spun Klein bottles. Then for any integer  $a$ , the projection  $(Kl^a(K))^*$  in  $\mathbf{R}^3$  is the 2-complex obtained from  $p(K)$ , i.e.,  $(Kl^a(K))^* = p(K)(2a + 1, 2)$ .

**DEFINITION 3.5.** Let  $\alpha_1$  be the 2-symmetric 1-complex as shown in Fig. 4 (1) with  $\alpha_1 \subset \overline{B}_0$ . Then there exist two 2-balls  $D_1, D_2$  in  $\overline{B}_0$  such that  $D_1 \cap D_2$  is the point  $(0, 2, 0)$  and  $\alpha_1 = \partial D_1 \cup \partial D_2$ . For an integer  $b$ , the 3-complex  $X_b$  is defined by

$$X_b = \{(x, y \cos \theta, y \sin \theta) \mid (x, y, 0) \in \overline{F}_{\{(2b+1)/2\}\theta}(D_1 \cup D_2), \theta \in [0, 2\pi]\}.$$

Note that the closure of one component of  $\mathbf{R}^3 \setminus \{\alpha_1(b, 2)\}$  is  $X_b$ . Let  $S^1$  be the unit 1-sphere. Then, the 1-sphere  $S^1$  is identified with  $[0, 2\pi]/0 \sim 2\pi$ . We have a natural embedding  $\psi$  of the solid torus  $\overline{B}_0 \times S^1$  in  $\mathbf{R}^3$  defined by  $\psi(x, y, \theta) = (x, y \cos \theta, y \sin \theta)$ . Let  $g: \overline{B}_0 \times S^1 \rightarrow \mathbf{R}^3$  be an embedding. Then  $g(\psi^{-1}(X_b))$  is also a 3-complex in  $\mathbf{R}^3$ . We call it a *coiled solid torus*. Let  $\alpha$  be a  $c$ -symmetric 1-complex. Then we also call  $g(\psi^{-1}(\alpha(b, c)))$  a 2-complex obtained from  $\alpha$  for any integer  $b$ .

Let  $F$  be an embedded surface in  $\mathbf{R}^4$  such that

(K0)  $F$  is the disjoint union of one Klein bottle and tori, or the disjoint union of tori,

(K1)  $\Gamma^*(F)$  consists only of double points, and

(K2) each component of  $\Gamma(F)$  is not homotopic to zero in  $F$ , and  $F^*$  is connected.

From Lemma 3.1, we have the following lemma.

**Lemma 3.6.** *Let  $F$  be as above. Then we have the following.*

- (1)  $F^* \setminus \Gamma^*(F)$  consists of open annuli.
- (2) Let  $C$  be a component of  $\Gamma^*(F)$ , and  $N(C)$  a regular neighborhood of  $C$  in  $\mathbf{R}^3$ . Then  $N(C) \cap F^*$  consists of two immersed annuli or two immersed Möbius bands.

A curve  $C$  is an  $A$ -curve if  $N(C) \cap F^*$  is two immersed annuli, and is an  $M$ -curve if  $N(C) \cap F^*$  is two immersed Möbius bands.

In the case of classical knots, any knot diagram in  $\mathbf{R}^2$  can be considered in the 2-sphere. Because, by ambient isotopies the bounded region of  $\mathbf{R}^2 \setminus \{\text{a knot projection}\}$  can be changed. Similarly, without loss of generality we may consider that the projection of knotted surfaces is in the 3-sphere  $S^3$ . Here, we consider the 3-sphere  $S^3$  as a one point compactification of  $\mathbf{R}^3$ . We discuss about a 2-complex which is the projection into  $\mathbf{R}^3$  of an embedded surface in  $\mathbf{R}^4$  satisfying (K0), (K1) and (K2). Note that the above 2-complex is called a 2-complex consisting of annuli in [14]. From now on, we assume that such a projection is in the 3-sphere  $S^3$  in this section.

**Lemma 3.7** ([16, Lemma 2.1]). *Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$  such that  $\Gamma^*(F)$  consists only of one simple closed curve, and each component of  $\Gamma(F)$  has a Möbius band neighborhood. Then there exists an odd integer  $b$  and an embedding  $g: \overline{B}_0 \times S^1 \rightarrow S^3$  such that  $F^*$  can be moved to the 2-complex  $g(\psi^{-1}(\alpha_1(b, 2)))$  by an ambient isotopy of  $S^3$ , where  $\alpha_1$  is the 2-symmetric 1-complex as shown in Fig. 4 (1).*

**3.3. Good solid tori sequences.** Let  $F$  be an embedded surface in  $\mathbf{R}^4$  satisfying the conditions (K0), (K1) and (K2). Then  $\Gamma^*(F)$  consists only of  $A$ -curves and at most one  $M$ -curve. Let  $V_1, V_2, \dots, V_k$  be solid tori in  $S^3$ , and  $\mathfrak{V} = \{V_1, V_2, \dots, V_k\}$ . We say that  $\mathfrak{V}$  is a *solid tori sequence for  $F^*$*  if  $\mathfrak{V}$  satisfies the following two conditions:

- (i)  $\partial V_i \subset F^*$  for all  $i$ .
- (ii) If  $i \neq j$ , then  $V_i \cap V_j = \partial V_i \cap \partial V_j$  is one simple closed curve, an annulus or empty. Let  $X$  be a coiled solid torus, and  $\mathfrak{V}$  as above. We say that  $\mathfrak{V} \cup \{X\}$  is an *almost solid tori sequence for  $F^*$*  if  $\mathfrak{V} \cup \{X\}$  satisfies the above conditions (i), (ii), and
- (iii) the intersection of  $X$  and  $\overline{S^3 \setminus X}$  is contained in  $F^*$ , and
- (iv)  $X \cap V_i$  is one simple closed curve, an annulus or empty for all  $i$ .

**EXAMPLE 3.8.** Let  $\alpha_3$  be a  $c$ -symmetric 1-complex as shown in Fig. 4 (3), and let  $D_1, D_2$  be 2-balls in  $\overline{B}_0$  such that  $D_1 \subset D_2$  and  $\alpha_3 = \partial D_1 \cup \partial D_2$ . For an integer  $b$  with  $(b, c) = 1$ , let  $W_i = \{(x, y \cos \theta, y \sin \theta) \mid (x, y, 0) \in \overline{F}_{(b/c)\theta}(D_i), \theta \in [0, 2\pi]\}$ . Then  $W_1, W_2$  are the solid tori in  $S^3$  with  $W_1 \subset W_2$  and  $\partial W_1 \cup \partial W_2 = \alpha_3(b, c)$ . We see that  $\{W_2\}$  is a solid tori sequence for the 2-complex  $\alpha_3(b, c)$ . Let  $V_2 = S^3 \setminus W_2$ . Then  $V_2$  is a solid torus,  $\partial W_1 \cup \partial V_2 = \alpha_3(b, c)$ , and  $W_1 \cap V_2 = \partial W_1 \cap \partial V_2$  is one simple closed curve, say  $L$ . The set  $\{W_1, V_2\}$  is a solid tori sequence for  $\alpha_3(a, b)$ . Let  $N$  be a



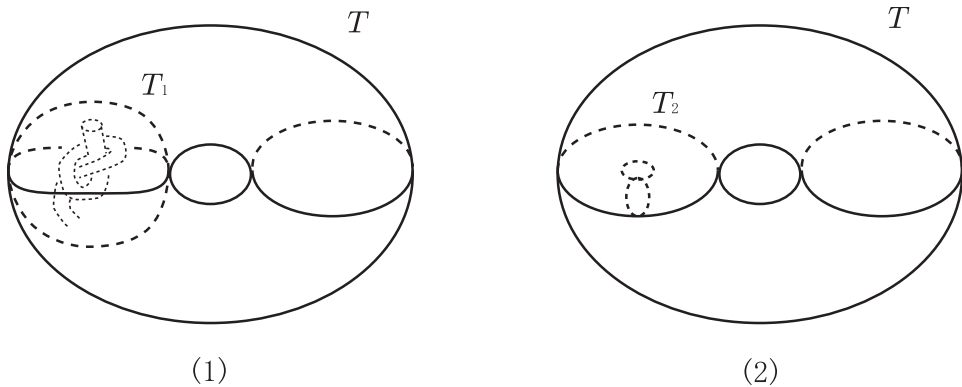


Fig. 5. (1) the 2-complex  $K_1$  (2) the 2-complex  $K_2$ .

regular neighborhood of  $L$  in  $S^3$ . Note that if  $L$  is not a trivial knot, then  $W_1 \cup V_2 \cup N$  is not a solid torus. Because,  $W_1 \cup V_2 \cup N$  is homeomorphic to the complement of an open regular neighborhood of  $L$ .

Let  $F$  be an embedded surface in  $\mathbf{R}^4$  satisfying (K0), (K1) and (K2). Let  $\mathfrak{V} = \{V_1, V_2, \dots, V_k\}$  be a solid tori sequence for the 2-complex  $F^*$ . Let  $c_i$  be a component of  $\Gamma^*(F)$  with  $c_i \subset \partial V_i$ . Let  $n$  be the minimal number of intersection points of  $c_i$  and a meridian disk of the solid torus  $V_i$ . For the solid torus  $V_i$  we define  $n(V_i)$  as follows:

$$n(V_i) = \begin{cases} n & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, V_i \text{ is non-standard,} \\ \infty & \text{if } n = 0, V_i \text{ is standard.} \end{cases}$$

Here, a standard solid torus means a regular neighborhood of a trivial knot in  $S^3$ . We would like to distinguish standard and non-standard solid tori. Let  $T_1, T_2, T$  be tori in  $S^3$  such that

- $T$  bounds a standard solid torus  $V$ ,
- $T_1, T_2 \subset V$ ,
- $T_i \cap T$  is a simple closed curve for  $i = 1, 2$ ,
- $T_1$  bounds the complement of an open regular neighborhood of a trefoil knot in  $V$ , and

- $T_2$  bounds a solid torus  $V_2$  in  $V$  so that  $V_2$  has a 2-ball  $D$  in  $V$  with  $D \cap V_2 = \partial D$ .

See Fig. 5. For the torus  $T_1$ , there exists a solid torus  $V_3$  with  $\partial V_3 = T_1$ . Let  $K_i = T_i \cup T$  for  $i = 1, 2$ . Then  $\{V\}$  is a solid tori sequence for  $K_i$  with  $K_i \subset V$  and  $n(V) = 0$ , and  $\{V_3\}$  is a solid tori sequence for  $K_1$  with  $n(V_3) = \infty$ . However,  $K_1$  is not a 2-complex  $\alpha(b, c)$  obtained from any symmetric 1-complex  $\alpha$ . If an embedded torus in  $\mathbf{R}^4$  has such a projection  $K_1$  into  $S^3$ , then by an ambient isotopy of  $\mathbf{R}^4$  we can assume that its projection in  $S^3$  is  $K_2$ . Let  $W = \overline{S^3 \setminus V}$ . Note that  $K_2$  has a solid tori sequence

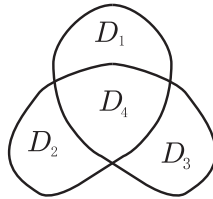


Fig. 6.

$\mathfrak{W} = \{V_2, W\}$  with  $K_2 \subset \cup \mathfrak{W}$ ,  $n(V_2) = 1$  and  $n(W) = 1$ . By Proposition 3.18, we see that  $K_2$  is a 2-complex obtained from some symmetric 1-complex. In this paper we discuss about immersed Klein bottles. It is not important a solid torus  $V$  with  $n(V) = 0$  or  $n(V) = \infty$ .

We construct the graph  $G(\mathfrak{V})$  obtained by a solid tori sequence  $\mathfrak{V}$  as follows. The vertices are in one to one correspondence with the solid tori  $\{V_i\}$ , and the edges are in one to one correspondence with the set  $\{V_i \cap V_j \neq \emptyset\}$ . If  $V_i \cap V_j \neq \emptyset$ , then we connect the vertices  $v(V_i)$  and  $v(V_j)$  by the edge  $e_{ij}$ .

DEFINITION 3.9. Let  $F$  be an embedded surface in  $\mathbf{R}^4$  satisfying (K0), (K1) and (K2), and  $\mathfrak{V} = \{V_1, V_2, \dots, V_k\}$  a solid tori sequence for the 2-complex  $F^*$ . A solid tori sequence  $\mathfrak{V}$  is *good*, if  $\mathfrak{V}$  satisfies the following four conditions:

- (i)  $G(\mathfrak{V})$  is a connected tree.
  - (ii) If  $B$  is an annulus with  $B \subset F^*$  and if  $(\cup \mathfrak{V}) \cap B = \partial B$ , then  $\partial B \subset \partial V_i$  for some  $i$ . Namely, for any annulus  $B$  in  $F^*$  with  $\partial B \cap (\cup \mathfrak{V}) = \partial B$ , the boundary of  $B$  is *not* contained in *different* two solid tori.
  - (iii) There exists a vertex  $v(V_1)$  of  $G(\mathfrak{V})$  such that if  $V_i \neq V_1$  then  $n(V_i) = 1$ .
  - (iv) If  $i \neq j$ , then  $V_i \cap V_j$  is either one simple closed curve or empty.
- The vertex  $v(V_1)$  is called the *special vertex*.

EXAMPLE 3.10. We give not good solid tori sequences as follows. Let  $M$  be the 1-complex in  $\overline{B}_0$  as shown in Fig. 6, and let  $D_1, D_2, D_3, D_4$  be the closures of the bounded components of  $\overline{B}_0 \setminus M$  as shown in Fig. 6. We naturally embed the 2-complex  $M \times S^1 \subset \overline{B}_0 \times S^1$  in  $S^3$  via  $\psi$ .

- (i) The solid tori sequence  $\mathfrak{V}_1 = \{D_1 \times S^1, D_2 \times S^1, D_3 \times S^1\}$  is not a good solid tori for  $M \times S^1$ , because  $G(\mathfrak{V}_1)$  is a circle.
- (ii) Let  $A$  be the closure of a component of  $M \setminus D_1 \cup D_2$ . Then  $A$  is an arc in  $\partial D_3$ . The solid tori sequence  $\mathfrak{V}_2 = \{D_1 \times S^1, D_2 \times S^1\}$  is not a good solid tori sequence for  $M \times S^1$ , because there exists the annulus  $A \times S^1$  with  $(\partial A \times S^1) \cap (\partial D_i \times S^1) \neq \emptyset$  for  $i = 1, 2$ .
- (iii) Let  $L, \alpha_3(b, c), W_1, V_2$  be as in Example 3.8. Suppose that  $b, c$  are integers with  $b > 1$  and  $c > 1$ . Then the knot  $L$  wraps  $b$  times in the longitudinal direction of  $W_1$ , and then  $L$  wraps  $c$  times in the longitudinal direction of  $V_2$ . Moreover,  $n(W_1) = b$

and  $n(V_2) = c$ . Since  $b > 1$  and  $c > 1$ ,  $\{W_1, V_2\}$  is not a good solid tori sequence for  $\alpha_3(b, c)$ .

However, there exist good solid tori sequences  $\mathfrak{V}$  and  $\mathfrak{W}$  for  $M \times S^1$  and  $\alpha_3(b, c)$ , respectively, such that  $\alpha_3(b, c) \subset (\cup \mathfrak{V})$  and  $M \times S^1 \subset (\cup \mathfrak{W})$ . In the case of  $M \times S^1$ , let  $D = D_1 \cup D_2 \cup D_3 \cup D_4$ , then  $\mathfrak{V} = \{D \times S^1\}$  is a desired solid tori sequence. In the case of  $\alpha_3(b, c)$ , since  $V_2$  is a standard solid torus,  $W = \overline{S^3 \setminus V_2}$  is a solid torus with  $W_1 \subset W$ . Hence,  $\mathfrak{W} = \{W\}$  is a desired solid tori sequence.

For a coiled solid torus  $X$ , we define  $n(X) = 2$ . For an almost solid tori sequence  $\mathfrak{V}$ , we construct the graph  $G(\mathfrak{V})$  in a similar way as above.

**DEFINITION 3.11.** Let  $F$  be an embedded surface in  $\mathbf{R}^4$  satisfying (K0), (K1) and (K2). Let  $X$  be a coiled solid torus, and  $\mathfrak{V} = \{X, V_1, V_2, \dots, V_k\}$  an almost solid tori sequence for  $F^*$ . An almost solid tori sequence  $\mathfrak{V}$  is *good*, if  $\mathfrak{V}$  satisfies the following four conditions:

- (i)  $G(\mathfrak{V})$  is a connected tree.
- (ii) If  $B$  is an annulus with  $B \subset F^*$  and if  $(\cup \mathfrak{V}) \cap B = \partial B$ , then  $\partial B \subset \partial V_i$  for some  $i$  or  $\partial B \subset X \cap \overline{S^3 \setminus X}$ .
- (iii)  $n(V_i) = 1$  for all solid tori  $V_i$ .
- (iv) If  $i \neq j$ , then  $V_i \cap V_j$  and  $X \cap V_i$  are one simple closed curve or empty.

The vertex  $v(X)$  is called the *special vertex*.

Let  $\mathfrak{V} = \{V_1, \dots, V_k\}$  be a (almost) solid tori sequence. If  $V_i \cap V_j$  is one simple closed curve, let  $N_{ij}$  be a regular neighborhood of  $V_i \cap V_j$  in  $S^3$ . If  $V_i \cap V_j = \emptyset$ , let  $N_{ij} = \emptyset$ . If  $V_i \cap V_j$  is an annulus, let  $N_{ij} = V_i \cap V_j$ . Then we say that  $(\cup \mathfrak{V}) \cup (\cup N_{ij})$  is a *shape* of  $\mathfrak{V}$ .

**Lemma 3.12** ([15, Lemma 3.4]). *Let  $\{V_1, V_2\}$  be a solid tori sequence. Let  $V$  be a shape of  $\mathfrak{V}$ .*

- (1) *If  $V$  is a solid torus, then  $n(V_1) = 1$  or  $n(V_2) = 1$ .*
- (2) *If  $V$  is not a solid torus, then  $n(V_1) > 1$ ,  $n(V_2) > 1$ , and  $V_1, V_2$  are standard solid tori in  $S^3$ .*

*Here a standard solid torus means a regular neighborhood of a trivial knot in  $S^3$ .*

**Lemma 3.13.** *Let  $\{V_1\}, \{V_2\}$  be solid tori sequences such that  $V_2 \subset V_1$ , and  $\partial V_1 \cap \partial V_2$  is one simple closed curve or an annulus. If  $n(V_2)$  is not equal to 0, 1, and  $\infty$ , then  $\partial V_1 \cup \partial V_2$  can be moved a 2-complex obtained from one of Fig. 7 (1), (3) by an ambient isotopy of  $S^3$ . Hence  $V_1$  can be moved to  $V_2$  by an ambient isotopy of  $S^3$ .*

**Proof.** In the case that  $\partial V_1 \cap \partial V_2$  is an annulus, by [12, Lemma 2.1] the annulus  $B = \overline{\text{Int } V_1} \cap \partial V_2$  is parallel to a boundary annulus in  $\partial V_2$ . The annulus  $B$  is decom-

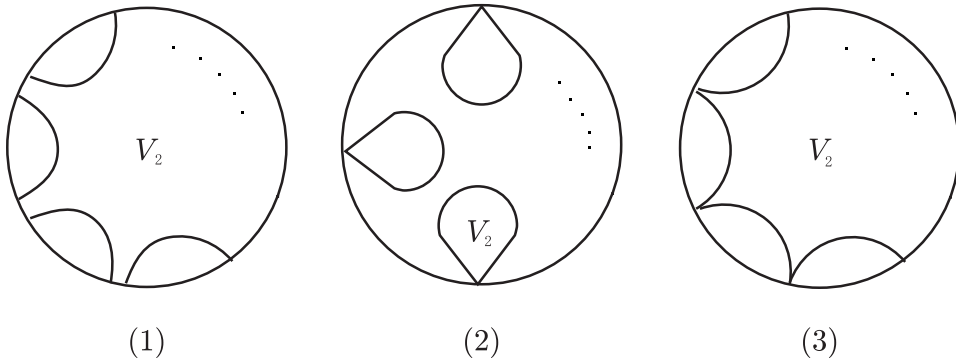


Fig. 7. Cutting a meridian disk.

posed  $V_1$  into two solid tori  $V_2$  and  $\overline{V_2 \setminus V_1}$ . Note that cutting a meridian disk of  $V_2$ , then we have Fig. 7 (1) which is the intersection of the meridian disk and  $\partial V_2$ . Since  $n(V_2) \neq 0, 1, \infty$ ,  $V_1$  can be moved to  $V_2$  by an ambient isotopy of  $S^3$ .

In the case that  $\partial V_1 \cap \partial V_2$  is one simple closed curve  $C$ , let  $N$  be a regular neighborhood of  $C$  in  $V_1$ . Let  $K = \overline{(\partial V_1 \cup \partial V_2) \setminus N} \cup \text{Int } V_1 \cap \partial N$ . Then, the solid tori sequence  $\{V_1 \setminus N, V_2 \setminus N\}$  for  $K$  satisfies the above condition. Cutting a meridian disk of  $V_1$ , then we have Fig. 7 (2) or (3) which is the intersection of the meridian disk and  $\partial V_2$ . If  $\partial V_2 \cap \partial V_1$  is a longitude curve of  $V_2$ , i.e.,  $n(V_2) = 1$ , then we see Fig. 7 (2). We have that  $V_2$  can be moved to  $V_1$  by an ambient isotopy of  $S^3$  if and only if we see Fig. 7 (3). Since  $n(V_2) \neq 0, 1, \infty$ ,  $V_1$  can be moved to  $V_2$  by an ambient isotopy of  $S^3$ .  $\square$

REMARK 3.14. Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$  satisfying (K1) and (K2). Let  $\mathfrak{V}$  be a good almost solid tori sequence for  $F^*$ ,  $C$  the M-curve in the coiled solid torus  $X$ . Let  $N$  be a regular neighborhood of  $C$  in  $S^3$ ,  $X' = X \cup N$ ,  $K = (F^* \setminus N) \cup (\partial N \cap \partial X')$ . Then  $X'$  is a solid torus,  $\mathfrak{V}' = \{X'\} \cup (\mathfrak{V} \setminus \{X\})$  is a good solid tori sequence for  $K$  with  $n(X') = 2$ .

**Lemma 3.15.** *Let  $F$  be an embedded surface satisfying (K0), (K1) and (K2). Let  $\mathfrak{V}$  be a good (almost) solid tori sequence for  $F^*$  such that  $n(V_1) = 2$ , where  $v(V_1)$  is the special vertex. Let  $C$  be an A-curve in  $\cup \mathfrak{V}$ , and  $V$  a shape of  $\mathfrak{V}$ . Then  $V$  is a coiled solid torus if  $\mathfrak{V}$  is almost, and  $V$  is a solid torus otherwise. Moreover,  $[C] = \pm 2 \in H_1(V)$ .*

Proof. In the case of a solid tori sequence, we showed in [14, Lemma 7.5]. So, we may assume that  $\mathfrak{V}$  is almost. By Remark 3.14,  $\mathfrak{V}$  can be changed a solid tori sequence. Given that  $N$  is a regular neighborhood of the M-curve in  $S^3$ , we have  $V \cup N$  is a solid torus, and  $[C] = \pm 2 \in H_1(V \cup N) \cong H_1(V)$ . This and Lemma 3.13 imply

that  $V$  can be moved to the coiled solid torus  $X$  in  $\mathfrak{V}$  by an ambient isotopy of  $S^3$ . □

**Lemma 3.16.** *Let  $F$  be an embedded surface satisfying (K0), (K1) and (K2). Let  $\mathfrak{V}$  be a good (almost) solid tori sequence for  $F^*$  with  $\cup\mathfrak{V} \not\subset F^*$ , and  $n(V_1) = 2$ , where  $v(V_1)$  is the special vertex. Then, there exists a solid torus  $V$  such that  $\partial V \subset F^*$  and  $\partial V \cap (\cup\mathfrak{V})$  is a simple closed curve or an annulus,  $n(V) = 2$  if  $V$  contains  $V_1$ , and  $n(V) = 1$  otherwise. Moreover, if the M-curve is a trivial knot in  $S^3$ , then there exists a coiled solid torus  $X$  with  $X \cap (\overline{S^3 \setminus X}) \subset F^*$  such that  $X$  can be moved to the 3-complex  $X_b$  for some integer  $b$  of an ambient isotopy of  $S^3$ , where  $X_b$  is the set in Definition 3.5. In particular, if  $b = 0$  or  $-1$ , then we can take a solid torus  $V$  with  $n(V) = 1$ .*

Proof. By Remark 3.14, it suffices to prove for a solid tori sequence. Let  $\mathfrak{V} = \{V_1, \dots, V_k\}$  be a good solid tori sequence. Since  $\cup\mathfrak{V} \not\subset F^*$ , by the definition of good, there exists a torus or an annulus,  $B$ , in  $F^*$  such that

$$B \cap (\cup\mathfrak{V}) = \begin{cases} \text{one simple closed curve,} & \text{if } B \text{ is a torus,} \\ B \cap \partial V_i = \partial B, & \text{if } B \text{ is an annulus.} \end{cases}$$

By the solid torus theorem in [10], there exists a solid torus  $V$  with  $B \subset \partial V \subset F^*$ . We see that  $\partial V \cap (\cup\mathfrak{V})$  is a simple closed curve or an annulus. Let  $C$  be a component of  $\Gamma^*(F)$  in  $\partial V \cap (\cup\mathfrak{V})$ .

CASE 1.  $V$  contains  $V_1$ .

Let  $\mathfrak{V}' = \{V_i \in \mathfrak{V} \mid V_i \subset V\}$ . Then  $\mathfrak{V}'$  is a good solid tori sequence for  $F^*$ . By Lemma 3.15, a shape  $V'$  of  $\mathfrak{V}'$  is a solid torus and  $V_1 \subset V'$ . By  $[C] = \pm 2 \in H_1(V')$  and Lemma 3.13, we can show that  $V'$  can be moved to  $V$  by an ambient isotopy of  $S^3$ . This implies  $n(V) = 2$ .

CASE 2.  $V$  does not contain  $V_1$ .

Let  $\mathfrak{V}' = \{V_i \in \mathfrak{V} \mid V_i \not\subset V\}$ , then  $\mathfrak{V}'$  is a good solid tori sequence for  $F^*$ . By Lemma 3.15, a shape  $V'$  of  $\mathfrak{V}'$  is a solid torus. Since  $V' \cap V = \partial V' \cap \partial V$  is a simple closed curve or an annulus, by Lemma 3.12,  $n(V) = 1$  or  $V$  is standard. If  $V$  is standard, then this case can be proved in a similar way to Case 1 by replacing  $V$  by  $\overline{S^3 \setminus V}$ . If  $n(V) = 1$ , then there is nothing to do.

Moreover, we assume that  $\mathfrak{V}$  is a good almost solid tori sequence and the M-curve is a trivial knot in  $S^3$ . Then there exists a 2-complex  $K \subset F^*$  such that  $K$  is a projection of an embedded Klein bottle satisfying (K1) and (K2),  $K$  contains only one M-curve and no A-curve. By Lemma 3.7, there exists a coiled solid torus  $X$ . Since the M-curve is a trivial knot, we can easily prove that  $X$  can be moved to the 3-complex  $X_b$  for some  $b$  of an ambient isotopy of  $S^3$ . Suppose that  $b = 0$  or  $-1$ . In the case of  $n(V) = 1$ , there is nothing to do. Suppose  $n(V) = 2$ . Let  $\mathfrak{V}' = \{V_i \in \mathfrak{V} \mid V_i \subset V\}$  and let  $V'$  be a shape of  $\mathfrak{V}'$ . Then  $V' \subset V$ ,  $\partial V' \cap \partial V$  is an

annulus or a simple closed curve, and  $V'$  is the coiled solid torus by Lemma 3.13. So we may assume  $V' = X_b$ . Since the M-curve is a trivial knot,  $V$  is a standard solid torus. Let  $W = \overline{S^3 \setminus V}$ . Since  $b = 0$  or  $-1$ , a simple closed curve of  $\partial V' \cap \partial V$  is homologous to  $\pm 2l \pm m \in H_1(\partial V)$ , where  $m$  is a meridian curve of  $V$ ,  $l$  is a preferred longitude of  $V$ . This implies  $n(W) = 1$ , and  $W$  is a desired solid torus.  $\square$

**Proposition 3.17.** *Let  $F$  be an embedded surface satisfying (K0), (K1) and (K2). Then there exists a good (almost) solid tori sequence  $\mathfrak{V}$  for  $F^*$  with  $\cup \mathfrak{V} \supset F^*$ . Moreover, suppose that the M-curve in  $F^*$  is a trivial knot in  $S^3$ , and suppose that there exists a good almost solid tori sequence  $\{X\}$  for  $F^*$  such that  $X$  can be moved to the 3-complexes  $X_0$  or  $X_{-1}$  of an ambient isotopy of  $S^3$ . Then we can take that  $\mathfrak{V}$  is almost.*

*Proof.* We only prove for the case that  $F^*$  contains an M-curve. There exists a good almost solid tori sequence  $\{X\}$  for  $F^*$  such that  $X$  is maximal, i.e.,  $X$  is not contained in another coiled solid torus. We prove by induction on the number of the components of  $F^* \setminus \Gamma^*(F)$  in a good (almost) solid tori sequence. Let  $\mathfrak{V}$  be a good (almost) solid tori sequence for  $F^*$ . If  $\cup \mathfrak{V} \not\supset F^*$ , then by Lemma 3.16 there exists a solid torus  $V$  satisfying the condition in Lemma 3.16. By Lemma 3.16, there exists only one solid torus  $V_j \in \mathfrak{V}$  such that  $(\cup \mathfrak{V}) \cap \partial V = V_j \cap \partial V$  is an annulus or a simple closed curve. Let  $\tilde{V} = V \cup V_j$  if  $V_j \cap \partial V$  is an annulus, let  $\tilde{V} = V$  otherwise. Since  $n(V_j) = 1$ ,  $\tilde{V}$  is a solid torus. We have a good solid tori sequence  $\mathfrak{W} = \{V_i \in \mathfrak{V} \mid V_i \not\subset \tilde{V}\} \cup \{\tilde{V}\}$  for  $F^*$  with  $\cup \mathfrak{V} \subset \cup \mathfrak{W}$ . In particular, if the M-curve is trivial, and if the coiled solid torus  $X \in \mathfrak{V}$  can be moved to the 3-complex  $X_0$  or  $X_{-1}$  of an ambient isotopy of  $S^3$ , by Lemma 3.16, then  $n(V) = 1$ , and  $\mathfrak{W}$  contains the coiled solid torus  $X$ . Inductively, this completes the proof of Proposition 3.17.  $\square$

**Proposition 3.18.** *Let  $F$  be an embedded surface satisfying (K0), (K1) and (K2). Let  $\mathfrak{V}$  be a good (almost) solid tori sequence for  $F^*$  with  $\cup \mathfrak{V} \supset F^*$ ,  $n(V_1) \neq 0$  and  $n(V_1) \neq \infty$ , where  $v(V_1)$  is the special vertex. Then  $F^*$  can be moved to a 2-complex obtained by a c-symmetric 1-complex by an ambient isotopy of  $S^3$ , where  $b = n(V_1)$ ,  $(b, c) = 1$ . In particular, if  $\mathfrak{V}$  is almost, then  $b = 2$ .*

*Proof.* In the case that  $\Gamma^*(F)$  consists only A-curves, we showed in [14, Proposition 7.15].

Assume that  $\Gamma^*(F)$  contains one M-curve. Let  $C$  be the M-curve,  $N$  a regular neighborhood of  $C$  in  $S^3$ ,  $V = V_1 \cup N$  and  $K = (F^* \setminus N) \cup (\partial N \cap \partial V)$ . By Remark 3.14,  $(\mathfrak{V} \setminus \{V_1\}) \cup \{V\}$  is a good solid tori sequence for  $K$ . Since it is true for the case of only A-curves, we see that  $K$  is a 2-complex obtained from some symmetric 1-complex. Hence,  $F^*$  is also a 2-complex obtained from some symmetric 1-complex.  $\square$

#### 4. Spun Klein bottles

**Proposition 4.1.** *Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$  such that  $\Gamma^*(F)$  consists only of double points, and each component of  $\Gamma(F)$  is not homotopic to zero in  $\pi_1(F)$ . Then  $F^*$  is the projection into  $\mathbf{R}^3$  of a spun Klein bottle in  $\mathbf{R}^4$ . In particular,  $F$  is ambient isotopic to a simple spun Klein bottle in  $\mathbf{R}^4$ .*

*Proof.* By [16, Remark 1.5], the number of components of  $\Gamma(F)$  is even. Hence, by Lemma 3.1,  $(\Gamma(F), F)$  is homeomorphic to  $(\Gamma_1, F)$ . We see that  $\Gamma^*(F)$  consists only of A-curves and one M-curve. By Proposition 3.17, there exists a good (almost) solid tori sequence  $\mathfrak{A}$  for  $F^*$  with  $F^* \subset \cup \mathfrak{A}$ . By Proposition 3.18 and Remark 3.4, there exists a spun Klein bottle  $KL^a(K)$  in  $\mathbf{R}^4$  such that  $F$  is ambient isotopic to  $KL^a(K)$ .

If  $a \neq 0$  and  $a \neq -1$ , by Remark 2.1 (2), then we may assume that  $a = 0$  or  $-1$ , and the M-curve of  $F^*$  is a trivial knot. Applying Proposition 3.18 again, we obtain a good *almost* solid tori sequence  $\mathfrak{A}$  for  $F^*$  with  $F^* \subset \cup \mathfrak{A}$ . Hence  $F$  is simple.  $\square$

#### 5. Diagrams for embedded surfaces

For an embedded surface, we define a ‘diagram’ in  $\mathbf{R}^3$ . In classical knots, it is convenient to represent by a diagram, i.e., an immersed closed curve in the plane that has crossing information indicated at its double points. A ‘diagram’ for an embedded surface is like a diagram of classical knots.

Let  $\varphi : F \rightarrow \mathbf{R}^3$  be an immersion of a closed surface  $F$  (possibly disconnected, non-orientable) such that the singular set of  $\varphi$  has only transverse double points; each component of its is a circle. Such a circle is called a *crossing circle*. A *diagram*  $D$  is an immersion of a union of 2-spheres and a Klein bottle with a mark at each crossing circle satisfying the two conditions:

(i) For any crossing circle  $C$ , let  $N$  be a regular neighborhood of  $C$  in  $\mathbf{R}^3$ . Then  $N \cap \text{Im } D$  consists of two annuli or two Möbius bands, say  $A_1, A_2$ .

(ii) One of  $A_1, A_2$  is marked either by ‘a’ (for ‘above’) or by ‘b’ (for ‘below’).

We define that there is a mark ‘a’ on  $A_i$  if and only if there is a mark ‘b’ on  $A_j$  ( $i \neq j$ ).

We usually place a mark ‘a’ or ‘b’ on only one  $A_i$ . A surface  $A_i$  with mark ‘a’ (resp. ‘b’) is called an *a-tube* (resp. a *b-tube*). We define the associated embedded surface  $L_D$  of a diagram  $D$  by the following properties.

(i)  $p(L_D) = \text{Im } D$ , where  $p : \mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R} \rightarrow \mathbf{R}^3$  is the projection onto  $\mathbf{R}^3$ .

(ii)  $L_D \cap (\mathbf{R}^3 \times \{0\}) = (\text{Im } D \setminus \text{Int}(a\text{-tubes in } D)) \times \{0\}$ , and  $L_D \subset \mathbf{R}^3 \times [0, \infty)$ .

These conditions determine an embedded surface up to ambient isotopy.

The mark ‘a’ and ‘b’ are used in [6] and [7]. Yajima [19] uses an arrow. Giller [4, p. 629] uses ‘+’ for our ‘a’. Carter and Saito [2, 3] define a broken surface diagram.

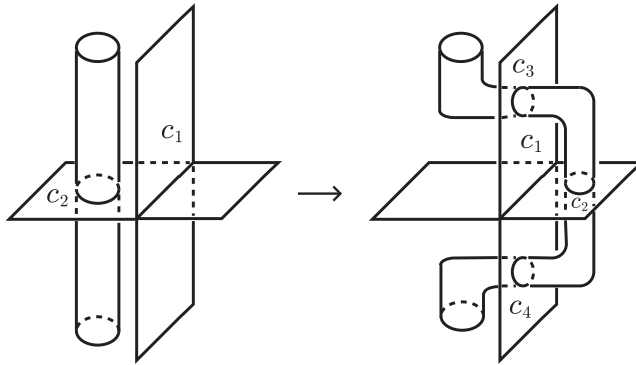


Fig. 8. Type (Ω1) move.

**5.1. 1-handles for diagrams.** In this subsection, we define a 1-handle for a diagram.

Let  $D$  be a diagram. Let  $h_i: B^2 \times I \rightarrow \mathbf{R}^3$ ,  $i = 1, 2, \dots, m$ , be a collection of embeddings with mutually disjoint images such that

$$h_i(B^2 \times I) \cap \text{Im } D = h_i(B^2 \times \{0, t_1, \dots, t_{i_k}, 1\})$$

for some  $t_{i_1}, t_{i_2}, \dots, t_{i_k}$  with  $0 < t_{i_1} < t_{i_2} < \dots < t_{i_k} < 1$ , where  $B^2$  is a 2-ball and  $I = [0, 1]$ . Define the immersed surface  $D + \sum_{i=1}^m h_i$  to be

$$\left( \text{Im } D \setminus \bigcup_{i=1}^m h_i(B^2 \times \partial I) \right) \cup \left( \bigcup_{i=1}^m h_i(\partial B^2 \times I) \right).$$

We call the embedding  $h_i$  *1-handle on the diagram  $D$* , and the diagram with  $D + \sum_{i=1}^m h_i$  a *diagram obtained from  $D$  by attaching 1-handles*. For a 1-handle  $h_i$ , we call the disks  $h_i(B^2 \times 0)$  and  $h_i(B^2 \times 1)$ , *attaching disks*, the disk  $h_i(B^2 \times t)$ ,  $0 < t < 1$ , a *cocore* of  $h_i$ , and the arc  $h_i(x \times I)$ ,  $x \in \text{Int } B^2$ , a *core*; see [7, Fig. 1].

**5.2. Local moves.** Local moves between diagrams are defined in [7]. They do not change the ambient isotopy classes of associated embedded surfaces of diagrams. Now, we define three of them.

(Ω1) Moving a 1-handle through a sheet as shown in Fig. 8, where  $c_1, c_2 \in \{a, b\}$  and

$$c_3 = c_4 = \begin{cases} c_1 & \text{if } c_1 = c_2, \\ \text{either } a \text{ or } b & \text{if } c_1 \neq c_2. \end{cases}$$

This move adds two crossing circles. (cf. Fig. 4 in [19])

(Ω2) Sliding a 1-handle through a sheet as shown in Fig. 9, where  $c_1 = c_2 \in \{a, b\}$ . This move adds one crossing circle.



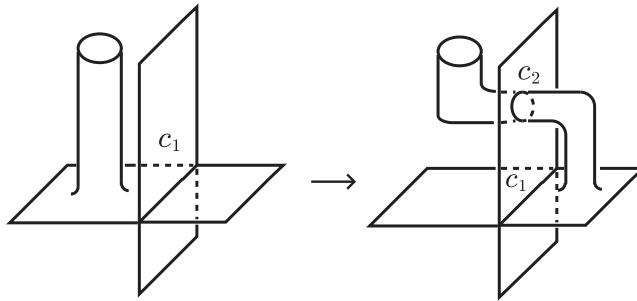


Fig. 9. Type  $(\Omega_2)$  move.

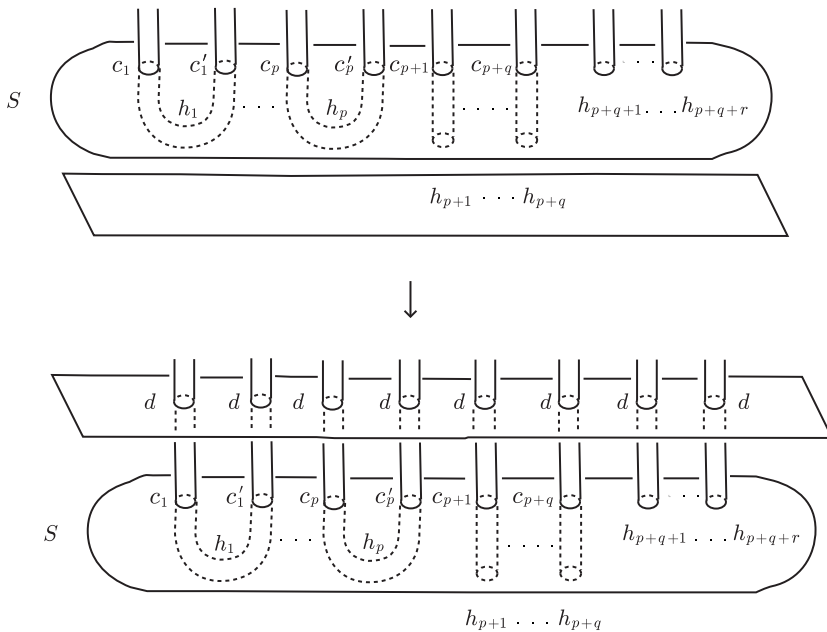


Fig. 10. Type  $(\Omega_6)$  move.

$(\Omega_6)$  Pulling out a 2-sphere with 1-handles across a sheet as shown in Fig. 10, where  $S$  is a 2-sphere bounding a 3-ball  $B$ , and  $h_i$ ,  $1 \leq i \leq p + q + r$ , are 1-handles such that

- (i)  $h_1, \dots, h_p$  are passing through  $S$ ,
- (ii)  $h_{p+1}, \dots, h_{p+q+r}$  are attached on  $S$  whose one attaching disks are in  $S$ ,
- (iii) the pair  $(B, B \cap (\bigcup_{i=1}^{p+q} \alpha_i))$ , where  $\alpha_i$  is a core of  $h_i$ , is a trivial tangle, meaning that it is homeomorphic to the pair  $(D^2, \{x_1, \dots, x_{p+q}\}) \times [0, 1]$ , where  $x_i$  are interior points of the 2-ball  $D^2$ , and

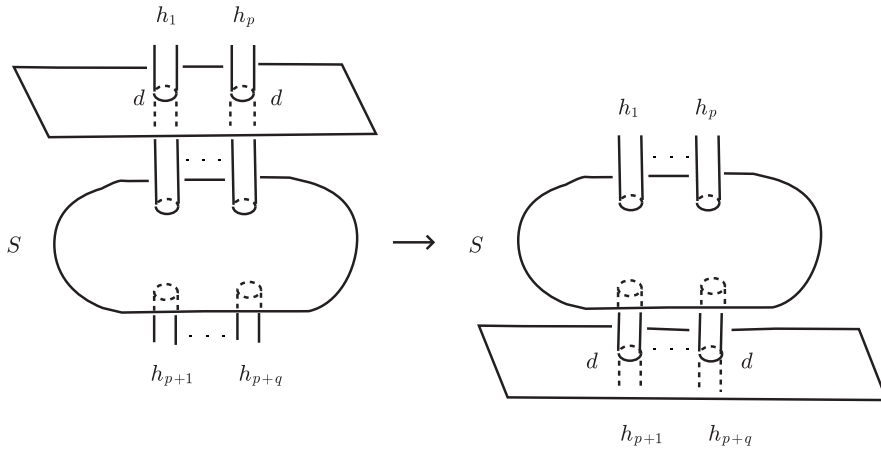


Fig. 11. Type  $(\Omega 6)'$  move.

(iv)  $c_i, c'_j, d \in \{a, b\}$ , where  $1 \leq i \leq p + q, 1 \leq j \leq p$ .

The following move is a generalization of the move  $(\Omega 6)$ .

$(\Omega 6)'$  Pulling out a 2-sphere  $S$  across a sheet as shown in Fig. 11, where  $S$  is bounding a 3-ball  $B$ , and  $h_i, 1 \leq i \leq p + q$ , are 1-handles such that

- (i)  $h_1, \dots, h_{p+q}$  are passing through  $S$  or are attached on  $S$ , and
- (ii)  $d \in \{a, b\}$ .

(cf. Lemma 4.6 in [19])

A diagram  $D$  is *with good position*, if it is obtained by attaching 1-handles from 2-spheres  $S_1, \dots, S_m$  and an immersed Klein bottle  $K$  in  $\mathbf{R}^3$  such that

- (i)  $K$  is the projection of an embedded Klein bottle in  $\mathbf{R}^3$  satisfying (K1) and (K2), and
- (ii) there exist disjoint 3-balls  $B_1, \dots, B_{m+1}$  in  $\mathbf{R}^3$  with  $S_i \subset \text{Int } B_i$  and  $K \subset \text{Int } B_{m+1}$ .

Observe that an associated surface as above is a Klein bottle obtained from a spun Klein bottle by  $m$ -fusion. Also, a diagram obtained by attaching 1-handles from only 2-spheres  $S_1, \dots, S_m$  is called a diagram with good position. Observe that an associated surface of its diagram is a ribbon surface.

**Proposition 5.1.** *Any diagram can be transformed into a diagram with good position by a sequence of moves  $(\Omega 1)$ ,  $(\Omega 2)$  and  $(\Omega 6)'$ .*

*Proof.* First of all, we show that any diagram can be transformed into a diagram by attaching 1-handles from disjoint 2-spheres in  $\mathbf{R}^3$ , or a diagram by attaching 1-handles from disjoint 2-spheres and the projection of a spun Klein bottle. Let  $D$  be a diagram obtained from a diagram  $D_0$  by attaching 1-handles  $h_1, \dots, h_m$ , where  $D_0$  is the image of an immersion of a surface  $F$ . Let  $R(D_0)$  be the components in the

singular set of  $D_0$  in  $\mathbf{R}^3$  such that one of the preimage bounds a disk in  $F$ . We use induction on the number of the components in  $R(D_0)$ , say  $n$ .

In case of  $n = 0$ , i.e.,  $R(D_0) = \emptyset$ , by Proposition 4.1,  $D_0$  is disjoint 2-spheres in  $\mathbf{R}^3$ , or disjoint 2-spheres and the projection of a spun Klein bottle in  $\mathbf{R}^3$ . This implies the desired result.

Assume it is true for less than  $n$ , and the number of the components in  $R(D_0)$  is  $n$ . Choose the disk  $E$  in  $D_0$  such that  $\partial E$  is a component of  $R(D_0)$ , and  $E$  is a non-singular disk in  $\mathbf{R}^3$ . If  $E$  intersects a cocore of a 1-handle, perform the 1-handle by the move  $(\Omega 1)$  in Fig. 8. See the first move in Fig. 12. By the move  $(\Omega 1)$ , two crossing circles appear, but the number of the components in  $R(D_0)$  does not change. If  $E$  intersects an attaching disk of a 1-handle, then perform the 1-handle by the move  $(\Omega 2)$  in Fig. 9. See the second move in Fig. 12. Similarly, we see that the number of the components in  $R(D_0)$  does not change. Hence, we may assume that  $E$  does not intersect 1-handles. A regular neighborhood of  $E$  in  $\mathbf{R}^3$  consists of an annulus  $A$  and a disk  $E'$  containing  $E$ . By replacing the annulus  $A$  with two disks, each of which is parallel to  $E$ . Then we obtain a diagram  $D_1$  such that  $D_0$  is obtained from  $D_1$  by attaching a 1-handle  $h$  such that  $h(\partial B^2 \times I) = A$ . Thus,  $D$  is obtained from  $D_1$  by attaching 1-handles  $h_1, \dots, h_m, h$ . The number of the components of  $R(D_1)$  is less than that of  $R(D_0)$ , which yields the result.

Next, we consider a diagram obtained by attaching 1-handles  $h_1, \dots, h_n$  on 2-spheres  $S_1, \dots, S_m$  and immersed Klein bottle  $K$  such that  $K$  is a 2-complex consisting of annuli. If the 2-spheres and  $K$  are contained in the interior of disjoint 3-balls, respectively, then the diagram is a desired diagram. Otherwise, take a 2-sphere, say  $S_i$ , such that  $S_i$  does not contain any other 2-sphere in  $\mathbf{R}^3$ . Let  $B, B_i$  be 3-balls in  $\mathbf{R}^3$  such that the interior of  $B$  contains  $K$ ,  $\partial B \cap S_i = \emptyset$  for all  $i$ , and  $\partial B_i = S_i$ . If  $B_i$  does not contain  $K$ , by a sequence of the move  $(\Omega 6)'$ , then we pull out  $S_i$  from the 2-sphere that contains  $S_i$ . If not, by a sequence of the move  $(\Omega 6)'$ , then we pull  $K$ , and then we pull out  $S_i$  from the 2-sphere that contains  $S_i$ . Inductively, we have a diagram with good position. Similarly, we can prove for the case of a diagram obtained by attaching 1-handles on 2-spheres.  $\square$

The technique in Proposition 5.1 was used in [7] and [19].

## 6. Proof of the main theorem

From Proposition 5.1, we have:

**Theorem 6.1** (Theorem 1.1). *Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$ . If  $\Gamma^*(F)$  consists of double points, then  $F$  is ambient isotopic to either a ribbon Klein bottle, or a Klein bottle obtained from a spun Klein bottle by  $m$ -fusion.*

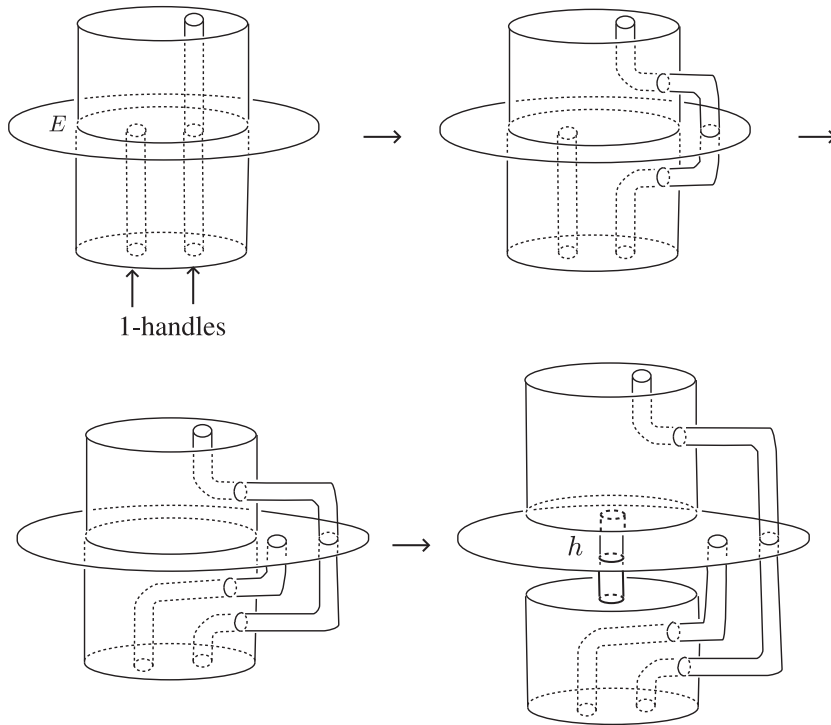


Fig. 12. A transformation for the case that  $E$  intersects one cocore and one attaching disk.

**Lemma 6.2.** *Let  $L$  be a knot in  $S^3$ . If  $\pi_1(S^3 \setminus L)/\langle m^2 = 1 \rangle$  is isomorphic to  $\mathbf{Z}_2$ , then  $L$  is trivial.*

Proof. Let  $N$  be a regular neighborhood of  $L$  in  $S^3$ ,  $E = \overline{S^3 \setminus N}$ ,  $E_2$  the 2-fold cover,  $X_2$  the 2-fold branch cover. Then we obtain the following exact sequences:

$$\begin{array}{ccccccc}
 1 & \rightarrow & \pi_1(E_2) & \xrightarrow{\tilde{m}=m^2} & \pi_1(E) & \longrightarrow & \mathbf{Z}_2 \rightarrow 1 \\
 & & \tilde{m}=1 \downarrow & & m^2=1 \downarrow & & \downarrow \\
 1 & \rightarrow & \pi_1(X_2) & \longrightarrow & \pi_1(E)/\langle m^2 = 1 \rangle & \xrightarrow{\cong} & \mathbf{Z}_2 \rightarrow 1
 \end{array}$$

where  $m$  is a meridian curve of  $L$ . By the above diagram, we have  $\pi_1(X_2) \cong 1$ . By the Smith Conjecture [9], if  $\pi_1(X_2) \cong 1$ , then the branch set of  $X_2$  is a trivial knot. And we can show that  $L$  is trivial.  $\square$

**Corollary 6.3** (Corollary 1.2). *Let  $F$  be an embedded Klein bottle in  $\mathbf{R}^4$ . Suppose that  $\Gamma^*(F)$  consists only of double points, and all components of the singular set  $\Gamma(F)$  are not homotopic to zero in  $\pi_1(F)$ . If  $\pi_1(\mathbf{R}^4 \setminus F)$  is isomorphic to  $\mathbf{Z}_2$ , then  $F$*

is trivial.

Proof. By assumption,  $F^*$  consists only of A-curves and one M-curve. By Proposition 4.1,  $F$  is ambient isotopic to a simple spun Klein bottle  $KL^a(L\#(-L))$ . By Lemma 6.2 and Remark 2.1 (2), if the fundamental group of the complement of  $KL^a(L\#(-L))$  is isomorphic to  $\mathbf{Z}_2$ , then the knot  $L$  is trivial in  $S^3$ . Hence  $KL^a(L\#(-L))$  is ambient isotopic to a Klein bottle  $F'$  such that  $\Gamma^*(F')$  consists only of one simple closed curve. Hence  $F'$  is a boundary of a solid Klein bottle in  $\mathbf{R}^4$ . Therefore  $F$  is trivial.  $\square$

**6.1. Example of a non-ribbon surface.** In [12], [13], and [14], we classified for an embedded torus  $T$  whose singular set  $\Gamma^*(T)$  consists of at most three disjoint simple closed curves. The twist spun torus of the trefoil knot has the projection into  $\mathbf{R}^3$  with the singular set consisting three disjoint simple closed curves. This example is given in [1] or [14].

**Proposition 6.4.** *The twist spun torus  $F$  is not a ribbon surface.*

Proof. Suppose that  $F$  is a ribbon surface. Let  $N$  be a regular neighborhood of  $F$  in  $\mathbf{R}^4$ . Boyle [1] defined the  $\mathbf{Z}_2$ -invariant  $q$  for a curve  $c$  in  $\partial N$  which is homologous to zero in  $\overline{\mathbf{R}^4 \setminus N}$ , this is modulo 2 to the intersection number of a surface with boundary  $c$  in  $\overline{\mathbf{R}^4 \setminus N}$ . Then, there exists a unique simple closed curve  $C$  on the boundary of  $N$  such that  $C$  is homotopic to zero in  $\overline{\mathbf{R}^4 \setminus N}$ . We see that  $q(C) = 1$ . However, a ribbon torus has a curve  $C'$  on  $\partial N$  such that  $C'$  is homotopic to zero in  $\overline{\mathbf{R}^4 \setminus N}$ , and  $q(C') = 0$ . This is a contradiction. Hence,  $F$  is not a ribbon surface.  $\square$

**Question 6.5.** For a trefoil knot  $L$ , is the spun Klein bottle  $KL^a(L\#(-L))$  a non-ribbon surface?

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