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α -PARABOLIC BERGMAN SPACES

Dedicated to the memory of Professor Isao Higuchi

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Abstract

The α -parabolic Bergman space b_α^p is the set of all p -th integrable solutions u of the equation $(\partial/\partial t + (-\Delta)^\alpha)u = 0$ on the upper half space, where $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. The Huygens property for the above u will be obtained. After verifying that the space b_α^p forms a Banach space, we discuss the fundamental properties. For example, as for the duality, $(b_\alpha^p)^* \cong b_\alpha^q$ for $p > 1$ and $(b_\alpha^1)^* \cong \mathcal{B}_\alpha/\mathbf{R}$ are shown, where q is the exponent conjugate to p and \mathcal{B}_α is the α -parabolic Bloch space.

1. Introduction

Let \mathbf{R}^{n+1} denote the $(n+1)$ -dimensional Euclidean space ($n \geq 2$) and H be its upper half space

$$H = \{(x, t) \in \mathbf{R}^{n+1}; x \in \mathbf{R}^n, t > 0\}.$$

For $0 < \alpha \leq 1$, we consider a parabolic operator

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta)^\alpha$$

on H , where Δ is the Laplace operator with respect to x . When $\alpha = 1$, $L^{(\alpha)}$ is the heat operator. Otherwise, $L^{(\alpha)}$ is a non-local operator.

For $1 \leq p \leq \infty$, we denote by b_α^p the set of all solutions of $L^{(\alpha)}u = 0$ on H such that

$$\|u\|_{L^p(H)} := \left(\int_0^\infty \int_{\mathbf{R}^n} |u(x, t)|^p dx dt \right)^{1/p} < \infty.$$

It is shown that b_α^p is a Banach space under the norm $\|\cdot\|_{L^p(H)}$. We call b_α^p the α -parabolic Bergman space (of order p), because $L^{(\alpha)}$ has parabolic nature.

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In this paper we study the properties of solutions of $L^{(\alpha)}u = 0$ on H in the framework of the Bergman space theory. One of our main results is to show the following identity: for $u \in \mathbf{b}_\alpha^p$,

$$(1.1) \quad u(x, t) = \int_{\mathbf{R}^n} u(x - y, t - s) W^{(\alpha)}(y, s) dy$$

whenever $t > s > 0$. According to the heat operator case [12], we call (1.1) the Huygens property for u . Since all solutions of $L^{(\alpha)}u = 0$ form a balayage space (cf. [2]), we make use of potential theory method for the proof of (1.1). In particular, the theory of α -harmonic measures is useful ([4] and [7]). In the sequel, we call a solution u of $L^{(\alpha)}u = 0$ an $L^{(\alpha)}$ -harmonic function.

Our study is motivated by recent results [10] and [13] of harmonic Bergman spaces on the upper half space. We remark that α -parabolic Bergman space is a generalization of the harmonic Bergman space. In fact, $(1/2)$ -parabolic Bergman spaces coincide with harmonic Bergman spaces, because in the case $\alpha = 1/2$, the fundamental solution of $L^{(1/2)}$ is equal to the Poisson kernel on H (see Corollary 4.4 below).

Based on the Huygens property, we shall discuss the following subjects: the boundedness of the point evaluations, the explicit form of the α -parabolic Bergman kernels, the dual space of \mathbf{b}_α^p , the α -parabolic little Bloch space and the pre-dual space of \mathbf{b}_α^1 . The estimates of the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ play crucial roles in various contexts.

2. $L^{(\alpha)}$ -harmonic functions

In this section, we discuss mainly in the case $0 < \alpha < 1$, because the corresponding results are well-known in the case $\alpha = 1$ (e.g. see [3] and [11]). For an open set D in \mathbf{R}^{n+1} , let $C_K^\infty(D)$ denote the set of all infinitely differentiable functions with compact support on D . In order to define $L^{(\alpha)}$ -harmonic functions, we shall recall how the adjoint operator $\tilde{L}^{(\alpha)} = -\partial/\partial t + (-\Delta)^\alpha$ acts on $C_K^\infty(\mathbf{R}^{n+1})$. For $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the convolution operator defined by $-c_{n,\alpha}$ p.f. $|x|^{-n-2\alpha}$, where

$$c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma\left(\frac{n+2\alpha}{2}\right) / \Gamma(-\alpha) > 0$$

and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Hence for $\varphi \in C_K^\infty(\mathbf{R}^{n+1})$,

$$\tilde{L}^{(\alpha)}\varphi(x, t) = -\frac{\partial}{\partial t}\varphi(x, t) - c_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y| > \delta} (\varphi(x + y, t) - \varphi(x, t)) |y|^{-n-2\alpha} dy.$$

It is easily seen that if $\text{supp}(\varphi)$, the support of φ , is contained in $\{|x| < r, t_1 < t < t_2\}$, then

$$(2.1) \quad |\tilde{L}^{(\alpha)}\varphi(x, t)| \leq 2^{n+2\alpha} c_{n,\alpha} \left(\sup_{t_1 < s < t_2} \int_{\mathbf{R}^n} |\varphi(y, s)| dy \right) \cdot |x|^{-n-2\alpha}$$

for (x, t) with $|x| \geq 2r$. Remark also that

$$\tilde{L}^{(\alpha)}(\partial_t \varphi) = \partial_t \tilde{L}^{(\alpha)}(\varphi) \quad \text{and} \quad \tilde{L}^{(\alpha)}(\partial_{x_j} \varphi) = \partial_{x_j} \tilde{L}^{(\alpha)}(\varphi) \quad \text{for } j = 1, \dots, n,$$

where $\partial_t = \partial/\partial t$ and $\partial_{x_j} = \partial/\partial x_j$.

Now we give the definition of $L^{(\alpha)}$ -harmonicity. For an open set D in \mathbf{R}^{n+1} , we put

$$s(D) := \{(x, t) \in \mathbf{R}^{n+1}; (y, t) \in D \text{ for some } y \in \mathbf{R}^n\}.$$

Since $\text{supp}(\tilde{L}^{(\alpha)}\varphi)$ extends to $s(D)$ even if $\text{supp}(\varphi) \subset D$, we can define $L^{(\alpha)}$ -harmonicity on D by duality only for the functions defined on $s(D)$.

DEFINITION 2.1. A function h is said to be $L^{(\alpha)}$ -harmonic on an open set D , when h is defined on $s(D)$ and satisfies the following conditions:

- (a) h is a Borel measurable function on $s(D)$,
- (b) h is continuous on D ,
- (c) for every $\varphi \in C_K^\infty(D)$, $\iint_{s(D)} |h \cdot \tilde{L}^{(\alpha)}\varphi| dx dt < \infty$ and $\iint_{s(D)} h \cdot \tilde{L}^{(\alpha)}\varphi dx dt = 0$.

REMARK 2.2. When $0 < \alpha < 1$, the inequality (2.1) implies that the integrability condition in (c) of Definition 2.1 is equivalent to the following: for any closed strip $[t_1, t_2] \times \mathbf{R}^n \subset s(D)$

$$(2.2) \quad \int_{t_1}^{t_2} \int_{\mathbf{R}^n} |h(x, t)|(1 + |x|)^{-n-2\alpha} dx dt < \infty.$$

The following lemma will be useful in the Section 4.

Lemma 2.3. *Let v be $L^{(\alpha)}$ -harmonic on H . If $v = 0$ continuously on the boundary $\partial H = \mathbf{R}^n \times \{0\}$ and if $\int_0^\delta \int_{\mathbf{R}^n} |v(x, t)|(1 + |x|)^{-n-2\alpha} dx dt < \infty$ for some $\delta > 0$, then the function V defined by*

$$V(x, t) = \int_0^t v(x, \tau) d\tau$$

is also $L^{(\alpha)}$ -harmonic on H .

Proof. If $\alpha = 1$, the lemma is clearly true, so we assume $0 < \alpha < 1$. Take arbitrary $\varphi \in C_K^\infty(H)$. Then

$$\begin{aligned} & \int_0^\infty \int_{\mathbf{R}^n} V(x, t) \tilde{L}^{(\alpha)}\varphi(x, t) dx dt \\ &= \int_0^\infty \int_{\mathbf{R}^n} \int_0^t v(x, \tau) d\tau \tilde{L}^{(\alpha)}\varphi(x, t) dx dt \end{aligned}$$

$$= \int_0^\infty \int_{\mathbf{R}^n} v(x, t) \varphi(x, t) dx dt + \int_0^\infty \int_0^t \int_{\mathbf{R}^n} v(x, \tau) (-\Delta)^\alpha \varphi(x, t) dx d\tau dt.$$

To calculate the second integral of the last line, fix $t > 0$. Considering a C^∞ approximation of the indicator function of the set $[0, t]$, we see

$$\int_0^t \int_{\mathbf{R}^n} v(x, \tau) (-\Delta)^\alpha \varphi(x, t) dx d\tau = \int_{\mathbf{R}^n} (v(x, 0) - v(x, t)) \varphi(x, t) dx.$$

Since $v(x, 0) = 0$, we have therefore

$$\int_0^\infty \int_{\mathbf{R}^n} V(x, t) \tilde{L}^{(\alpha)} \varphi(x, t) dx dt = 0$$

and $L^{(\alpha)}$ -harmonicity of V follows. □

The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is

$$(2.3) \quad W^{(\alpha)}(x, t) = \begin{cases} (2\pi)^{-n} \int_{\mathbf{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1} x \cdot \xi) d\xi & t > 0 \\ 0 & t \leq 0, \end{cases}$$

where $x \cdot \xi$ is the inner product of x and ξ and $|\xi| = (\xi \cdot \xi)^{1/2}$. Then

$$\tilde{W}^{(\alpha)}(x, t) := W^{(\alpha)}(x, -t)$$

is the fundamental solution of $\tilde{L}^{(\alpha)}$.

In the case $\alpha = 1$, $W^{(1)}$ is the Gauss-Weierstrass kernel

$$W^{(1)}(x, t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \leq 0. \end{cases}$$

In the case $\alpha = 1/2$, $W^{(1/2)}$ is the Poisson kernel (cf. [1, p.74])

$$(2.4) \quad W^{(1/2)}(x, t) = \begin{cases} \Gamma\left(\frac{n+1}{2}\right) \frac{t}{(|x|^2 + t^2)^{(n+1)/2}} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

The harmonicity of $W^{(1/2)}$ derives a close connection between $L^{(1/2)}$ -harmonic functions and usual harmonic functions on H (see Corollary 4.4 below). For other $\alpha \in (0, 1)$ any simple explicit expressions for $W^{(\alpha)}$ are not known.

Note also that $W^{(\alpha)}(x, t) \geq 0$,

$$(2.5) \quad \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t - s) dx = 1$$

and for every $0 < s < t$,

$$(2.6) \quad W^{(\alpha)}(x, t) = \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t - s)W^{(\alpha)}(y, s) dy.$$

When we put

$$(2.7) \quad \phi_\alpha(|x|) := W^{(\alpha)}(x, 1),$$

then for $t > 0$,

$$(2.8) \quad W^{(\alpha)}(x, t) = t^{-n/(2\alpha)}\phi_\alpha(t^{-1/(2\alpha)}|x|)$$

and $\phi_\alpha(r) = O(r^{-n-2\alpha})$ when $0 < \alpha < 1$ (use (3.3) below), and $\phi_1(r) = O(\exp(-r^2/4))$ as $r \rightarrow +\infty$. Further estimates of $W^{(\alpha)}$ will be given in next section.

Since $W^{(\alpha)}(x - y, t) dy$ converges vaguely to the Dirac measure at x as $t \rightarrow +0$, we see the following convergence result.

Lemma 2.4. *Let f be a continuous function on \mathbf{R}^n . If f belongs to $L^p(\mathbf{R}^n)$ with $1 \leq p \leq \infty$, then for every $x \in \mathbf{R}^n$,*

$$\lim_{t \rightarrow +0} \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t)f(y) dy = f(x).$$

The fact that $W^{(\alpha)}$ is $L^{(\alpha)}$ -harmonic off $(0, 0)$ is important. In fact the following assertion follows from this.

Proposition 2.5 (see [9, Proposition 10]). *If u satisfies the Huygens property, that is, for every $x \in \mathbf{R}^n$ and $0 < s < t$,*

$$u(x, t) = \int_{\mathbf{R}^n} u(x - y, t - s)W^{(\alpha)}(y, s) dy,$$

then u is an $L^{(\alpha)}$ -harmonic function on H .

3. Estimates of fundamental solutions

In the sequel, we use the following notations. For $\delta > 0$ and a function f on H , we write

$$T_\delta f(x, t) := f(x, t + \delta).$$

Then $T_\delta f$ is a function on $\mathbf{R}^n \times (-\delta, \infty)$. Let k be a nonnegative integer and $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}_0^n$ be a multi-index, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Then $|\beta| := \beta_1 + \dots + \beta_n$ and

$$\partial_x^\beta \partial_t^k f(x, t) := \frac{\partial^{|\beta|+k}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n} \partial t^k} f(x, t).$$

Using the above notation, we start with the following equality which follows from (2.3) easily.

$$(3.1) \quad \partial_x^\beta \partial_t^k W^{(\alpha)}(x, t) = t^{-((n+|\beta|)/(2\alpha)+k)} (\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)}x, 1).$$

The following estimate of $W^{(\alpha)}$ plays an important role in our later argument.

Lemma 3.1. *Let $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$. Then there is a constant $C > 0$ such that for every $(x, t) \in H$,*

$$(3.2) \quad |\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t)| \leq Ct^{1-k} (t + |x|^{2\alpha})^{-(n+|\beta|)/(2\alpha)-1}$$

Proof. For $x_0 = (1, 0, \dots, 0) \in \mathbf{R}^n$, we put

$$\psi_\alpha(t) := W^{(\alpha)}(x_0, t).$$

Then it was shown that

$$(3.3) \quad \psi_\alpha(t) = O(t) \quad \text{as } t \rightarrow 0$$

in [5, Lemma 2.1]. The argument which was done there gives that for every $k \in \mathbf{N}$,

$$(3.4) \quad \psi_\alpha^{(k)}(t) \quad \text{is bounded on } (0, \infty).$$

In fact, as in [5] we have

$$\psi_\alpha^{(k)}(t) = (-1)^k (2\pi)^{-n/2} \int_0^\infty \left(\int_{\mathbf{R}^n} |\xi|^{2\alpha k} e^{-s|\xi|^2} \hat{\nu}(\xi) d\xi \right) d\sigma_t^\alpha(s)$$

where $\hat{\nu}$ is the Fourier transform of the normalized uniform measure ν on the unit sphere and $(\sigma_t^\alpha)_{t \geq 0}$ is the one-side stable semi-group on $(0, \infty)$ (see [1, p.74]). Thus (3.4) follows if we prove that

$$\Psi(s) := \int_{\mathbf{R}^n} |\xi|^{2\alpha k} e^{-s|\xi|^2} \hat{\nu}(\xi) d\xi$$

is bounded on $(0, \infty)$.

In the case that αk is an integer, we have

$$\Psi(s) = (2\pi)^{n/2} (-\Delta)^{\alpha k} (g_s * \nu)(0),$$

where $g_s(x) = W^{(1)}(x, s)$ is the Gauss-Weierstrass kernel. This formula shows the boundedness of Ψ .

If αk is not an integer, we take $l \in \mathbf{N}$ such that $-2 < 2\alpha k - 2l < 0$. Then

$$\Psi(s) = (2\pi)^{n/2} C_{n, \alpha k - l} (-\Delta)^l (|x|^{-n+2l-2\alpha k} * g_s * \nu)(0)$$

$$= (2\pi)^{n/2} c_{n,\alpha k-l} \{(\varphi(x)|x|^{-n+2l-2\alpha k}) * ((-\Delta)^l g_s) * \nu)(0) \\ + (-\Delta)^l ((1 - \varphi(x))|x|^{-n+2l-2\alpha k}) * g_s * \nu)(0)\}$$

where $\varphi \in C_K^\infty(\mathbf{R}^n)$ with $0 \leq \varphi \leq 1$, $\text{supp}(\varphi) \subset \{|x| < 1\}$ and $\varphi = 1$ on $\{|x| < 1/3\}$, and $c_{n,\alpha k-l} = -4^{\alpha k-l} \pi^{-n/2} \Gamma((n+2\alpha k-2l)/2) / \Gamma(l-\alpha k)$. The boundedness of Ψ follows even if $\alpha k \notin \mathbf{N}$.

Now we return to the proof of (3.2). Since $W^{(\alpha)}(x, t) = |x|^{-n} \psi_\alpha(|x|^{-2\alpha} t)$, we have

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t) = \partial_x^\beta (|x|^{-n-2\alpha k} \psi_\alpha^{(k)}(|x|^{-2\alpha} t)),$$

so that

$$(\partial_x^\beta \partial_t^k W^{(\alpha)})(x, 1) = \partial_x^\beta (|x|^{-n-2\alpha k} \psi_\alpha^{(k)}(|x|^{-2\alpha})) \\ = \sum_{\beta=\beta'+\beta''} \binom{\beta}{\beta'} \partial_x^{\beta'} (|x|^{-n-2\alpha k}) \partial_x^{\beta''} (\psi_\alpha^{(k)}(|x|^{-2\alpha})).$$

It is easily seen that $\partial_x^{\beta'} (|x|^{-n-2\alpha k}) = O(|x|^{-n-2\alpha k-|\beta'|})$ and $\partial_x^{\beta''} (\psi_\alpha^{(k)}(|x|^{-2\alpha})) = O(|x|^{-|\beta''|})$ as $|x| \rightarrow \infty$. As a result, we have

$$(3.5) \quad |(\partial_x^\beta \partial_t^k W^{(\alpha)})(x, 1)| \leq C|x|^{-n-2\alpha-|\beta|} \quad \text{as } |x| \rightarrow \infty.$$

Remark that (3.5) remains true for the case $k = 0$ because of (3.3). Hence (3.1) shows that if $|x| \geq t^{1/(2\alpha)}$, then

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t)| = t^{-(n+|\beta|)/(2\alpha+k)} |(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)} x, 1)| \\ \leq C t^{1-k} |x|^{-n-2\alpha-|\beta|}$$

by (3.5), and if $|x| \leq t^{1/(2\alpha)}$, then

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t)| = t^{-(n+|\beta|)/(2\alpha)-k} |(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)} x, 1)| \\ \leq C t^{-(n+|\beta|)/(2\alpha)-k}$$

by the boundedness of $|(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)} x, 1)|$. These inequalities imply (3.2). □

We note here that T. Takehi and K. Sakai gave an alternative proof of (3.5) ([6]).

As for the L^q -norm of derivatives of $W^{(\alpha)}$, the homogeneity (3.1) gives us the following identity.

Lemma 3.2. *Let $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$ and let $q \geq 1$. If $q > (n+2\alpha)/(n+|\beta|+2\alpha k)$, then there is a constant $C > 0$ such that for any $\delta > 0$*

$$(3.6) \quad \|\partial_x^\beta \partial_t^k T_\delta W^{(\alpha)}\|_{L^q(H)} = C \delta^{-(n+|\beta|)/(2\alpha+k)+(n/(2\alpha)+1)(1/q)}.$$

Proof. Noting that Lemma 3.1 ensures the integrability, we obtain the equality immediately. \square

4. Huygens property

We have seen in Proposition 2.5 that every Borel measurable function satisfying the Huygens property is $L^{(\alpha)}$ -harmonic on H . In this section, we shall prove the converse assertion for p -th integrable $L^{(\alpha)}$ -harmonic functions. This result will be very useful in other contexts as well.

Theorem 4.1. *Let $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. If an $L^{(\alpha)}$ -harmonic function u on H belongs to $L^p(H)$, then u satisfies the Huygens property, that is,*

$$(4.1) \quad u(x, t) = \int_{\mathbf{R}^n} u(x - y, t - s) W^{(\alpha)}(y, s) dy$$

holds for every $x \in \mathbf{R}^n$ and $0 < s < t < \infty$.

The next two lemmas will be used in the proof of the above theorem. The first lemma is concerning $L^{(\alpha)}$ -harmonic measures. For $0 < \alpha < 1$ and $r > 0$, put

$$w_r^\alpha(x) = \begin{cases} 0 & \text{if } |x| \leq r \\ \frac{a_{n,\alpha} r^{2\alpha}}{(|x|^2 - r^2)^\alpha |x|^n} & \text{if } |x| > r, \end{cases}$$

where $a_{n,\alpha} = \Gamma(n/2)\pi^{-n/2-1} \sin(\pi\alpha)$. We know that $w_r^\alpha(x) dx$ is the balayaged measure on $\{|x| \geq r\}$ of the Dirac measure at the origin with respect to the Riesz kernel $|x|^{2\alpha-n}$ (see [4]). Recalling the equality

$$(4.2) \quad c_{n,-\alpha} |x|^{2\alpha-n} = \int_0^\infty W^{(\alpha)}(x, t) dt = \int_{-\infty}^0 \tilde{W}^{(\alpha)}(x, s) ds,$$

where $c_{n,-\alpha} = 4^{-\alpha} \pi^{-n/2} \Gamma((n-2\alpha)/2) / \Gamma(\alpha)$ (cf. [1]), we see the following relation between the above balayaged measure and the $L^{(\alpha)}$ -harmonic measure.

Lemma 4.2. *Let $0 < \alpha < 1$ and let ν_r^α be the $L^{(\alpha)}$ -harmonic measure at the origin on $B_r(0) \times \mathbf{R}$, where $B_r(0)$ is the ball of radius r and center 0 in \mathbf{R}^n . Then*

$$(4.3) \quad \int_A w_r^\alpha(x) dx = \nu_r^\alpha(A \times (-\infty, 0])$$

for every Borel set A in \mathbf{R}^n .

Proof. Since the $L^{(\alpha)}$ -harmonic measure ν_r^α is the balayaged measure on $\{|x| \geq r\} \times (-\infty, 0]$ of the Dirac measure at the origin with respect to $\tilde{W}^{(\alpha)}$,

$$\tilde{W}^{(\alpha)}(y, s) = \int_{|x| \geq r} \int_{-\infty}^0 \tilde{W}^{(\alpha)}(y - x, s - t) d\nu_r^\alpha(x, t)$$

holds for $|y| > r$. Furthermore, by [5, Proposition 4.2 (2)], this equality holds for $|y| = r$, because every boundary point is regular with respect to $\tilde{L}^{(\alpha)}$. Now we denote by μ_r the measure on \mathbf{R}^n defined by $\mu_r(A) = \nu_r^\alpha(A \times (-\infty, 0])$. Then by (4.2), for $|y| \geq r$,

$$\begin{aligned} c_{n,-\alpha}|y|^{2\alpha-n} &= \int_{-\infty}^0 \tilde{W}^{(\alpha)}(y, s) ds \\ &= \int_{-\infty}^0 \left(\int_{|x| \geq r} \int_s^0 \tilde{W}^{(\alpha)}(y - x, s - t) d\nu_r^\alpha(x, t) \right) ds \\ &= \int_{|x| \geq r} \int_{-\infty}^0 \left(\int_{-\infty}^t \tilde{W}^{(\alpha)}(x - y, s - t) ds \right) d\nu_r^\alpha(x, t) \\ &= c_{n,-\alpha} \int_{|x| \geq r} |x - y|^{2\alpha-n} d\mu_r(x). \end{aligned}$$

On the other hand, since $w_r^\alpha(x) dx$ is the balayaged measure on $\{|x| \geq r\}$ with respect to $|x|^{2\alpha-n}$, we have

$$|y|^{2\alpha-n} = \int_{|x| \geq r} |x - y|^{2\alpha-n} w_r^\alpha(x) dx$$

on $|y| > r$, and by the reason similar to above, this equality also holds on its boundary $\{|y| = r\}$. Hence

$$(4.4) \quad \int_{|x| \geq r} |x - y|^{2\alpha-n} d\mu_r(x) = \int_{|x| \geq r} |x - y|^{2\alpha-n} w_r^\alpha(x) dx$$

on $|y| \geq r$. Since the support of both measures μ_r and $w_r^\alpha(x) dx$ is contained in $\{|x| \geq r\}$, the domination principle ([2, Corollary 4.13]) implies that (4.4) holds for all $y \in \mathbf{R}^n$. Finally the unicity principle for the Riesz kernel ([7, Theorem 1.12]) gives the equality (4.3). \square

The next lemma is an estimate of the function \tilde{w}_R^α defined by

$$\tilde{w}_R^\alpha(x) = \frac{1}{R} \int_R^{2R} w_r^\alpha(x) dr \quad (R > 0).$$

Lemma 4.3. *Let $1 \leq p \leq \infty$. Then there is a constant $C > 0$ such that for every $R > 0$,*

$$\|\widetilde{w}_R^\alpha\|_{L^q(\mathbf{R}^n)} \leq CR^{-n/p},$$

where q is the exponent conjugate to p .

Proof. We take x with $|x| \geq R$. If $R \leq |x| \leq 3R$, then we have

$$\widetilde{w}_R^\alpha(x) \leq \frac{a_{n,\alpha}}{R|x|^n} \int_R^{|x|} \frac{r^{2\alpha}}{(|x|^2 - r^2)^\alpha} dr \leq \frac{2a_{n,\alpha}3^{n+4\alpha}}{2^{2\alpha}(1-\alpha)} R^{2\alpha} |x|^{-n-2\alpha}.$$

Next if $|x| \geq 3R$, then $w_r^\alpha(x) \leq (9/5)^\alpha a_{n,\alpha} R^{2\alpha} |x|^{-n-2\alpha}$ because $R < r < 2R$. Hence when $p = 1$, then $q = \infty$ and the lemma holds clearly. When $1 < p \leq \infty$, using the above estimates, we have

$$\int_{\mathbf{R}^n} \widetilde{w}_R^\alpha(x)^q dx \leq C \int_R^\infty R^{2\alpha q} r^{-(n+2\alpha)q} r^{n-1} dr \leq CR^{-n(q-1)}$$

with some constant $C > 0$. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. In the case that $\alpha = 1$, the assertion is known (see for example [11, Theorem 3.6, p.76]) and in the case $p = \infty$, the assertion follows from [9, Proposition 11]. Hence we may assume that $0 < \alpha < 1$ and $1 \leq p < \infty$. Remark that for any $\delta_0 > 0$, there exists $0 < \delta < \delta_0$ such that

$$u_\delta(\cdot, 0) \in L^p(\mathbf{R}^n),$$

where $u_\delta(x, t) := T_\delta u(x, t) = u(x, t + \delta)$. Define the function v by

$$v(x, t) := u_\delta(x, t) - \tilde{u}_\delta(x, t),$$

where

$$\tilde{u}_\delta(x, t) := \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t) u_\delta(y, 0) dy.$$

Then v is clearly $L^{(\alpha)}$ -harmonic on H and by Lemma 2.4, v vanishes continuously on the lower boundary $\mathbf{R}^n \times \{0\}$. By the Minkowski inequality, $\|\tilde{u}_\delta(\cdot, t)\|_{L^p(\mathbf{R}^n)} \leq \|u_\delta(\cdot, 0)\|_{L^p(\mathbf{R}^n)}$, so that the Hölder inequality shows

$$\int_0^\delta \int_{\mathbf{R}^n} |\tilde{u}_\delta(x, t)| (1 + |x|)^{-n-2\alpha} dx dt < \infty.$$

By definition u_δ also satisfies the same inequality, so that v fulfills the assumption in Lemma 2.3. Hence

$$V(x, t) := \int_0^t v(x, \tau) d\tau$$

is $L^{(\alpha)}$ -harmonic on H . Let $(x, t) \in H$ be fixed. Then the $L^{(\alpha)}$ -harmonic measure $\nu_\omega^{(x,t)}$ of a cylinder $\omega = B_r(x) \times (0, t + 1)$ can be written as

$$\nu_\omega^{(x,t)} = \nu_\omega^{(x,t)}|_{B_r(x)^c \times [0,t]} + \nu_\omega^{(x,t)}|_{B_r(x) \times \{0\}},$$

where the first term in the right hand side is

$$\nu_r^\alpha(y - x, s - t)|_{\{|y-x| \geq r, -t \leq s-t \leq 0\}}$$

and the second term is absolutely continuous with respect to the n -dimensional Lebesgue measure dy whose density is bounded by $W^{(\alpha)}(x - y, t)$. Since $V(y, 0) = 0$, by (4.3),

$$\begin{aligned} |V(x, t)| &= \left| \int_{|y| \geq r} \int_{-t}^0 V(x + y, t + s) d\nu_r^\alpha(y, s) \right| \\ &\leq \int_{|y| \geq r} \int_{-t}^0 \left(\int_0^{s+t} |v(x + y, \tau)| d\tau \right) d\nu_r^\alpha(y, s) \\ &= \int_0^t \left(\int_{|y| \geq r} \int_{\tau-t}^0 |v(x + y, \tau)| d\nu_r^\alpha(y, s) \right) d\tau \\ &\leq \int_0^t \left(\int_{|y| \geq r} |v(x + y, \tau)| w_r^\alpha(y) dy \right) d\tau \end{aligned}$$

so that

$$|V(x, t)| \leq \int_0^t \int_{\mathbf{R}^n} |v(y, \tau)| \widetilde{w}_R^\alpha(x - y) dy d\tau.$$

Therefore by the Hölder inequality and Lemma 4.3, letting $R \rightarrow \infty$, we have $V(x, t) \equiv 0$. Since $v(x, t) = \partial_t V(x, t) = 0$,

$$u_\delta(x, t) = \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t) u(y, \delta) dy.$$

By (2.6), the right hand side satisfies the Huygens property, so does u because δ_0 is arbitrary. □

Recalling (2.4) and Proposition 2.5, we have the following interesting corollary of the theorem above.

Corollary 4.4. *Let $1 \leq p \leq \infty$ and suppose that $u \in L^p(H)$. Then u is an $L^{(1/2)}$ -harmonic function if and only if u is a usual harmonic function on H .*

REMARK 4.5. Throughout this paper we always assume that $n \geq 2$. The reason is that some arguments in this section are not valid for the case $n = 1$. For example, (4.2) does not hold if $n = 1$ and $1/2 \leq \alpha < 1$ (cf. [1, p.135]).

5. α -parabolic Bergman spaces

In this section, we shall define α -parabolic Bergman spaces and discuss some basic properties.

DEFINITION 5.1. For $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we denote by \mathbf{b}_α^p the set of all $L^{(\alpha)}$ -harmonic functions on H which belong to $L^p(H)$. The space \mathbf{b}_α^p is called the α -parabolic Bergman space (of order p).

To show the closedness of \mathbf{b}_α^p in $L^p(H)$, we use the following boundedness of point evaluations.

Proposition 5.2. *Let $1 \leq p \leq \infty$. Then, there is a constant $C > 0$ such that for every $u \in \mathbf{b}_\alpha^p$ and every $(x, t) \in H$,*

$$(5.1) \quad |u(x, t)| \leq C \|u\|_{L^p(H)} t^{-(n/(2\alpha)+1)(1/p)}.$$

Proof. If $p = \infty$, then $|u(x, t)| \leq \|u\|_{L^\infty(H)}$, which is the assertion of the lemma. We suppose $1 \leq p < \infty$. For fixed $0 < a_1 < a_2 < 1$, the Huygens property (4.1) gives

$$\begin{aligned} u(x, t) &= \int_{\mathbf{R}^n} u(x - y, t - s) W^{(\alpha)}(y, s) dy && (t > s > 0) \\ &= \frac{1}{(a_2 - a_1)t} \int_{a_1 t}^{a_2 t} \int_{\mathbf{R}^n} u(x - y, t - s) W^{(\alpha)}(y, s) dy ds. \end{aligned}$$

Then using (3.2), we have

$$|u(x, t)| \leq C \|u\|_{L^p(H)} t^{-(n/(2\alpha)+1)(1/p)}. \quad \square$$

The next theorem implies that \mathbf{b}_α^p is a Banach space under the L^p -norm.

Theorem 5.3. *Let $1 \leq p \leq \infty$. Then \mathbf{b}_α^p is a closed subspace of $L^p(H)$.*

Proof. By Proposition 5.2, the L^p -convergence implies the uniform convergence on $\mathbf{R}^n \times [t_1, \infty) \subset H$ for every $t_1 > 0$. Hence the limit function of any L^p -convergent sequence in \mathbf{b}_α^p is continuous and satisfies the Huygens property. The result follows

from Proposition 2.5. □

It follows from the Huygens property that $\mathbf{b}_\alpha^p \subset C^\infty(H)$, where $C^\infty(H)$ is the set of all C^∞ -functions on H . Furthermore, as in the proof of Proposition 5.2, we have the following estimate for point evaluations of derivatives.

Theorem 5.4. *Let $1 \leq p \leq \infty$ and $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$. Then there is a constant $C > 0$ such that*

$$(5.2) \quad |\partial_x^\beta \partial_t^k u(x, t)| \leq C \|u\|_{L^p(H)} t^{-(|\beta|/(2\alpha)+k)-(n/(2\alpha)+1)(1/p)}$$

for any $u \in \mathbf{b}_\alpha^p$ and $(x, t) \in H$.

The following norm inequality is also established.

Proposition 5.5. *Let $1 \leq p \leq \infty$ and $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$. Then there is a constant $C > 0$ such that for every $u \in \mathbf{b}_\alpha^p$,*

$$(5.3) \quad \|t^{|\beta|/(2\alpha)+k} \partial_x^\beta \partial_t^k u\|_{L^p(H)} \leq C \|u\|_{L^p(H)}.$$

Proof. By the Huygens property,

$$\partial_x^\beta \partial_t^k u(x, t) = \int_{\mathbf{R}^n} u(x - y, s) (\partial_x^\beta \partial_t^k W^{(\alpha)})(y, t - s) dy$$

for every $t > s > 0$. Hence, taking $0 < \gamma < 1$ and $s = \gamma t$, we have

$$\begin{aligned} \partial_x^\beta \partial_t^k u(x, t) &= \int_{\mathbf{R}^n} u(x - y, \gamma t) (\partial_x^\beta \partial_t^k W^{(\alpha)})(y, (1 - \gamma)t) dy \\ &= ((1 - \gamma)t)^{-(|\beta|/(2\alpha)+k)} \int_{\mathbf{R}^n} u(x - ((1 - \gamma)t)^{1/(2\alpha)} z, \gamma t) (\partial_x^\beta \partial_t^k W^{(\alpha)})(z, 1) dz. \end{aligned}$$

Thus the Minkowski inequality yields

$$\|t^{|\beta|/(2\alpha)+k} \partial_x^\beta \partial_t^k u\|_{L^p(H)} \leq (1 - \gamma)^{-(|\beta|/(2\alpha)+k)} \gamma^{-1/p} \left(\int_{\mathbf{R}^n} |(\partial_x^\beta \partial_t^k W^{(\alpha)})(z, 1)| dz \right) \|u\|_{L^p(H)}. \quad \square$$

Finally we discuss the integrals over the hyperplanes $\{t = \text{constant}\}$. The following lemma is interesting in itself.

Lemma 5.6. *Let $1 \leq p \leq \infty$. For $u \in \mathbf{b}_\alpha^p$, the function $t \mapsto \|u(\cdot, t)\|_{L^p(\mathbf{R}^n)}$ is decreasing on $(0, \infty)$.*

Proof. Take $t_2 > t_1 > 0$. By the Huygens property,

$$u(x, t_2) = \int_{\mathbf{R}^n} u(x - y, t_1) W^{(\alpha)}(y, t_2 - t_1) dy.$$

The Minkowski inequality gives that

$$\|u(\cdot, t_2)\|_{L^p(\mathbf{R}^n)} \leq \int_{\mathbf{R}^n} \|u(\cdot, t_1)\|_{L^p(\mathbf{R}^n)} W^{(\alpha)}(y, t_2 - t_1) dy = \|u(\cdot, t_1)\|_{L^p(\mathbf{R}^n)}. \quad \square$$

REMARK 5.7. For $1 \leq p \leq \infty$, we define the α -parabolic Hardy space \mathbf{h}_α^p on H as follows:

$$\mathbf{h}_\alpha^p := \left\{ v; L^{(\alpha)}\text{-harmonic on } H \text{ and } \sup_{t>0} \|v(\cdot, t)\|_{L^p(\mathbf{R}^n)} < \infty \right\}.$$

Then as a corollary to Lemma 5.6, we see that $T_\delta u \in \mathbf{h}_\alpha^p$ for every $u \in \mathbf{b}_\alpha^p$ and $\delta > 0$.

The next result is called the cancellation property.

Proposition 5.8. *For every $u \in \mathbf{b}_\alpha^1$ and every $t > 0$,*

$$(5.4) \quad \int_{\mathbf{R}^n} u(x, t) dx = 0.$$

Proof. By the Huygens property, we have

$$u(y, t + s) = \int_{\mathbf{R}^n} u(x, t) W^{(\alpha)}(y - x, s) dx.$$

Integrating the both sides by y and then s , we find

$$\int_0^T \int_{\mathbf{R}^n} u(y, t + s) dy ds = \int_0^T \int_{\mathbf{R}^n} u(x, t) dx ds = T \int_{\mathbf{R}^n} u(x, t) dx.$$

Since the left hand side converges as $T \rightarrow \infty$, (5.4) follows. □

REMARK 5.9. This proposition shows that \mathbf{b}_α^1 does not contain any nonzero non-negative element. More generally, \mathbf{b}_α^p contains a nonnegative u such that $u \neq 0$ if and only if $p > (n + 2\alpha)/n$. This condition is related to (3.6) in Lemma 3.2 for $(\beta, k) = (0, 0)$. Using Lemma 3.2 again for $(\beta, k) = (0, 2)$, we have

$$\frac{\|\partial_t^2 T_\delta W^{(\alpha)}\|_{L^p(H)}}{\|\partial_t^2 T_\delta W^{(\alpha)}\|_{L^q(H)}} = C \delta^{(n/(2\alpha)+1)(1/p-1/q)}$$

for all $\delta > 0$. Hence the closed graph theorem tells us that there is no inclusion relation between \mathbf{b}_α^p and \mathbf{b}_α^q for $p \neq q$.

6. α -parabolic Bergman kernel

Since the point evaluation is bounded, b_α^2 has the reproducing kernel. In this section, we shall prove that the kernel

$$(6.1) \quad R_\alpha(x, t; y, s) = -2\partial_t W^{(\alpha)}(x - y, t + s)$$

is the desired reproducing kernel of b_α^2 (see Remark 6.5 below). We call R_α the α -parabolic Bergman kernel.

For $m = 0, 1, 2, \dots$, we also use the kernel R_α^m defined by

$$R_\alpha^m(x, t; y, s) = c_m s^m \partial_s^m R_\alpha(x, t; y, s),$$

where $c_m = (-2)^m/m!$. Note that $R_\alpha^0 = R_\alpha$ and it is a symmetric kernel.

We begin with two lemmas concerning these kernels. The first one is an estimate of their growth order, which follows from Lemma 3.1 immediately.

Lemma 6.1. *Let $m \geq 0$ be an integer. Then there is a constant $C > 0$ such that for any $(x, t), (y, s) \in H$,*

$$|R_\alpha^m(x, t; y, s)| \leq C s^m (s + t)^{-m} (s + t + |x - y|^{2\alpha})^{-n/(2\alpha)-1}.$$

In particular, $R_\alpha^m(x, t; \cdot, \cdot) \in L^q(H)$ for every $q > 1$ and $(x, t) \in H$.

The second one is an estimate of growth order for their integrals.

Lemma 6.2. *Let $m \geq 0$ be an integer. If $-1 - m < \gamma < 0$, then there exists a constant $c_1(\gamma) > 0$ such that, for every $t > 0$,*

$$\iint_H s^\gamma |R_\alpha^m(x, t; y, s)| dy ds = c_1(\gamma) t^\gamma.$$

If $-1 < \gamma < m$, then there exists a constant $c_2(\gamma) > 0$ such that, for every $s > 0$,

$$\iint_H t^\gamma |R_\alpha^m(x, t; y, s)| dx dt = c_2(\gamma) s^\gamma.$$

Proof. By (3.1) we have

$$\begin{aligned} & \iint_H s^\gamma |R_\alpha^m(x, t; y, s)| dy ds \\ &= 2|c_m| \int_0^\infty \int_{\mathbf{R}^n} s^\gamma s^m |\partial_t^{m+1} W^{(\alpha)}(x - y, t + s)| dy ds \\ &= 2|c_m| \int_0^\infty \int_{\mathbf{R}^n} s^{\gamma+m} (t + s)^{-n/(2\alpha)-m-1} |(\partial_t^{m+1} W^{(\alpha)})((t + s)^{-1/(2\alpha)} y, 1)| dy ds \end{aligned}$$

$$= c_1(\gamma)t^\gamma,$$

where

$$c_1(\gamma) = 2|c_m| \left(\int_{\mathbf{R}^n} |(\partial_t^{m+1} W^{(\alpha)})(y, 1)| dy \right) \left(\int_0^\infty u^{\gamma+m}(1+u)^{-m-1} du \right).$$

Remark that the second integral in the above is finite if and only if $-1 - m < \gamma < 0$. The second assertion follows similarly. \square

In the sequel, we use the same symbol R_α^m for the integral operator defined by the kernel R_α^m :

$$R_\alpha^m f(x, t) := \iint_H R_\alpha^m(x, t; y, s) f(y, s) dy ds.$$

Then the following interesting relation holds.

Theorem 6.3. *Let $m \geq 0$ be an integer and let $1 \leq p < \infty$. Then $R_\alpha^m u = u$ for every $u \in \mathbf{b}_\alpha^p$, that is*

$$(6.2) \quad u(x, t) = \iint_H R_\alpha^m(x, t; y, s) u(y, s) dy ds.$$

Proof. Let $(x, t) \in H$ be fixed. We shall show the theorem by induction on m . Let $m = 0$. Take $\delta > 0$ and put $u_\delta = T_\delta u$. Then, by the Fubini theorem, we have

$$\begin{aligned} & \iint_H R_\alpha(x, t; y, s) u_\delta(y, s) dy ds \\ &= 2 \int_{\mathbf{R}^n} u_\delta(y, 0) W^{(\alpha)}(x - y, t) dy + 2 \int_{\mathbf{R}^n} \int_0^\infty \partial_s u_\delta(y, s) W^{(\alpha)}(x - y, t + s) ds dy. \end{aligned}$$

Here we use the estimate (5.1). Then by the Huygens property for u_δ and $\partial_s u_\delta$, the first term is equal to $2u_\delta(x, t)$ and the second term is equal to $-u_\delta(x, t)$ respectively. Thus (6.2) holds for u_δ . Since u_δ converges to u in $L^p(H)$ as δ tends to zero, Lemma 6.1 shows the theorem in the case $m = 0$.

Next we assume that the theorem holds for $m - 1 \geq 0$. Take $u \in \mathbf{b}_\alpha^p$ and put $u_\delta = T_\delta u$ as before. Then

$$\begin{aligned} R_\alpha^m u_\delta(x, t) &= \iint_H R_\alpha^m(x, t; y, s) u_\delta(y, s) dy ds \\ &= -2c_m \int_{\mathbf{R}^n} \int_0^\infty u_\delta(y, s) s^m \partial_s^{m+1} W^{(\alpha)}(x - y, t + s) ds dy \\ &= 2c_m \int_{\mathbf{R}^n} \int_0^\infty \{m u_\delta(y, s) s^{m-1} + \partial_s u_\delta(y, s) \cdot s^m\} \partial_s^m W^{(\alpha)}(x - y, t + s) ds dy \end{aligned}$$

$$= 2u_\delta(x, t) + 2c_m \int_{\mathbf{R}^n} \int_0^\infty \partial_s u_\delta(y, s) \cdot s^m \partial_s^m W^{(\alpha)}(x - y, t + s) ds dy,$$

here we use the induction assumption for $m - 1$. Denoting by I the inner integral of the second term, integrating by parts m times and applying the Leibniz rule, we obtain

$$\begin{aligned} I &= (-1)^m \int_0^\infty \partial_s^m [\partial_s u_\delta(y, s) \cdot s^m] W^{(\alpha)}(x - y, t + s) ds \\ &= (-1)^m \sum_{j=0}^m \binom{m}{j} \frac{m!}{(m-j)!} \int_0^\infty \partial_s^{m+1-j} u_\delta(y, s) s^{m-j} W^{(\alpha)}(x - y, t + s) ds. \end{aligned}$$

Therefore, since $\partial_s^{m+1-j} u_\delta$ also satisfies the Hyugens property, by change the order of the integral, we have

$$\begin{aligned} &2c_m \int_{\mathbf{R}^n} I dy \\ &= 2(-1)^m c_m \sum_{j=0}^m \binom{m}{j} \frac{m!}{(m-j)!} \int_0^\infty s^{m-j} \partial_t^{m+1-j} u_\delta(x, t + 2s) ds \\ &= 2(-1)^m c_m \sum_{j=0}^m \binom{m}{j} \frac{m!}{(m-j)!} \frac{1}{2^{m-j}} (-1)^{m-j} (m-j)! \int_0^\infty \partial_s u_\delta(x, t + 2s) ds \\ &= -(-2)^m \sum_{j=0}^m \binom{m}{j} \frac{1}{2^{m-j}} (-1)^j u_\delta(x, t) \\ &= -u_\delta(x, t). \end{aligned}$$

Letting $\delta \downarrow 0$, we complete the induction. □

The main result of this section is the following theorem.

Theorem 6.4. (1) For $1 < p < \infty$, R_α is a bounded operator from $L^p(H)$ onto \mathfrak{b}_α^p .

(2) Let $m \geq 1$ and $1 \leq p < \infty$. Then R_α^m is a bounded operator from $L^p(H)$ onto \mathfrak{b}_α^p .

Proof. First we show (1). By Lemma 6.2 for $\gamma = -1/p$, we have

$$\begin{aligned} &|R_\alpha f(x, t)| \\ &\leq \iint_H |f(y, s) R_\alpha(x, t; y, s)| dy ds \\ &\leq \left(\iint_H |f(y, s)|^p s^{1/q} |R_\alpha(x, t; y, s)| dy ds \right)^{1/p} \left(\iint_H s^{-1/p} |R_\alpha(x, t; y, s)| dy ds \right)^{1/q} \end{aligned}$$

$$= c_1(-1/p)^{1/q} t^{-1/(pq)} \left(\iint_H |f(y, s)|^p s^{1/q} |R_\alpha(x, t; y, s)| dy ds \right)^{1/p}.$$

Therefore using the first estimate of Lemma 6.2 for $\gamma = -1/q$ again, we have

$$\begin{aligned} & \iint_H |R_\alpha f(x, t)|^p dx dt \\ & \leq c_1 \left(-\frac{1}{p} \right)^{p/q} \iint_H \left(\iint_H t^{-1/q} |f(y, s)|^p s^{1/q} |R_\alpha(x, t; y, s)| dy ds \right) dx dt \\ & = c_1 \left(-\frac{1}{p} \right)^{p/q} c_1 \left(-\frac{1}{p} \right) \iint_H s^{-1/q} |f(y, s)|^p s^{1/q} dy ds \\ & = c_1 \left(-\frac{1}{p} \right)^{p/q} c_1 \left(-\frac{1}{p} \right) \|f\|_{L^p(H)}^p, \end{aligned}$$

because R_α is symmetric. The surjectivity of R_α follows from Theorem 6.3. Thus (1) is shown. Similarly, using Lemma 6.2, we have (2). Note that Lemma 6.2 is applicable for $\gamma = 0$ in the case $m \geq 1$ and $q = \infty$. \square

REMARK 6.5. By Theorems 6.3 and 6.4, we see that the kernel R_α is the reproducing kernel for \mathfrak{b}_α^2 . Furthermore, the operator R_α on $L^2(H)$ is the orthogonal projection to \mathfrak{b}_α^2 , because R_α is real-valued and symmetric. Thus R_α is called the α -parabolic Bergman projection.

We generalize (6.2) in the following lemma.

Lemma 6.6. *Let $1 \leq p < \infty$ and $m, k \in \mathbf{N}_0$ with $m + k \geq 1$. Then for $u \in \mathfrak{b}_\alpha^p$ and $\delta > 0$,*

$$\iint_H \partial_s^k T_\delta u(y, s) \cdot s^{m+k-1} \partial_s^m W^{(\alpha)}(x - y, t + s) dy ds = \frac{(m+k-1)!}{(-2)^{m+k}} T_\delta u(x, t).$$

Proof. We remark that the integral is well-defined by (3.2) and (5.2). To prove the formula by induction, we first consider the case $(k, m) = (0, m)$. Then $m \geq 1$ and, by Theorem 6.3,

$$\begin{aligned} & \iint_H T_\delta u(y, s) \cdot s^{m-1} \partial_s^m W^{(\alpha)}(x - y, t + s) dy ds \\ & = -\frac{1}{2c_{m-1}} (R_\alpha^{m-1} T_\delta u)(x, t) = -\frac{1}{2c_{m-1}} T_\delta u(x, t) \end{aligned}$$

which is the desired equality, because $c_{m-1} = (-2)^{m-1}/(m-1)!$.

Next let $(k, m) = (1, 0)$. Then

$$\iint_H \partial_s T_\delta u(y, s) W^{(\alpha)}(x - y, t + s) dy ds = \int_0^\infty (\partial_t T_\delta u)(x, t + 2s) ds = -\frac{1}{2} T_\delta u(x, t).$$

Finally we consider the general case with $k + m \geq 2$. Assuming that the lemma holds for $(k - 1, m)$ and $(k - 1, m + 1)$, we have

$$\begin{aligned} & \int_{\mathbf{R}^n} \int_0^\infty \partial_s^k T_\delta u(y, s) \cdot s^{m+k-1} \partial_s^m W^{(\alpha)}(x - y, t + s) dy ds \\ &= - \int_{\mathbf{R}^n} \left(\int_0^\infty \partial_s^{k-1} T_\delta u(y, s) [(m+k-1)s^{m+k-2} \partial_s^m + s^{m+k-1} \partial_s^{m+1}] W^{(\alpha)}(x - y, t + s) ds \right) dy \\ &= \frac{(m+k-1)!}{(-2)^{m+k}} T_\delta u(x, t), \end{aligned}$$

which completes the induction. □

The boundedness of the kernel R_α^m and the above lemma give the following formula.

Theorem 6.7. *Let $k, m \in \mathbf{N}_0$. Then for every $u \in \mathbf{b}_\alpha^p$ with $1 \leq p < \infty$,*

$$(6.3) \quad R_\alpha^m(t^k \partial_t^k u) = \frac{c_m}{c_{m+k}} u.$$

Proof. Recall that $c_m = (-2)^m/m!$. By Lemma 6.6, (6.3) holds for $T_\delta u$. Thus letting $\delta \downarrow 0$, we have the assertion. □

Proposition 6.8. *Let $1 \leq p < \infty$ and $k \in \mathbf{N}$. Then there is a constant $C \geq 1$ such that for every $u \in \mathbf{b}_\alpha^p$,*

$$C^{-1} \|t^k \partial_t^k u\|_{L^p(H)} \leq \|u\|_{L^p(H)} \leq C \|t^k \partial_t^k u\|_{L^p(H)}.$$

Proof. The first inequality follows from Proposition 5.5. Theorems 6.4 (2) and 6.7 give the second inequality. □

7. α -parabolic Bloch Space

In this section we define the α -parabolic Bloch space.

DEFINITION 7.1. We denote by \mathcal{B}_α the set of all $L^{(\alpha)}$ -harmonic function u on H such that u is of C^1 class and that

$$(7.1) \quad \|u\|_{\mathcal{B}_\alpha} := |u(0, 1)| + \sup_{(x,t) \in H} \{t^{1/(2\alpha)} |\nabla_x u(x, t)| + t |\partial_t u(x, t)|\} < \infty,$$

where ∇_x denotes the gradient operator with respect to the space variable, and $0 = (0, \dots, 0) \in \mathbf{R}^n$. As seen later, \mathcal{B}_α is a Banach space under the Bloch norm $\|\cdot\|_{\mathcal{B}_\alpha}$. We call \mathcal{B}_α the α -parabolic Bloch space.

We begin with the boundedness of point evaluation on \mathcal{B}_α .

Proposition 7.2. *There is a constant $C > 0$ such that for $u \in \mathcal{B}_\alpha$ and $(x, t) \in H$,*

$$(7.2) \quad |u(x, t)| \leq C \|u\|_{\mathcal{B}_\alpha} (1 + |\log t| + \log(1 + |x|)).$$

Proof. For an $x \in \mathbf{R}^n$, we set $\tau = ((1 + |x|)/(1 + \log(1 + |x|)))^{2\alpha} \geq 1$. Then we have

$$\begin{aligned} |u(x, t)| &\leq |u(0, 1)| + \int_1^\tau |\partial_t u(0, s)| ds + \int_0^{|x|} \left| \nabla_x u \left(r \frac{x}{|x|}, \tau \right) \right| dr + \left| \int_\tau^t \partial_t u(x, s) ds \right| \\ &\leq \|u\|_{\mathcal{B}_\alpha} \left(1 + \int_1^\tau \frac{ds}{s} + \tau^{-1/(2\alpha)} |x| + \left| \int_\tau^t \frac{ds}{s} \right| \right) \\ &\leq \|u\|_{\mathcal{B}_\alpha} \left(1 + \log \tau + \frac{|x|(1 + \log(1 + |x|))}{1 + |x|} + |\log t| + \log \tau \right). \end{aligned}$$

Since $\log \tau \leq 2\alpha \log(1 + |x|)$, the assertion follows. \square

By the same manner as in Theorem 5.4, we have the following

Theorem 7.3. *For $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0 \setminus \{(0, 0)\}$, there is a constant $C > 0$ such that*

$$(7.3) \quad |\partial_x^\beta \partial_t^k u(x, t)| \leq C \|u\|_{\mathcal{B}_\alpha} t^{-(|\beta|/(2\alpha)+k)}$$

for $u \in \mathcal{B}_\alpha$ and any $(x, t) \in H$. In particular, $\mathcal{B}_\alpha \subset C^\infty(H)$.

Proof. We first remark that $\mathbf{b}_\alpha^\infty \subset C^\infty(H)$. Let $(x_0, t_0) \in H$ be fixed. If $k \neq 0$, applying Theorem 5.4 to $T_{t_0/2} \partial_t u \in \mathbf{b}_\alpha^\infty$, we have

$$\begin{aligned} |\partial_x^\beta \partial_t^k u(x_0, t_0)| &= \left| \partial_x^\beta \partial_t^{k-1} (T_{t_0/2} \partial_t u) \left(x_0, \frac{t_0}{2} \right) \right| \\ &\leq C \|T_{t_0/2} \partial_t u\|_{L^\infty(H)} t_0^{-(|\beta|/(2\alpha)+k-1)} \\ &\leq 2C \|u\|_{\mathcal{B}_\alpha} t_0^{-(|\beta|/(2\alpha)+k)}. \end{aligned}$$

Similarly, we can obtain the theorem when the case $\beta \neq 0$. \square

Theorem 7.4. *Every element in \mathcal{B}_α satisfies the Huygens property, and \mathcal{B}_α is a Banach space under the Bloch norm (7.1).*

Proof. Take $u \in \mathcal{B}_\alpha$. Since $T_s \partial_t u$ belongs to \mathbf{b}_α^∞ for every $s > 0$, we have

$$\partial_t u(x, t + s) = \int_{\mathbf{R}^n} \partial_t u(x - y, t) W^{(\alpha)}(y, s) dy$$

and hence for $t_2 > t_1 > 0$,

$$\begin{aligned} u(x, t_2 + s) - u(x, t_1 + s) &= \int_{t_1}^{t_2} \int_{\mathbf{R}^n} \partial_t u(x - y, t) W^{(\alpha)}(y, s) dy dt \\ &= \int_{\mathbf{R}^n} (u(x - y, t_2) - u(x - y, t_1)) W^{(\alpha)}(y, s) dy. \end{aligned}$$

This implies $v(x, t, s)$ is a constant function with respect to t , where

$$v(x, t, s) = u(x, t + s) - \int_{\mathbf{R}^n} u(x - y, t) W^{(\alpha)}(y, s) dy.$$

A similar argument with respect to the variable x gives that v does not depend on x either. For fixed $t > 0$, since $v(\cdot, t, \cdot)$ is $L^{(\alpha)}$ -harmonic, we have $\partial_s v = L_{(x,s)}^{(\alpha)} v = 0$, which implies v is a constant. Further this constant is equal to

$$\lim_{s \rightarrow 0} v(x, t, s) = 0,$$

so that the Huygens property for u follows.

To show the completeness of \mathcal{B}_α , consider any Cauchy sequence in \mathcal{B}_α with respect to the Bloch norm. By Proposition 7.2, it converges locally uniformly to a continuous function u on H . It is not difficult to show that this limit function also satisfies the Huygens property, so that u is $L^{(\alpha)}$ -harmonic on H and is of C^∞ class. Theorem 7.3 gives $\|u\|_{\mathcal{B}_\alpha} < \infty$. □

Since \mathcal{B}_α contains constant functions, we may identify $\mathcal{B}_\alpha/\mathbf{R} \cong \tilde{\mathcal{B}}_\alpha$, where

$$\tilde{\mathcal{B}}_\alpha = \{u \in \mathcal{B}_\alpha; u(0, 1) = 0\}.$$

The α -parabolic Bergman kernel R_α is not bounded on $L^\infty(H)$, so that we consider the modified α -parabolic Bergman kernel \tilde{R}_α , which is inspired by [10]:

$$\tilde{R}_\alpha(x, t; y, s) := R_\alpha(x, t; y, s) - R_\alpha(0, 1; y, s).$$

Lemma 7.5. *There is a constant $C > 0$ such that for every $(x, t) \in H$,*

$$\iint_H |\tilde{R}_\alpha(x, t; y, s)| dy ds \leq C(1 + \log(1 + |x|) + |\log t|).$$

Proof. Put $\tau = ((1 + |x|)/(1 + \log(1 + |x|)))^{2\alpha}$. Then

$$\begin{aligned} & \|\tilde{R}_\alpha(x, t; \cdot, \cdot)\|_{L^1(H)} \\ & \leq \|R_\alpha(x, t; \cdot, \cdot) - R_\alpha(x, \tau; \cdot, \cdot)\|_{L^1(H)} + \|R_\alpha(x, \tau; \cdot, \cdot) - R_\alpha(0, \tau; \cdot, \cdot)\|_{L^1(H)} \\ & \quad + \|R_\alpha(0, \tau; \cdot, \cdot) - R_\alpha(0, 1; \cdot, \cdot)\|_{L^1(H)}. \end{aligned}$$

The Minkowski inequality and Lemma 3.2 show that the first term of the right hand side is bounded by

$$2 \left| \int_\tau^t \|T_\delta \partial_t^2 W^{(\alpha)}\|_{L^1(H)} d\delta \right| \leq C \left| \int_\tau^t \delta^{-1} d\delta \right| \leq C(|\log t| + \log \tau),$$

and the second term is less than

$$\begin{aligned} 2 \int_0^1 \iint_H \left| \frac{\partial}{\partial r} (\partial_t W^{(\alpha)}(rx - y, \tau + s)) \right| dy ds dr & \leq 2 \int_0^1 |x| \|T_\tau \nabla_x \partial_t W^{(\alpha)}\|_{L^1(H)} dr \\ & \leq C|x| \tau^{-1/(2\alpha)} \end{aligned}$$

and the third term is bounded by

$$2 \left| \int_1^\tau \|T_\delta \partial_t^2 W^{(\alpha)}\|_{L^1(H)} d\delta \right| \leq C \log \tau,$$

which show the required estimate as in the proof of Proposition 7.2. \square

Lemma 7.6. For every $(x, t) \in H$ and for every $0 < \delta < 1$,

$$\iint_H \frac{1}{s+\delta} |W^{(\alpha)}(x+y, t+s) - W^{(\alpha)}(y, s+1)| dy ds < \infty.$$

Proof. For fixed $x = (x_1, \dots, x_n)$, the equality

$$W^{(\alpha)}(x+y, s+1) - W^{(\alpha)}(y, s+1) = \int_0^1 x \cdot \nabla_x W^{(\alpha)}(rx+y, s+1) dr$$

and (3.2) give

$$\begin{aligned} & \iint_H \frac{1}{s+\delta} |W^{(\alpha)}(x+y, s+1) - W^{(\alpha)}(y, s+1)| dy ds \\ & \leq C|x| \int_0^1 \int_0^\infty \left(\int_{\mathbf{R}^n} |\nabla_x W^{(\alpha)}((s+1)^{-1/(2\alpha)}(rx+y), 1)| dy \right) (s+1)^{-(n+1)/(2\alpha)} (s+\delta)^{-1} ds dr \\ & \leq C'|x| \int_0^\infty (s+1)^{-1/(2\alpha)} (s+\delta)^{-1} ds < \infty, \end{aligned}$$

and since

$$W^{(\alpha)}(x + y, t + s) - W^{(\alpha)}(x + y, s + 1) = \int_1^t \partial_t W^{(\alpha)}(x + y, s + \tau) d\tau,$$

we also have

$$\begin{aligned} & \iint_H \frac{1}{s + \delta} |W^{(\alpha)}(x + y, t + s) - W^{(\alpha)}(x + y, s + 1)| dy ds \\ & \leq \left| \int_1^t \int_0^\infty \left(\int_{\mathbf{R}^n} |\partial_t W^{(\alpha)}((s + \tau)^{-1/(2\alpha)}(x + y), 1)| dy \right) (s + \tau)^{-n/(2\alpha)-1} (s + \delta)^{-1} ds d\tau \right| \\ & \leq C \left| \int_1^t \int_0^\infty (s + \tau)^{-1} (s + \delta)^{-1} ds d\tau \right| < \infty. \end{aligned}$$

Thus our assertion follows from the triangle inequality. □

Theorem 7.7. *The kernel \tilde{R}_α is a bounded linear operator from $L^\infty(H)$ to $\tilde{\mathcal{B}}_\alpha$.*

Proof. For every $f \in L^\infty(H)$, we can define $\tilde{R}_\alpha f(x, t)$ by Lemma 7.5. Further since $\tilde{R}_\alpha(x, t; \cdot, \cdot)$ is $L^{(\alpha)}$ -harmonic, so is $\tilde{R}_\alpha f$. Clearly $\tilde{R}_\alpha f(0, 1) = 0$. For every $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$ with $(\beta, k) \neq (0, 0)$, we have

$$|\partial_x^\beta \partial_t^k [\tilde{R}_\alpha f(x, t)]| = \left| \iint_H \partial_x^\beta \partial_t^k R_\alpha(x, t; y, s) f(y, s) dy ds \right| \leq C \|f\|_{L^\infty(H)} t^{-(1|\beta|/(2\alpha)+k)},$$

by Lemma 3.2. In particular, $\|\tilde{R}_\alpha f\|_{\mathcal{B}_\alpha} \leq C \|f\|_{L^\infty(H)}$ holds. □

Similarly to Lemma 6.6, Theorem 6.7 and Proposition 6.8, we can obtain the following results for α -parabolic Bloch spaces. Remark that Lemma 7.6 assures the necessary integrability in the following results.

Lemma 7.8. *Let m, k be nonnegative integers with $m + k \geq 1$. Then for every $u \in \mathcal{B}_\alpha$ and every $\delta > 0$, we have*

$$\begin{aligned} (7.4) \quad & \iint_H \partial_s^k T_\delta u(y, s) \cdot s^{m+k-1} \partial_s^m (W^{(\alpha)}(x - y, t + s) - W^{(\alpha)}(y, s + 1)) dy ds \\ & = \frac{(m + k - 1)!}{(-2)^{m+k}} (T_\delta u(x, t) - T_\delta u(0, 1)). \end{aligned}$$

Theorem 7.9. *For any $u \in \tilde{\mathcal{B}}_\alpha$, $u = -2\tilde{R}_\alpha(t\partial_t u)$ holds. More generally, for any $k \in \mathbf{N}$, we have*

$$\tilde{R}_\alpha(t^k \partial_t^k u) = \frac{k!}{(-2)^k} u.$$

Proposition 7.10. *Let $k \geq 1$ be an integer. Then there is a constant $C \geq 1$ such that for every $u \in \mathcal{B}_\alpha$*

$$C^{-1} \|t^k \partial_t^k u\|_{L^\infty(H)} \leq \|u\|_{\mathcal{B}_\alpha} \leq C \|t^k \partial_t^k u\|_{L^\infty(H)}.$$

8. Dual Spaces

In this section, we characterize the dual space of \mathbf{b}_α^p for $1 \leq p < \infty$. In the following, we use the following convention: write $X = (x, t) \in H$ and for an integrable function f on H ,

$$\int_H f(X) dX = \iint_H f(x, t) dx dt.$$

Theorem 8.1. *Let $1 < p < \infty$. Then $(\mathbf{b}_\alpha^p)^* \cong \mathbf{b}_\alpha^q$, that is, the dual space of \mathbf{b}_α^p can be identified with \mathbf{b}_α^q , where q is the exponent conjugate to p .*

Proof. For $v \in \mathbf{b}_\alpha^q$, we define a functional on \mathbf{b}_α^p by

$$\Lambda_v(u) = \int_H u(X)v(X) dX.$$

Then $\Lambda_v \in (\mathbf{b}_\alpha^p)^*$ and $\|\Lambda_v\| \leq \|v\|_{L^q(H)}$. Put $\iota(v) = \Lambda_v$. By the open mapping theorem, it is sufficient to show that the mapping $\iota: \mathbf{b}_\alpha^q \rightarrow (\mathbf{b}_\alpha^p)^*$ is bijective.

Assuming $\Lambda_v = 0$, we have

$$v(X) = \int_H R_\alpha(X; Y)v(Y) dY = \Lambda_v(R_\alpha(X; \cdot)) = 0$$

because $R_\alpha(X; \cdot) \in \mathbf{b}_\alpha^p$, which implies ι is injective.

Next for $\Lambda \in (\mathbf{b}_\alpha^p)^*$, using the Hahn-Banach theorem, there exists f in $L^q(H)$ such that

$$\Lambda(u) = \int_H u(X)f(X) dX$$

for all $u \in \mathbf{b}_\alpha^p$. Since R_α is symmetric, Theorems 6.3 and 6.4 show

$$\Lambda(u) = \int_H (R_\alpha u)(X)f(X) dX = \int_H u(Y)(R_\alpha f)(Y) dY = \Lambda_{R_\alpha f}(u).$$

This implies ι is surjective and the proof of Theorem completes. □

To determine the dual space for $p = 1$, we use a subspace of \mathbf{b}_α^∞ . We put

$$(8.1) \quad \mathcal{D} := \{u \in \mathbf{b}_\alpha^\infty; (1+t)(1+t+|x|^{2\alpha})^{n/(2\alpha)+1}u(x, t) \text{ is bounded on } H\}.$$

Lemma 8.2. \mathcal{D} is dense in b_α^p for $1 \leq p < \infty$.

Proof. Let $u \in b_\alpha^p$. Taking an exhaustion $\{K_j\}_{j=1}^\infty$ of H , we see that $R_\alpha^1(u \cdot \chi_{K_j})$ converges to u by Theorems 6.3 and 6.4 (2), where χ_{K_j} denotes the indicator function of K_j . Further, Lemma 6.1 shows $R_\alpha^1(u \cdot \chi_{K_j}) \in \mathcal{D}$. \square

Lemma 8.3. For $u \in \mathcal{D}$ and $v \in \tilde{\mathcal{B}}_\alpha$,

$$(8.2) \quad \int_H u(X)v(X) dX = -2 \int_H u(X)\Phi v(X) dX,$$

where $\Phi v(X) = t\partial_t u(x, t)$. In particular

$$(8.3) \quad \left| \int_H u(X)v(X) dX \right| \leq 2\|u\|_{L^1(H)}\|v\|_{\mathcal{B}_\alpha}.$$

Proof. We first observe the following integrability. Since Φv is bounded, Lemma 7.5 shows that there is a constant $C > 0$ such that

$$\begin{aligned} & \int_H \left(\int_H |u(X)\tilde{R}_\alpha(X; Y)\Phi v(Y)| dY \right) dX \\ & \leq C \iint_H \frac{1 + \log(1 + |x|) + |\log t|}{(1+t)(1+t+|x|^{2\alpha})^{n/(2\alpha)+1}} dx dt \\ & \leq C \left(\int_0^\infty \frac{1 + |\log t|}{(1+t)^{3/2}} dt \right) \left(\int_{\mathbb{R}^n} \frac{1 + \log(1 + |x|)}{(1 + |x|^{2\alpha})^{(n/(2\alpha))+1/2}} dx \right) \\ & < \infty. \end{aligned}$$

We also observe that since R_α is symmetric and u has the cancelation property,

$$\begin{aligned} u(Y) &= \int_H R_\alpha(Y; X)u(X) dX = \int_H R_\alpha(X; Y)u(X) dX \\ &= \int_H \{R_\alpha(X; Y) - R_\alpha(X_0; Y)\}u(X) dX \\ &= \int_H \tilde{R}_\alpha(X; Y)u(X) dX, \end{aligned}$$

where $X_0 = (0, 1)$. Hence these observations and Theorem 7.9 ensure that

$$\begin{aligned} \int_H u(X)v(X) dX &= -2 \int_H u(X)\tilde{R}_\alpha\Phi v(X) dX \\ &= -2 \int_H \left(\int_H u(X)\tilde{R}_\alpha(X; Y) dX \right) \Phi v(Y) dY \\ &= -2 \int_H u(Y)\Phi v(Y) dY. \end{aligned}$$

The inequality (8.3) follows from Definition 7.1. \square

Now we shall characterize the dual space of b_α^p for the case $p = 1$.

Theorem 8.4. *The dual space of b_α^1 can be identified with $\mathcal{B}_\alpha/\mathbf{R} \cong \tilde{\mathcal{B}}_\alpha$.*

Proof. For any $v \in \tilde{\mathcal{B}}_\alpha$, we define a linear functional on b_α^1 by

$$\Lambda_v(u) = -2 \int_H u(X) \Phi v(X) dX.$$

Then since $|\Lambda_v(u)| \leq 2\|u\|_{L^1(H)}\|v\|_{\mathcal{B}_\alpha}$ by Lemma 8.3, $\Lambda_v \in (b_\alpha^1)^*$. Put $\iota(v) = \Lambda_v$. As in the proof of Theorem 8.1, it is sufficient to show that the mapping $\iota: \tilde{\mathcal{B}}_\alpha \rightarrow (b_\alpha^1)^*$ is bijective. Since $\tilde{R}_\alpha(X; \cdot) \in b_\alpha^1$, the injectivity follows from Theorem 7.9.

To show the surjectivity, we take $\Lambda \in (b_\alpha^1)^*$ arbitrarily. Then by the Hahn-Banach theorem, there exists $f \in L^\infty(H)$ such that $\|f\|_{L^\infty(H)} = \|\Lambda\|$ and

$$\Lambda(u) = \int_H u(X) f(X) dX$$

for every $u \in b_\alpha^1$. Then Theorem 7.7 gives us that $\tilde{R}_\alpha f \in \tilde{\mathcal{B}}_\alpha$ and $\|\tilde{R}_\alpha f\|_{\mathcal{B}_\alpha} \leq C\|f\|_{L^\infty(H)} = C\|\Lambda\|$ with some constant $C > 0$. Hence by the same reason as in the proof of Lemma 8.3, we have

$$\begin{aligned} \Lambda(u) &= \int_H u(Y) f(Y) dY \\ &= \int_H \left(\int_H R_\alpha(Y; X) u(X) dX \right) f(Y) dY \\ &= \int_H u(X) \tilde{R}_\alpha f(X) dX \\ &= -2 \int_H u(X) \Phi(\tilde{R}_\alpha f)(X) dX = \Lambda_{\tilde{R}_\alpha f}(u) \end{aligned}$$

provided that $u \in \mathcal{D}$. Since \mathcal{D} is dense in b_α^1 , the mapping ι is surjective. \square

9. α -parabolic Little Bloch Space

In this section we define the α -parabolic little Bloch space, which turns out to be the predual of b_α^1 . The argument here is inspired by [13].

DEFINITION 9.1. A function $u \in \mathcal{B}_\alpha$ is said to be an α -parabolic little Bloch function, if

$$(9.1) \quad \lim_{(x,t) \rightarrow \partial H \cup \{\infty\}} \{t|\partial_t u(x,t)| + t^{1/(2\alpha)}|\nabla_x u(x,t)|\} = 0.$$

We denote by $\mathcal{B}_{\alpha,0}$ the set of all α -parabolic little Bloch functions on H and call $\mathcal{B}_{\alpha,0}$ the α -parabolic little Bloch space.

Let $\tilde{\mathcal{B}}_{\alpha,0} := \{u \in \mathcal{B}_{\alpha,0}; u(0, 1) = 0\}$. Since $\mathcal{B}_{\alpha,0}$ and $\tilde{\mathcal{B}}_{\alpha,0}$ are closed subspace of \mathcal{B}_{α} , they are both Banach spaces with the Bloch norm $\|\cdot\|_{\mathcal{B}_{\alpha}}$.

We let $C_0(H)$ denote the set of all continuous functions on H which vanish continuously on $\partial H \cup \{\infty\}$.

Lemma 9.2. $\tilde{\mathcal{B}}_{\alpha,0} = \{u \in \tilde{\mathcal{B}}_{\alpha}; \Phi u \in C_0(H)\} = \{\tilde{R}_{\alpha}f; f \in C_0(H)\}$.

Proof. For the first equality it is sufficient to show that if $\Phi u = t\partial_t u$ belongs to $C_0(H)$ then so does $t^{1/(2\alpha)}|\nabla_x u|$. Since $u = -2\tilde{R}_{\alpha}(\Phi u)$ by Theorem 7.9, we have for $j = 1, \dots, n$

$$\partial_{x_j} u(x, t) = -2 \iint_H \partial_{x_j} \partial_t W^{(\alpha)}(x - y, t + s) \cdot s \partial_s u(y, s) dy ds.$$

Given $\varepsilon > 0$, there is a compact set K in H such that $|s\partial_s u| < \varepsilon$ outside K . Then

$$\begin{aligned} |t^{1/(2\alpha)} \partial_{x_j} u(x, t)| &\leq 2\varepsilon t^{1/(2\alpha)} \iint_{K^c} |\partial_{x_j} \partial_t W^{(\alpha)}(x - y, t + s)| dy ds \\ &\quad + 2t^{1/(2\alpha)} \iint_K |\partial_{x_j} \partial_t W^{(\alpha)}(x - y, t + s)| \cdot |s\partial_s u(y, s)| dy ds. \end{aligned}$$

The first term in the right hand side is less than $2C\varepsilon$ by Lemma 3.2, while the second term tends to 0 provided that (x, t) tends to $\partial H \cup \{\infty\}$ (use (3.2)). We therefore conclude $t^{1/(2\alpha)}|\nabla_x u| \in C_0(H)$.

To show the second equality in the lemma, take $f \in C_0(H)$ arbitrarily. Then $\tilde{R}_{\alpha}f$ is in $\tilde{\mathcal{B}}_{\alpha}$ by Theorem 7.7. The same argument as above shows $\Phi(\tilde{R}_{\alpha}f) \in C_0(H)$, which implies $\tilde{\mathcal{B}}_{\alpha,0} \supset \{\tilde{R}_{\alpha}f; f \in C_0(H)\}$. The converse inclusion follows easily from the equality $u = -2\tilde{R}_{\alpha}(\Phi u)$. \square

We can now prove the main result of this section.

Theorem 9.3. *The pre-dual space of \mathfrak{b}_{α}^1 can be identified with $\mathcal{B}_{\alpha,0}/\mathbf{R}$.*

Proof. As in Theorem 8.4, we may identify $\mathcal{B}_{\alpha,0}/\mathbf{R}$ with $\tilde{\mathcal{B}}_{\alpha,0}$. For $u \in \mathfrak{b}_{\alpha}^1$, we define a functional on $\tilde{\mathcal{B}}_{\alpha,0}$ by

$$\Lambda_u(v) := \iint_H u(x, t)\Phi v(x, t) dx dt.$$

Then by Lemma 8.3 the mapping $\iota: \mathfrak{b}_{\alpha}^1 \rightarrow (\tilde{\mathcal{B}}_{\alpha,0})^*$, defined by $\iota(u) = \Lambda_u$, is bounded. To show the injectivity of ι , we assume that $\Lambda_u = 0$. Then for every $f \in C_0(H)$, since

$\partial_t \tilde{R}_\alpha(x, t; y, s) = \partial_t R_\alpha(x, t; y, s) = \partial_t R_\alpha(y, s; x, t)$, we have

$$\begin{aligned} 0 &= \Lambda_u(\tilde{R}_\alpha(f)) \\ &= \iint_H \left(u(x, t) \iint_H t \partial_t \tilde{R}_\alpha(x, t; y, s) f(y, s) dy ds \right) dx dt \\ &= \iint_H \left(\iint_H u(x, t) t \partial_t R_\alpha(y, s; x, t) dx dt \right) f(y, s) dy ds \\ &= -\frac{1}{2} \iint_H R_\alpha^1 u(y, s) f(y, s) dy ds = -\frac{1}{2} \iint_H u(y, s) f(y, s) dy ds, \end{aligned}$$

which implies $u = 0$. Note that all the above double integrals converge. In fact, by Lemma 6.1

$$\begin{aligned} &\iint_H \iint_H |u(x, t) t \partial_t R_\alpha(x, t; y, s) f(y, s)| dy ds dx dt \\ &\leq \|f\|_{L^\infty(H)} \iint_H |u(x, t)| \left(\iint_H \frac{t}{(t+s)(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+1}} dy ds \right) dx dt \\ &\leq C \|f\|_{L^\infty(H)} \|u\|_{L^1(H)} < \infty. \end{aligned}$$

Next, to show the surjectivity of ι , take $\Lambda \in (\tilde{\mathcal{B}}_{\alpha,0})^*$ arbitrarily. Then because of Theorem 7.7 and Lemma 9.2, $f \mapsto \Lambda(\tilde{R}_\alpha f)$ defines a bounded linear functional on $C_0(H)$. Hence by the Riesz representation theorem, there exists a bounded signed measure μ on H such that

$$\Lambda(\tilde{R}_\alpha f) = \iint_H f(x, t) d\mu(x, t),$$

for every $f \in C_0(H)$. We define a function u on H by

$$u(y, s) = 4 \iint_H t \partial_t \tilde{R}_\alpha(x, t; y, s) d\mu(x, t).$$

Then $u \in \mathfrak{b}_\alpha^1$. In fact, since $t \partial_t \tilde{R}_\alpha(x, t; y, s)$ is $L^{(\alpha)}$ -harmonic with respect to (y, s) , so is u . Furthermore

$$\begin{aligned} \|u\|_{L^1(H)} &\leq 4 \iint_H \left(\iint_H |t \partial_t \tilde{R}_\alpha(x, t; y, s)| d|\mu|(x, t) \right) dy ds \\ &\leq 8 \iint_H \left(\iint_H |t T_t \partial_s^2 W^{(\alpha)}(x-y, s)| dy ds \right) d|\mu|(x, t) \\ &= 8 \iint_H t \|\partial_s^2 T_t W^{(\alpha)}\|_{L^1(H)} d|\mu|(x, t) = 8C \| \mu \|, \end{aligned}$$

where we use Lemma 3.2 for the last equality. Now for every $v \in \tilde{\mathcal{B}}_{\alpha,0}$ the equality

$v = -2\tilde{R}_\alpha(\Phi v)$ gives $\Phi v = -2\Phi(\tilde{R}_\alpha(\Phi v))$ so that

$$\begin{aligned} \Lambda(v) &= -2\Lambda(\tilde{R}_\alpha(\Phi v)) = -2 \iint_H \Phi v(x, t) d\mu(x, t) \\ &= 4 \iint_H \Phi(\tilde{R}_\alpha(\Phi v))(x, t) d\mu(x, t) \\ &= 4 \iint_H \left(\iint_H t \partial_t \tilde{R}_\alpha(x, t; y, s) d\mu(x, t) \right) \Phi v(y, s) dy ds \\ &= \iint_H u(y, s) \Phi v(y, s) dy ds = \Lambda_u(v). \end{aligned}$$

This implies that the map ι is surjective, and hence $\mathfrak{b}_\alpha^1 \cong (\tilde{\mathcal{B}}_{\alpha,0})^*$. □

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