Nishio, M., Shimomura, K. and Suzuki, N.
Osaka J. Math.
42 (2005), 133–162

α-PARABOLIC BERGMAN SPACES

Dedicated to the memory of Professor Isao Higuchi

MASAHARU NISHIO, KATSUNORI SHIMOMURA and NORIAKI SUZUKI

(Received August 29, 2002)

Abstract

The α-parabolic Bergman space $\boldsymbol{b}_\alpha^p$ is the set of all p-th integrable solutions u of the equation $(\partial/\partial t + (-\Delta)^\alpha)u = 0$ on the upper half space, where $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. The Huygens property for the above u will be obtained. After verifying that the space $\boldsymbol{b}_\alpha^p$ forms a Banach space, we discuss the fundamental properties. For example, as for the duality, $(\boldsymbol{b}_\alpha^p)^* \cong \boldsymbol{b}_\alpha^q$ for $p > 1$ and $(\boldsymbol{b}_\alpha^1)^* \cong \mathcal{B}_\alpha/\mathbf{R}$ are shown, where q is the exponent conjugate to p and $\mathcal{B}_\alpha$ is the α-parabolic Bloch space.

1. Introduction

Let $\mathbf{R}^{n+1}$ denote the $(n+1)$-dimensional Euclidean space $(n \geq 2)$ and H be its upper half space

$$H = \{(x,t) \in \mathbf{R}^{n+1};\, x \in \mathbf{R}^n,\ \ t > 0\}.$$

For $0 < \alpha \leq 1$, we consider a parabolic operator

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta)^\alpha$$

on H, where Δ is the Laplace operator with respect to x. When $\alpha = 1$, $L^{(\alpha)}$ is the heat operator. Otherwise, $L^{(\alpha)}$ is a non-local operator.

For $1 \leq p \leq \infty$, we denote by $\boldsymbol{b}_\alpha^p$ the set of all solutions of $L^{(\alpha)}u = 0$ on H such that

$$\|u\|_{L^p(H)} := \left(\int_0^\infty \int_{\mathbf{R}^n} |u(x,t)|^p\, dx\, dt\right)^{1/p} < \infty.$$

It is shown that $\boldsymbol{b}_\alpha^p$ is a Banach space under the norm $\|\cdot\|_{L^p(H)}$. We call $\boldsymbol{b}_\alpha^p$ the α-parabolic Bergman space (of order p), because $L^{(\alpha)}$ has parabolic nature.

2000 *Mathematics Subject Classification* : 35K05, 31B05, 31B10, 46E15, 46E22.

This work was supported in part by Grant-in-Aid for Scientific Research (C) No. 13640186, No. 13640151 and No. 15540163, Japan Society for the Promotion of Science.

In this paper we study the properties of solutions of $L^{(\alpha)}u = 0$ on H in the framework of the Bergman space theory. One of our main results is to show the following identity: for $u \in \boldsymbol{b}_\alpha^p$,

$$u(x,t) = \int_{\mathbf{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)\, dy \tag{1.1}$$

whenever $t > s > 0$. According to the heat operator case [12], we call (1.1) the Huygens property for u. Since all solutions of $L^{(\alpha)}u = 0$ form a balayage space (cf. [2]), we make use of potential theory method for the proof of (1.1). In particular, the theory of α-harmonic measures is useful ([4] and [7]). In the sequel, we call a solution u of $L^{(\alpha)}u = 0$ an $L^{(\alpha)}$-harmonic function.

Our study is motivated by recent results [10] and [13] of harmonic Bergman spaces on the upper half space. We remark that α-parabolic Bergman space is a generalization of the harmonic Bergman space. In fact, $(1/2)$-parabolic Bergman spaces coincide with harmonic Bergman spaces, because in the case $\alpha = 1/2$, the fundamental solution of $L^{(1/2)}$ is equal to the Poisson kernel on H (see Corollary 4.4 below).

Based on the Huygens property, we shall discuss the following subjects: the boundedness of the point evaluations, the explicit form of the α-parabolic Bergman kernels, the dual space of $\boldsymbol{b}_\alpha^p$, the α-parabolic little Bloch space and the pre-dual space of $\boldsymbol{b}_\alpha^1$. The estimates of the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ play crucial roles in various contexts.

2. $L^{(\alpha)}$-harmonic functions

In this section, we discuss mainly in the case $0 < \alpha < 1$, because the corresponding results are well-known in the case $\alpha = 1$ (e.g. see [3] and [11]). For an open set D in $\mathbf{R}^{n+1}$, let $C_K^\infty(D)$ denote the set of all infinitely differentiable functions with compact support on D. In order to define $L^{(\alpha)}$-harmonic functions, we shall recall how the adjoint operator $\tilde{L}^{(\alpha)} = -\partial/\partial t + (-\Delta)^\alpha$ acts on $C_K^\infty(\mathbf{R}^{n+1})$. For $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the convolution operator defined by $-c_{n,\alpha}$ p.f. $|x|^{-n-2\alpha}$, where

$$c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma\left(\frac{n+2\alpha}{2}\right) \Big/ \Gamma(-\alpha) > 0$$

and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Hence for $\varphi \in C_K^\infty(\mathbf{R}^{n+1})$,

$$\tilde{L}^{(\alpha)}\varphi(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} (\varphi(x+y,t) - \varphi(x,t))|y|^{-n-2\alpha}\, dy.$$

It is easily seen that if $\operatorname{supp}(\varphi)$, the support of φ, is contained in $\{|x| < r,\ t_1 < t < t_2\}$, then

$$|\tilde{L}^{(\alpha)}\varphi(x,t)| \le 2^{n+2\alpha} c_{n,\alpha} \left(\sup_{t_1 < s < t_2} \int_{\mathbf{R}^n} |\varphi(y,s)|\, dy \right) \cdot |x|^{-n-2\alpha} \tag{2.1}$$

for (x, t) with $|x| \geq 2r$. Remark also that

$$\tilde{L}^{(\alpha)}(\partial_t \varphi) = \partial_t \tilde{L}^{(\alpha)}(\varphi) \quad \text{and} \quad \tilde{L}^{(\alpha)}(\partial_{x_j} \varphi) = \partial_{x_j} \tilde{L}^{(\alpha)}(\varphi) \quad \text{for} \quad j = 1, \ldots, n,$$

where $\partial_t = \partial/\partial t$ and $\partial_{x_j} = \partial/\partial x_j$.

Now we give the definition of $L^{(\alpha)}$-harmonicity. For an open set D in $\mathbf{R}^{n+1}$, we put

$$s(D) := \{(x, t) \in \mathbf{R}^{n+1};\ (y, t) \in D \text{ for some } y \in \mathbf{R}^n\}.$$

Since $\operatorname{supp}(\tilde{L}^{(\alpha)}\varphi)$ extends to $s(D)$ even if $\operatorname{supp}(\varphi) \subset D$, we can define $L^{(\alpha)}$-harmonicity on D by duality only for the functions defined on $s(D)$.

DEFINITION 2.1. A funcion h is said to be $L^{(\alpha)}$-harmonic on an open set D, when h is defined on $s(D)$ and satisfies the following conditions:
(a) h is a Borel measurable function on $s(D)$,
(b) h is continuous on D,
(c) for every $\varphi \in C_K^\infty(D)$, $\iint_{s(D)} |h \cdot \tilde{L}^{(\alpha)}\varphi|\, dx\, dt < \infty$ and $\iint_{s(D)} h \cdot \tilde{L}^{(\alpha)}\varphi\, dx\, dt = 0$.

REMARK 2.2. When $0 < \alpha < 1$, the inequality (2.1) implies that the integrability condition in (c) of Definition 2.1 is equivalent to the following: for any closed strip $[t_1, t_2] \times \mathbf{R}^n \subset s(D)$

$$\int_{t_1}^{t_2} \int_{\mathbf{R^n}} |h(x, t)|(1 + |x|)^{-n-2\alpha}\, dx\, dt < \infty. \tag{2.2}$$

The following lemma will be useful in the Section 4.

Lemma 2.3. *Let v be $L^{(\alpha)}$-harmonic on H. If $v = 0$ continuously on the boundary $\partial H = \mathbf{R}^n \times \{0\}$ and if $\int_0^\delta \int_{\mathbf{R}^n} |v(x, t)|(1 + |x|)^{-n-2\alpha}\, dx\, dt < \infty$ for some $\delta > 0$, then the function V defined by*

$$V(x, t) = \int_0^t v(x, \tau)\, d\tau$$

is also $L^{(\alpha)}$-harmonic on H.

Proof. If $\alpha = 1$, the lemma is clearly true, so we assume $0 < \alpha < 1$. Take arbitrary $\varphi \in C_K^\infty(H)$. Then

$$\int_0^\infty \int_{\mathbf{R}^n} V(x, t)\tilde{L}^{(\alpha)}\varphi(x, t)\, dx\, dt$$
$$= \int_0^\infty \int_{\mathbf{R}^n} \int_0^t v(x, \tau)\, d\tau\, \tilde{L}^{(\alpha)}\varphi(x, t)\, dx\, dt$$

$$= \int_0^\infty \int_{\mathbf{R}^n} v(x,t)\varphi(x,t)\,dx\,dt + \int_0^\infty \int_0^t \int_{\mathbf{R}^n} v(x,\tau)(-\Delta)^\alpha \varphi(x,t)\,dx\,d\tau\,dt.$$

To calculate the second integral of the last line, fix $t > 0$. Considering a C^∞ approximation of the indicator function of the set $[0, t]$, we see

$$\int_0^t \int_{\mathbf{R}^n} v(x,\tau)(-\Delta)^\alpha \varphi(x,t)\,dx\,d\tau = \int_{\mathbf{R}^n} (v(x,0) - v(x,t))\,\varphi(x,t)\,dx.$$

Since $v(x, 0) = 0$, we have therefore

$$\int_0^\infty \int_{\mathbf{R}^n} V(x,t)\tilde{L}^{(\alpha)}\varphi(x,t)\,dx\,dt = 0$$

and $L^{(\alpha)}$-harmonicity of V follows. □

The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n} \displaystyle\int_{\mathbf{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}\,x\cdot\xi)\,d\xi & t > 0 \\ 0 & t \le 0, \end{cases} \tag{2.3}$$

where $x \cdot \xi$ is the inner product of x and ξ and $|\xi| = (\xi\cdot\xi)^{1/2}$. Then

$$\tilde{W}^{(\alpha)}(x,t) := W^{(\alpha)}(x,-t)$$

is the fundamental solution of $\tilde{L}^{(\alpha)}$.

In the case $\alpha = 1$, $W^{(1)}$ is the Gauss-Weierstrass kernel

$$W^{(1)}(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left(-\dfrac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \le 0. \end{cases}$$

In the case $\alpha = 1/2$, $W^{(1/2)}$ is the Poisson kernel (cf. [1, p.74])

$$W^{(1/2)}(x,t) = \begin{cases} \Gamma\left(\dfrac{n+1}{2}\right) \dfrac{t}{(|x|^2 + t^2)^{(n+1)/2}} & t > 0 \\ 0 & t \le 0. \end{cases} \tag{2.4}$$

The harmonicity of $W^{(1/2)}$ derives a close connection between $L^{(1/2)}$-harmonic functions and usual harmonic functions on H (see Corollary 4.4 below). For other $\alpha \in (0, 1)$ any simple explicit expressions for $W^{(\alpha)}$ are not known.

Note also that $W^{(\alpha)}(x,t) \ge 0$,

$$\int_{\mathbf{R}^n} W^{(\alpha)}(x-y, t-s)\,dx = 1 \tag{2.5}$$

and for every $0 < s < t$,

$$W^{(\alpha)}(x,t) = \int_{\mathbf{R}^n} W^{(\alpha)}(x-y,t-s)W^{(\alpha)}(y,s)\,dy. \tag{2.6}$$

When we put

$$\phi_\alpha(|x|) := W^{(\alpha)}(x,1), \tag{2.7}$$

then for $t > 0$,

$$W^{(\alpha)}(x,t) = t^{-n/(2\alpha)}\phi_\alpha(t^{-1/(2\alpha)}|x|) \tag{2.8}$$

and $\phi_\alpha(r) = O(r^{-n-2\alpha})$ when $0 < \alpha < 1$ (use (3.3) below), and $\phi_1(r) = O(\exp(-r^2/4))$ as $r \to +\infty$. Further estimates of $W^{(\alpha)}$ will be given in next section.

Since $W^{(\alpha)}(x-y,t)\,dy$ converges vaguely to the Dirac measure at x as $t \to +0$, we see the following convergence result.

Lemma 2.4. *Let f be a continuous function on $\mathbf{R}^n$. If f belongs to $L^p(\mathbf{R}^n)$ with $1 \le p \le \infty$, then for every $x \in \mathbf{R}^n$,*

$$\lim_{t\to+0}\int_{\mathbf{R}^n} W^{(\alpha)}(x-y,t)f(y)\,dy = f(x).$$

The fact that $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic off $(0,0)$ is important. In fact the following assertion follows from this.

Proposition 2.5 (see [9, Proposition 10]). *If u satisfies the Huygens property, that is, for every $x \in \mathbf{R}^n$ and $0 < s < t$,*

$$u(x,t) = \int_{\mathbf{R}^n} u(x-y,t-s)W^{(\alpha)}(y,s)\,dy,$$

then u is an $L^{(\alpha)}$-harmonic function on H.

3. Estimates of fundamental solutions

In the sequel, we use the following notations. For $\delta > 0$ and a function f on H, we write

$$T_\delta f(x,t) := f(x,t+\delta).$$

Then $T_\delta f$ is a function on $\mathbf{R}^n \times (-\delta,\infty)$. Let k be a nonnegative integer and $\beta = (\beta_1,\dots,\beta_n) \in \mathbf{N}_0^n$ be a multi-index, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Then $|\beta| := \beta_1 + \cdots + \beta_n$ and

$$\partial_x^\beta \partial_t^k f(x,t) := \frac{\partial^{|\beta|+k}}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}\partial t^k} f(x,t).$$

Using the above notation, we start with the following equality which follows from (2.3) easily.

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t) = t^{-((n+|\beta|)/(2\alpha)+k)}(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)}x, 1). \tag{3.1}$$

The following estimate of $W^{(\alpha)}$ plays an important role in our later argument.

Lemma 3.1. *Let $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$. Then there is a constant $C > 0$ such that for every $(x,t) \in H$,*

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| \le Ct^{1-k}(t + |x|^{2\alpha})^{-(n+|\beta|)/(2\alpha)-1} \tag{3.2}$$

Proof. For $x_0 = (1, 0, \ldots, 0) \in \mathbf{R}^n$, we put

$$\psi_\alpha(t) := W^{(\alpha)}(x_0, t).$$

Then it was shown that

$$\psi_\alpha(t) = O(t) \quad \text{as} \quad t \to 0 \tag{3.3}$$

in [5, Lemma 2.1]. The argument which was done there gives that for every $k \in \mathbf{N}$,

$$\psi_\alpha^{(k)}(t) \quad \text{is bounded on} \quad (0, \infty). \tag{3.4}$$

In fact, as in [5] we have

$$\psi_\alpha^{(k)}(t) = (-1)^k (2\pi)^{-n/2} \int_0^\infty \left(\int_{\mathbf{R}^n} |\xi|^{2\alpha k} e^{-s|\xi|^2} \hat{\nu}(\xi)\, d\xi \right) d\sigma_t^\alpha(s)$$

where $\hat{\nu}$ is the Fourier transform of the normalized uniform measure ν on the unit sphere and $(\sigma_t^\alpha)_{t \ge 0}$ is the one-side stable semi-group on $(0, \infty)$ (see [1, p.74]). Thus (3.4) follows if we prove that

$$\Psi(s) := \int_{\mathbf{R}^n} |\xi|^{2\alpha k} e^{-s|\xi|^2} \hat{\nu}(\xi)\, d\xi$$

is bounded on $(0, \infty)$.

In the case that αk is an integer, we have

$$\Psi(s) = (2\pi)^{n/2}(-\Delta)^{\alpha k}(g_s * \nu)(0),$$

where $g_s(x) = W^{(1)}(x, s)$ is the Gauss-Weierstrass kernel. This formula shows the boundedness of Ψ.

If αk is not an integer, we take $l \in \mathbf{N}$ such that $-2 < 2\alpha k - 2l < 0$. Then

$$\Psi(s) = (2\pi)^{n/2} c_{n,\alpha k - l}(-\Delta)^l((|x|^{-n+2l-2\alpha k}) * g_s * \nu)(0)$$

$$
\begin{aligned}
&= (2\pi)^{n/2} c_{n,\alpha k-l}\{(\varphi(x)(|x|^{-n+2l-2\alpha k}) * ((-\Delta)^l g_s) * \nu)(0) \\
&\qquad + (-\Delta)^l((1-\varphi(x))(|x|^{-n+2l-2\alpha k})) * g_s * \nu)(0)\}
\end{aligned}
$$

where $\varphi \in C_K^\infty(\mathbf{R}^n)$ with $0 \le \varphi \le 1$, $\mathrm{supp}(\varphi) \subset \{|x| < 1\}$ and $\varphi = 1$ on $\{|x| < 1/3\}$, and $c_{n,\alpha k-l} = -4^{\alpha k-l}\pi^{-n/2}\Gamma((n+2\alpha k-2l)/2)/\Gamma(l-\alpha k)$. The boundedness of Ψ follows even if $\alpha k \notin \mathbf{N}$.

Now we return to the proof of (3.2). Since $W^{(\alpha)}(x,t) = |x|^{-n}\psi_\alpha(|x|^{-2\alpha}t)$, we have

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t) = \partial_x^\beta(|x|^{-n-2\alpha k}\psi_\alpha^{(k)}(|x|^{-2\alpha}t)),$$

so that

$$
\begin{aligned}
(\partial_x^\beta \partial_t^k W^{(\alpha)})(x,1) &= \partial_x^\beta(|x|^{-n-2\alpha k}\psi_\alpha^{(k)}(|x|^{-2\alpha})) \\
&= \sum_{\beta=\beta'+\beta''} \binom{\beta}{\beta'} \partial_x^{\beta'}(|x|^{-n-2\alpha k})\partial_x^{\beta''}(\psi_\alpha^{(k)}(|x|^{-2\alpha})).
\end{aligned}
$$

It is easily seen that $\partial_x^{\beta'}(|x|^{-n-2\alpha k}) = O(|x|^{-n-2\alpha k-|\beta'|})$ and $\partial_x^{\beta''}(\psi_\alpha^{(k)}(|x|^{-2\alpha})) = O(|x|^{-|\beta''|})$ as $|x| \to \infty$. As a result, we have

$$|(\partial_x^\beta \partial_t^k W^{(\alpha)})(x,1)| \le C|x|^{-n-2\alpha-|\beta|} \quad \text{as} \quad |x| \to \infty. \tag{3.5}$$

Remark that (3.5) remains true for the case $k = 0$ because of (3.3). Hence (3.1) shows that if $|x| \ge t^{1/(2\alpha)}$, then

$$
\begin{aligned}
|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| &= t^{-((n+|\beta|)/(2\alpha)+k)}|(\partial_x^\beta \partial_t^k W^{(\alpha)}(t^{-1/(2\alpha)}x,1)| \\
&\le Ct^{1-k}|x|^{-n-2\alpha-|\beta|}
\end{aligned}
$$

by (3.5), and if $|x| \le t^{1/(2\alpha)}$, then

$$
\begin{aligned}
|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| &= t^{-(n+|\beta|)/(2\alpha)-k}|(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)}x,1)| \\
&\le Ct^{-(n+|\beta|)/(2\alpha)-k}
\end{aligned}
$$

by the boundedness of $|(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)}x,1)|$. These inequalities imply (3.2). □

We note here that T. Kakehi and K. Sakai gave an alternative proof of (3.5) ([6]).

As for the L^q-norm of derivatives of $W^{(\alpha)}$, the homogeneity (3.1) gives us the following identity.

Lemma 3.2. *Let $(\beta,k) \in \mathbf{N}_0^n \times \mathbf{N}_0$ and let $q \ge 1$. If $q > (n+2\alpha)/(n+|\beta|+2\alpha k)$, then there is a constant $C > 0$ such that for any $\delta > 0$*

$$\|\partial_x^\beta \partial_t^k T_\delta W^{(\alpha)}\|_{L^q(H)} = C\delta^{-((n+|\beta|)/(2\alpha)+k)+(n/(2\alpha)+1)(1/q)}. \tag{3.6}$$

Proof. Noting that Lemma 3.1 ensures the integrability, we obtain the equality immediately. □

4. Huygens property

We have seen in Proposition 2.5 that every Borel measurable function satisfying the Huygens property is $L^{(\alpha)}$-harmonic on H. In this section, we shall prove the converse assertion for p-th integrable $L^{(\alpha)}$-harmonic functions. This result will be very useful in other contexts as well.

Theorem 4.1. *Let $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. If an $L^{(\alpha)}$-harmonic function u on H belongs to $L^p(H)$, then u satisfies the Huygens property, that is,*

$$u(x,t) = \int_{\mathbf{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)\, dy \tag{4.1}$$

holds for every $x \in \mathbf{R}^n$ and $0 < s < t < \infty$.

The next two lemmas will be used in the proof of the above theorem. The first lemma is concerning $L^{(\alpha)}$-harmonic measures. For $0 < \alpha < 1$ and $r > 0$, put

$$w_r^\alpha(x) = \begin{cases} 0 & \text{if } |x| \leq r \\ \dfrac{a_{n,\alpha} r^{2\alpha}}{(|x|^2 - r^2)^\alpha |x|^n} & \text{if } |x| > r, \end{cases}$$

where $a_{n,\alpha} = \Gamma(n/2)\pi^{-n/2-1}\sin(\pi\alpha)$. We know that $w_r^\alpha(x)\,dx$ is the balayaged measure on $\{|x| \geq r\}$ of the Dirac measure at the origin with respect to the Riesz kernel $|x|^{2\alpha-n}$ (see [4]). Recalling the equality

$$c_{n,-\alpha}|x|^{2\alpha-n} = \int_0^\infty W^{(\alpha)}(x,t)\, dt = \int_{-\infty}^0 \tilde{W}^{(\alpha)}(x,s)\, ds, \tag{4.2}$$

where $c_{n,-\alpha} = 4^{-\alpha}\pi^{-n/2}\Gamma((n-2\alpha)/2)/\Gamma(\alpha)$ (cf. [1]), we see the following relation between the above balayaged measure and the $L^{(\alpha)}$-harmonic measure.

Lemma 4.2. *Let $0 < \alpha < 1$ and let ν_r^α be the $L^{(\alpha)}$-harmonic measure at the origin on $B_r(0) \times \mathbf{R}$, where $B_r(0)$ is the ball of radius r and center 0 in $\mathbf{R}^n$. Then*

$$\int_A w_r^\alpha(x)\, dx = \nu_r^\alpha(A \times (-\infty, 0]) \tag{4.3}$$

for every Borel set A in $\mathbf{R}^n$.

Proof. Since the $L^{(\alpha)}$-harmonic measure ν_r^α is the balayaged measure on $\{|x| \geq r\} \times (-\infty, 0]$ of the Dirac measure at the origin with respect to $\tilde{W}^{(\alpha)}$,

$$\tilde{W}^{(\alpha)}(y, s) = \int_{|x|\geq r} \int_{-\infty}^{0} \tilde{W}^{(\alpha)}(y - x, s - t)\, d\nu_r^\alpha(x, t)$$

holds for $|y| > r$. Furthermore, by [5, Proposition 4.2 (2)], this equality holds for $|y| = r$, because every boundary point is regular with respect to $\tilde{L}^{(\alpha)}$. Now we denote by μ_r the measure on $\mathbf{R}^n$ defined by $\mu_r(A) = \nu_r^\alpha(A \times (-\infty, 0])$. Then by (4.2), for $|y| \geq r$,

$$\begin{aligned} c_{n,-\alpha}|y|^{2\alpha-n} &= \int_{-\infty}^{0} \tilde{W}^{(\alpha)}(y, s)\, ds \\ &= \int_{-\infty}^{0} \left(\int_{|x|\geq r} \int_{s}^{0} \tilde{W}^{(\alpha)}(y - x, s - t)\, d\nu_r^\alpha(x, t) \right) ds \\ &= \int_{|x|\geq r} \int_{-\infty}^{0} \left(\int_{-\infty}^{t} \tilde{W}^{(\alpha)}(x - y, s - t)\, ds \right) d\nu_r^\alpha(x, t) \\ &= c_{n,-\alpha} \int_{|x|\geq r} |x - y|^{2\alpha-n}\, d\mu_r(x). \end{aligned}$$

On the other hand, since $w_r^\alpha(x)\, dx$ is the balayaged measure on $\{|x| \geq r\}$ with respect to $|x|^{2\alpha-n}$, we have

$$|y|^{2\alpha-n} = \int_{|x|\geq r} |x - y|^{2\alpha-n} w_r^\alpha(x)\, dx$$

on $|y| > r$, and by the reason similar to above, this equality also holds on its boundary $\{|y| = r\}$. Hence

$$\int_{|x|\geq r} |x - y|^{2\alpha-n}\, d\mu_r(x) = \int_{|x|\geq r} |x - y|^{2\alpha-n} w_r^\alpha(x)\, dx \tag{4.4}$$

on $|y| \geq r$. Since the support of both measures μ_r and $w_r^\alpha(x)\, dx$ is contained in $\{|x| \geq r\}$, the domination principle ([2, Corollary 4.13]) implies that (4.4) holds for all $y \in \mathbf{R}^n$. Finally the unicity principle for the Riesz kernel ([7, Theorem 1.12]) gives the equality (4.3). □

The next lemma is an estimate of the function $\widetilde{w_R^\alpha}$ defined by

$$\widetilde{w_R^\alpha}(x) = \frac{1}{R} \int_{R}^{2R} w_r^\alpha(x)\, dr \qquad (R > 0).$$

Lemma 4.3. *Let $1 \leq p \leq \infty$. Then there is a constant $C > 0$ such that for every $R > 0$,*

$$\|\widetilde{w_R^{\alpha}}\|_{L^q(\mathbf{R}^n)} \leq C R^{-n/p},$$

where q is the exponent conjugate to p.

Proof. We take x with $|x| \geq R$. If $R \leq |x| \leq 3R$, then we have

$$\widetilde{w_R^{\alpha}}(x) \leq \frac{a_{n,\alpha}}{R|x|^n} \int_R^{|x|} \frac{r^{2\alpha}}{(|x|^2 - r^2)^{\alpha}}\, dr \leq \frac{2a_{n,\alpha} 3^{n+4\alpha}}{2^{2\alpha}(1-\alpha)} R^{2\alpha} |x|^{-n-2\alpha}.$$

Next if $|x| \geq 3R$, then $w_r^{\alpha}(x) \leq (9/5)^{\alpha} a_{n,\alpha} R^{2\alpha} |x|^{-n-2\alpha}$ because $R < r < 2R$. Hence when $p = 1$, then $q = \infty$ and the lemma holds clearly. When $1 < p \leq \infty$, using the above estimates, we have

$$\int_{\mathbf{R}^n} \widetilde{w_R^{\alpha}}(x)^q\, dx \leq C \int_R^{\infty} R^{2\alpha q} r^{-(n+2\alpha)q} r^{n-1}\, dr \leq C R^{-n(q-1)}$$

with some constant $C > 0$. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. In the case that $\alpha = 1$, the assertion is known (see for example [11, Theorem 3.6, p.76]) and in the case $p = \infty$, the assertion follows from [9, Proposition 11]. Hence we may assume that $0 < \alpha < 1$ and $1 \leq p < \infty$. Remark that for any $\delta_0 > 0$, there exists $0 < \delta < \delta_0$ such that

$$u_{\delta}(\,\cdot\,, 0) \in L^p(\mathbf{R}^n),$$

where $u_{\delta}(x, t) := T_{\delta} u(x, t) = u(x, t + \delta)$. Define the function v by

$$v(x, t) := u_{\delta}(x, t) - \tilde{u}_{\delta}(x, t),$$

where

$$\tilde{u}_{\delta}(x, t) := \int_{\mathbf{R}^n} W^{(\alpha)}(x - y, t) u_{\delta}(y, 0)\, dy.$$

Then v is clearly $L^{(\alpha)}$-harmonic on H and by Lemma 2.4, v vanishes continuously on the lower boundary $\mathbf{R}^n \times \{0\}$. By the Minkowski inequality, $\|\tilde{u}_{\delta}(\,\cdot\,, t)\|_{L^p(\mathbf{R}^n)} \leq \|u_{\delta}(\,\cdot\,, 0)\|_{L^p(\mathbf{R}^n)}$, so that the Hölder inequality shows

$$\int_0^{\delta} \int_{\mathbf{R}^n} |\tilde{u}_{\delta}(x, t)| (1 + |x|)^{-n-2\alpha}\, dx\, dt < \infty.$$

By definition u_δ also satisfies the same inequality, so that v fulfills the assumption in Lemma 2.3. Hence

$$V(x,t) := \int_0^t v(x,\tau)\,d\tau$$

is $L^{(\alpha)}$-harmonic on H. Let $(x,t) \in H$ be fixed. Then the $L^{(\alpha)}$-harmonic measure $\nu_\omega^{(x,t)}$ of a sylinder $\omega = B_r(x) \times (0, t+1)$ can be written as

$$\nu_\omega^{(x,t)} = \nu_\omega^{(x,t)}|_{B_r(x)^c \times [0,t]} + \nu_\omega^{(x,t)}|_{B_r(x)\times\{0\}},$$

where the first term in the right hand side is

$$\nu_r^\alpha(y-x, s-t)|_{\{|y-x|\geq r,\, -t\leq s-t\leq 0\}}$$

and the second term is absolutely continuous with respect to the n-dimensional Lebesgue measure dy whose density is bounded by $W^{(\alpha)}(x-y,t)$. Since $V(y,0) = 0$, by (4.3),

$$\begin{aligned}
|V(x,t)| &= \left| \int_{|y|\geq r} \int_{-t}^0 V(x+y, t+s)\,d\nu_r^\alpha(y,s) \right| \\
&\leq \int_{|y|\geq r} \int_{-t}^0 \left(\int_0^{s+t} |v(x+y,\tau)|\,d\tau \right) d\nu_r^\alpha(y,s) \\
&= \int_0^t \left(\int_{|y|\geq r} \int_{\tau-t}^0 |v(x+y,\tau)|\,d\nu_r^\alpha(y,s) \right) d\tau \\
&\leq \int_0^t \left(\int_{|y|\geq r} |v(x+y,\tau)| w_r^\alpha(y)\,dy \right) d\tau
\end{aligned}$$

so that

$$|V(x,t)| \leq \int_0^t \int_{\mathbf{R}^n} |v(y,\tau)| \widetilde{w_R^\alpha}(x-y)\,dy\,d\tau.$$

Therefore by the Hölder inequality and Lemma 4.3, letting $R \to \infty$, we have $V(x,t) \equiv 0$. Since $v(x,t) = \partial_t V(x,t) = 0$,

$$u_\delta(x,t) = \int_{\mathbf{R}^n} W^{(\alpha)}(x-y,t) u(y,\delta)\,dy.$$

By (2.6), the right hand side satisfies the Huygens property, so does u because δ_0 is arbitrary. $\square$

Recalling (2.4) and Proposition 2.5, we have the following interesting corollary of the theorem above.

Corollary 4.4. *Let $1 \leq p \leq \infty$ and suppose that $u \in L^p(H)$. Then u is an $L^{(1/2)}$-harmonic function if and only if u is a usual harmonic function on H.*

REMARK 4.5. Throughout this paper we always assume that $n \geq 2$. The reason is that some arguments in this section are not valid for the case $n = 1$. For example, (4.2) does not hold if $n = 1$ and $1/2 \leq \alpha < 1$ (cf. [1, p.135]).

5. α-parabolic Bergman spaces

In this section, we shall define α-parabolic Bergman spaces and discuss some basic properties.

DEFINITION 5.1. For $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we denote by $\boldsymbol{b}_\alpha^p$ the set of all $L^{(\alpha)}$-harmonic functions on H which belong to $L^p(H)$. The space $\boldsymbol{b}_\alpha^p$ is called the α-parabolic Bergman space (of order p).

To show the closedness of $\boldsymbol{b}_\alpha^p$ in $L^p(H)$, we use the following boundedness of point evaluations.

Proposition 5.2. *Let $1 \leq p \leq \infty$. Then, there is a constant $C > 0$ such that for every $u \in \boldsymbol{b}_\alpha^p$ and every $(x, t) \in H$,*

$$|u(x,t)| \leq C\|u\|_{L^p(H)} t^{-(n/(2\alpha)+1)(1/p)}. \tag{5.1}$$

Proof. If $p = \infty$, then $|u(x,t)| \leq \|u\|_{L^\infty(H)}$, which is the assertion of the lemma. We suppose $1 \leq p < \infty$. For fixed $0 < a_1 < a_2 < 1$, the Huygens property (4.1) gives

$$\begin{aligned} u(x,t) &= \int_{\mathbf{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)\, dy \qquad (t > s > 0) \\ &= \frac{1}{(a_2 - a_1)t} \int_{a_1 t}^{a_2 t} \int_{\mathbf{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)\, dy\, ds. \end{aligned}$$

Then using (3.2), we have

$$|u(x,t)| \leq C\|u\|_{L^p(H)} t^{-(n/(2\alpha)+1)(1/p)}. \qquad \square$$

The next theorem implies that $\boldsymbol{b}_\alpha^p$ is a Banach space under the L^p-norm.

Theorem 5.3. *Let $1 \leq p \leq \infty$. Then $\boldsymbol{b}_\alpha^p$ is a closed subspace of $L^p(H)$.*

Proof. By Proposition 5.2, the L^p-convergence implies the uniform convergence on $\mathbf{R}^n \times [t_1, \infty) \subset H$ for every $t_1 > 0$. Hence the limit function of any L^p-convergent sequence in $\boldsymbol{b}_\alpha^p$ is continuous and satisfies the Huygens property. The result follows from Proposition 2.5. □

It follows from the Huygens property that $\boldsymbol{b}_\alpha^p \subset C^\infty(H)$, where $C^\infty(H)$ is the set of all C^∞-functions on H. Furthermore, as in the proof of Proposition 5.2, we have the following estimate for point evaluations of derivatives.

Theorem 5.4. *Let $1 \le p \le \infty$ and $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$. Then there is a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k u(x,t)| \le C\|u\|_{L^p(H)} t^{-(|\beta|/(2\alpha)+k)-(n/(2\alpha)+1)(1/p)} \tag{5.2}$$

for any $u \in \boldsymbol{b}_\alpha^p$ and $(x,t) \in H$.

The following norm inequality is also established.

Proposition 5.5. *Let $1 \le p \le \infty$ and $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0$. Then there is a constant $C > 0$ such that for every $u \in \boldsymbol{b}_\alpha^p$,*

$$\|t^{|\beta|/(2\alpha)+k} \partial_x^\beta \partial_t^k u\|_{L^p(H)} \le C\|u\|_{L^p(H)}. \tag{5.3}$$

Proof. By the Hyugens property,

$$\partial_x^\beta \partial_t^k u(x,t) = \int_{\mathbf{R}^n} u(x-y,s)(\partial_x^\beta \partial_t^k W^{(\alpha)})(y, t-s)\, dy$$

for every $t > s > 0$. Hence, taking $0 < \gamma < 1$ and $s = \gamma t$, we have

$$\begin{aligned}\partial_x^\beta \partial_t^k u(x,t) &= \int_{\mathbf{R}^n} u(x-y,\gamma t)(\partial_x^\beta \partial_t^k W^{(\alpha)})(y,(1-\gamma)t)\, dy \\ &= ((1-\gamma)t)^{-(|\beta|/(2\alpha)+k)} \int_{\mathbf{R}^n} u(x-((1-\gamma)t)^{1/(2\alpha)}z, \gamma t)(\partial_x^\beta \partial_t^k W^{(\alpha)})(z,1)\, dz.\end{aligned}$$

Thus the Minkowski inequality yields

$$\|t^{|\beta|/(2\alpha)+k} \partial_x^\beta \partial_t^k u\|_{L^p(H)} \le (1-\gamma)^{-(|\beta|/(2\alpha)+k)} \gamma^{-1/p} \left(\int_{\mathbf{R}^n} |(\partial_x^\beta \partial_t^k W^{(\alpha)})(z,1)|\, dz \right) \|u\|_{L^p(H)}.$$

□

Finally we discuss the integrals over the hyperplanes $\{t = \text{constant}\}$. The following lemma is interesting in itself.

Lemma 5.6. *Let $1 \le p \le \infty$. For $u \in \boldsymbol{b}_\alpha^p$, the function $t \mapsto \|u(\,\cdot\,,t)\|_{L^p(\mathbf{R}^n)}$ is decreasing on $(0,\infty)$.*

Proof. Take $t_2 > t_1 > 0$. By the Huygens property,

$$u(x,t_2) = \int_{\mathbf{R}^n} u(x-y,t_1) W^{(\alpha)}(y,t_2-t_1)\,dy.$$

The Minkowski inequality gives that

$$\|u(\,\cdot\,,t_2)\|_{L^p(\mathbf{R}^n)} \le \int_{\mathbf{R}^n} \|u(\,\cdot\,,t_1)\|_{L^p(\mathbf{R}^n)} W^{(\alpha)}(y,t_2-t_1)\,dy = \|u(\,\cdot\,,t_1)\|_{L^p(\mathbf{R}^n)}. \qquad \square$$

REMARK 5.7. For $1 \le p \le \infty$, we define the α-parabolic Hardy space $\boldsymbol{h}_\alpha^p$ on H as follows:

$$\boldsymbol{h}_\alpha^p := \left\{ v;\ L^{(\alpha)}\text{-harmonic on } H \text{ and } \sup_{t>0} \|v(\,\cdot\,,t)\|_{L^p(\mathbf{R}^n)} < \infty \right\}.$$

Then as a corollary to Lemma 5.6, we see that $T_\delta u \in \boldsymbol{h}_\alpha^p$ for every $u \in \boldsymbol{b}_\alpha^p$ and $\delta > 0$.

The next result is called the cancelation property.

Proposition 5.8. *For every $u \in \boldsymbol{b}_\alpha^1$ and every $t > 0$,*

$$\int_{\mathbf{R}^n} u(x,t)\,dx = 0. \tag{5.4}$$

Proof. By the Huygens property, we have

$$u(y,t+s) = \int_{\mathbf{R}^n} u(x,t) W^{(\alpha)}(y-x,s)\,dx.$$

Integrating the both sides by y and then s, we find

$$\int_0^T \int_{\mathbf{R}^n} u(y,t+s)\,dy\,ds = \int_0^T \int_{\mathbf{R}^n} u(x,t)\,dx\,ds = T \int_{\mathbf{R}^n} u(x,t)\,dx.$$

Since the left hand side converges as $T \to \infty$, (5.4) follows. $\square$

REMARK 5.9. This proposition shows that $\boldsymbol{b}_\alpha^1$ does not contain any nonzero nonnegative element. More generally, $\boldsymbol{b}_\alpha^p$ contains a nonnegative u such that $u \not\equiv 0$ if and only if $p > (n+2\alpha)/n$. This condition is related to (3.6) in Lemma 3.2 for $(\beta,k) = (0,0)$. Using Lemma 3.2 again for $(\beta,k) = (0,2)$, we have

$$\frac{\|\partial_t^2 T_\delta W^{(\alpha)}\|_{L^p(H)}}{\|\partial_t^2 T_\delta W^{(\alpha)}\|_{L^q(H)}} = C\delta^{(n/(2\alpha)+1)(1/p-1/q)}$$

for all $\delta > 0$. Hence the closed graph theorem tells us that there is no inclusion relation between $\boldsymbol{b}_\alpha^p$ and $\boldsymbol{b}_\alpha^q$ for $p \ne q$.

6. α-parabolic Bergman kernel

Since the point evaluation is bounded, $\boldsymbol{b}_\alpha^2$ has the reproducing kernel. In this section, we shall prove that the kernel

$$R_\alpha(x,t;y,s) = -2\partial_t W^{(\alpha)}(x-y,t+s) \tag{6.1}$$

is the desired reproducing kernel of $\boldsymbol{b}_\alpha^2$ (see Remark 6.5 below). We call R_α the α-parabolic Bergman kernel.

For $m = 0, 1, 2, \ldots,$ we also use the kernel R_α^m defined by

$$R_\alpha^m(x,t;y,s) = c_m s^m \partial_s^m R_\alpha(x,t;y,s),$$

where $c_m = (-2)^m/m!$. Note that $R_\alpha^0 = R_\alpha$ and it is a symmetric kernel.

We begin with two lemmas concerning these kernels. The first one is an estimate of their growth order, which follows from Lemma 3.1 immediately.

Lemma 6.1. *Let $m \geq 0$ be an integer. Then there is a constant $C > 0$ such that for any (x,t), $(y,s) \in H$,*

$$|R_\alpha^m(x,t;y,s)| \leq C s^m (s+t)^{-m} (s+t+|x-y|^{2\alpha})^{-n/(2\alpha)-1}.$$

In particular, $R_\alpha^m(x,t;\,\cdot\,,\,\cdot\,) \in L^q(H)$ for every $q > 1$ and $(x,t) \in H$.

The second one is an estimate of growth order for their integrals.

Lemma 6.2. *Let $m \geq 0$ be an integer. If $-1-m < \gamma < 0$, then there exists a constant $c_1(\gamma) > 0$ such that, for every $t > 0$,*

$$\iint_H s^\gamma |R_\alpha^m(x,t;y,s)|\, dy\, ds = c_1(\gamma) t^\gamma.$$

If $-1 < \gamma < m$, then there exists a constant $c_2(\gamma) > 0$ such that, for every $s > 0$,

$$\iint_H t^\gamma |R_\alpha^m(x,t;y,s)|\, dx\, dt = c_2(\gamma) s^\gamma.$$

Proof. By (3.1) we have

$$\begin{aligned}
&\iint_H s^\gamma |R_\alpha^m(x,t;y,s)|\, dy\, ds \\
&= 2|c_m| \int_0^\infty \int_{\mathbf{R}^n} s^\gamma s^m |\partial_t^{m+1} W^{(\alpha)}(x-y,t+s)|\, dy\, ds \\
&= 2|c_m| \int_0^\infty \int_{\mathbf{R}^n} s^{\gamma+m} (t+s)^{-n/(2\alpha)-m-1} |(\partial_t^{m+1} W^{(\alpha)})((t+s)^{-1/(2\alpha)}y, 1)|\, dy\, ds
\end{aligned}$$

$$= c_1(\gamma)t^\gamma,$$

where

$$c_1(\gamma) = 2|c_m| \left(\int_{\mathbf{R}^n} |(\partial_t^{m+1} W^{(\alpha)})(y,1)|\, dy \right) \left(\int_0^\infty u^{\gamma+m}(1+u)^{-m-1}\, du \right).$$

Remark that the second integral in the above is finite if and only if $-1-m < \gamma < 0$. The second assertion follows similarly. $\square$

In the sequel, we use the same symbol R_α^m for the integral operator defined by the kernel R_α^m:

$$R_\alpha^m f(x,t) := \iint_H R_\alpha^m(x,t;y,s) f(y,s)\, dy\, ds.$$

Then the following interesting relation holds.

Theorem 6.3. *Let $m \geq 0$ be an integer and let $1 \leq p < \infty$. Then $R_\alpha^m u = u$ for every $u \in \boldsymbol{b}_\alpha^p$, that is*

$$u(x,t) = \iint_H R_\alpha^m(x,t;y,s) u(y,s)\, dy\, ds. \tag{6.2}$$

Proof. Let $(x,t) \in H$ be fixed. We shall show the theorem by induction on m. Let $m = 0$. Take $\delta > 0$ and put $u_\delta = T_\delta u$. Then, by the Fubini theorem, we have

$$\begin{aligned}
&\iint_H R_\alpha(x,t;y,s) u_\delta(y,s)\, dy\, ds \\
&= 2\int_{\mathbf{R}^n} u_\delta(y,0) W^{(\alpha)}(x-y,t)\, dy + 2\int_{\mathbf{R}^n}\int_0^\infty \partial_s u_\delta(y,s) W^{(\alpha)}(x-y,t+s)\, ds\, dy.
\end{aligned}$$

Here we use the estimate (5.1). Then by the Huygens property for u_δ and $\partial_s u_\delta$, the first term is equal to $2u_\delta(x,t)$ and the second term is equal to $-u_\delta(x,t)$ respectively. Thus (6.2) holds for u_δ. Since u_δ converges to u in $L^p(H)$ as δ tends to zero, Lemma 6.1 shows the theorem in the case $m = 0$.

Next we assume that the theorem holds for $m-1 \geq 0$. Take $u \in \boldsymbol{b}_\alpha^p$ and put $u_\delta = T_\delta u$ as before. Then

$$\begin{aligned}
R_\alpha^m u_\delta(x,t) &= \iint_H R_\alpha^m(x,t;y,s) u_\delta(y,s)\, dy\, ds \\
&= -2c_m \int_{\mathbf{R}^n}\int_0^\infty u_\delta(y,s) s^m \partial_s^{m+1} W^{(\alpha)}(x-y,t+s)\, ds\, dy \\
&= 2c_m \int_{\mathbf{R}^n}\int_0^\infty \{m u_\delta(y,s) s^{m-1} + \partial_s u_\delta(y,s)\cdot s^m\} \partial_s^m W^{(\alpha)}(x-y,t+s)\, ds\, dy
\end{aligned}$$

$$= 2u_\delta(x,t) + 2c_m \int_{\mathbf{R}^n}\int_0^\infty \partial_s u_\delta(y,s)\cdot s^m \partial_s^m W^{(\alpha)}(x-y,t+s)\,ds\,dy,$$

here we use the induction assumption for $m-1$. Denoting by I the inner integral of the second term, integrating by parts m times and applying the Leibniz rule, we obtain

$$\begin{aligned} I &= (-1)^m \int_0^\infty \partial_s^m[\partial_s u_\delta(y,s)\cdot s^m] W^{(\alpha)}(x-y,t+s)\,ds \\ &= (-1)^m \sum_{j=0}^m \binom{m}{j}\frac{m!}{(m-j)!}\int_0^\infty \partial_s^{m+1-j}u_\delta(y,s)s^{m-j}W^{(\alpha)}(x-y,t+s)\,ds. \end{aligned}$$

Therefore, since $\partial_s^{m+1-j}u_\delta$ also satisfies the Hyugens property, by change the order of the integral, we have

$$\begin{aligned} &2c_m\int_{\mathbf{R}^n} I\,dy \\ &= 2(-1)^m c_m \sum_{j=0}^m \binom{m}{j}\frac{m!}{(m-j)!}\int_0^\infty s^{m-j}\partial_t^{m+1-j}u_\delta(x,t+2s)\,ds \\ &= 2(-1)^m c_m \sum_{j=0}^m \binom{m}{j}\frac{m!}{(m-j)!}\frac{1}{2^{m-j}}(-1)^{m-j}(m-j)!\int_0^\infty \partial_s u_\delta(x,t+2s)\,ds \\ &= -(-2)^m \sum_{j=0}^m \binom{m}{j}\frac{1}{2^{m-j}}(-1)^j u_\delta(x,t) \\ &= -u_\delta(x,t). \end{aligned}$$

Letting $\delta \downarrow 0$, we complete the induction. □

The main result of this section is the following theorem.

Theorem 6.4. (1) *For $1 < p < \infty$, R_α is a bounded operator from $L^p(H)$ onto $\boldsymbol{b}^p_\alpha$.*

(2) *Let $m \geq 1$ and $1 \leq p < \infty$. Then R^m_α is a bounded operator from $L^p(H)$ onto $\boldsymbol{b}^p_\alpha$.*

Proof. First we show (1). By Lemma 6.2 for $\gamma = -1/p$, we have

$$\begin{aligned} &|R_\alpha f(x,t)| \\ &\leq \iint_H |f(y,s)R_\alpha(x,t;y,s)|\,dy\,ds \\ &\leq \left(\iint_H |f(y,s)|^p s^{1/q}|R_\alpha(x,t;y,s)|\,dy\,ds\right)^{1/p}\left(\iint_H s^{-1/p}|R_\alpha(x,t;y,s)|\,dy\,ds\right)^{1/q} \end{aligned}$$

$$= c_1(-1/p)^{1/q} t^{-1/(pq)} \left(\iint_H |f(y,s)|^p s^{1/q} |R_\alpha(x,t;y,s)|\, dy\, ds \right)^{1/p}.$$

Therefore using the first estimate of Lemma 6.2 for $\gamma = -1/q$ again, we have

$$\begin{aligned}
&\iint_H |R_\alpha f(x,t)|^p\, dx\, dt \\
&\le c_1\left(-\frac{1}{p}\right)^{p/q} \iint_H \left(\iint_H t^{-1/q} |f(y,s)|^p s^{1/q} |R_\alpha(x,t;y,s)|\, dy\, ds \right) dx\, dt \\
&= c_1\left(-\frac{1}{p}\right)^{p/q} c_1\left(-\frac{1}{p}\right) \iint_H s^{-1/q} |f(y,s)|^p s^{1/q}\, dy\, ds \\
&= c_1\left(-\frac{1}{p}\right)^{p/q} c_1\left(-\frac{1}{p}\right) \|f\|^p_{L^p(H)},
\end{aligned}$$

because R_α is symmetric. The surjectivity of R_α follows from Theorem 6.3. Thus (1) is shown. Similarly, using Lemma 6.2, we have (2). Note that Lemma 6.2 is applicable for $\gamma = 0$ in the case $m \ge 1$ and $q = \infty$. □

REMARK 6.5. By Theorems 6.3 and 6.4, we see that the kernel R_α is the reproducing kernel for $\boldsymbol{b}^2_\alpha$. Furthermore, the operator R_α on $L^2(H)$ is the orthogonal projection to $\boldsymbol{b}^2_\alpha$, because R_α is real-valued and symmetric. Thus R_α is called the α-parabolic Bergman projection.

We generalize (6.2) in the following lemma.

Lemma 6.6. *Let $1 \le p < \infty$ and $m, k \in \mathbf{N}_0$ with $m + k \ge 1$. Then for $u \in \boldsymbol{b}^p_\alpha$ and $\delta > 0$,*

$$\iint_H \partial_s^k T_\delta u(y,s) \cdot s^{m+k-1} \partial_s^m W^{(\alpha)}(x-y, t+s)\, dy\, ds = \frac{(m+k-1)!}{(-2)^{m+k}} T_\delta u(x,t).$$

Proof. We remark that the integral is well-defined by (3.2) and (5.2). To prove the formula by induction, we first consider the case $(k, m) = (0, m)$. Then $m \ge 1$ and, by Theorem 6.3,

$$\begin{aligned}
&\iint_H T_\delta u(y,s) \cdot s^{m-1} \partial_s^m W^{(\alpha)}(x-y, t+s)\, dy\, ds \\
&= -\frac{1}{2c_{m-1}} (R_\alpha^{m-1} T_\delta u)(x,t) = -\frac{1}{2c_{m-1}} T_\delta u(x,t)
\end{aligned}$$

which is the desired equality, because $c_{m-1} = (-2)^{m-1}/(m-1)!$.

Next let $(k, m) = (1, 0)$. Then

$$\iint_H \partial_s T_\delta u(y, s) W^{(\alpha)}(x - y, t + s)\, dy\, ds = \int_0^\infty (\partial_t T_\delta u)(x, t + 2s)\, ds = -\frac{1}{2} T_\delta u(x, t).$$

Finally we consider the general case with $k + m \geq 2$. Assuming that the lemma holds for $(k - 1, m)$ and $(k - 1, m + 1)$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} \int_0^\infty \partial_s^k T_\delta u(y, s) \cdot s^{m+k-1} \partial_s^m W^{(\alpha)}(x-y, t+s)\, dy\, ds \\
&= -\int_{\mathbf{R}^n} \left(\int_0^\infty \partial_s^{k-1} T_\delta u(y, s) [(m+k-1) s^{m+k-2} \partial_s^m + s^{m+k-1} \partial_s^{m+1}] W^{(\alpha)}(x-y, t+s)\, ds \right) dy \\
&= \frac{(m+k-1)!}{(-2)^{m+k}} T_\delta u(x, t),
\end{aligned}$$

which completes the induction. □

The boundedness of the kernel R_α^m and the above lemma give the following formula.

Theorem 6.7. *Let $k, m \in \mathbf{N}_0$. Then for every $u \in \boldsymbol{b}_\alpha^p$ with $1 \leq p < \infty$,*

$$R_\alpha^m (t^k \partial_t^k u) = \frac{c_m}{c_{m+k}} u. \tag{6.3}$$

Proof. Recall that $c_m = (-2)^m / m!$. By Lemma 6.6, (6.3) holds for $T_\delta u$. Thus letting $\delta \downarrow 0$, we have the assertion. □

Proposition 6.8. *Let $1 \leq p < \infty$ and $k \in \mathbf{N}$. Then there is a constant $C \geq 1$ such that for every $u \in \boldsymbol{b}_\alpha^p$,*

$$C^{-1} \|t^k \partial_t^k u\|_{L^p(H)} \leq \|u\|_{L^p(H)} \leq C \|t^k \partial_t^k u\|_{L^p(H)}.$$

Proof. The first inequality follows from Proposition 5.5. Theorems 6.4 (2) and 6.7 give the second inequality. □

7. α-parabolic Bloch Space

In this section we define the α-parabolic Bloch space.

DEFINITION 7.1. We denote by $\mathcal{B}_\alpha$ the set of all $L^{(\alpha)}$-harmonic function u on H such that u is of C^1 class and that

$$\|u\|_{\mathcal{B}_\alpha} := |u(0, 1)| + \sup_{(x,t) \in H} \{t^{1/(2\alpha)} |\nabla_x u(x, t)| + t |\partial_t u(x, t)|\} < \infty, \tag{7.1}$$

where ∇_x denotes the gradient operator with respect to the space variable, and $0 = (0, \dots, 0) \in \mathbf{R}^n$. As seen later, $\mathcal{B}_\alpha$ is a Banach space under the Bloch norm $\|\cdot\|_{\mathcal{B}_\alpha}$. We call $\mathcal{B}_\alpha$ the α-parabolic Bloch space.

We begin with the boundedness of point evaluation on $\mathcal{B}_\alpha$.

Proposition 7.2. *There is a constant $C > 0$ such that for $u \in \mathcal{B}_\alpha$ and $(x, t) \in H$,*

$$|u(x,t)| \le C\|u\|_{\mathcal{B}_\alpha}(1 + |\log t| + \log(1 + |x|)). \tag{7.2}$$

Proof. For an $x \in \mathbf{R}^n$, we set $\tau = ((1 + |x|)/(1 + \log(1 + |x|)))^{2\alpha} \ge 1$. Then we have

$$\begin{aligned}
|u(x,t)| &\le |u(0,1)| + \int_1^\tau |\partial_t u(0,s)|\,ds + \int_0^{|x|} \left|\nabla_x u\left(r\frac{x}{|x|}, \tau\right)\right| dr + \left|\int_\tau^t \partial_t u(x,s)\,ds\right| \\
&\le \|u\|_{\mathcal{B}_\alpha}\left(1 + \int_1^\tau \frac{ds}{s} + \tau^{-1/(2\alpha)}|x| + \left|\int_\tau^t \frac{ds}{s}\right|\right) \\
&\le \|u\|_{\mathcal{B}_\alpha}\left(1 + \log\tau + \frac{|x|(1 + \log(1 + |x|))}{1 + |x|} + |\log t| + \log\tau\right).
\end{aligned}$$

Since $\log\tau \le 2\alpha\log(1 + |x|)$, the assertion follows. □

By the same manner as in Theorem 5.4, we have the following

Theorem 7.3. *For $(\beta, k) \in \mathbf{N}_0^n \times \mathbf{N}_0 \setminus \{(0,0)\}$, there is a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k u(x,t)| \le C\|u\|_{\mathcal{B}_\alpha} t^{-(|\beta|/(2\alpha)+k)} \tag{7.3}$$

for $u \in \mathcal{B}_\alpha$ and any $(x, t) \in H$. In particular, $\mathcal{B}_\alpha \subset C^\infty(H)$.

Proof. We first remark that $\boldsymbol{b}_\alpha^\infty \subset C^\infty(H)$. Let $(x_0, t_0) \in H$ be fixed. If $k \ne 0$, applying Theorem 5.4 to $T_{t_0/2}\partial_t u \in \boldsymbol{b}_\alpha^\infty$, we have

$$\begin{aligned}
|\partial_x^\beta \partial_t^k u(x_0, t_0)| &= \left|\partial_x^\beta \partial_t^{k-1}(T_{t_0/2}\partial_t u)\left(x_0, \frac{t_0}{2}\right)\right| \\
&\le C\|T_{t_0/2}\partial_t u\|_{L^\infty(H)} t_0^{-(|\beta|/(2\alpha)+k-1)} \\
&\le 2C\|u\|_{\mathcal{B}_\alpha} t_0^{-(|\beta|/(2\alpha)+k)}.
\end{aligned}$$

Similarly, we can obtain the theorem when the case $\beta \ne 0$. □

Theorem 7.4. *Every element in $\mathcal{B}_\alpha$ satisfies the Huygens property, and $\mathcal{B}_\alpha$ is a Banach space under the Bloch norm* (7.1).

Proof. Take $u \in \mathcal{B}_\alpha$. Since $T_s \partial_t u$ belongs to $\boldsymbol{b}_\alpha^\infty$ for every $s > 0$, we have

$$\partial_t u(x, t+s) = \int_{\mathbf{R}^n} \partial_t u(x-y, t) W^{(\alpha)}(y, s)\, dy$$

and hence for $t_2 > t_1 > 0$,

$$\begin{aligned} u(x, t_2+s) - u(x, t_1+s) &= \int_{t_1}^{t_2} \int_{\mathbf{R}^n} \partial_t u(x-y, t) W^{(\alpha)}(y, s)\, dy\, dt \\ &= \int_{\mathbf{R}^n} (u(x-y, t_2) - u(x-y, t_1))\, W^{(\alpha)}(y, s)\, dy. \end{aligned}$$

This implies $v(x, t, s)$ is a constant function with respect to t, where

$$v(x, t, s) = u(x, t+s) - \int_{\mathbf{R}^n} u(x-y, t) W^{(\alpha)}(y, s)\, dy.$$

A similar argument with respect to the variable x gives that v does not depend on x either. For fixed $t > 0$, since $v(\,\cdot\,, t, \,\cdot\,)$ is $L^{(\alpha)}$-harmonic, we have $\partial_s v = L^{(\alpha)}_{(x,s)} v = 0$, which implies v is a constant. Further this constant is equal to

$$\lim_{s \to 0} v(x, t, s) = 0,$$

so that the Huygens property for u follows.

To show the completeness of $\mathcal{B}_\alpha$, consider any Cauchy sequence in $\mathcal{B}_\alpha$ with respect to the Bloch norm. By Proposition 7.2, it converges locally uniformly to a continuous function u on H. It is not difficult to show that this limit function also satisfies the Huygens property, so that u is $L^{(\alpha)}$-harmonic on H and is of C^∞ class. Theorem 7.3 gives $\|u\|_{\mathcal{B}_\alpha} < \infty$. □

Since $\mathcal{B}_\alpha$ contains constant functions, we may identify $\mathcal{B}_\alpha/\mathbf{R} \cong \tilde{\mathcal{B}}_\alpha$, where

$$\tilde{\mathcal{B}}_\alpha = \{u \in \mathcal{B}_\alpha\,;\, u(0,1) = 0\}.$$

The α-parabolic Bergman kernel R_α is not bounded on $L^\infty(H)$, so that we consider the modified α-parabolic Bergman kerenl $\tilde{R}_\alpha$, which is inspired by [10]:

$$\tilde{R}_\alpha(x, t; y, s) := R_\alpha(x, t; y, s) - R_\alpha(0, 1; y, s).$$

Lemma 7.5. *There is a constant $C > 0$ such that for every $(x, t) \in H$,*

$$\iint_H |\tilde{R}_\alpha(x, t; y, s)|\, dy\, ds \le C(1 + \log(1 + |x|) + |\log t|).$$

Proof. Put $\tau = ((1+|x|)/(1+\log(1+|x|)))^{2\alpha}$. Then

$$
\begin{aligned}
&\|\tilde{R}_\alpha(x,t;\ \cdot\ ,\ \cdot\ )\|_{L^1(H)} \\
&\le \|R_\alpha(x,t;\ \cdot\ ,\ \cdot\ ) - R_\alpha(x,\tau;\ \cdot\ ,\ \cdot\ )\|_{L^1(H)} + \|R_\alpha(x,\tau;\ \cdot\ ,\ \cdot\ ) - R_\alpha(0,\tau;\ \cdot\ ,\ \cdot\ )\|_{L^1(H)} \\
&\quad + \|R_\alpha(0,\tau;\ \cdot\ ,\ \cdot\ ) - R_\alpha(0,1;\ \cdot\ ,\ \cdot\ )\|_{L^1(H)}.
\end{aligned}
$$

The Minkowski inequality and Lemma 3.2 show that the first term of the right hand side is bounded by

$$
2\left|\int_\tau^t \|T_\delta \partial_t^2 W^{(\alpha)}\|_{L^1(H)}\, d\delta\right| \le C\left|\int_\tau^t \delta^{-1}\, d\delta\right| \le C(|\log t| + \log\tau),
$$

and the second term is less than

$$
\begin{aligned}
2\int_0^1 \iint_H \left|\frac{\partial}{\partial r}(\partial_t W^{(\alpha)}(rx-y,\tau+s))\right| dy\,ds\,dr &\le 2\int_0^1 |x|\|T_\tau \nabla_x \partial_t W^{(\alpha)}\|_{L^1(H)}\, dr \\
&\le C|x|\tau^{-1/(2\alpha)}
\end{aligned}
$$

and the third term is bounded by

$$
2\left|\int_1^\tau \|T_\delta \partial_t^2 W^{(\alpha)}\|_{L^1(H)}\, d\delta\right| \le C\log\tau,
$$

which show the required estimate as in the proof of Proposition 7.2. □

Lemma 7.6. *For every $(x,t)\in H$ and for every $0<\delta<1$,*

$$
\iint_H \frac{1}{s+\delta}\left|W^{(\alpha)}(x+y,t+s) - W^{(\alpha)}(y,s+1)\right| dy\,ds < \infty.
$$

Proof. For fixed $x=(x_1,\dots,x_n)$, the equality

$$
W^{(\alpha)}(x+y,s+1) - W^{(\alpha)}(y,s+1) = \int_0^1 x\cdot\nabla_x W^{(\alpha)}(rx+y,s+1)\,dr
$$

and (3.2) give

$$
\begin{aligned}
&\iint_H \frac{1}{s+\delta}\left|W^{(\alpha)}(x+y,s+1) - W^{(\alpha)}(y,s+1)\right| dy\,ds \\
&\le C|x|\int_0^1\int_0^\infty\left(\int_{\mathbf{R}^n}\left|\nabla_x W^{(\alpha)}((s+1)^{-1/(2\alpha)}(rx+y),1)\right| dy\right)(s+1)^{-(n+1)/(2\alpha)}(s+\delta)^{-1}\,ds\,dr \\
&\le C'|x|\int_0^\infty (s+1)^{-1/(2\alpha)}(s+\delta)^{-1}\,ds < \infty,
\end{aligned}
$$

and since

$$W^{(\alpha)}(x+y,t+s)-W^{(\alpha)}(x+y,s+1)=\int_1^t \partial_t W^{(\alpha)}(x+y,s+\tau)\,d\tau,$$

we also have

$$\begin{aligned}
&\iint_H \frac{1}{s+\delta}\left|W^{(\alpha)}(x+y,t+s)-W^{(\alpha)}(x+y,s+1)\right|\,dy\,ds\\
&\le \left|\int_1^t\int_0^\infty \left(\int_{\mathbf{R}^n}\left|\partial_t W^{(\alpha)}((s+\tau)^{-1/(2\alpha)}(x+y),1))\right|\,dy\right)(s+\tau)^{-n/(2\alpha)-1}(s+\delta)^{-1}\,ds\,d\tau\right|\\
&\le C\left|\int_1^t\int_0^\infty (s+\tau)^{-1}(s+\delta)^{-1}\,ds\,d\tau\right|<\infty.
\end{aligned}$$

Thus our assertion follows from the triangle inequality. □

Theorem 7.7. *The kernel $\tilde{R}_\alpha$ is a bounded linear operator from $L^\infty(H)$ to $\tilde{\mathcal{B}}_\alpha$.*

Proof. For every $f\in L^\infty(H)$, we can define $\tilde{R}_\alpha f(x,t)$ by Lemma 7.5. Further since $\tilde{R}_\alpha(x,t;\,\cdot\,,\,\cdot\,)$ is $L^{(\alpha)}$-harmonic, so is $\tilde{R}_\alpha f$. Clearly $\tilde{R}_\alpha f(0,1)=0$. For every $(\beta,k)\in\mathbf{N}_0^n\times\mathbf{N}_0$ with $(\beta,k)\ne(0,0)$, we have

$$|\partial_x^\beta\partial_t^k[\tilde{R}_\alpha f(x,t)]|=\left|\iint_H \partial_x^\beta\partial_t^k R_\alpha(x,t;y,s)f(y,s)\,dy\,ds\right|\le C\|f\|_{L^\infty(H)}t^{-(|\beta|/(2\alpha)+k)},$$

by Lemma 3.2. In particular, $\|\tilde{R}_\alpha f\|_{\mathcal{B}_\alpha}\le C\|f\|_{L^\infty(H)}$ holds. □

Similarly to Lemma 6.6, Theorem 6.7 and Proposition 6.8, we can obtain the following results for α-parabolic Bloch spaces. Remark that Lemma 7.6 assures the necessary integrabilty in the following results.

Lemma 7.8. *Let m,k be nonnegative integers with $m+k\ge 1$. Then for evey $u\in\mathcal{B}_\alpha$ and every $\delta>0$, we have*

$$\begin{aligned}
&\iint_H \partial_s^k T_\delta u(y,s)\cdot s^{m+k-1}\partial_s^m(W^{(\alpha)}(x-y,t+s)-W^{(\alpha)}(y,s+1))\,dy\,ds\\
&=\frac{(m+k-1)!}{(-2)^{m+k}}(T_\delta u(x,t)-T_\delta u(0,1)).
\end{aligned}\tag{7.4}$$

Theorem 7.9. *For any $u\in\tilde{\mathcal{B}}_\alpha$, $u=-2\tilde{R}_\alpha(t\partial_t u)$ holds. More generally, for any $k\in\mathbf{N}$, we have*

$$\tilde{R}_\alpha(t^k\partial_t^k u)=\frac{k!}{(-2)^k}u.$$

Proposition 7.10. *Let $k \geq 1$ be an integer. Then thers is a constant $C \geq 1$ such that for every $u \in \mathcal{B}_\alpha$*

$$C^{-1}\|t^k \partial_t^k u\|_{L^\infty(H)} \leq \|u\|_{\mathcal{B}_\alpha} \leq C\|t^k \partial_t^k u\|_{L^\infty(H)}.$$

8. Dual Spaces

In this section, we characterize the dual space of $\boldsymbol{b}_\alpha^p$ for $1 \leq p < \infty$. In the following, we use the following convention: write $X = (x, t) \in H$ and for an integrable function f on H,

$$\int_H f(X)\,dX = \iint_H f(x, t)\,dx\,dt.$$

Theorem 8.1. *Let $1 < p < \infty$. Then $(\boldsymbol{b}_\alpha^p)^* \cong \boldsymbol{b}_\alpha^q$, that is, the dual space of $\boldsymbol{b}_\alpha^p$ can be identified with $\boldsymbol{b}_\alpha^q$, where q is the exponent conjugate to p.*

Proof. For $v \in \boldsymbol{b}_\alpha^q$, we define a functional on $\boldsymbol{b}_\alpha^p$ by

$$\Lambda_v(u) = \int_H u(X)v(X)\,dX.$$

Then $\Lambda_v \in (\boldsymbol{b}_\alpha^p)^*$ and $\|\Lambda_v\| \leq \|v\|_{L^q(H)}$. Put $\iota(v) = \Lambda_v$. By the open mapping theorem, it is sufficient to show that the mapping $\iota\colon \boldsymbol{b}_\alpha^q \to (\boldsymbol{b}_\alpha^p)^*$ is bijective.

Assuming $\Lambda_v = 0$, we have

$$v(X) = \int_H R_\alpha(X; Y)v(Y)\,dY = \Lambda_v(R_\alpha(X;\ \cdot\ )) = 0$$

because $R_\alpha(X;\ \cdot\ ) \in \boldsymbol{b}_\alpha^p$, which implies ι is injective.

Next for $\Lambda \in (\boldsymbol{b}_\alpha^p)^*$, using the Hahn-Banach theorem, there exists f in $L^q(H)$ such that

$$\Lambda(u) = \int_H u(X)f(X)\,dX$$

for all $u \in \boldsymbol{b}_\alpha^p$. Since R_α is symmetric, Theorems 6.3 and 6.4 show

$$\Lambda(u) = \int_H (R_\alpha u)(X)f(X)\,dX = \int_H u(Y)(R_\alpha f)(Y)\,dY = \Lambda_{R_\alpha f}(u).$$

This implies ι is surjective and the proof of Theorem completes. $\square$

To determine the dual space for $p = 1$, we use a subspace of $\boldsymbol{b}_\alpha^\infty$. We put

$$\mathcal{D} := \{u \in \boldsymbol{b}_\alpha^\infty;\ (1 + t)(1 + t + |x|^{2\alpha})^{n/(2\alpha)+1}u(x, t) \quad \text{is bounded on } H\}. \tag{8.1}$$

Lemma 8.2. *$\mathcal{D}$ is dense in $\boldsymbol{b}_\alpha^p$ for $1 \le p < \infty$.*

Proof. Let $u \in \boldsymbol{b}_\alpha^p$. Taking an exhaustion $\{K_j\}_{j=1}^\infty$ of H, we see that $R_\alpha^1(u \cdot \chi_{K_j})$ converges to u by Theorems 6.3 and 6.4 (2), where χ_{K_j} denotes the indicator function of K_j. Further, Lemma 6.1 shows $R_\alpha^1(u \cdot \chi_{K_j}) \in \mathcal{D}$. $\square$

Lemma 8.3. *For $u \in \mathcal{D}$ and $v \in \tilde{\mathcal{B}}_\alpha$,*

$$\int_H u(X)v(X)\,dX = -2\int_H u(X)\Phi v(X)\,dX, \tag{8.2}$$

where $\Phi v(X) = t\partial_t u(x,t)$. In particular

$$\left|\int_H u(X)v(X)\,dX\right| \le 2\|u\|_{L^1(H)}\|v\|_{\mathcal{B}_\alpha}. \tag{8.3}$$

Proof. We first observe the following integrability. Since Φv is bounded, Lemma 7.5 shows that there is a constant $C > 0$ such that

$$\begin{aligned}
&\int_H \left(\int_H |u(X)\tilde{R}_\alpha(X;Y)\Phi v(Y)|\,dY\right) dX \\
&\le C \iint_H \frac{1+\log(1+|x|)+|\log t|}{(1+t)(1+t+|x|^{2\alpha})^{n/(2\alpha)+1}}\,dx\,dt \\
&\le C\left(\int_0^\infty \frac{1+|\log t|}{(1+t)^{3/2}}\,dt\right)\left(\int_{\mathbf{R}^n} \frac{1+\log(1+|x|)}{(1+|x|^{2\alpha})^{(n/(2\alpha))+1/2}}\,dx\right) \\
&< \infty.
\end{aligned}$$

We also observe that since R_α is symmetric and u has the cancelation property,

$$\begin{aligned}
u(Y) &= \int_H R_\alpha(Y;X)u(X)\,dX = \int_H R_\alpha(X;Y)u(X)\,dX \\
&= \int_H \{R_\alpha(X;Y) - R_\alpha(X_0;Y)\}u(X)\,dX \\
&= \int_H \tilde{R}_\alpha(X;Y)u(X)\,dX,
\end{aligned}$$

where $X_0 = (0,1)$. Hence these observations and Theorem 7.9 ensure that

$$\begin{aligned}
\int_H u(X)v(X)\,dX &= -2\int_H u(X)\tilde{R}_\alpha\Phi v(X)\,dX \\
&= -2\int_H \left(\int_H u(X)\tilde{R}_\alpha(X;Y)\,dX\right)\Phi v(Y)\,dY \\
&= -2\int_H u(Y)\Phi v(Y)\,dY.
\end{aligned}$$

The inequality (8.3) follows from Definition 7.1. □

Now we shall characterize the dual space of $\boldsymbol{b}_\alpha^p$ for the case $p = 1$.

Theorem 8.4. *The dual space of $\boldsymbol{b}_\alpha^1$ can be identified with $\mathcal{B}_\alpha/\mathbf{R} \cong \tilde{\mathcal{B}}_\alpha$.*

Proof. For any $v \in \tilde{\mathcal{B}}_\alpha$, we define a linear functional on $\boldsymbol{b}_\alpha^1$ by

$$\Lambda_v(u) = -2\int_H u(X)\Phi v(X)\,dX.$$

Then since $|\Lambda_v(u)| \le 2\|u\|_{L^1(H)}\|v\|_{\mathcal{B}_\alpha}$ by Lemma 8.3, $\Lambda_v \in (\boldsymbol{b}_\alpha^1)^*$. Put $\iota(v) = \Lambda_v$. As in the proof of Theorem 8.1, it is sufficient to show that the mapping $\iota\colon \tilde{\mathcal{B}}_\alpha \to (\boldsymbol{b}_\alpha^1)^*$ is bijective. Since $\tilde{R}_\alpha(X;\,\cdot\,) \in \boldsymbol{b}_\alpha^1$, the injectivity follows from Theorem 7.9.

To show the surjectivity, we take $\Lambda \in (\boldsymbol{b}_\alpha^1)^*$ arbitrarily. Then by the Hahn-Banach theorem, there exists $f \in L^\infty(H)$ such that $\|f\|_{L^\infty(H)} = \|\Lambda\|$ and

$$\Lambda(u) = \int_H u(X)f(X)\,dX$$

for every $u \in \boldsymbol{b}_\alpha^1$. Then Theorem 7.7 gives us that $\tilde{R}_\alpha f \in \tilde{\mathcal{B}}_\alpha$ and $\|\tilde{R}_\alpha f\|_{\mathcal{B}_\alpha} \le C\|f\|_{L^\infty(H)} = C\|\Lambda\|$ with some constant $C > 0$. Hence by the same reason as in the proof of Lemma 8.3, we have

$$\begin{aligned}
\Lambda(u) &= \int_H u(Y)f(Y)\,dY \\
&= \int_H \left(\int_H R_\alpha(Y;X)u(X)\,dX\right) f(Y)\,dY \\
&= \int_H u(X)\tilde{R}_\alpha f(X)\,dX \\
&= -2\int_H u(X)\Phi(\tilde{R}_\alpha f)(X)\,dX = \Lambda_{\tilde{R}_\alpha f}(u)
\end{aligned}$$

provided that $u \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $\boldsymbol{b}_\alpha^1$, the mapping ι is surjective. □

9. α-parabolic Little Bloch Space

In this section we define the α-parabolic little Bloch space, which turns out to be the predual of $\boldsymbol{b}_\alpha^1$. The argument here is inspired by [13].

DEFINITION 9.1. A function $u \in \mathcal{B}_\alpha$ is said to be an α-parabolic little Bloch function, if

(9.1) $$\lim_{(x,t)\to\partial H\cup\{\infty\}} \{t|\partial_t u(x,t)| + t^{1/(2\alpha)}|\nabla_x u(x,t)|\} = 0.$$

We denote by $\mathcal{B}_{\alpha,0}$ the set of all α-parabolic little Bloch functions on H and call $\mathcal{B}_{\alpha,0}$ the α-parabolic little Bloch space.

Let $\tilde{\mathcal{B}}_{\alpha,0} := \{u \in \mathcal{B}_{\alpha,0}\,;\, u(0,1) = 0\}$. Since $\mathcal{B}_{\alpha,0}$ and $\tilde{\mathcal{B}}_{\alpha,0}$ are closed subspace of $\mathcal{B}_\alpha$, they are both Banach spaces with the Bloch norm $\|\cdot\|_{\mathcal{B}_\alpha}$.

We let $C_0(H)$ denote the set of all continuous functions on H which vanish continuously on $\partial H \cup \{\infty\}$.

Lemma 9.2. $\tilde{\mathcal{B}}_{\alpha,0} = \{u \in \tilde{\mathcal{B}}_\alpha\,;\, \Phi u \in C_0(H)\} = \{\tilde{R}_\alpha f\,;\, f \in C_0(H)\}$.

Proof. For the first equality it is sufficient to show that if $\Phi u = t\partial_t u$ belongs to $C_0(H)$ then so does $t^{1/(2\alpha)}|\nabla_x u|$. Since $u = -2\tilde{R}_\alpha(\Phi u)$ by Theorem 7.9, we have for $j = 1, \ldots, n$

$$\partial_{x_j} u(x,t) = -2 \iint_H \partial_{x_j}\partial_t W^{(\alpha)}(x-y, t+s) \cdot s\partial_s u(u,s)\,dy\,ds.$$

Given $\varepsilon > 0$, there is a compact set K in H such that $|s\partial_s u| < \varepsilon$ outside K. Then

$$\begin{aligned} |t^{1/(2\alpha)}\partial_{x_j} u(x,t)| &\le 2\varepsilon t^{1/(2\alpha)} \iint_{K^c} |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)|\,dy\,ds \\ &\quad + 2t^{1/(2\alpha)} \iint_K |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)| \cdot |s\partial_s u(y,s)|\,dy\,ds. \end{aligned}$$

The first term in the right hand side is less than $2C\varepsilon$ by Lemma 3.2, while the second term tends to 0 provided that (x,t) tends to $\partial H \cup \{\infty\}$ (use (3.2)). We therefore conclude $t^{1/(2\alpha)}|\nabla_x u| \in C_0(H)$.

To show the second equality in the lemma, take $f \in C_0(H)$ arbitrarily. Then $\tilde{R}_\alpha f$ is in $\tilde{\mathcal{B}}_\alpha$ by Theorem 7.7. The same argument as above shows $\Phi(\tilde{R}_\alpha f) \in C_0(H)$, which implies $\tilde{\mathcal{B}}_{\alpha,0} \supset \{\tilde{R}_\alpha f\,;\, f \in C_0(H)\}$. The converse inclusion follows easily from the equality $u = -2\tilde{R}_\alpha(\Phi u)$. ☐

We can now prove the main result of this section.

Theorem 9.3. *The pre-dual space of $\boldsymbol{b}^1_\alpha$ can be identified with $\mathcal{B}_{\alpha,0}/\mathbf{R}$.*

Proof. As in Theorem 8.4, we may identify $\mathcal{B}_{\alpha,0}/\mathbf{R}$ with $\tilde{\mathcal{B}}_{\alpha,0}$. For $u \in \boldsymbol{b}^1_\alpha$, we define a functional on $\tilde{\mathcal{B}}_{\alpha,0}$ by

$$\Lambda_u(v) := \iint_H u(x,t)\Phi v(x,t)\,dx\,dt.$$

Then by Lemma 8.3 the mapping $\iota\colon \boldsymbol{b}^1_\alpha \to (\tilde{\mathcal{B}}_{\alpha,0})^*$, defined by $\iota(u) = \Lambda_u$, is bounded. To show the injectivity of ι, we assume that $\Lambda_u = 0$. Then for every $f \in C_0(H)$, since

$\partial_t \tilde{R}_\alpha(x, t; y, s) = \partial_t R_\alpha(x, t; y, s) = \partial_t R_\alpha(y, s; x, t)$, we have

$$\begin{aligned}
0 &= \Lambda_u(\tilde{R}_\alpha(f)) \\
&= \iint_H \left(u(x, t) \iint_H t\partial_t \tilde{R}_\alpha(x, t; y, s) f(y, s) dyds \right) dx\,dt \\
&= \iint_H \left(\iint_H u(x, t) t\partial_t R_\alpha(y, s; x, t)\, dx\,dt \right) f(y, s)\, dy\,ds \\
&= -\frac{1}{2} \iint_H R_\alpha^1 u(y, s) f(y, s)\, dy\,ds = -\frac{1}{2} \iint_H u(y, s) f(y, s)\, dy\,ds,
\end{aligned}$$

which implies $u = 0$. Note that all the above double integrals converge. In fact, by Lemma 6.1

$$\begin{aligned}
&\iint_H \iint_H |u(x, t) t\partial_t R_\alpha(x, t; y, s) f(y, s)|\, dy\,ds\,dx\,dt \\
&\leq \|f\|_{L^\infty(H)} \iint_H |u(x, t)| \left(\iint_H \frac{t}{(t + s)(t + s + |x - y|^{2\alpha})^{n/(2\alpha)+1}}\, dy\,ds \right) dx\,dt \\
&\leq C\|f\|_{L^\infty(H)} \|u\|_{L^1(H)} < \infty.
\end{aligned}$$

Next, to show the surjectivity of ι, take $\Lambda \in (\tilde{\mathcal{B}}_{\alpha,0})^*$ arbitrarily. Then because of Theorem 7.7 and Lemma 9.2, $f \mapsto \Lambda(\tilde{R}_\alpha f)$ defines a bounded linear functional on $C_0(H)$. Hence by the Riesz representation theorem, there exists a bounded signed measure μ on H such that

$$\Lambda(\tilde{R}_\alpha f) = \iint_H f(x, t)\, d\mu(x, t),$$

for every $f \in C_0(H)$. We define a function u on H by

$$u(y, s) = 4 \iint_H t\partial_t \tilde{R}_\alpha(x, t; y, s)\, d\mu(x, t).$$

Then $u \in \boldsymbol{b}_\alpha^1$. In fact, since $t\partial_t \tilde{R}_\alpha(x, t; y, s)$ is $L^{(\alpha)}$-harmonic with respect to (y, s), so is u. Furthermore

$$\begin{aligned}
\|u\|_{L^1(H)} &\leq 4 \iint_H \left(\iint_H |t\partial_t \tilde{R}_\alpha(x, t; y, s)|\, d|\mu|(x, t) \right) dyds \\
&\leq 8 \iint_H \left(\iint_H |tT_t \partial_s^2 W^{(\alpha)}(x - y, s)|\, dy\,ds \right) d|\mu|(x, t) \\
&= 8 \iint_H t\|\partial_s^2 T_t W^{(\alpha)}\|_{L^1(H)}\, d|\mu|(x, t) = 8C\|\mu\|,
\end{aligned}$$

where we use Lemma 3.2 for the last equality. Now for every $v \in \tilde{\mathcal{B}}_{\alpha,0}$ the equality

$v = -2\tilde{R}_\alpha(\Phi v)$ gives $\Phi v = -2\Phi(\tilde{R}_\alpha(\Phi v))$ so that

$$\begin{aligned}\Lambda(v) &= -2\Lambda(\tilde{R}_\alpha(\Phi v)) = -2\iint_H \Phi v(x,t)\,d\mu(x,t)\\ &= 4\iint_H \Phi(\tilde{R}_\alpha(\Phi v))(x,t)\,d\mu(x,t)\\ &= 4\iint_H \left(\iint_H t\partial_t \tilde{R}_\alpha(x,t;y,s)\,d\mu(x,t)\right)\Phi v(y,s)\,dy\,ds\\ &= \iint_H u(y,s)\Phi v(y,s)\,dy\,ds = \Lambda_u(v).\end{aligned}$$

This implies that the map ι is surjective, and hence $\boldsymbol{b}_\alpha^1 \cong (\tilde{\mathcal{B}}_{\alpha,0})^*$. $\square$

References

[1] C. Berg and G. Forst: Potential theory on locally compact abelian groups, Springer-Verlag, New York-Heidelberd, 1975.

[2] J. Bliedtner and W. Hansen: Potential theory, Springer-Verlag, Berlin, 1986.

[3] J.L. Doob: Classical potential theory and its probabilistic counterpart, Springer-Verlag, New York, 1984.

[4] M. Itô: *On α-harmonic functions*, Nagoya Math. J. **26** (1966), 205–221.

[5] M. Itô and M. Nishio: *Poincaré type conditions of the regularity for the parabolic operator of order α*, Nagoya Math. J. **115** (1989), 1–22.

[6] T. Kakehi and K. Sakai: *Behavior of solutions to linear and semilinear parabolic pseudo-differential equations*, Preprint.

[7] N.S. Landkof: Foundations of modern potential theory, Springer-Verlag, New York-Heidelberg, 1972.

[8] M. Nishio: *The Wiener criterion of regular points for the parabolic operator of order α*, Nagoya Math. J. **116** (1989), 163–179.

[9] M. Nishio and N. Suzuki: *A characterization of strip domains by a mean value property for the parabolic operator of order α*, New Zealand J. Math. **29** (2000), 47–54.

[10] W.C. Ramey and H. Yi: *Harmonic Bergman functions on half-spaces*, Trans. Amer. Math. Soc. **348** (1996), 633–660.

[11] N.A. Watson: Parabolic equations on an infinite strip, Marcel Dekker, 1989.

[12] D.V. Widder: The heat equation, Academic Press, 1975.

[13] H. Yi: *Harmonic little Bloch functions on half-spaces*, Math. Japon. **47** (1998), 21–28.

Masaharu Nishio
Department of Mathematics
Graduate School of Science
Osaka City University
Osaka 558-8585, Japan
e-mail: nishio@sci.osaka-cu.ac.jp

Katsunori Shimomura
Department of Mathematical Sciences
Faculty of Science
Ibaraki University
Mito 310-8512, Japan
e-mail: shimomur@mx.ibaraki.ac.jp

Noriaki Suzuki
Graduate School of Mathematics
Nagoya University
Nagoya 464-8602, Japan
e-mail: nsuzuki@math.nagoya-u.ac.jp