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## AN EXTENTION OF MILNOR'S INEQUALITY

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### 1. Introduction

Milnor showed the following theorem.

**Theorem 1.1** (Milnor [4]). *Let  $\rho : \pi_1(\Sigma_g) \rightarrow SL(2, \mathbb{R})$  be a representation of the surface group and let  $E$  be the corresponding flat vector bundle of rank 2 over  $\Sigma_g$ . Then the Euler number  $\langle \chi(E), [\Sigma_g] \rangle$  satisfies the following inequality:*

$$|\langle \chi(E), [\Sigma_g] \rangle| \leq g - 1.$$

*Conversely any integer  $\chi$  such that  $|\chi| \leq g - 1$  is realized as the Euler number of a flat  $SL(2, \mathbb{R})$ -vector bundle.*

There are various generalizations of this theorem, for which see [5].

In this paper we will prove an extension of this theorem to a representation of surface group to the symplectic group  $Sp(2p, \mathbb{R})$ . More precisely, let  $\rho : \pi_1(\Sigma_g) \rightarrow Sp(2p, \mathbb{R})$  be a representation of the surface group and let  $V$  be the corresponding flat symplectic vector bundle of rank  $2p$  over  $\Sigma_g$ . Since a maximal compact subgroup of  $Sp(2p, \mathbb{R})$  is isomorphic to  $U(p)$ ,  $V$  admits a positive compatible complex structure with the symplectic structure which always exists and is unique up to homotopy. Hence the first Chern class

$$c_1(V) \in H^2(\Sigma_g; \mathbb{Z})$$

is uniquely defined.

The main theorem of this paper is the following.

**Theorem 1.2.** *Let  $g, p \geq 1$  be integers and  $\Sigma_g$  a closed oriented surface of genus  $g$ . For any representation  $\rho : \pi_1(\Sigma_g) \rightarrow Sp(2p, \mathbb{R})$ , let  $V$  be the corresponding flat symplectic vector bundle of rank  $2p$  over  $\Sigma_g$ . Then the following inequality holds:*

$$|\langle c_1(V), [\Sigma_g] \rangle| \leq p(g - 1).$$

Conversely any integer  $c_1$  such that  $|c_1| \leq p(g-1)$  is realized as the first Chern class of a flat symplectic vector bundle of rank  $2p$ .

## 2. The signature of local coefficient systems

In this section we shall review the result due to Lusztig and Atiyah ([1, 3]) which is needed to prove the main theorem.

Let  $\Sigma_g$  be a closed oriented surface of genus  $g \geq 1$ . Let  $E \rightarrow \Sigma_g$  be a flat (possibly indefinite) hermitian vector bundle with a hermitian form  $h$ . It corresponds to a homomorphism

$$\rho: \pi_1(\Sigma_g) \rightarrow U(p, q),$$

where  $U(p, q)$  is the unitary group of the hermitian form  $\sum_{i=1}^p |z_i|^2 - \sum_{i=p+1}^{p+q} |z_i|^2$  on  $(z_i) \in \mathbb{C}^{p+q}$ . Since a flat vector bundle is identified with a local coefficient system, we have the cohomology group  $H^*(\Sigma_g; E)$ . Moreover the first cohomology group  $H^1(\Sigma_g; E)$  has a skew-hermitian form  $A$  induced by the cup product and the hermitian form  $h$ . Since  $\sqrt{-1}A$  is a hermitian form, we can consider its signature. Let  $\text{sign}(\Sigma_g, E)$  denote the signature of  $\sqrt{-1}A$ .

Take a splitting  $E = E^+ \oplus E^-$  of  $E$ . Here a splitting of the hermitian vector bundle  $E$  is defined as a smooth decomposition  $E = E^+ \oplus E^-$  of  $E$  which is orthogonal relative to the hermitian form and such that the form is positive on  $E^+$  and negative on  $E^-$ . Such a splitting is identified with a reduction of the structure group  $U(p, q)$  of  $E$  to its maximal compact subgroup  $U(p) \times U(q)$ . It always exists and is unique up to homotopy.

**Theorem 2.1** (Lusztig [3], Atiyah [1]). *If  $E$  is a flat hermitian vector bundle over the closed surface  $\Sigma_g$  then*

$$\text{sign}(\Sigma_g, E) = 2(c_1(L^+) - c_1(L^-)),$$

where  $L^\pm = \det E^\pm$  and  $E = E^+ \oplus E^-$  is a splitting of  $E$ .

## 3. Proof of the main theorem

In this section we shall prove the main theorem.

Let  $V \rightarrow \Sigma_g$  be a flat symplectic vector bundle of rank  $2p$  over  $\Sigma_g$  and  $\omega$  its symplectic form.

Put  $E = V \otimes \mathbb{C}$  and extend the symplectic form  $\omega$  on  $V$  to a complex symplectic form on  $E$  which is also denoted by  $\omega$ . The hermitian form  $h$  on  $E$  is defined by

$$h_x(u, v) = -\sqrt{-1}\omega_x(u, \bar{v})$$

for any  $x \in \Sigma_g$  and  $u, v \in E_x$ .

Let  $J: V \rightarrow V$  be a positive compatible complex structure with the symplectic form  $\omega$ , i.e.  $J: V \rightarrow V$  is a bundle homomorphism which satisfies  $J^2 = -1$  and such that  $\omega(\cdot, J\cdot)$  is a Euclidean metric of  $V$ . Such a complex structure always exists and is unique up to homotopy. The complex vector bundle  $(V, J)$  of rank  $p$  over  $\mathbb{C}$  is denoted by  $V_J$ . Extend  $J$  to a complex linear map on  $E$ , then by  $J^2 = -1$ , we have the decomposition  $E = E' \oplus E''$  of  $E$  by the eigenspaces of  $J$ , where  $J|_{E'} = \sqrt{-1}id_{E'}$  and  $J|_{E''} = -\sqrt{-1}id_{E''}$ . It is well known that the complex isomorphisms  $E' \cong V_J$  and  $E'' \cong \bar{E}' \cong \bar{V}_J$  hold, where  $\bar{\cdot}$  denotes the complex conjugation. Thus we have  $E = V_J \oplus \bar{V}_J$ . Moreover it is easy to see that this decomposition gives a splitting of  $E$  where  $E^+ \cong V_J$  and  $E^- \cong \bar{V}_J$  (see [6]).

By Theorem 2.1, we have

$$\text{sign}(\Sigma_g, E) = 2(c_1(V_J) - c_1(\bar{V}_J)) = 4c_1(V_J).$$

Note that, for any complex vector bundle  $F$ , the equalities  $c_1(F) = c_1(\det F)$  and  $c_1(\bar{F}) = -c_1(F)$  hold.

On the other hand, since  $E$  is a flat vector bundle, we have the following de Rham complex with local coefficients in  $E$ :

$$d_E^\bullet: 0 \rightarrow \Omega^0(\Sigma_g; E) \xrightarrow{d_E} \Omega^1(\Sigma_g; E) \xrightarrow{d_E} \Omega^2(\Sigma_g; E) \rightarrow 0.$$

By the Atiyah-Singer index theorem ([2]), we obtain

$$\text{ind}(d_E^\bullet) = \text{rank}_{\mathbb{C}} E \cdot \chi(\Sigma_g) = 4p(1 - g).$$

We shall prove the inequality in the main theorem in the following two cases.

In the first case we suppose that  $H^0(\Sigma_g; E) = 0$ . We then have  $H^2(\Sigma_g; E) = 0$  by the duality after identifications  $E \cong \bar{E} \cong E^*$ . Thus we have  $\dim_{\mathbb{C}} H^1(\Sigma_g; E) = 4p(g - 1)$ , and hence obtain

$$|\langle c_1(V_J), [\Sigma_g] \rangle| = \frac{1}{4} |\text{sign}(\Sigma_g, E)| \leq p(g - 1).$$

In the second case we suppose that  $H^0(\Sigma_g; E) \neq 0$  and hence  $H^0(\Sigma_g; V)$  is also nontrivial. Thus there exists a nonzero element  $s$  of  $H^0(\Sigma_g; V)$  which does not vanish at any point of  $\Sigma_g$ .  $W := \mathbb{R}s$  defines an  $\mathbb{R}$ -line subbundle of  $V$ . Let  $W^{\perp\omega} \subset V$  be the subbundle of  $V$  orthogonal to  $W$  with respect to the symplectic form  $\omega$ , then  $W$  is also a subbundle of  $W^{\perp\omega}$ . The quotient bundle  $W^{\perp\omega}/W$  has the symplectic form  $\underline{\omega}$  induced from  $\omega$ . From the construction, it is clear that this bundle is a flat symplectic vector bundle of rank  $2(p - 1)$ .

Put  $W_J = \mathbb{R}s \oplus J\mathbb{R}s$ , then it is a trivial complex line subbundle of  $V_J$ . Let  $W_J^{\perp h'}$  be the subbundle of  $V_J$  orthogonal to  $W_J$  with respect to the hermitian form  $h'(\cdot, \cdot) := \omega(\cdot, J\cdot) - \sqrt{-1}\omega(\cdot, \cdot)$ , which is also a symplectic vector bundle with a positive compatible complex structure. It is easy to see that the inclusion  $W_J^{\perp h'} \hookrightarrow W^{\perp\omega}$

induces the isomorphism

$$(W_J^{\perp h'}, \omega|_{W_J^{\perp h'}}) \cong (W^{\perp \omega}/W, \underline{\omega})$$

as smooth symplectic bundles. So a positive compatible complex structure with  $\underline{\omega}$  on  $W^{\perp \omega}/W$  is induced from the one on  $W_J^{\perp h'}$  via the isomorphism. Thus we have

$$c_1(V_J) = c_1(W_J^{\perp h'} \oplus W_J) = c_1(W_J^{\perp h'}) = c_1(W^{\perp \omega}/W).$$

In particular if  $p = 1$ , then we obtain  $c_1(V_J) = 0$ . So we complete the proof of the inequality in Theorem 1.2 for  $p = 1$ .

For  $p > 1$ , we may suppose that  $c_1(V) \neq 0$ . If necessary, repeating the above argument, we can construct a flat symplectic vector bundle  $Z$  over  $\Sigma_g$  of rank  $q < p$  such that  $H^0(\Sigma_g; Z \otimes \mathbb{C}) = 0$  and  $c_1(V) = c_1(Z)$ . From the first case, we have

$$|\langle c_1(V), [\Sigma_g] \rangle| = |\langle c_1(Z), [\Sigma_g] \rangle| \leq q(g-1) \leq p(g-1).$$

This completes the proof of the inequality.

Conversely for any integer  $c_1$  such that  $|c_1| \leq p(g-1)$ , since  $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$ , the existence of a flat symplectic vector bundle  $V$  over  $\Sigma_g$  of rank  $2p$  with  $\langle c_1(V), [\Sigma_g] \rangle = c_1$  is true for  $p = 1$  by Theorem 1.1.

For  $p > 1$ , suppose that  $c_1 = p(g-1)$  since the other cases can easily be obtained by the same method. Let  $V_1$  be a flat symplectic vector bundle over  $\Sigma_g$  of rank 2 with  $c_1(V_1) = g-1$ . The direct sum  $V = \bigoplus^p V_1$  of  $p$  copies of  $V_1$  is a flat symplectic vector bundle of rank  $2p$  with

$$\langle c_1(V), [\Sigma_g] \rangle = p \langle c_1(V_1), [\Sigma_g] \rangle = p(g-1).$$

This vector bundle  $V$  is a required one.

This finishes the proof of the main theorem.

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