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## A VARIATION ON THE GLAUBERMAN CORRESPONDENCE

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### 1. Introduction

Suppose that  $G$  is a finite  $p$ -solvable group, where  $p$  is a prime. Let  $\text{IBr}(G)$  be the set of irreducible Brauer characters of  $G$ , and let  $\text{IBr}_{p'}(G)$  be those  $\varphi \in \text{IBr}(G)$  of degree not divisible by  $p$ .

The Glauberman correspondence, in the important case where a  $p$ -group acts on a  $p'$ -group, can be viewed as a natural correspondence between  $\text{IBr}_{p'}(G)$  and  $\text{IBr}(\mathbf{N}_G(P))$ , where  $P \in \text{Syl}_p(G)$  and  $G$  is a group with a normal  $p$ -complement. Our point in this note is to show that it is not necessary to assume that  $G$  has a normal  $p$ -complement: it suffices to assume that  $\mathbf{N}_G(P)$  does.

**Theorem A.** *Suppose that  $G$  is  $p$ -solvable, and let  $P \in \text{Syl}_p(G)$ . Assume that  $\mathbf{N}_G(P)$  has a normal  $p$ -complement. Then for every  $\varphi \in \text{IBr}_{p'}(G)$ , there is a unique  $\varphi^* \in \text{IBr}(\mathbf{N}_G(P))$  such that*

$$\varphi_{\mathbf{N}_G(P)} = e\varphi^* + p\Delta,$$

where  $e$  is not divisible by  $p$  and  $\Delta$  is some Brauer character of  $\mathbf{N}_G(P)$  or zero. Also, the map  $\text{IBr}_{p'}(G) \rightarrow \text{IBr}(\mathbf{N}_G(P))$  given by  $\varphi \mapsto \varphi^*$  is a bijection. On the other hand, if  $\tau \in \text{IBr}(G)$  has degree divisible by  $p$ , then

$$\tau_{\mathbf{N}_G(P)} = p\Xi,$$

where  $\Xi$  is some Brauer character of  $\mathbf{N}_G(P)$ .

Even in the case where  $\mathbf{N}_G(P) = P$ , Theorem A above tells us something non-trivial (although well-known): a Sylow  $p$ -subgroup  $P$  of a  $p$ -solvable group  $G$  is self-normalizing, if and only if all nontrivial irreducible Brauer characters of  $G$  have degree divisible by  $p$ .

The condition of  $\mathbf{N}_G(P)$  having a normal  $p$ -complement is natural enough that can be read off from the character table of  $G$  (whenever  $G$  is  $p$ -solvable).

**Theorem B.** *Suppose that  $G$  is  $p$ -solvable and let  $P \in \text{Syl}_p(G)$ . Then  $\mathbf{N}_G(P)$  has a normal  $p$ -complement iff the number of  $p$ -regular classes of  $G$  of size not divis-*

ible by  $p$  is the number of irreducible Brauer characters of  $G$  of degree not divisible by  $p$ .

Theorem B is already false for  $G = A_5$  and  $p = 2$ . In this case,  $G$  has only one irreducible Brauer character of odd degree and only one 2-regular class of odd size. However the Sylow 2-normalizer of  $G$  does not have a normal 2-complement.

## 2. Proofs

We begin with a lemma.

**Lemma 2.1.** *Suppose that  $G$  is a  $p$ -solvable and let  $P$  be a Sylow  $p$ -subgroup of  $G$ . Suppose that  $N \triangleleft G$  and that  $\theta \in \text{IBr}(N)$  is  $P$ -invariant of  $p'$ -degree. Then there exists  $\varphi \in \text{IBr}(G)$  of  $p'$ -degree lying over  $\theta$ .*

*Proof.* We argue by induction on  $|G : N|$ . If  $N = G$ , we let  $\varphi = \theta$  and the proof of the lemma follows. Now, let  $M/N$  be a chief factor of  $G$ . If  $M/N$  is a  $p$ -group, then  $\theta$  is  $M$ -invariant since  $M \subseteq NP$ . By Green's Theorem (8.11) of [3], there exists a unique  $\eta \in \text{IBr}(M)$  lying over  $\theta$ . Furthermore,  $\eta$  extends  $\theta$ . In particular,  $\eta$  has  $p'$ -degree and by uniqueness is  $P$ -invariant. Now,  $|G : M| < |G : N|$  and by induction there is some  $\varphi \in \text{IBr}(G)$  of  $p'$ -degree lying over  $\eta$ . Then  $\varphi$  lies over  $\theta$  and the proof of the lemma is complete. Suppose now that  $M/N$  is a  $p'$ -group. In this case, all irreducible constituents of  $\theta^M$  have  $p'$ -degree by Theorem (8.30) of [3]. Now,  $P$  acts on the irreducible constituents of the Brauer character  $\theta^M$ . Since this character has  $p'$ -degree, necessarily it follows that  $P$  fixes some irreducible constituent  $\xi \in \text{IBr}(M)$  of  $\theta^M$ . Now,  $\xi$  lies over  $\theta$  (by Corollary (8.7) of [3]) and the proof of the lemma follows by induction (as in the previous case).  $\square$

*Proof of Theorem A.* Let  $N = \mathbf{O}_{p'}(G)$  and let  $C = \mathbf{C}_N(P)$ . If  $N = G$ , then there is nothing to prove. We claim that  $\mathbf{N}_G(P) = P \times C$ . Write  $M = \mathbf{N}_G(P)$ . By hypothesis, we know that  $M = P \times K$ . Hence,  $K = \mathbf{O}_{p'}(M) \subseteq N$ , by a well-known group theoretical fact. Hence, the claim easily follows.

Let  $\varphi \in \text{IBr}_{p'}(G)$ . We claim that  $\varphi_N$  has a unique irreducible  $P$ -invariant constituent  $\theta \in \text{Irr}(N)$ . Let  $\nu \in \text{Irr}(N)$  be an irreducible constituent of  $\varphi_N$ . Since  $\varphi$  has  $p'$ -degree it follows that the inertia group of  $\nu$  in  $G$  has  $p'$ -index (by the Clifford correspondence, Theorem (8.9) of [3]). Hence, some conjugate  $\theta$  of  $\nu$  has stabilizer  $T$  containing  $P$ . Therefore  $\theta$  is  $P$ -invariant. Suppose that  $\mu \in \text{Irr}(N)$  is some other  $P$ -invariant irreducible constituent of  $\varphi_N$ . Then  $\mu = \theta^g$ , by Clifford's theorem. Now, we have that  $P$  and  $P^{g^{-1}}$  are inside  $T$ . Therefore,  $P^{t^g} = P$  for some  $t \in T$ , and we deduce that  $\mu$  and  $\theta$  are  $M$ -conjugate. However  $M = CP$ , and therefore  $\mu = \theta$ , as claimed.

Now, let  $\theta \in \text{Irr}(N)$  be  $P$ -invariant. We claim that there is a unique  $\varphi \in \text{IBr}_{p'}(G)$

over  $\theta$ . By Lemma (2.1), we see that there is some  $\varphi \in \text{IBr}(G)$  of  $p'$ -degree lying over  $\theta$ . We prove that  $\varphi$  is unique by induction on  $|G|$ . By hypothesis, we have that  $MN/N \subseteq PN/N \subseteq \mathbf{O}^{p'}(G/N)$ . Hence, by the Frattini argument, we have that  $\mathbf{O}^{p'}(G/N) = G/N$ . So let  $K/N = \mathbf{O}^p(G/N) < G/N$ , and let  $L/N = \mathbf{O}^{p'}(K/N)$ . Write  $U = LP$ . Hence,  $G = KU$  and  $K \cap U = L$ . Since  $M = CP \subseteq NP$ , we have that  $M \subseteq U$ . In particular,  $\mathbf{C}_{K/L}(P) = 1$ . Now,  $N = \mathbf{O}_{p'}(L)$  since  $L \triangleleft G$ . Since  $U/L$  is a  $p$ -group, it follows that  $N = \mathbf{O}_{p'}(U)$ . If  $U = G$ , then  $K = L = N$  and we have that  $G = NP$ . In this case,  $\varphi$  is unique by Green's Theorem (8.11) of [3]. Hence, we may assume that  $U$  is proper in  $G$ . By induction, there is a unique  $\eta \in \text{IBr}_{p'}(U)$  lying over  $\theta$ . Suppose now that  $\delta \in \text{IBr}_{p'}(G)$  also lies over  $\theta$  and has  $p'$ -degree. Now,  $\delta_U$  has a  $p'$ -degree irreducible constituent  $\xi$ . Also,  $\xi_N$  has a  $P$ -invariant constituent (by the second paragraph, for instance). Since by the second paragraph,  $\delta_N$  has a unique  $P$ -invariant irreducible constituent, we deduce that  $\xi_N$  contains  $\theta$ . By induction, we have that  $\eta = \xi$ . By the same reason,  $\varphi_U$  contains  $\eta$ . Now, by using repeatedly Corollary (8.22) of [3], we have that  $\delta_K$  and  $\varphi_K$  are  $P$ -invariant irreducible Brauer characters of  $K$  lying over  $\eta_L$ . Now, let  $\delta_1, \varphi_1 \in B_{p'}(K)$  and  $\eta_1 \in B_{p'}(L)$  be the canonical Isaacs liftings of  $\delta_K, \varphi_K$  and  $\eta_L$ , respectively (see Corollary (10.3) of [1]). By uniqueness, we have that these three characters are  $P$ -invariant. Also, by Corollary (7.5) and Corollary (10.3) of [1], it easily follows that  $\delta_1$  and  $\varphi_1$  lie over  $\eta_1$ . By Problem (13.10) of [2], we have that  $\delta_1 = \varphi_1$ . Hence  $\varphi_K = \delta_K$ . By Theorem (8.11) of [3], we have that  $\varphi = \delta$ , and the claim is proven.

Now, given  $\varphi \in \text{IBr}_{p'}(G)$ , we have that  $\varphi_N$  has a unique  $P$ -invariant irreducible constituent  $\theta \in \text{Irr}(N)$ , and that  $\theta$  and  $\varphi$  uniquely determine one each other. In particular, we have proven that

$$|\text{IBr}_{p'}(G)| = |\text{Irr}_P(N)|,$$

where, as usual,  $\text{Irr}_P(N)$  denotes the irreducible  $P$ -invariant characters of  $N$ . Let  $\Omega$  be the set of  $G$ -conjugates of  $\theta$ . Hence,  $P$  acts on  $\Omega$  fixing only  $\theta$ , and we may write

$$\varphi_N = d \left( \theta + \sum_{\mathcal{O}} \left( \sum_{\eta \in \mathcal{O}} \eta \right) \right),$$

where  $\mathcal{O}$  runs over the different  $P$ -orbits not equal  $\{\theta\}$ . Also, since  $\varphi(1)$  is not divisible by  $p$ , we have that  $d$  is not divisible by  $p$ . Now, since  $C = C_N(P)$ , notice that  $\eta_C = (\eta^x)_C$  for  $x \in P$  and  $\eta \in \text{Irr}(N)$ . Therefore we may write  $\varphi_C = d\theta_C + p\Psi$ , where  $\Psi$  is some character of  $C$  or zero. Now, by Theorem (13.14) of [2], we have that  $\theta_C = e\theta^* + p\Delta$ , where  $\theta^* \in \text{Irr}(C)$  is the Glauberman correspondent of  $\theta$ ,  $p$  does not divide  $e$  and  $\Delta$  is a character of  $C$  or zero. Since  $\mathbf{N}_G(P) = P \times C$ , and the irreducible Brauer characters of  $\mathbf{N}_G(P)$  are naturally identifiable with the irreducible characters of  $C$ , the first part of the theorem easily follows. Now, since

$$|\text{IBr}_{p'}(G)| = |\text{Irr}_P(N)| = |\text{Irr}(C)| = |\text{Irr}(\mathbf{N}_G(P)/P)| = |\text{IBr}(\mathbf{N}_G(P))|,$$

(where the equality  $|\text{Irr}_P(N)| = |\text{Irr}(C)|$  follows from the Glauberman correspondence) to prove that the map  $\varphi \mapsto \varphi^*$  is bijective, it suffices to show that  $*$  is one to one. Assume that  $\varphi^* = \delta^*$ , where  $\varphi, \delta \in \text{IBr}_P(G)$ . By how our map is constructed and using that the Glauberman correspondence is one to one, we easily deduce that  $\varphi$  and  $\delta$  lie over the same  $P$ -invariant irreducible character of  $N$ . Hence, by the third paragraph of this proof, we have that  $\varphi = \delta$ , as required.

Suppose now that  $\tau \in \text{IBr}(G)$  has degree divisible by  $p$ . We distinguish two cases. Suppose first that  $\tau_N$  contains a  $P$ -invariant irreducible constituent  $\theta \in \text{Irr}(N)$ . Let  $T$  be the inertia group of  $\theta$  in  $G$ , and let  $\mu \in \text{IBr}(T \mid \theta)$  be the Clifford correspondent of  $\tau$  over  $\theta$  (Theorem (8.9) of [3]). Since  $|G : T|$  is not divisible by  $p$ , we conclude that  $p$  divides  $\mu(1)$  since  $\tau(1) = |G : T|\mu(1)$ . Now, since  $\mu_N = d\theta$  and  $p$  does not divide  $\theta(1)$ , we conclude that  $p$  divides  $d$ . Since  $d$  is the multiplicity of  $\theta$  in  $\tau_N$  (again, by Theorem (8.9) of [3]), by Clifford's theorem, we deduce that  $\tau_C = p\Xi$ , for some ordinary character  $\Xi$  of  $C$ . In this case, the last part of the theorem follows. Finally, suppose that  $\tau_N$  does not contain any  $P$ -invariant irreducible constituent. In this case, we may write

$$\varphi_N = d \left( \sum_{\mathcal{O}} \left( \sum_{\eta \in \mathcal{O}} \eta \right) \right),$$

where  $\mathcal{O}$  runs over the different  $P$ -orbits on the action of  $P$  on the irreducible constituents of  $\tau_N$ . Since elements in the same  $P$ -orbit have the same restriction to  $P$ , the proof of the theorem is completed. □

To prove Theorem B, we use the following notation. We denote by  $\text{cl}(G)$  the set of conjugacy classes of  $G$ . Also,  $\text{cl}(G^0)$  is the set of conjugacy classes of  $p$ -regular elements of  $G$ , and  $\text{cl}(G^0 \mid P)$  is the set of  $p$ -regular classes of  $G$  with defect group  $P$ .

Proof of Theorem B. First, we prove that in a group  $G$  with a normal Sylow  $p$ -subgroup  $P$ , we have that  $G$  has a normal  $p$ -complement iff

$$|\text{cl}(G/P)| = |\text{cl}(G^0 \mid P)|.$$

Let  $K$  be a  $p$ -complement of  $G$ . If  $K \triangleleft G$ , then  $G = P \times K$ , and  $|\text{cl}(G/P)| = |\text{cl}(K)|$ . Also, if  $x \in G$  is  $p$ -regular, then  $x \in K$  and  $P \subseteq C_G(x)$ . So

$$|\text{cl}(G^0 \mid P)| = |\text{cl}(G^0)| = |\text{cl}(K)|,$$

and one direction is proven. Conversely, assume now that

$$|\text{cl}(G/P)| = |\text{cl}(G^0 \mid P)|.$$

Hence, we have that

$$|\text{cl}(K)| = |\text{cl}(G^0 \mid P)| \leq |\text{cl}(G^0)| \leq |\text{cl}(K)|,$$

and we conclude that all  $p$ -regular classes of  $G$  have defect group  $P$ . Hence, we have that  $K \subseteq \mathbf{C}_G(P)$ , and the claim is proven.

Since  $G$  is  $p$ -solvable, it is well known that

$$|\mathrm{IBr}_{p'}(G)| = |\mathrm{Irr}(\mathbf{N}_G(P)/P)|.$$

(This follows, for instance, from Corollary (1.16) of [4], Lemma (5.4) and Corollary (10.3) of [1]). Now, by Lemma (4.16) of [3], it follows that

$$|\mathrm{cl}(G^0 \mid P)| = |\mathrm{cl}(\mathbf{N}_G(P^0) \mid P)|.$$

Hence

$$|\mathrm{IBr}_{p'}(G)| = |\mathrm{cl}(G^0 \mid P)|$$

iff

$$|\mathrm{cl}(\mathbf{N}_G(P)/P)| = |\mathrm{cl}(\mathbf{N}_G(P^0) \mid P)|$$

which happens iff  $\mathbf{N}_G(P)$  has a normal  $p$ -complement, by the first paragraph.  $\square$

Of course, the numbers  $|\mathrm{IBr}_{p'}(G)|$  and  $|\mathrm{cl}(G^0 \mid P)|$  can be read off from the character table of  $G$ , whenever  $G$  is  $p$ -solvable. Higman's theorem (8.21) of [2], allows us to distinguish if an element  $x \in G$  is  $p$ -regular. In this case, the class of  $x$  has defect group a Sylow  $p$ -subgroup of  $G$  iff  $|\mathbf{C}_G(x)|$  is divisible by  $|G|_p$ . On the other hand, Corollary (10.4) of [3], allows to construct the Brauer character table of  $G$  from its ordinary one, and we can easily count how many irreducible Brauer characters of  $G$  have degree not divisible by  $p$ .

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